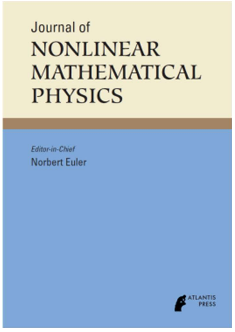

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Finite genus solutions for Geng hierarchy

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The Geng hierarchy is derived with the aid of Lenard recursion sequences. Based on the Lax matrix, a hyperelliptic curve $\mathscr{K}_{n+1}$ of arithmetic genus $n+1$ is introduced, from which meromorphic function ϕ is defined. The finite genus solutions for Geng hierarchy are achieved according to asymptotic properties of ϕ and the algebro-geometric characters of $\mathscr{K}_{n+1}$.

Keywords: Hyperelliptic curve; meromorphic function; finite genus solutions.

2000 Mathematics Subject Classification: 35Q51, 35C08, 14H55

1. Introduction

The soliton equations describe various nonlinear phenomena in natural and applied sciences such as fluid dynamics, plasma physics, optical fibers and other sciences. It is of great importance to solve nonlinear soliton equations from both theoretical and practical points of view. Due to the nonlinearity of soliton equations, it is a difficult job for us to determine whatever exact solutions to soliton equations, but with the development of soliton theory several systematic methods has been developed to obtain explicit solutions of soliton equations, such as the inverse scattering transformation [1], the Hirota bilinear transformation [2], the B*ä*cklund and the Darboux transformation [3,4], the algebro-geometric method [5], the nonlinearization approach of Lax pairs [6], the homogeneous balance method [7], etc [8-12].

The nonlinear diffusion equation [13-15]

$$u_t = \left(\frac{u_x}{u^2}\right)_x, \tag{1.1}$$

which has a lot of applications in plasma physics, laser physics, semiconductor physics, astrophysics, population dynamics in biology and helium combustion process in chemistry, can be obtained in the following equations in case of $v = 1$ [16]

$$\begin{cases} u_t = \left(\dfrac{1}{u^2}\right)_x - \left(\dfrac{v^2}{u^2}\right)_x - \left(\dfrac{v}{u}\right)_{xx}, \\ v_t = \dfrac{1}{2}\left(\dfrac{1}{u^2}\right)_{xx} - \dfrac{1}{2}\left(\dfrac{v^2}{u^2}\right)_{xx}. \end{cases} \tag{1.2}$$

In this paper we will concentrate primarily on constructing the finite genus solutions of the entire Geng hierarchy related to (1.2) based on the approaches in Refs. [17-32]. The finite genus solutions are associated to nonłlinear flows in the Jacobian of a hyperelliptic curve. This phenomenon is connected to the existence of integrable hierarchies with nonłlinear dependence on the spectral parameter. Such problem was first considered in Refs. [24] and [25]. The algebraic geometric approach was proposed in [26]. Key examples are the Camassa-Holm [27] and Harry dym equations [28] whose algebraic geometric solutions produce nonłlinear flows in the generalized Jacobian of hyperelliptic curves in Refs. [29] and [30]. After separation of variables, the appearance of nonłlinear flows in the (generalized) Jacobians of algebraic curves, also appears in ODEs and was first considered in Refs [31] and [32].

The outline of this paper is as follows. In Section 2, we obtain the coupled diffusion equation hierarchy based on Lenard recursion sequences. In Section 3, with the aid of Lax matrix we shall introduce hyperelliptic curve $\mathscr{K}_{n+1}$ of arithmetic genus $n+1$. In Section 4, we define the meromorphic function ϕ and investigate the asymptotic properties of ϕ. Moreover, we construct the finite genus solutions of the whole hierarchy by use of the Riemann theta functions according to the asymptotic properties of ϕ and the algebro-geometric characters of $\mathscr{K}_{n+1}$.

2. Hierarchy of nonlinear evolution equations

In this section, we shall derive the Geng hierarchy associated with the 2×2 spectral problem [16]

$$\varphi_x = U\varphi, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad U = \begin{pmatrix} \lambda u & v-1 \\ \lambda(v+1) & -\lambda u \end{pmatrix}, \tag{2.1}$$

where u and v are two potentials, and λ is a constant spectral parameter. To this end, we first introduce the Lenard recursion sequences

$$KL_j = JL_{j+1}, \quad L_j = (a_j, b_j)^T, \quad j \geq 0 \tag{2.2}$$

with three starting points

$$L_0 = \left(\frac{1-v^2}{2u^2}, \frac{v}{u}\right)^T, \quad \bar{L}_0 = (1,0)^T, \quad \tilde{L}_0 = (0,1)^T, \tag{2.3}$$

where K and J are two operators defined by

$$K = \begin{pmatrix} 2\partial & -\partial^2 \\ \partial^2 & 0 \end{pmatrix}, \quad J = 2\begin{pmatrix} \partial u\partial^{-1}u\partial & \partial u\partial^{-1}v\partial \\ \partial v\partial^{-1}u\partial & \partial v\partial^{-1}v\partial - \partial \end{pmatrix}, \quad \partial\partial^{-1} = \partial^{-1}\partial = 1. \tag{2.4}$$

It is easy to see that $\mathrm{Ker}J = \{\alpha_0 L_0 + \bar{\alpha}_0 \bar{L}_0 + \tilde{\alpha}_0 \tilde{L}_0 \mid \forall \alpha_0, \bar{\alpha}_0, \tilde{\alpha}_0 \in \mathbb{R}\}$. Hence L_j are uniquely determined by the recursion relation (2.2) up to a term const.L_0+const.$\bar{L}_0$+const.$\tilde{L}_0$, which is always assumed to be zero. Assume that the eigenfunction φ satisfies an auxiliary problem

$$\varphi_{t_m} = V^{(m)}\varphi, \quad V^{(m)} = \begin{pmatrix} V_{11}^{(m)} & V_{12}^{(m)} \\ V_{21}^{(m)} & -V_{11}^{(m)} \end{pmatrix}, \tag{2.5}$$

where $V_{11}^{(m)}, V_{12}^{(m)}, V_{21}^{(m)}$ are defined as follows;

$$\begin{aligned} V_{11}^{(m)} &= \sum_{j=0}^{m} (-\partial b_j - 2u\partial^{-1}u\partial a_j\lambda - 2u\partial^{-1}v\partial b_j\lambda)\lambda^{m+1-j}, \\ V_{12}^{(m)} &= \sum_{j=0}^{m} (\partial a_j + 2\partial^{-1}u\partial a_j\lambda - 2v\partial^{-1}u\partial a_j\lambda + 2\partial^{-1}v\partial b_j\lambda - 2v\partial^{-1}v\partial b_j\lambda)\lambda^{m-j}, \\ V_{21}^{(m)} &= \sum_{j=0}^{m} (\partial a_j - 2\partial^{-1}u\partial a_j\lambda - 2v\partial^{-1}u\partial a_j\lambda - 2\partial^{-1}v\partial b_j\lambda - 2v\partial^{-1}v\partial b_j\lambda)\lambda^{m+1-j}. \end{aligned} \tag{2.6}$$

Then the compatibility condition of (2.1) and (2.5) yields the zero curvature equation, $U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0$, which is equivalent to a class of nonlinear evolution equations

$$u_{t_m} = 2a_{m,x} - b_{m,xx}, \quad v_{t_m} = a_{m,xx}. \tag{2.7}$$

The first two nontrivial flows in (2.7) are (1.2) and

$$\begin{aligned} u_t &= \frac{1}{u}\left(\frac{1-v^2+vu_x-uv_x}{u^3}\right)_x - \frac{v}{u}\left(\frac{2u_x-v^2u_x+uvv_x+v-v^3}{u^3}\right)_x \\ &\quad - \frac{1}{2}\left(\frac{2u_x-v^2u_x+uvv_x+v-v^3}{u^3}\right)_{xx}, \\ v_t &= \frac{1}{2}\left(\frac{1}{u}\left(\frac{1-v^2+vu_x-uv_x}{u^3}\right)_x - \frac{v}{u}\left(\frac{2u_x-v^2u_x+uvv_x+v-v^3}{u^3}\right)_x\right)_x. \end{aligned} \tag{2.8}$$

3. Hyperelliptic curve

Let $\chi = (\chi_1, \chi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ be two basic solutions of (2.1) and (2.5). We introduce a Lax matrix

$$W = \frac{1}{2}(\chi\psi^T + \psi\chi^T)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} G & F \\ H & -G \end{pmatrix} \tag{3.1}$$

which satisfies the Lax equations

$$W_x = [U, W], \quad W_{t_m} = [V^{(m)}, W]. \tag{3.2}$$

Therefore, $\det W$ is a constant independent of x and t_m. Equation (3.2) can be written as

$$\begin{aligned} G_x &= (v-1)H - \lambda(v+1)F, \\ F_x &= 2\lambda uF - 2(v-1)G, \\ H_x &= 2\lambda(v+1)G - 2\lambda uH, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} G_{t_m} &= V_{12}^{(m)}H - V_{21}^{(m)}F, \\ F_{t_m} &= 2(V_{11}^{(m)}F - V_{12}^{(m)}G), \\ H_{t_m} &= 2(V_{21}^{(m)}G - V_{11}^{(m)}H). \end{aligned} \tag{3.4}$$

Suppose functions F, G and H are finite-order polynomials in λ

$$G=\sum_{j=0}^{n+1} G_j\lambda^{n+2-j}, \quad F=\sum_{j=0}^{n+1} F_j\lambda^{n+1-j}, \quad H=\sum_{j=0}^{n+1} H_j\lambda^{n+2-j}, \tag{3.5}$$

where

$$\begin{aligned}
&G_0=-2u\partial^{-1}u\partial c_0-2u\partial^{-1}v\partial d_0, \quad F_0=2(1-v)\partial^{-1}u\partial c_0+2(1-v)\partial^{-1}v\partial d_0,\\
&H_0=-2(1+v)\partial^{-1}u\partial c_0-2(1+v)\partial^{-1}v\partial d_0, \quad F_{n+1}=H_{n+1}=c_{n,x},\\
&G_j=-\partial d_{j-1}-2u\partial^{-1}u\partial c_j-2u\partial^{-1}v\partial d_j, \quad 1\le j\le n, \quad G_{n+1}=-d_{n,x},\\
&F_j=\partial c_{j-1}+2(1-v)\partial^{-1}u\partial c_j+2(1-v)\partial^{-1}v\partial d_j, \quad 1\le j\le n,\\
&H_j=\partial c_{j-1}-2(1+v)\partial^{-1}u\partial c_j-2(1+v)\partial^{-1}v\partial d_j, \quad 1\le j\le n.
\end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.3) yields

$$KE_j=JE_{j+1}, \quad JE_0=0, \tag{3.7}$$

$$KE_n=0, \tag{3.8}$$

where $E_j=(c_j,d_j)^T$, $0\le j\le n-1$. It is easy to see that the equation $JE_0=0$ has a special general solution

$$E_0=L_0. \tag{3.9}$$

By induction, we obtain from recursive relations (3.7) and (2.2) that

$$E_k=\sum_{j=0}^{k}\alpha_j L_{k-j}, \quad 0\le k\le n, \tag{3.10}$$

which are special solutions of (3.7), where $\alpha_1,\alpha_2,\dots,\alpha_k$ are constants of integration and $\alpha_0=1$. Moreover, from (3.8) we can get

$$c_n=\beta_0 x+\beta_1, \quad d_n=\beta_0 x^2+\beta_2 x+\beta_3, \tag{3.11}$$

where $\beta_0,\beta_1,\beta_2,\beta_3$ are constants of integration.

Since $\det W$ is a $(2n+4)$th-order polynomial in λ, whose coefficients are constants independent of x and t_m, we have

$$-\det W=G^2+FH=4\lambda\prod_{j=1}^{2n+3}(\lambda-\lambda_j)=4R(\lambda), \tag{3.12}$$

one is naturally led to introduce the hyperelliptic curve $\mathcal{K}_{n+1}$ of arithmetic genus $n+1$ defined by

$$\mathcal{K}_{n+1}: y^2-R(\lambda)=0. \tag{3.13}$$

The curve $\mathcal{K}_{n+1}$ can be compactified by joining two points at infinity, $P_{\infty\pm}$, where $P_{\infty+}\neq P_{\infty-}$. For notational simplicity the compactification of the curve $\mathcal{K}_{n+1}$ is also denoted by $\mathcal{K}_{n+1}$. Here we assume that the zeros λ_j of $R(\lambda)$ in (3.12) are mutually distinct. Then the hyperelliptic curve $\mathcal{K}_{n+1}$ becomes nonsingular and irreducible.

We write F and H as finite products which take the form

$$F = \frac{2(1-v)}{u}\prod_{j=1}^{n+1}(\lambda-\mu_j), \quad H = -\frac{2(1+v)}{u}\lambda\prod_{j=1}^{n+1}(\lambda-\nu_j), \tag{3.14}$$

where $\{\mu_j\}_{j=1}^{n+1}$ and $\{\nu_j\}_{j=1}^{n+1}$ are called elliptic variables. According to the definition of $\mathscr{K}_{n+1}$, we can lift the roots μ_j and ν_j to $\mathscr{K}_{n+1}$ by introducing

$$\hat{\mu}_j(x,t_m) = \left(\mu_j(x,t_m), -\frac{1}{2}G(\mu_j(x,t_m),x,t_m)\right), \quad j=1,\ldots,n+1, \tag{3.15}$$

$$\hat{\nu}_j(x,t_m) = \left(\nu_j(x,t_m), \frac{1}{2}G(\nu_j(x,t_m),x,t_m)\right), \quad j=1,\ldots,n+1, \tag{3.16}$$

where $(x,t_m)\in\mathbb{R}^2$.

From the following lemma, we can explicitly represent α_l $(0\le l\le n)$ by the constants $\lambda_1,\ldots,\lambda_{2n+3}$.

Lemma 3.1.

$$\alpha_l = c_l(\underline{\Lambda}), \qquad l=0,\ldots,n, \tag{3.17}$$

where

$$\begin{aligned} &\underline{\Lambda} = (\lambda_1,\ldots,\lambda_{2n+3}), \quad c_0(\underline{\Lambda})=1, \quad c_1(\underline{\Lambda}) = -\frac{1}{2}\sum_{j=1}^{2n+3}\lambda_j,\ldots, \\ &c_l(\underline{\Lambda}) = -\sum_{\substack{j_1,\ldots,j_{2n+3}=0\\ j_1+\ldots+j_{2n+3}=l}}^{l} \frac{(2j_1)!\ldots(2j_{2n+3})!\lambda_1^{j_1}\ldots\lambda_{2n+3}^{j_{2n+3}}}{2^{2l}(j_1!)^2\ldots(j_{2n+3}!)^2(2j_1-1)\ldots(2j_{2n+3}-1)}. \end{aligned} \tag{3.18}$$

Proof. Assume that

$$\hat{F}_j = F_j|_{\alpha_1=\ldots=\alpha_j=0}, \quad \hat{H}_j = H_j|_{\alpha_1=\ldots=\alpha_j=0}, \quad \hat{G}_j = G_j|_{\alpha_1=\ldots=\alpha_j=0}. \tag{3.19}$$

It will be convenient to introduce the notion of a degree, deg(.), to effectively distinguish between homogeneous and nonhomogeneous quantities. Define

$$\deg(u)=-1, \quad \deg(v)=0, \quad \deg(\partial_x)=1, \tag{3.20}$$

thus from (3.7) it can be implied that

$$\deg(\hat{F}_k)=\deg(\hat{H}_k)=2k+1, \quad \deg(\hat{G}_k)=2k, \quad k\in\mathbb{N}_0. \tag{3.21}$$

Temporarily fixed the branch of $R(\lambda)^{1/2}$ as λ^{n+2} near infinity, $R(\lambda)^{-1/2}$ has the following expansion

$$R(\lambda)^{-1/2} \underset{\lambda\to\infty}{=} \sum_{l=0}^{\infty}\hat{c}_l(\underline{\Lambda})\lambda^{-n-2-l}, \tag{3.22}$$

where

$$\begin{aligned} &\underline{\Lambda} = (\lambda_1,\ldots,\lambda_{2n+3}), \quad \hat{c}_0(\underline{\Lambda})=1, \quad \hat{c}_1(\underline{\Lambda}) = \tfrac{1}{2}\sum_{j=1}^{2n+3}\lambda_j,\ldots, \\ &\hat{c}_l(\underline{\Lambda}) = \sum_{\substack{j_1,\ldots,j_{2n+3}=0\\ j_1+\ldots+j_{2n+3}=l}}^{l} \frac{(2j_1)!\ldots(2j_{2n+3})!\lambda_1^{j_1}\ldots\lambda_{2n+3}^{j_{2n+3}}}{2^{2l}(j_1!)^2\ldots(j_{2n+3}!)^2}. \end{aligned} \tag{3.23}$$

Dividing $F(\lambda)$, $H(\lambda)$, $G(\lambda)$ by $R(\lambda)^{1/2}$ near infinity respectively, we obtain

$$\begin{aligned}
\frac{F(\lambda)}{R(\lambda)^{1/2}} &\underset{\lambda\to\infty}{=} \sum_{l=0}^{\infty} \hat{c}_l(\underline{\Delta})\lambda^{-n-2-l} \sum_{l=0}^{n+1} F_l \lambda^{n+1-l} = \sum_{l=0}^{\infty} \check{F}_l \lambda^{-l-1}, \\
\frac{H(\lambda)}{R(\lambda)^{1/2}} &\underset{\lambda\to\infty}{=} \sum_{l=0}^{\infty} \hat{c}_l(\underline{\Delta})\lambda^{-n-2-l} \sum_{l=0}^{n+1} H_l \lambda^{n+2-l} = \sum_{l=0}^{\infty} \check{H}_l \lambda^{-l}, \\
\frac{G(\lambda)}{R(\lambda)^{1/2}} &\underset{\lambda\to\infty}{=} \sum_{l=0}^{\infty} \hat{c}_l(\underline{\Delta})\lambda^{-n-2-l} \sum_{l=0}^{n+1} G_l \lambda^{n+2-l} = \sum_{l=0}^{\infty} \check{G}_l \lambda^{-l},
\end{aligned} \tag{3.24}$$

for some coefficients $\check{F}_l$, $\check{F}_l$, $\check{G}_l$ to be determined next. From (3.3) and (3.24), we have

$$\begin{aligned}
&\check{G}_{k,x} = (v-1)\check{H}_k - (v+1)\check{F}_k, \\
&\check{F}_{k,x} = 2u\check{F}_{k+1} - 2(v-1)\check{G}_{k+1}, \\
&\check{H}_{k,x} = 2(v+1)\check{G}_{k+1} - 2u\check{H}_{k+1},
\end{aligned} \tag{3.25}$$

for $k \in \mathbb{N}_0$. The initial values of $\check{F}_0$, $\check{H}_0$ and $\check{G}_0$ have been chosen as $\check{F}_0 = \frac{2(1-v)}{u}, \check{H}_0 = -\frac{2(1+v)}{u}, \check{G}_0 = -2$ such that $\check{F}_0 = \hat{F}_0$, $\check{H}_0 = \hat{H}_0$ and $\check{G}_0 = \hat{G}_0$. Moreover, we can prove inductively that

$$\deg(\check{F}_k) = \deg(\check{H}_k) = 2k+1, \quad \deg(\check{G}_k) = 2k, \quad k \in \mathbb{N}_0. \tag{3.26}$$

Hence, $\check{F}_l$, $\check{H}_l$ and $\check{G}_l$ are equal to $\hat{F}_l$, $\hat{H}_l$ and $\hat{G}_l$ respectively for all $l \in \mathbb{N}_0$. Thus we proved

$$\frac{F(\lambda)}{R(\lambda)^{1/2}} = \sum_{l=0}^{\infty} \hat{F}_l \lambda^{-l-1}, \quad \frac{H(\lambda)}{R(\lambda)^{1/2}} = \sum_{l=0}^{\infty} \hat{H}_l \lambda^{-l}, \quad \frac{G(\lambda)}{R(\lambda)^{1/2}} = \sum_{l=0}^{\infty} \hat{G}_l \lambda^{-l}. \tag{3.27}$$

Considering

$$R(\lambda)^{1/2} \underset{\lambda\to\infty}{=} \sum_{l=0}^{\infty} c_l(\underline{\Delta})\lambda^{n+2-l}, \tag{3.28}$$

a comparison of the coefficients of λ^{-k} in the following equation

$$1 = R(\lambda)^{1/2} \times R(\lambda)^{-1/2} = \left(\sum_{l=0}^{\infty} c_l(\underline{\Delta})\lambda^{n+2-l}\right)\left(\sum_{l=0}^{\infty} \hat{c}_l(\underline{\Delta})\lambda^{-n-2-l}\right) \tag{3.29}$$

yields

$$\sum_{l=0}^{k} c_{k-l}(\underline{\Delta})\hat{c}_l(\underline{\Delta}) = \delta_{k,0}, \quad k \in \mathbb{N}_0. \tag{3.30}$$

Therefore, we compute that

$$\sum_{m=0}^{k} c_{k-m}(\underline{\Delta})\hat{F}_m = \sum_{m=0}^{k} c_{k-m}(\underline{\Delta}) \sum_{l=0}^{m} F_l \hat{c}_{m-l}(\underline{\Delta}) = \sum_{l=0}^{k} F_l \sum_{p=0}^{k-l} c_{k-l-p}(\underline{\Delta})\hat{c}_p(\underline{\Delta}) = F_k, \tag{3.31}$$

where $k = 0, \dots, n$. □

4. Finite genus solutions

Equip the $\mathscr{K}_{n+1}$ with canonical basis cycles: $\tilde{a}_1,\ldots,\tilde{a}_{n+1}$; $\tilde{b}_1,\ldots,\tilde{b}_{n+1}$, which are independent and have intersection numbers as follows

$$\tilde{a}_j \circ \tilde{a}_k = 0, \quad \tilde{b}_j \circ \tilde{b}_k = 0, \quad \tilde{a}_j \circ \tilde{b}_k = \delta_{jk}. \tag{4.1}$$

For the present, we will choose our basis as the following set [17]

$$\tilde{\omega}_l = \frac{\lambda^{l-1} d\lambda}{y(P)}, \quad 1 \le l \le n+1, \tag{4.2}$$

which are $n+1$ linearly independent homomorphic differentials on $\mathscr{K}_{n+1}$. Then the period matrices A and B can be constructed from

$$A_{kj} = \int_{\tilde{a}_j} \tilde{\omega}_k, \quad B_{kj} = \int_{\tilde{b}_j} \tilde{\omega}_k. \tag{4.3}$$

It is possible to show that matrices A and B are invertible [33,34]. Now we define the matrices C and τ by $C = A^{-1}$, $\tau = A^{-1}B$. The matrix τ can be shown to be symmetric ($\tau_{kj} = \tau_{jk}$), and it has positive definite imaginary part ($\mathrm{Im}\tau > 0$). If we normalize $\tilde{\omega}_l$ into the new basis ω_j,

$$\omega_j = \sum_{l=1}^{n+1} C_{jl} \tilde{\omega}_l, \quad 1 \le j \le n+1, \tag{4.4}$$

then we obtain

$$\int_{\tilde{a}_k} \omega_j = \sum_{l=1}^{n+1} C_{jl} \int_{\tilde{a}_k} \tilde{\omega}_l = \delta_{jk}, \int_{\tilde{b}_k} \omega_j = \tau_{jk}. \tag{4.5}$$

Let $\mathscr{T}_{n+1}$ be the period lattice $\mathscr{T}_{n+1} = \{\underline{z} \in \mathbb{C}^{n+1} | \underline{n} + \underline{m}\tau;\ \underline{m},\underline{n} \in \mathbb{Z}^{n+1}\}$. The complex torus $\mathscr{T} = \mathbb{C}^{n+1}/\mathscr{T}_{n+1}$ is called the Jacobian variety of $\mathscr{K}_{n+1}$. Now we introduce the Abel map $\underline{\mathscr{A}}(P)$: $\mathrm{Div}(\mathscr{K}_{n+1}) \to \mathscr{T}$

$$\underline{\mathscr{A}}(P) = \left(\int_{Q_0}^{P} \underline{\omega}\right) (\mathrm{mod} \mathscr{T}_{n+1}), \quad \underline{\mathscr{A}}\left(\sum n_k P_k\right) = \sum n_k \underline{\mathscr{A}}(P_k), \tag{4.6}$$

where P, $P_k \in \mathscr{K}_{n+1}$, $\underline{\omega} = (\omega_1, \ldots, \omega_{n+1})$.

Let $\theta(z)$ denote the Riemann theta function associated with $\mathscr{K}_{n+1}$ [33-35]:

$$\theta(\underline{z}) = \sum_{\underline{N} \in \mathbb{Z}^{n+1}} \exp\{2\pi i < \underline{N}, \underline{z} > + \pi i < \underline{N}\tau, \underline{N} >\}, \tag{4.7}$$

where $\underline{z} = (z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1}$, $< \underline{N}, \underline{z} >= \sum_{k=1}^{n+1} N_k z_k$, $< \underline{N}\tau, \underline{N} >= \sum_{k,j=1}^{n+1} \tau_{kj} N_k N_j$. For brevity, define the function $\underline{z} : \mathscr{K}_{n+1} \times \sigma^{n+1} \mathscr{K}_{n+1} \to \mathbb{C}^{n+1}$ by

$$\underline{z}(P, \underline{Q}) = \underline{\Lambda} - \underline{\mathscr{A}}(P) + \sum_{Q' \in \underline{Q}} \mathscr{D}(Q') \underline{\mathscr{A}}(Q'), \tag{4.8}$$

where $P \in \mathcal{K}_{n+1}$, $\underline{Q} = \{Q_1, \cdots, Q_{n+1}\} \in \sigma^{n+1}\mathcal{K}_{n+1}$, $\sigma^{n+1}\mathcal{K}_{n+1}$ denotes the $(n+1)$th symmetric power of $\mathcal{K}_{n+1}$, and $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_{n+1})$ is the vector of Riemann constant defined by

$$\Lambda_j = \frac{1}{2}(1+\tau_{jj}) - \sum_{\substack{k=1\\k\neq j}}^{n+1} \int_{\widetilde{a}_k} \omega_k \int_{Q_0}^{P} \omega_j, \quad j = 1, \dots, n+1. \tag{4.9}$$

Without loss of generality, we choose the branch point $Q_0 = (\lambda_{j_0}, 0)$, $j_0 \in \{1, \dots, 2n+3\}$ as a convenient base point, and $\lambda(Q_0)$ is its local coordinate.

By virtue of (3.12) and (3.13) we can define the meromorphic function $\phi(P,x,t_m)$ on $\mathcal{K}_{n+1}$:

$$\phi(P,x,t_m) = \frac{2y-G}{F} = \frac{H}{2y+G}, \tag{4.10}$$

where $P = (\lambda, y) \in \mathcal{K}_{n+1} \backslash \{P_{\infty\pm}\}$.

Lemma 4.1. *Suppose that $u(x,t_m), v(x,t_m) \in C^\infty(\mathbb{R}^2)$ satisfy the hierarchy (2.7). Let $\lambda_j \in \mathbb{C}\backslash\{0\}$, $1 \le j \le 2n+3$, and $P = (\lambda, y) \in \mathcal{K}_{n+1} \backslash \{P_{\infty+}, P_{\infty+}, P_0\}$, where, $P_0 = (0,0)$. Then*

$$\phi(P,x,t_m) \underset{\zeta\to 0}{=} \begin{cases} \dfrac{1+v}{2u} + O(\zeta), & as\ P \to P_{\infty+},\ \zeta = \lambda^{-1}, \\ \dfrac{2u}{1-v}\zeta^{-1} + O(1), & as\ P \to P_{\infty-},\ \zeta = \lambda^{-1}, \end{cases} \tag{4.11}$$

and

$$\phi(P,x,t_m) \underset{\zeta\to 0}{=} \zeta + O(\zeta^2), \ \ as\ P \to P_0,\ \zeta = \sigma\lambda^{\frac{1}{2}},\ \sigma = \pm 1. \tag{4.12}$$

Proof. From (3.12) and (3.13), we have

$$y \underset{\zeta\to 0}{=} \mp\zeta^{-n-2}\left(1 + \frac{2G_0G_1 + F_0H_0}{8}\zeta + O(\zeta^2)\right), \ \ as\ P \to P_{\infty\pm}. \tag{4.13}$$

From (3.5), we obtain

$$G \underset{\zeta\to 0}{=} \zeta^{-n-2}(G_0 + G_1\zeta + O(\zeta^2)), \ \ as\ P \to P_{\infty\pm}, \tag{4.14}$$

$$F \underset{\zeta\to 0}{=} \zeta^{-n-1}(F_0 + F_1\zeta + O(\zeta^2)), \ \ as\ P \to P_{\infty\pm}. \tag{4.15}$$

Then according to the definition of $\phi(P,x,t_m)$ in (4.10), we have

$$\begin{aligned} \phi(P,x,t_m) &= \frac{2y-G}{F} \\ &\underset{\zeta\to 0}{=} \frac{\mp 2 - G_0 + \left(\mp\frac{2G_0G_1+F_0H_0}{4} - G_1\right)\zeta + O(\zeta^2)}{\zeta(F_0 + F_1\zeta + O(\zeta^2))} \\ &\underset{\zeta\to 0}{=} \begin{cases} \dfrac{1+v}{2u} + O(\zeta), & as\ P \to P_{\infty+}, \\ \dfrac{2u}{1-v}\zeta^{-1} + O(1), & as\ P \to P_{\infty-}. \end{cases} \end{aligned} \tag{4.16}$$

To prove (4.12), we introduce the local coordinate $\zeta = \sigma\lambda^{\frac{1}{2}}$ near P_0. Similarly we have

$$y \underset{\zeta\to 0}{=} \frac{1}{2}F_{n+1}\zeta + O(\zeta^3), \ \ as \ \ P \to P_0, \tag{4.17}$$

$$G \underset{\zeta\to 0}{=} G_{n+1}\zeta^2 + O(\zeta^4), \ \ as \ \ P \to P_0, \tag{4.18}$$

$$F \underset{\zeta\to 0}{=} F_{n+1} + O(\zeta^2), \ \ as \ \ P \to P_0, \tag{4.19}$$

then (4.12) is given by virtue of (4.10) and (4.17)-(4.19). □

The divisor of $\phi(P,x,t_m)$ is given by

$$(\phi(P,x,t_m)) = \mathscr{D}_{P_0,\hat{\nu}_1(x,t_m),\ldots,\hat{\nu}_{n+1}(x,t_m)} - \mathscr{D}_{P_{\infty-},\hat{\mu}_1(x,t_m),\ldots,\hat{\mu}_{n+1}(x,t_m)} \tag{4.20}$$

from the Lemma 4.1 and the definition of $\phi(P,x,t_m)$ in (4.10).

Let $\omega^{(3)}_{P_0,P_{\infty-}}(P)$ denote the normalized Abelian differentials of the third kind holomorphic on $\mathscr{K}_{n+1}\setminus \{P_0,P_{\infty-}\}$ with simple poles at P_0 and $P_{\infty-}$ with residues ± 1, respectively, which can be expressed as

$$\omega^{(3)}_{P_0,P_{\infty-}}(P) = \frac{1}{2\lambda}d\lambda + \frac{1}{2y}\prod_{j=1}^{n+1}(\lambda-\delta_j)d\lambda, \tag{4.21}$$

where $\gamma_j \in \mathbb{C}$, $j = 1,\ldots,n+1$, are constants that are determined by

$$\int_{\tilde{a}_j} \omega^{(3)}_{P_0,P_{\infty-}}(P) = 0, \quad j = 1,\ldots,n+1. \tag{4.22}$$

The explicit formula (4.21) then implies

$$\omega^{(3)}_{P_0,P_{\infty-}}(P) \underset{\zeta\to 0}{=} \begin{cases} (\zeta^{-1}+O(1))d\zeta, \ \ as \ \ P \to P_0, \ \ \zeta = \sigma\lambda^{\frac{1}{2}}, \\ O(1)d\zeta, \ \ as \ \ P \to P_{\infty+}, \ \ \zeta = \lambda^{-1}, \\ (-\zeta^{-1}+O(1))d\zeta, \ \ as \ \ P \to P_{\infty-}, \ \ \zeta = \lambda^{-1}. \end{cases} \tag{4.23}$$

Therefore,

$$\int_{Q_0}^{P} \omega^{(3)}_{P_0,P_{\infty-}}(P) \underset{\zeta\to 0}{=} \begin{cases} \ln\zeta + \ln(\omega^0) + O(\zeta), \ \ as \ \ P \to P_0, \\ \ln(\omega^{\infty+}) + O(\zeta), \ \ as \ \ P \to P_{\infty+}, \\ -\ln\zeta + \ln(\omega^{\infty-}) + O(\zeta), \ \ as \ \ P \to P_{\infty-}, \end{cases} \tag{4.24}$$

for some constants $\omega^{\infty+}$, $\omega^{\infty-}$, $\omega^0 \in \mathbb{C}$.

Theorem 4.1. *Let $P = (\lambda,y) \in \mathscr{K}_{n+1}\setminus\{P_{\infty+},P_{\infty-},P_0\}$, $(x,t_m) \in M$, where $M \subseteq \mathbb{R}^2$ is open and connected. Suppose $u(x,t_m), v(x,t_m) \in C^\infty(M)$ satisfy the hierarchy of equations (2.7), and assume* *that λ_j, $1 \leq j \leq 2n+3$, in (3.12) satisfy $\lambda_j \in \mathbb{C}\backslash\{0\}$, and $\lambda_j \neq \lambda_k$ as $j \neq k$. Moreover, suppose that $\mathscr{D}_{\underline{\hat{\mu}}(x,t_m)}$ or equivalently, $\mathscr{D}_{\underline{\hat{\nu}}(x,t_m)}$, is nonspecial for $(x,t_m) \in M$. Then*

$$\frac{1}{u} = \frac{\omega^{\infty+}}{\omega^0}\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\nu}}(x,t_m)))\theta(\underline{z}(P_0,\underline{\hat{\mu}}(x,t_m)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(x,t_m)))\theta(\underline{z}(P_0,\underline{\hat{\nu}}(x,t_m)))} + \frac{\omega^0}{\omega^{\infty-}}\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(x,t_m)))\theta(\underline{z}(P_0,\underline{\hat{\nu}}(x,t_m)))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\nu}}(x,t_m)))\theta(\underline{z}(P_0,\underline{\hat{\mu}}(x,t_m)))}, \tag{4.25}$$

$$\frac{1-v}{1+v} = \frac{(\omega^0)^2}{\omega^{\infty+}\omega^{\infty-}}\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(x,t_m)))\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(x,t_m)))\theta^2(\underline{z}(P_0,\underline{\hat{\nu}}(x,t_m)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\nu}}(x,t_m)))\theta(\underline{z}(P_{\infty-},\underline{\hat{\nu}}(x,t_m)))\theta^2(\underline{z}(P_0,\underline{\hat{\mu}}(x,t_m)))}. \tag{4.26}$$

Proof. According to Riemann's vanishing theorem [17,33], the definition and asymptotic properties of $\phi(P,x,t_m)$, $\phi(P,x,t_m)$ has expression of the following type

$$\phi(P,x,t_m) = N(x,t_m)\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t_m)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_m)))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_0,P_{\infty-}}(P)\right), \tag{4.27}$$

where $N(x,t_m)$ is independent of $P \in \mathscr{K}_{n+1}$, $\underline{\hat{\mu}}(x,t_m) = \{\hat{\mu}_1(x,t_m),\dots,\hat{\mu}_{n+1}(x,t_m)\}, \underline{\hat{\nu}}(x,t_m) = \{\hat{\nu}_1(x,t_m),\dots,\hat{\nu}_{n+1}(x,t_m)\} \in \sigma^{n+1}\mathscr{K}_{n+1}$. Considering the asymptotic expansions of $\phi(P,x,t_m)$ near $P_{\infty\pm}$ and P_0, we have

$$\frac{1+v}{2u} = N(x,t_m)\omega^{\infty+}\frac{\theta(\underline{z}(P_{\infty+},\underline{\hat{\nu}}(x,t_m)))}{\theta(\underline{z}(P_{\infty+},\underline{\hat{\mu}}(x,t_m)))}, \tag{4.28}$$

$$\frac{2u}{1-v} = N(x,t_m)\omega^{\infty-}\frac{\theta(\underline{z}(P_{\infty-},\underline{\hat{\nu}}(x,t_m)))}{\theta(\underline{z}(P_{\infty-},\underline{\hat{\mu}}(x,t_m)))}, \tag{4.29}$$

$$1 = N(x,t_m)\omega^0\frac{\theta(\underline{z}(P_0,\underline{\hat{\nu}}(x,t_m)))}{\theta(\underline{z}(P_0,\underline{\hat{\mu}}(x,t_m)))}, \tag{4.30}$$

Then combining (4.28)-(4.30) yields (4.25) and (4.26). □

5. Conclusions

In this paper, Finite genus solutions for Geng hierarchy are constructed, which are very important because they reveal inherent structure mechanism of solutions and describe the quasi-periodic behavior of nonlinear phenomenon or characteristic for the integrability of soliton equations. Moreover, they can be used to find multi-soliton solutions, elliptic function solutions, and others. However, we can't straighten the flows of the entire soliton hierarchy under the Abel-Jacobi coordinates, we will study it in the future.

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