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The Mixed Kuper-Camassa-Holm-Hunter-Saxton Equations

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In this paper, a mixed Kuper-CH-HS equation by a Kupershmidt deformation is introduced and its integrable properties are studied. Moreover, that the equation can be viewed as a constraint Hamiltonian flow on the coadjoint orbit of Neveu-Schwarz superalgebra is shown.

Keywords: Lax Pair, mixed Kuper-CH-HS equation, Neveu-Schwarz superalgebra, Hamiltonian structure.

2000 Mathematics Subject Classification: 37K10, 35Q51, 35Q53

1. Introduction

There are many interesting differential equations in mathematics and physics, such as the Camassa-Holm (CH in brief) equation \cite{1} which is the model for the propagation of shallow water waves of moderate amplitude

\[ u_t - u_{txx} = 2u_x u_{xx} + uu_{xxx} - 3uu_x, \]

the Hunter-Saxton (HS in brief) equation \cite{9} which is used as a progressive equation of liquid crystal rotator

\[ -u_{xx} = 2u_x u_{xx} + uu_{xxx}, \]

and the \( \mu \)HS equation \cite{16} which is closely related to the HS equation

\[ -u_{xx} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx}, \]

with

\[ \mu(u) = \int_{S^1} u dx. \]

It is worth mentioning that the above three equations can be expressed as

\[ m_t = -2mu_x - um_x, \quad (1.1) \]

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where
\[ m = u_{xx} - c \mu(u) - 4ku. \]

The CH-HS equation (1.1) is just CH equation, HS equation and \( \mu \)HS equation when the values of \( c \) and \( k \) are given respectively
\[
\begin{align*}
&\begin{cases} 
  c = 0, k = \frac{1}{4}; \\
  c = 0, k = 0; \\
  c = 1, k = 0.
\end{cases}
\end{align*}
\]

The CH-HS equation (1.1) admits a Lax pair, a bi-Hamiltonian structure
\[
m_t = P_1 \frac{\delta H_2}{\delta m} = P_2 \frac{\delta H_1}{\delta m},
\]

where the \( P_1 \) and \( P_2 \)
\[
P_1 = \partial^3 - 4k \partial,
\]
\[
P_2 = m \partial + \partial m,
\]

are two compatible operators of the CH-HS equation (1.1) and
\[
H_1 = -\frac{1}{2} \int m udx,
\]
\[
H_2 = \int \left( -\frac{1}{2} u^3 + \frac{1}{6} uu_x^2 + \frac{1}{3} u^2 u_{xx} \right) dx;
\]

are the first two conserved quantities of the CH equation, and
\[
H_1 = \frac{1}{2} \int m udx,
\]
\[
H_2 = -\int \left( \frac{1}{6} uu_x^2 + \frac{1}{3} u^2 u_{xx} \right) dx;
\]

are the first two conserved quantities of the HS equation, and
\[
H_1 = \frac{1}{2} \int m udx,
\]
\[
H_2 = \int \left( -\frac{1}{6} uu_x^2 - \frac{1}{3} u^2 u_{xx} + \frac{2}{3} u^2 \mu(u) u \right) dx.
\]

are the first two conserved quantities of the \( \mu \)HS equation. And CH-HS equation (1.1) is formally integrable through the inverse scattering method and can be regarded as geodesic equations on the diffeomorphism group of the circle (or of the line) for the right-invariant \( H^1 \) metric, see [1–5,11,12] and references therein.

The Kuper-CH equation [6] and Kuper-\( \mu \)HS equation [23] we firstly proposed and further researched in [23, 24] as Euler equation related to the Neveu-Schwarz superalgebra, especially, when taking \( H^1 \)-metric and \( \mu H^1 \)-metric, two new super-integrable systems—Kuper-CH system and Kuper-\( \mu \)HS system with Lax pair and local super-biHamiltonian structures, which are fermionic versions of the CH equation and \( \mu \)HS equation in (1|1) superspace are given. The Super-HS equation ([17,23]) is supersymmetric extensions of HS equation, super-bi-Hamiltonian. The Kuper-CH
equation, Super-HS equation and the Kuper-\(\mu\)HS equation can also be rewritten as a unified form, which is called Kuper-CH-HS equation here.

For an arbitrary integrable equation

\[ u_t = P_1 \frac{\delta H_n}{\delta u} \]

with two compatible Hamiltonian operators \(P_1\) and \(P_2\), Kupershmidt [15] proposed a nonholonomic deformation as follows

\[
\begin{cases}
  u_t = P_1 \frac{\delta H_n}{\delta u} - P_1 f, \\
  P_2 f = 0.
\end{cases}
\] (1.2)

Zhou [21] proposed the concept of mixed hierarchy of soliton equations based on Lenard scheme and defined the nonholonomic deformation as Kupershmidt deformation. Naturally we want to consider the fermionic cases of the CH-HS equation (1.1) and the mixed Kuper-CH-HS equation and their properties.

In this paper, motivated by the work about sKdV6 [22], we will study the mixed kuper-CH equation and its integrable properties, and study the relation to the corresponding Neveu-Schwarz superalgebra.

2. The Kuper-CH-HS equation and the mixed Kuper-CH-HS equation

The Kuper-CH-HS equation can be rewritten as

\[
\begin{cases}
  m_t = 2k_1 m_{xx} + k_1 m_x u + \frac{1}{2} k_1 \alpha \eta_x + \frac{1}{2} k_1 \alpha \eta_{xx}, \\
  \alpha_t = \frac{1}{2} k_1 u \alpha + k_1 u \alpha_x + \frac{1}{2} k_1 m \eta_x.
\end{cases}
\] (2.1)

which is a fermionic extension of the CH-HS equation (1.1), \(m = u_{xx} - c \mu(u) - 4ku\) is a bosonic function and \(\alpha = \eta_{xx} - k \eta\) is a fermionic function.

\[
\begin{cases}
  c = 0, k = \frac{1}{4}, k_1 = -1, \text{ Kuper-CH equation}; \\
  c = 0, k = 0, k_1 = 1, \text{ Super-HS equation}; \\
  c = 1, k = 0, k_1 = 1, \text{ Kuper-\(\mu\)HS equation}.
\end{cases}
\]

The Kuper-CH-HS equation (2.1) has the spectral problem

\[ \Phi_x = U \Phi, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ k + \frac{1}{2} \lambda m & 0 & \frac{1}{2} \lambda \alpha \\ \frac{1}{2} \lambda \alpha & 0 & 0 \end{pmatrix}. \] (2.2)

The Kuper-CH-HS equation (2.1) has super-bi-Hamiltonian structures, which can be rewritten as

\[
\begin{pmatrix} m \\ \alpha \end{pmatrix}_t = K \begin{pmatrix} \frac{\delta H_1}{\delta m} \\ \frac{\delta H_1}{\delta \alpha} \end{pmatrix} = J \begin{pmatrix} \frac{\delta H_1}{\delta m} \\ \frac{\delta H_1}{\delta \alpha} \end{pmatrix},
\]

the \(K\) and \(J\)

\[ K = \begin{pmatrix} \partial^3 - 4k \partial & 0 \\ 0 & \partial^2 - k \end{pmatrix}, \]

\[ J = \begin{pmatrix} \frac{1}{\lambda} \alpha & 0 \\ 0 & \frac{1}{\lambda} m \end{pmatrix}. \]
are two compatible operators of Kuper-CH-HS equation and

\[ H_1 = \frac{k_1}{2} \int (\mu u + \alpha \eta_x) dx, \]
\[ H_2 = -\frac{k_1}{3} \int \left( -6k u^3 + \frac{1}{2} uu_x^2 + u^2 u_{xx} - 2c\mu(u)u_x^2 - \frac{3k}{2} u\eta_x \right. \]
\[ \left. -\frac{1}{2} u\eta_x \eta_{xx} + \frac{1}{2} u\eta \alpha_x + \frac{3}{2} u_x \eta \alpha + \frac{m}{2} u \eta \phi_x \right) dx. \]

are the first two conserved quantities \( H_1, H_2 \) of the Kuper-CH-HS equation (2.1). Motivated by the Kupershmidt deformation (1.2), we propose a mixed Kuper-CH-HS equation as a nonholonomic deformation of the Kuper-CH-HS equation (2.1).

**Definition 2.1.** The mixed Kuper-CH-HS equation is defined as

\[
J = \left( \frac{m}{\alpha} \right)_t = K \left( \frac{\delta H_2}{\delta \mu} \right) - K \left( \frac{f}{\phi} \right),
\]

which is equivalent to

\[
m_t = 2k_1 \mu u_x + k_1 m_x u + \frac{1}{2} k_1 \alpha_x \eta_x + \frac{3}{2} k_1 \alpha \eta_{xx} - 4k f_x + f_{xxx},
\]
\[
\alpha_t = \frac{3}{2} k_1 u_x \alpha + k_1 u \alpha_x + \frac{1}{2} k_1 m_x \alpha - k \phi + \phi_{xx},
\]
\[
2m f_x + m_x f + \frac{3}{2} \alpha \phi_x + \frac{1}{2} \alpha_x \phi = 0,
\]
\[
\frac{3}{2} \alpha f_x + \alpha_x f + \frac{1}{2} m \phi = 0.
\]

where \( m = u_{xx} - c\mu(u) - 4ku \) and \( f \) are bosonic functions and \( \alpha = \eta_{xx} - k\eta \) and \( \phi \) are fermionic functions.

Corresponding we can get:

\[
c = 0, \quad k = \frac{1}{4}, \quad k_1 = -1, \text{ mixed Kuper-CH equation;}
\]
\[
c = 0, \quad k = 0, \quad k_1 = 1, \text{ mixed Super-HS equation;}
\]
\[
c = 1, \quad k = 0, \quad k_1 = 1, \text{ mixed Kuper-\mu HS equation.}
\]

It’s easy to prove that

\[
\frac{dH_n}{dt} = \nabla H_n K \nabla H_2 - \nabla H_n K \left( \frac{f}{\phi} \right)
\]

\[
= -\nabla H_{n-1} J \left( \frac{f}{\phi} \right)
\]
where functional gradient
\[ \nabla = \left( \frac{\delta}{\delta m}, \frac{\delta}{\delta \alpha} \right)^T. \]

So we have following proposition

**Proposition 2.2.** The mixed Kuper-CH-HS equation (2.5) has infinite many conserved quantities.

### 3. Lax Pair of mixed Kuper-CH Equation

Zero curvature representation and Lax pairs are two kinds of Commutator representations. A systematic approach for constructing zero curvature representation has been well developed by several papers, see [18, 20] and references therein for details. In this section we adopt direct method to construct the Lax pair of the mixed Kuper-CH-HS equation (2.5). From the spectral problem of Kuper-CH-HS equation (2.1), We have got its the Lax pair [24]

\[
\begin{align*}
\Phi_x &= U \Phi \\
\Phi_t &= V_0 \Phi \\
\Phi_t &= V_1 \Phi
\end{align*}
\]

Motivated by the (3.1), we assume that the Lax pair of the mixed Kuper-CH-HS equation has the following form

\[
\begin{align*}
\Phi_x &= U \Phi, \\
\Phi_t &= V \Phi
\end{align*}
\] (3.2)

with

\[ V = V_0 + \lambda V_1 \]

and

\[ V_1 = \begin{pmatrix} a & b & \xi \\ c_1 + c_2 \lambda & -a \beta & 0 \\ \beta & -\xi & 0 \end{pmatrix}, \]

where \( a, b, c_1, c_2 \) are bosonic fields, \( \xi, \beta \) are fermionic fields. From the compatibility condition

\[ U_t - V_x + [U, V] = 0, \]

considering its componentwise elements, we have

\[ a_x = c_1 - kb + \lambda (c_2 - \frac{1}{2}mb - \frac{1}{2} \xi \alpha), \]

\[ a = -\frac{1}{2} b_x, \]
\[ \beta = \frac{1}{2} \lambda b \alpha + \xi, \]
\[ m_t = 2k_1 m_u + k_1 m_u u + \frac{1}{2} k_1 \alpha \eta_u + \frac{3}{2} k_1 \alpha \eta_{xx} + 2c_{1x} - 4ka + 2 \lambda (c_{2x} - ma - \alpha \beta), \]
\[ \alpha_t = \frac{3}{2} k_1 u \alpha + k_1 u \alpha_x + \frac{1}{2} k_1 m \eta_x + 2 \beta_x - 2k \xi - \lambda (m \xi + a \alpha). \] (3.4)

Furthermore, from (3.4) and the first two terms in (2.5), the following two relations are given
\[ -4k f_x + f_{xxx} - 2c_{1x} + 4ka = 2 \lambda (c_{2x} - ma - \alpha \beta), \]
\[ -k \phi + \phi_{xx} - 2 \beta_x + 2k \xi = -\lambda (m \xi + a \alpha). \] (3.5)

By choosing \( b = -f \) and \( \xi = \frac{1}{2} \phi \), we can get
\[ a = \frac{1}{2} f_x, \]
\[ c_1 = \frac{1}{2} f_{xx} - kf, \]
\[ c_2 = \frac{1}{4} \phi \alpha - \frac{1}{2} m f, \]
\[ \beta = -\frac{1}{2} \lambda f \alpha + \frac{1}{2} \phi \alpha. \]

Meanwhile the system (3.5) reduces to
\[ 4mf_x + 2m_x f + 3 \alpha \phi_x + \alpha_x \phi = 0, \]
\[ 3\alpha f_x + 2 \alpha_x f + m \phi = 0. \] (3.6)

which are exactly the last terms of the mixed Kuper-CH-HS equation (2.5). So we have

**Proposition 3.1.** The mixed Kuper-CH-HS equation (2.5) has the Lax pair (3.2), where \( U \) and \( V = V_0 + \lambda V_1 \)

are defined above.

4. Geodesic Flow

The CH equation can be described as the geodesic flow on the Bott-Virasoro group for the right-invariant \( H^1 \)-metric on the group of diffeomorphisms \([3, 11, 13]\). The HS equation can be regarded as geodesic equations on the quotient space \( \text{Diff}(S^1)/S^1 \) of the group \( \text{Diff}(S^1) \) of orientation-preserving diffeomorphisms of the unit circle \( S^1 \) modulo the subgroup of rigid rotations \([12]\). Furthermore, the \( \mu \)HS equation can also be regarded as a member of this frame. Zuo \([22]\) described Euler equations associated to the generalized Neveu-Schwarz algebra and got many super-bi-Hamiltonian structure or supersymmetric equation, such as Kuper-CH equation, Kuper-\( \mu \)HS equation, super-CH equation, super-HS equation and Kuper-2CH equation etc. In this section, we want to describe the relation between Neveu-Schwarz algebra and the mixed Kuper-CH equation (2.5).

The Neveu-Schwarz superalgebra \([19, 23]\) is an algebra
\[ \mathcal{G} = \text{Vect}(S^1) \oplus \mathbb{C}^\omega(S^1) \oplus \mathbb{R} \]

with the bilinear operation
\[ [f, g] = (A, B, C), \]
where
\[ A = (fg' - f'g - \frac{1}{2} \phi \chi) \frac{d}{dx}, \]
\[
B = (f' - \frac{1}{2} f'' - g' + \frac{1}{2} g'')dx - \frac{1}{2},
\]
\[
C = \int_{S^1} (f'' - 4g')dx,
\]
with
\[
\hat{f} = (f(x,t) \frac{d}{dx}, \phi(x,t)dx^{-\frac{1}{2}}, a),
\]
\[
\hat{g} = (g(x,t) \frac{d}{dx}, \chi(x,t)dx^{-\frac{1}{2}}, b),
\]
\[
f' = \frac{\partial f}{\partial x}.
\]

Let us denote
\[
\mathcal{G}^* = C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbb{R}
\]
to be the dual space of \( \mathcal{G} \), under the following pair
\[
\langle \hat{m}, \hat{f} \rangle^* = \int_{S^1} (mf + \alpha \phi)dx + \varsigma a,
\]
where
\[
\hat{m} = (m(x,t)dx^2, \alpha(x,t)dx^{\frac{1}{2}}, \varsigma) \in \mathcal{G}^*.
\]
By the definition,
\[
\langle ad^*_f(\hat{m}), \hat{g} \rangle^* = -\langle \hat{m}, [\hat{f}, \hat{g}] \rangle^*,
\]
using integration by parts
\[
\langle ad^*_f(\hat{m}), \hat{g} \rangle^* = -\langle \hat{m}, [\hat{f}, \hat{g}] \rangle^*
\]
\[
= \int_{S^1} (2mf' + m'f - \varsigma f''' + \frac{3}{2} \alpha \phi' + \frac{1}{2} \alpha' \phi)gdx
\]
\[
+ \int_{S^1} \left( \frac{m}{2} \phi - \varsigma \phi'' + \frac{3}{2} f' \alpha + f \alpha' \right) \chi dx
\]
\[
= \left( \left( 2mf' + m'f - \varsigma f''' + \frac{3}{2} \alpha \phi' + \frac{1}{2} \alpha' \phi \right)dx^2, (\frac{m}{2} \phi - \varsigma \phi'' + \frac{3}{2} f' \alpha + f \alpha')dx^{\frac{1}{2}}, 0 \right), \hat{g} \right) ^*.
\]

Observe that the stabilizer space of the coadjoint action of the Neveu-Schwarz superalgebra \( \mathcal{G} \) on the hyperplane \( \varsigma = 0 \) of \( \mathcal{G}^* \) is given by
\[
2mf' + m'f - \varsigma f''' + \frac{3}{2} \alpha \phi' + \frac{1}{2} \alpha' \phi = 0,
\]
\[
\frac{m}{2} \phi - \varsigma \phi'' + \frac{3}{2} f' \alpha + f \alpha' = 0.
\]
which are exactly the latter two equations in (2.5). Thus we have

**Proposition 4.1.** The mixed Kuper-CH-HS equation (2.5) is the constraint Hamiltonian flow on the Neveu-Schwarz coadjoint orbit, that is to say
\[
\left( \begin{array}{c}
m \\
\alpha
\end{array} \right) = ad^*_f(\hat{m}),
\]
with
\[
\left( \begin{array}{c}
m \\
\alpha
\end{array} \right) = ad^*_f(\hat{m}) = 0,
\]

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\[ H = \frac{k_1}{2} \int (mu + \alpha \eta x)dx, \]

where

\[
\hat{m} = (m(x,t)x^2, \alpha(x,t)x^2, 0) \in \mathcal{G}^+,
\]

\[
\hat{f} = (f(x,t) \frac{d}{dx}, \phi(x,t)x^{-\frac{1}{2}}, a) \in \mathcal{G}.
\]

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