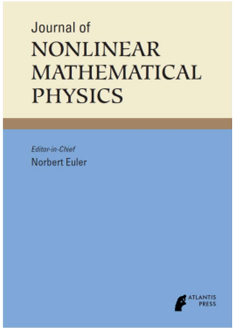

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

On the global dynamics of the Newell–Whitehead system

Claudia Valls

To cite this article: Claudia Valls (2019) On the global dynamics of the Newell–Whitehead system, Journal of Nonlinear Mathematical Physics 26:4, 569–578, DOI: https://doi.org/10.1080/14029251.2019.1640466

To link to this article: https://doi.org/10.1080/14029251.2019.1640466

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 26, No. 4 (2019) 569–578

On the global dynamics of the Newell–Whitehead system

Claudia Valls
Departamento de Matemática, Instituto Superior Técnico,
Universidade Técnica de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal
cvalls@math.ist.utl.pt

Received 18 November 2018

Accepted 22 May 2019

In this paper by using the Poincaré compactification in $\mathbb{R}^3$ we make a global analysis of the model $x' = z$, $y' = b(x-dy)$, $z' = x(x^2-1)+y+cz$ with $b \in \mathbb{R}$ and $c, d \in \mathbb{R}^+$, here known as the three-dimensional Newell–Whitehead system. We give the complete description of its dynamics on the sphere at infinity. For some values of the parameters this system has invariant algebraic surfaces and for these values we provide the dynamics of the system restricted to these surfaces and its global phase portrait in the Poincaré ball. We also include the description of the α-limit and ω-limit set of its orbits in the Poincaré ball including its boundary, that is, in the compactification of $\mathbb{R}^3$ with the sphere at the infinity. We recall that the restricted systems are not analytic and so in this paper we overcome this difficulty by using the blow-up technique.

Keywords: Global dynamics; Poincaré compactification; Newell–Whitehead system; invariant algebraic curve; invariant.

2000 Mathematics Subject Classification: 34C05, 34C07, 34C08

1. Introduction and statement of the results

The FitzHugh-Nagumo system is written as

$$u_t = u_{xx} - f(u) - v, \quad v_t = \varepsilon(u - \gamma v)$$

where $f(u) = u(u-1)(u-a)$, $0 < a < 1/2$, $\varepsilon > 0$ and $\gamma > 0$ are biological parameters. In this formula the variable u is the voltage inside the axon at position $x \in \mathbb{R}$ and time t and v is a part of trans–membrance current that is passing slowly adapting iron channels. When $a = -1$ it is the so-called Newell–Whitehead system.

These equations were introduced by FitzHugh [5] and Nagumo [9]. In [5], the author simplified the four dimensional Hodgkin-Huxley system into a planar system (called Bonhoeffer-Van der Por system) and he also considered the excitable and oscillatory behavior of the Bonhoeffer-Van der Por system and showed the underlying relationship between the Bonhoeffer-Van der Por system and the BVP system and Hodgkin-Huxley system. By the method used in [5] and the Kirchhoff's law, the authors in [9] considered the propagation of the excitation along the nerve axon into the Hodgkin-Huxley system, and the Hodgkin-Huxley system becomes the FitzHugh-Nagumo partial differential equation. This system has been intensively studied in the literature mainly due to its simplicity for describing the excitation of neural membranes and the propagation of nerve impulses along an axon. This has caused the attention of many authors that studied, among others aspects of the system, the existence, uniqueness and stability of its traveling wave solutions (see [2,3,6,7,9–11]). If we assume the existence of a traveling wave solution for the FitzHugh-Nagumo partial differential equation that

Co-published by Atlantis Press and Taylor & Francis

is a bounded solution $(u(x,t),v(x,t))$ with $x,\,t \in \mathbb{R}$ satisfying $(u(x,t),v(x,t)) = (u(x+ct),v(x+ct))$ being $c > 0$ the constant denoting the wave speed and we substitute it into the FitzHugh-Nagumo partial differential equation, we get the following ordinary differential equation

$$\begin{aligned}\dot{x} &= z = P(x,y,z),\\ \dot{y} &= b(x-dy) = Q(x,y,z),\\ \dot{z} &= x(x-1)(x-a)+y+z = R(x,y,z),\end{aligned}$$

where the dot denotes derivative with respect to τ with $\tau = x+ct$, $x = u$, $y = v$, $z = \dot{u}$, $b = \varepsilon$ and $d = \gamma/c$. Note that since γ, $c > 0$ we have that $d > 0$. In this paper, we focus on the global dynamics of equation (1.1) with $a = -1$ and arbitrary parameters $c,\,d \in \mathbb{R}^+$ and $b \in \mathbb{R}$, that is, we describe the global behavior of the Newell–Whitehead system which can be written in the form

$$\begin{aligned}\dot{x} &= z = P(x,y,z),\\ \dot{y} &= b(x-dy) = Q(x,y,z),\\ \dot{z} &= x(x^2-1)+y+cz = R(x,y,z),\end{aligned} \tag{1.1}$$

with $c,\,d \in \mathbb{R}^+$ and $b \in \mathbb{R}$.

The integrability of system (1.1) has been studied in [12] where the authors give the description of all the invariant algebraic surfaces of the system. Let U be an open subset of $\mathbb{R}^3$. A first integral $H\colon U \to \mathbb{R}$ of system (1.1) is a function which is constant on the trajectories of the system. A function $I(x,y,z,t)$ is an invariant of system (1.1) if $dI/dt = 0$ on the trajectories of the system, that is, an invariant is a first integral which depends on time. The following proposition proved in [12] summarizes the results on the integrability and on the existence of invariants for system (1.1).

Proposition 1.1. *The following holds for system* (1.1).

(i) *if* $bd = -c$, $b = \frac{2}{27}c^3 - \frac{c}{3}$ *with* $0 < c < \frac{3}{\sqrt{2}}$ *it has the invariant* $H_2 = F_2 e^{-4ct/3}$, *where* $F_2 = \frac{1}{2}x^4 - z^2 + 2xy + \frac{2}{3}cxz + \left(\frac{1}{9}c^2 - 1\right)x^2$;

(ii) *if* $bd = -\frac{2}{3}c$, $b = \frac{2}{27}c^3 - \frac{1}{3}c$ *with* $0 < c < \frac{3}{\sqrt{2}}$, *it has the invariant* $H_3 = F_3 e^{-4ct/3}$, *where* $F_3 = \frac{1}{2}x^4 - z^2 + 2xy + \frac{2}{3}cxz + \left(\frac{1}{9}c^2 - 1\right)x^2 - \frac{1}{2}dy^2$.

Proposition 1.1 will be used in the following sections for studying the global dynamics behavior of system (1.1) having an invariant. As any polynomial differential system, system (1.1) can be extended to an analytic system on a closed ball of radius one, whose interior is diffeomorphic to $\mathbb{R}^3$ and its invariant boundary, a two-dimensional sphere $\mathbb{S}^2$ plays the role of infinity. This ball will be denoted by B and called the *Poincaré ball*, due to the fact that the technique for doing so is the Poincaré compactification, which is well established (see, for instance [4]). Two polynomial vector fields are said to be topologically equivalent if there exists a homeomorphism on the closed Poincaré ball preserving the infinity (that is the boundary of the Poincaré ball) carrying orbits of the flow induced on the Poincaré ball by the first vector field into orbits of the flow induced in the Poincaré ball by the second vector field.

By using this compactification technique we can describe the dynamics of system (1.1) at infinity. This is the first main result of the paper.

Theorem 1.1. *For all values of the parameters* $b \in \mathbb{R}$ *and* $c,\,d \in \mathbb{R}^+$, *the phase portrait of system* (1.1) *on the Poincaré sphere is topologically equivalent to the one shown in Figure* 1.

Note that the dynamics at infinity do not depend on the parameter values. The proof of Theorem 1.1 is given in Section 3 by means of the Poincaré compactification technique.

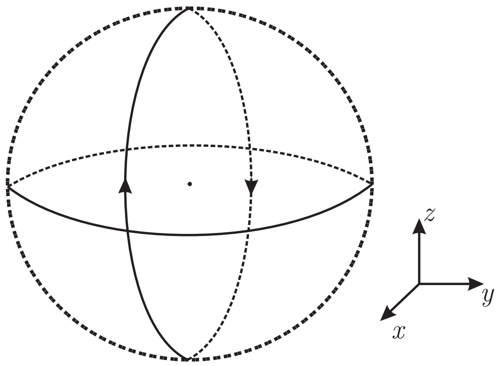

Fig. 1. Phase portrait at infinity on the Poincaré ball of system (1.1). Note that system (1.1) has one closed curve of equilibria which is $x = 0$, $y^2 + z^2 = 1$ and there are no equilibrium points in the sphere.

Now we continue the study of the dynamics of system (1.1) for the values of the parameters in Proposition 1.1. We provide the description of the global dynamics of the this polynomial differential system not only on $\mathbb{R}^3$ but also in its compactification for some values of the parameters. In particular, we will study the dynamics on the whole $\mathbb{R}^3$ including the behavior on the sphere at infinity, that is, on the Poincaré ball and we describe the α-limit sets and the ω-limit sets of all bounded orbits of the system (1.1) for some values of the parameters.

Theorem 1.2. *The global phase portraits of system* (1.1) *on the Poincaré ball for the values of the parameters in Proposition* 1.1 *are topologically equivalent to the ones described in Figure* 2*.*

The proof of Theorem 1.2 is given in Section 4.

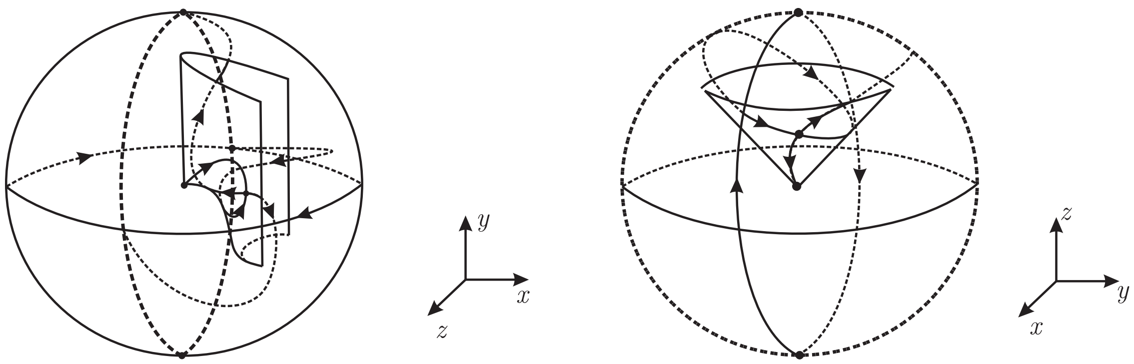

Fig. 2. Global phase portraits of system (1.1) under the assumptions of Proposition 1.1(i) on the left and under the assumptions of Proposition 1.1(ii) on the right.

2. Preliminaries

First we recall a well-known result that was proved in [8].

Lemma 2.1. *Let $F(x,y,z)=0$ be a degree m algebraic surface of $\mathbb{R}^3$. The extension of this surface to the boundary of the Poincaré ball is contained in the curve defined by*

$$w^m F\left(\frac{x}{w},\frac{y}{w},\frac{z}{w}\right)=0, \quad w=0.$$

Lemma 2.2. *If $\phi(t)=(x(t),y(t),v(t))$, $t\in\mathbb{R}$ is a bounded trajectory of system* (1.1) *satisfying the assumptions of statements* (i) *and* (ii) *in Proposition* 1.1 *with invariants $I_i=H_i e^{-\frac{4}{3}ct}$, $i=2,3$, then its α-limit set $\alpha(\phi)$ and its ω-limit set $\omega(\phi)$ are contained in the sets $\{H_2=0\}$* (*under the assumptions* (i)) *and $\{H_3=0\}$* (*under the assumptions* (ii)).

Proof. Let $q_1\in\omega(\phi)$. Thus there exists $t_n\to\infty$ such that $\phi(t_n)\to q_1$. Thus

$$H_i(\phi(t_n))e^{-\frac{4}{3}ct_n}\to H_i(q_1)\cdot 0=K \quad \text{as } t_n\to\infty.$$

Thus $K=0$ and then $H_i(\phi(t_n))\cdot e^{-\frac{4}{3}ct_n}=0$. This implies that $H_i(\phi(t_n))=0$ and so $H_i(q_1)=0$. In short $q_1\in\{H_i=0\}$. Alternatively, note that all trajectories ϕ of system (1.1) possessing the invariant F_i (and so under the assumptions (i) or (ii) of Proposition 1.1) and not contained in $\{H_i=0\}$ are unbounded at $t>0$ (otherwise considering the limit as $t\to\infty$ we would have $F_i=0$ and so $H_i=0$, a contradiction).

Assume now that $q_2\in\alpha(\phi)$. Then there exists $t_n\to-\infty$ such that $\phi(t_n)\to q_2$. Hence

$$H_i(\phi(t_n))=\frac{c}{e^{-\frac{4}{3}ct_n}}\to 0 \quad \text{as } t_n\to-\infty$$

and so $H_i(\phi(t_n))\to H_i(q_2)$ as $n\to-\infty$. It follows that $H_i(q_2)=0$ and so $q_2\in\{H_i=0\}$. □

Now we study the limit cycles of this system under the assumptions of Proposition 1.1.

Lemma 2.3. *Under the assumptions of Proposition* 1.1*, system* (1.1) *has no limit cycles.*

Proof. Since the cofactors of the invariant algebraic surfaces are constant, if the limit cycle exists, it must be located on the invariant surface. Taking into account that the divergence of system (1.1) is $c-bd$ and that on the values of the parameters provided by Proposition 1.1 the divergence is different from zero we conclude that such limit cycle cannot exist. □

Now we study the finite singular points under the assumptions of Proposition 1.1. We recall that the stability index of a hyperbolic point is the number of eigenvalues with negative real part.

Lemma 2.4. *The following holds for system* (1.1) *under the assumptions* (i) *or* (ii) *in Proposition* 1.1.

(a) *If $d\in(0,1]$ the origin is the unique singular point;*
(b) *If $d>1$ besides the origin there are the two additional singular points $S_\pm=\pm\sqrt{(d-1)/d}(1,1/d,0)$.*
(c) *The Jacobian matrix at the origin of system* (1.1) *under the assumptions* (i) *has eigenvalues $2c/3$ and $(2c\pm\sqrt{3(c^2-1)})/3$ and under the assumptions* (ii) *it has the eigenvalues $c/3$ and $2c\pm\sqrt{2c^2-9})/3$. So the origin is a global repeller.*
(d) *The Jacobian matrix at the two points $S_\pm$ of system* (1.1) *under the assumptions* (i) *has the eigenvalues $4c/3$ and $(c\pm\sqrt{9+2c^2})/3$* (*so both points are saddles with stability index one*), *and under the assumptions* (ii) *it has the eigenvalues $4c/3$ and $(c\pm\sqrt{18+5c^2})/6$* (*and again both points are saddles with stability index one*).

Proof. The proof of the lemma follows by direct calculations. □

3. Proof of Theorem 1.1

For studying the infinity of the Poincaré ball B we analyze the flow at infinity for the local charts U_1, U_2 and U_3. In the next subsecti ons we study the Poincaré compactification of system (1.1) in the local charts U_1, U_2, U_3 and V_1, V_2, V_3.

3.1. *Study of the infinite in the local charts U_1 and V_1*

The Poincaré compactification of system (1.1) in the local chart U_1 is given by

$$\begin{aligned} \dot{z}_1 &= bz_3^2 - bdz_1z_3^2 - z_1z_2z_3^2, \\ \dot{z}_2 &= 1 - z_3^2 + z_1z_3^2 + cz_2z_3^2 - z_2^2z_3^2, \\ \dot{z}_3 &= -z_2z_3^3. \end{aligned} \tag{3.1}$$

For $z_3 = 0$ (which corresponds to the points on the sphere $\mathbb{S}^2$ at infinity) system (3.1) becomes

$$\dot{z}_1 = 0, \quad \dot{z}_2 = 1. \tag{3.2}$$

This system has no equilibria. The solution is given by parallel straight lines to the z_2-axis. The flow on the local chart V_1 is the same than the flow of U_1, because the compactified vector field in V_1 coincides with the vector field in U_1 multiplied by $(-1)^{3-1} = 1$.

3.2. *Study of the infinite in the local charts U_2 and V_2*

The expression of the Poincaré compactification in the local chart U_2 of system (1.1) writes as

$$\begin{aligned} \dot{z}_1 &= bdz_1z_3^2 - bz_1^2z_3^2 + z_2z_3^2, \\ \dot{z}_2 &= z_1^3 + z_3^2 - z_1z_3^2 + cz_2z_3^2 + bdz_2z_3^2 - bz_1z_2z_3^2, \\ \dot{z}_3 &= bdz_3^3 - bz_1z_3^3. \end{aligned} \tag{3.3}$$

System (3.3) restricted to $z_3 = 0$ becomes

$$\dot{z}_1 = 0, \quad \dot{z}_2 = z_1^3. \tag{3.4}$$

System (3.4) has the plane $z_1 = 0$ of equilibria. Considering the invariance of the z_1z_2-plane under the flow of (3.3) we can completely describe the dynamics on the sphere at infinity, which is shown in Figure 1. Note that this system for $z_1 \neq 0$ is equivalent to system (3.2) and the plane $z_1 = 0$ is a plane of equilibria. Again the flow on the local chart V_2 is the same as the flow on the local chart U_2.

3.3. *Study of the infinite in the local charts U_3 and V_3*

The expression of the Poincaré compactification of system (1.1) in the local chart U_3 is given by

$$\begin{aligned} \dot{z}_1 &= -z_1^4 + z_3^2 - cz_1z_3^2 + z_1^2z_3^2 - z_1z_2z_3^2, \\ \dot{z}_2 &= -z_1^3z_2 + bz_1z_3^2 - cz_2z_3^2 - bdz_2z_3^2 + z_1z_2z_3^2 - z_2^2z_3^2, \\ \dot{z}_3 &= -z_1^3z_3 - cz_3^3 + z_1z_3^3 - z_2z_3^3. \end{aligned} \tag{3.5}$$

Observe that system (3.5) restricted to the invariant z_1z_2-plane reduces to

$$\dot{z}_1 = -z_1^4, \quad \dot{z}_2 = -z_1^2 z_2.$$

The solution of this system corresponds to the dynamics of system (3.3) in the local chart U_3. Note that $z_1 = 0$ is a line of equilibria. For $z_1 \neq 0$ the system is equivalent to

$$\dot{z}_1 = -z_1, \quad \dot{z}_2 = -z_2$$

whose origin is an improper node. The flow at infinity in the local chart V_3 is the same as the flow on the local chart U_3.

Proof of Theorem 1.1. Considering the analysis made in the previous subsections and gluing the flow in the three studied charts, we have a global picture of the dynamical behavior of system (1.1) at infinity given in Figure 1. The system has one closed curve of equilibria which is $x = 0$, $y^2 + z^2 = 1$ and there are no equilibrium points in the sphere. We observe that the description of the complete phase portrait of system (1.1) on the sphere at infinity (the Poincaré ball) was possible because of the invariance of these sets under the flow of the compactified system. This proves Theorem 1.1. We remark that the behavior of the flow at infinity does not depend on the parameters of the system. □

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. We consider the invariants given in Proposition 1.1 and how the surfaces end in the Poincaré sphere at infinity.

4.1. *Case* $bd = -c$, $c \neq 0$, $b = \frac{2}{27}c^3 - \frac{1}{3}c$ *with* $0 < c < \frac{3}{\sqrt{2}}$

In this case $F_2 = \frac{1}{2}x^4 - z^2 + 2xy + \frac{2}{3}cxz + \left(\frac{1}{9}c^2 - 1\right)x^2 = 0$ is an invariant algebraic surface. On $x = 0$ then $z = 0$ and the surface reduces to the y-axis. If $x \neq 0$ then the invariant algebraic surface can be written as the graphic of the function

$$y = y(x,z) = \frac{18x^2 - 2c^2x^2 - 9x^4 - 12cxz + 18z^2}{36x}.$$

According to Lemma 2.1 the boundary of this surface on the sphere $\mathbb{S}^2$ of the infinity is given by the system

$$\frac{1}{2}x^4 - z^2w^2 + 2xyw^2 + \frac{2}{3}cxzw^2 + \left(\frac{1}{9}c^2 - 1\right)x^2w^2 = 0, \quad w = 0,$$

from which we get $x = 0$. This means that the boundary at infinity of this surface is the great circle $\{x = 0\} \cap \{x^2 + y^2 + z^2 = 1\}$. Since for $x = 0$ we get $z = 0$, in fact, the boundary at infinity of this surface is $(0, \pm 1, 0)$.

Now we study the dynamics of equation (1.1) on this invariant surface when $x \neq 0$. If $x \neq 0$ the restriction of system (1.1) to the invariant surface is given by

$$\dot{x} = z, \quad \dot{z} = \frac{-18x^2 - 2c^2x^2 + 27x^4 + 24cxz + 18z^2}{36x}$$

and parameterizing the time we get the equivalent system

$$\dot{x} = 36xz, \quad \dot{z} = -18x^2 - 2c^2x^2 + 27x^4 + 24cxz + 18z^2. \tag{4.1}$$

This system has, among the origin, two finite singular points with $x \neq 0$ which are

$$\Big(\pm\frac{\sqrt{2}}{3\sqrt{3}}\sqrt{c^2+1}\,,0\Big).$$

The eigenvalues of the Jacobian matrix at these points are $(c \pm \sqrt{2c^2+9})/3$ and so both points are saddles. The origin is a degenerate singular point. So, in order to obtain its local behavior at infinity we need to do a blow up (see [1] for more details on this well-known technique). Doing so, we get that the local phase portrait near the origin is the one shown in Figure 3.

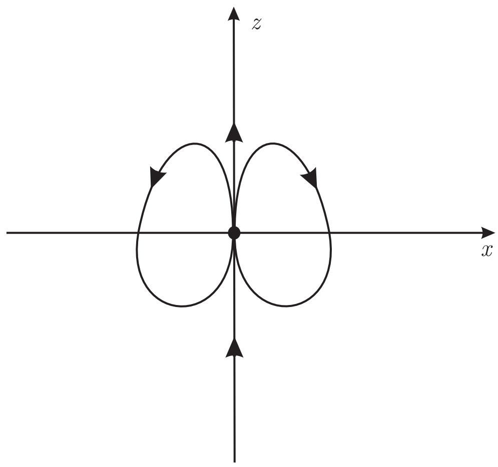

Fig. 3. Local phase portrait of system (4.1) at the origin.

Now we study the dynamics at infinity by means of the Poincaré compactification for system (4.1). On the local chart U_1 we get the system

$$u' = -\frac{1}{36}(-27+(2c^2+18)v^2-24cv^2u+18u^2v^2), \quad v' = -v^3u.$$

Note that this system has no singular points on $v = 0$ and so there are no singular points in the local chart U_1. On the local chart U_2 we get

$$\begin{aligned} u' &= u(18v^2-24cv^2u-27u^4+(2c^2+18)u^2v^2),\\ v' &= v(-18v^2-24cv^2u-27u^4+(2c^2+18)u^2v^2). \end{aligned}$$

Note that the origin of the local chart U_2 is a singular point which is degenerate. Doing again a blow-up we get that its local phase portrait is topologically equivalent to a node.

Taking into account the analysis on the surface together with the infinity and Lemma 2.3 we get the global phase portrait of system (4.1) on the Poincaré disc shown in Figure 4.

To obtain the global phase portrait on the Poincaré sphere given in Figure 4 we use the information above (on the boundary at infinity and on the surface $F_2 = 0$) together with Lemma 2.4 which guarantees that the origin is in this case a global repeller.

According to Lemma 2.2 for any trajectory not contained in the invariant surface we have that the ω-limit is contained in $\{F_2 = 0\}$ and the α-limit is contained in $\mathbb{S}^2$ (that is, it is the point $(0,\pm 1,0)$).

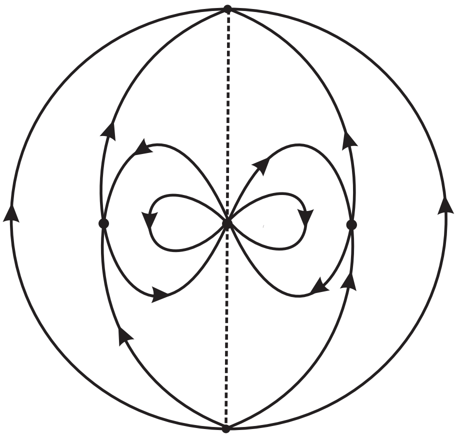

Fig. 4. Phase portrait of system (4.1) on the Poincaré disc.

The complete phase portrait given in Figure 2 can be obtained with the union of the Figure 4 with the infinity sphere.

4.2. *Case* $bd = -\frac{2}{3}c$, $c \neq 0$, $b = \frac{2}{27}c^3 - \frac{c}{3}$ *with* $0 < c < \frac{3}{\sqrt{2}}$

The invariant algebraic surface is

$$F_3 = \frac{1}{2}x^4 - z^2 + 2xy + \frac{2}{3}cxz + \big(\frac{1}{9}c^2 - 1\big)x^2 - \frac{1}{2}dy^2 = 0.$$

According to Lemma 2.1 the boundary of this surface on the sphere $\mathbb{S}^2$ of the infinity is given by

$$\frac{1}{2}x^4 - z^2w^2 + 2xyw^2 + \frac{2}{3}cxzw^2 + \big(\frac{1}{9}c^2 - 1\big)x^2w^2 - \frac{1}{2}dy^2w^2 = 0, \quad w = 0,$$

from which we get that $x = 0$ and so the boundary at infinity of the surface is the great circle $\{x = 0\} \cap \{x^2 + y^2 + z^2 = 1\}$.

Now we will study the dynamics of system (1.1) on the surface $F_3 = 0$ projected onto the (x, y)-plane. The surface $F_3 = 0$ yields

$$z_{\pm} = \frac{1}{6}\big(2cx \mp \sqrt{4(2c^2-9)x^2 + 18x^4 + 72xy + \frac{324}{2c^2-9}y^2}\big).$$

The projected systems taking $z_{\pm}$ is given by

$$\begin{aligned} \dot{x} &= \frac{1}{6}\big(2cx \mp \sqrt{4(2c^2-9)x^2 + 18x^4 + 72xy + \frac{324}{2c^2-9}y^2}\big), \\ \dot{y} &= \frac{c}{27}((2c^2-9)x + 18y), \end{aligned} \tag{4.2}$$

(note that $c \neq 3/\sqrt{2}$).

Among the origin, this system has the singular points

$$R_{\pm} = \pm\Big(\frac{1}{3\sqrt{2}}\sqrt{9+2c^2}, \frac{1}{54\sqrt{2}}(9-2c^2)\sqrt{9+2c^2}\Big).$$

The eigenvalues of R_+ for the restriction on z_+ (respectively of R_- for the restriction on z_-) are $\frac{5c \pm \sqrt{5c^2-18}}{6}$. So, it is a repeller node if $c \geq 3\sqrt{2}/\sqrt{5}$ or a repelling focus if $c < 3\sqrt{2}/\sqrt{5}$. In this paper we do not distinguish nodi and foci since their local phase portraits are topologically equivalent. On the other hand, the eigenvalues of R_- for the restriction on z_+ (respectively of R_+ on the restriction on z_-) are $\frac{c\pm\sqrt{18+5c^2}}{6}$ and so it is a saddle.

Note that systems (4.2) are not analytic at the origin. In order to study its local behavior at the origin we analyze the original system (1.1). In view of Lemma 2.4, taking into account that $0 < c < \frac{3}{\sqrt{2}}$, we conclude that the origin is an unstable focus (we recall that the stability at the origin along the eigenvectors is the same).

Now we consider the dynamics at the infinity. Note that the topological structure of both systems in (4.2) are the same and so we will only work with the projection on z_+. Taking the Poincaré transformation $x = 1/v$, $y = u/v$ with the scaling $d\tau = v\,dt$ we get the system

$$u' = \frac{cv}{27}(-9+2c^2+9u) - \frac{u}{3\sqrt{2}}\sqrt{T_1}, \quad v' = -\frac{cv^2}{3} - \frac{v}{3\sqrt{2}}\sqrt{T_1}$$

where

$$T_1 = \frac{9(-9+2c^2)+2(-9+2c^2+9u)^2v^2}{2c^2-9}.$$

Note that the origin is a singular point which is a stable node. On the other hand, taking the Poincaré transformation $x = u/v$, $y = 1/v$ with the time scaling $d\tau = v\,dt$, the system becomes

$$u' = -\frac{1}{6}(u-1)(2cuv+\sqrt{2}\sqrt{T_2}), \quad v' = -\frac{cv^2}{27}(19+(2c^2-9)u), \tag{4.3}$$

where

$$T_2 = \frac{9(-9+2c^2)u^4+2(9+(-9+2c^2)u)^2v^2}{2c^2-9}.$$

The origin is a degenerate singular point. Doing a blow-up we conclude that its local phase portrait is topologically equivalent to the one of Figure 5.

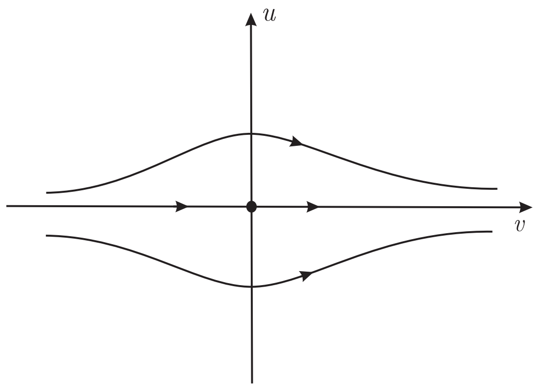

Fig. 5. Local phase portrait of the origin of system (4.3).

Combining the study on the finite plane and at infinity together with Lemma 2.3 we get that the global phase portrait of system (1.1) on $F_3 = 0$ is the one given in Figure 6.

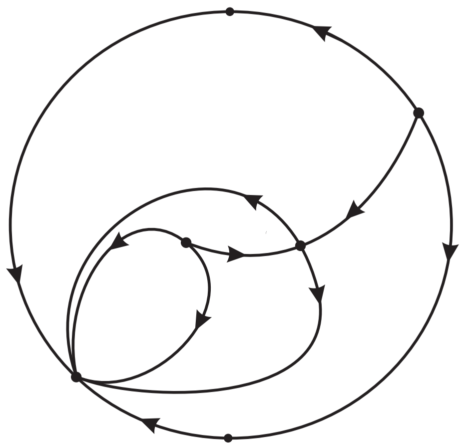

Fig. 6. Global phase portrait of system (1.1) on $\{F_3 = 0\}$.

According to Lemma 2.2 for any trajectory not contained in the invariant surface we have that the ω-limit of any orbit is contained in $\{F_3 = 0\}$ and the α-limit is contained in $\mathbb{S}^2$. The complete phase portrait given in Figure 2 can be obtained with the union of the Figure 6 with the infinity sphere.

Acknowledgements

Partially supported by FCT/Portugal through UID/MAT/04459/2013.

References

[1] M.J. Álvarez, A. Ferragut and X. Jarque, A survey on the blow up technique, *Int. J. Bifurcation and Chaos* **21** (2011) 3103–3118.
[2] G. Arioli and H. Koch, Existence and stability of traveling pulse solutions of the FitzHugh-Nagumo equation, *Nonlinear Analysis* **113** (2015) 51–70.
[3] D. Bleecker and G. Csordas, *Basic partial differential equations*, Van Nostrand Reinhold, New York, 1992.
[4] A. Cima and J. Llibre, Bounded polynomial systems, *Trans. Amer. Math. Soc.* **318** (1990) 557–579.
[5] R. FitzHugh, Impulses and physiological state in theoretical models of nerve membrane, *Biophys. J.* **12** (1961) 445–467.
[6] C.K.R.T. Jones, Stability of the traveling wave solution of the FitzHugh-Nagumo system, *Trans. Amer. Math. Soc.* **286** (1984) 431–469.
[7] W. Liu and E. Van Vleck, Turning points and traveling waves in FitzHugh–Nagumo type equations, *J. Differential Equations* **225** (2006) 381–410.
[8] J. Llibre, M. Messias and R.P. da Silva, On the global dynamics of the Rabinovich system, *J. Phys. A* **41** (2008) (275210) 21 pp.
[9] J.S. Nagumo, S. Arimoto and S. Yoshizawa, An active pulse transmission line simulating nerve axon, *Proc. IRE* **12** (1963) 2061–2070.
[10] C. Valls, Rikitake system: analytic and Darboux integrals, *Prco. Roy. Soc. Edinburgh Sect. A* **135** (2005) 1309–1326.
[11] C. Valls, Invariant algebraic surfaces for generalized Raychaudhuri Equation, *Commun. Math. Phys.* **308** (2011) 133–146.
[12] L. Zhang and J. Yu, *Invariant algebraic surfaces of the FitzHugh-Nagumo system*, preprint 2017.