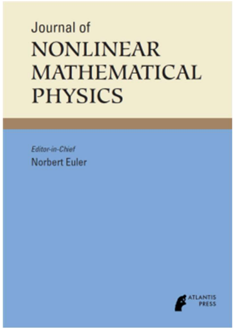

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Systems of Hamilton-Jacobi equations

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In this article we develop a generalization of the Hamilton-Jacobi theory, by considering in the cotangent bundle an involutive system of dynamical equations.

Keywords: Hamilton-Jacobi equation; Lagrangian submanifolds.

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1. Introduction

Hamilton-Jacobi theory arises with William Hamilton in the 1820's of the XIX century, who carries his unification purpose of particle and wave concepts of light, in the geometric optics, to crystallize (towards 1835) in the method of canonical transformations to determine the trajectories of systems. Later on Carl Gustav Jacobi interprets the dynamics of mechanical systems in terms of the complete solutions of the associated Partial Differential Equation.

Beyond classical mechanics, the Hamilton-Jacobi theory lets feel its influence in Quantum mechanics, not only under the principle that a classical system should be obtained as an appropriate limit of the quantum one, (v.g.r. [1] or [2], where it is considered how the equations of the characteristics, in the short-wave limit of evolutionary wave equation on the configuration space, produce the Hamilton equation on the cotangent bundle), but the consideration of Hamilton-Jacobi theory as a tool itself in Quantum Systems. Thus, a complete solution of the Hamilton-Jacobi equation for the Hamiltonian $H(t,q_i,p_i)$, is a function $S(t,q_i,x_i)$ satisfying the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}(t,q_i,x_i)+H\left(t,q_i,\frac{\partial S}{\partial q_i}(t,q_i,x_i)\right)=0. \tag{1.1}$$

Under the non-degeneracy condition $\det\left(\partial^2 S/\partial q_i \partial x_j\right)\neq 0$, we can define the canonical transformation by

$$\frac{\partial S}{\partial q_i}=p_i,\ \ \frac{\partial S}{\partial x_i}=y_i$$

in such a way that in the new coordinates (t, x_i, y_i), the dynamical system governed by the Hamiltonian H turns into a trivial dynamical system. Nevertheless, canonical transformations do not preserve neither the quantum Hilbert space nor the phase space path integral of Feynman's formulations of quantum mechanics. This fact has aroused a great interest establishing the frame which can be called the quantum Hamilton-Jacobi equation (v.g.r. see [3] for a discrete version of classical transformations in the path integral formulation, [4] for the determination of the quantum mechanical amplitude by means of a single momentum integration form a complete solution of the classical Hamilton-Jacobi equation, or [5] for a modification of the Hamilton-Jacobi equation that has suitable covariance properties in such a way that the function S is related to solutions of the Schrödinger equations).

Thus, Hamilton-Jacobi theory is not only a stepping stone in our comprehension of the quantum theory from classical terms but also it aims to provide a powerful quantum tool.

In the present paper, that goes back to the classical theory, we consider systems of Hamilton-Jacobi equations submitted to a compatibility condition – the involutive character – and consider the classical geometric problem of finding foliations transverse to the fibers of T^*Q and invariant under every dynamical evolution, this way extending the standard theory (see [6], [7], [8]). An argument that leads us to the consideration of Darboux coordinates, will allow us to raise this problem locally.

The importance of the Marsden-Weinstein reduction procedure along with the technique of generating functions motivated the consideration of the reduction of the Hamilton-Jacobi theory (see [9], [8] and [10] for a complete reference for Hamilton reduction). It is therefore, section 4 is devoted to extending the reduction and reconstruction procedures for the involutive systems frame.

2. Preliminaries

Important aspects of the modern geometric formulation of the Hamilton-Jacobi theory were established in [11] with the Poisson geometry of T^*Q and more recently in [6] by considering Lagrangian foliations transverse to the fibers of T^*Q that are invariant under the dynamical evolution associated to the symplectic structure.

Let us settle the fundaments of the theory in the most self-contained, brief and clear possible way.

Let (M, w_2) be a $2n$-dimensional symplectic manifold. A vector field $X \in \mathfrak{X}(M)$ is called an infinitesimal symmetry of the symplectic structure if $L_X w_2 = 0$. This condition is equivalent to the fact that $i_X w_2$ is a closed 1-form. Even more, if $i_X w_2$ is exact, that is, of the form df for certain smooth function f defined on M, then X (usually written as X_f) is called the Hamiltonian vector field associated to f. The Poisson bracket of two smooth functions f and g on M is defined by

$$[f, g] = w_2(X_f, X_g)$$

and endows $\mathscr{C}^\infty(M)$ with a Lie algebra structure.

Let $\mathscr{X}$ be a submanifold of M. For every $p \in \mathscr{X}$, we write $(T_p\mathscr{X})^\perp$ for the w_2-orthogonal complement of $T_p\mathscr{X}$ in T_pM. We say that $\mathscr{X}$ is a coisotropic submanifold of M dimension m, if for every $p \in \mathscr{X}$, $(T_p\mathscr{X})^\perp \subseteq T_p\mathscr{X}$. It is clear that then $2n - m \leq m$ and hence $n \leq m$. If $\dim \mathscr{X} = n$, then $\mathscr{X}$ is called a Lagrangian submanifold of M. It is immediate that, in this case, the condition $(T_p\mathscr{X})^\perp = T_p\mathscr{X}$ for every $p \in \mathscr{X}$ means $w_2\big|_{\mathscr{X}} = 0$.

Proposition 2.1. *Let $\mathscr{X}$ be a submanifold of M of dimension $m = 2n - r$.*

(i) *Let $\mathscr{I}$ be the sheaf of ideals of $\mathscr{X}$. Let $f_1,\dots,f_r$ be local generators of $\mathscr{I}$ on the open set $U \subset M$. Then the Hamiltonian vector fields X_{f_i} $(1 \le i \le r)$ constitute a basis for $(T_p\mathscr{X})^\perp$ for every $p \in U \cap \mathscr{X}$.*

(ii) *$\mathscr{X}$ is coisotropic if and only if $\mathscr{I}$ is stable by Poisson bracket.*

Proof. (i) It suffices to take into account that if $X \in T_p\mathscr{X}$ then

$$0 = Xf_i = df_{i,p}(X) = w_{2,p}(X_{f_i}, X), \quad 1 \le i \le r.$$

(ii) By (i), $\mathscr{X}$ is coisotropic if and only if $X_{f_i} \in \mathfrak{X}(\mathscr{X} \cap U)$, for $1 \le i \le r$. This means that

$$X_{f_i}(f_k) = [f_k, f_i] = 0, \quad 1 \le i, k \le r.$$

□

We say that the symplectic structure (M, w_2) is homogeneous (or, also, exact) if there exists a 1-form w_1 on M such that $w_2 = dw_1$. The vector field $X \in \mathfrak{X}(M)$ is Hamiltonian with respect to this homogeneous structure if and only if $L_X w_1$ is exact. This fact trivially follows from the relation

$$\begin{aligned} L_X w_1 &= d\,(i_X w_1) + i_X dw_1 \\ &= dw_1(X) + i_X w_2. \end{aligned}$$

Let G be a connected Lie group and $\mathscr{G}$ the corresponding Lie algebra. Let us suppose that there is a free and proper left action on (M, w_2). We say that this action is symplectic if for each element $g \in G$, we have

$$g^* w_2 = w_2 \tag{2.1}$$

(where g^* denotes the action of g by pull-back on differential forms on M). If for each element $A \in \mathscr{G}$ we denote by A^* its fundamental vector field on M (that is, the infinitesimal generator of the 1-parameter group of transformations of M : $\gamma(t) = \exp(tA)$), then (2.1) implies

$$L_{A^*} w_2 = 0 \ (\Leftrightarrow i_{A^*} w_2 \text{ is a closed 1-form}).$$

Moreover, if for each $A \in \mathscr{G}$, we have

$$i_{A^*} w_2 = df_{A^*}, \text{ for some smooth function } f_{A^*} \text{ on } Q$$

we say that the action of G on M is Hamiltonian. In that case, we define the *momentum mapping*

$$J : Q \to \mathscr{G}^* : J(x)(A) = f_{A^*}(x), \ \ A \in \mathscr{G} \ \ (x \in Q).$$

Let us now denote with Q any smooth manifold, set $M = T^*Q$ for its cotangent bundle, and let $\pi : M \to Q$ be the natural projection. There is an intrinsic way of define a 1-form w_1 on M as follows: let $q \in Q$, $p \in T_q^*Q$ and $X \in T_pM$, then

$$w_1 : X \mapsto \langle p, \pi' X \rangle$$

where π' is the tangent linear map $\pi' : T_pM \to T_qQ$. It is easily seen that $w_2 = -dw_1$ is a non-degenerate 2-form, thus defining the so called canonical symplectic structure on T^*Q. The 1-form w_1 is canonically defined and hence invariant under the induced action of any diffeomorphism of

Q. Consequently, if there is an action of a Lie group G on Q then under the induced action, we have $g^*w_1 = w_1$ for every $g \in G$. In this way, for every $A \in \mathscr{G}$ we have

$$L_{A^*}w_1 = 0. \tag{2.2}$$

Hence, the action of G on T^*G is Hamiltonian, and in fact, it follows from (2.2)

$$i_{A^*}w_2 = d\left(w_1(A^*)\right).$$

There is an explicit and intrinsic way of expressing the function $w_1(A^*)$. Given $X \in \mathfrak{X}(Q)$ we define $P_X : T^*Q \to \mathbb{R}$: $\alpha_q \mapsto \alpha_q(X)$. Now

$$w_1(A^*)(\alpha_q) = \left\langle \alpha_q, \pi' A^* \right\rangle = \alpha_q\left(A^*\right) = P_{A^*}\left(\alpha_q\right)$$

(we use the same notation A^* for the fundamental vector field associated to the action of G on Q and on T^*Q and that the latter projects onto the former). In this way, the momentum mapping $J : T^*Q \to \mathscr{G}^*$ is given by

$$J(\alpha_q)(A) = P_{A^*}(\alpha_q).$$

It is not difficult to see the G-equivariance of J, that is, $J(g^*\alpha_q) = ad^*_{g^{-1}}J(\alpha_q)$ and that for any $\mu \in \mathscr{G}^*$, $J^{-1}(\mu)$ is a submanifold of T^*Q. So, if $\mu \in \mathscr{G}^*$ is a fixed point for the coadjoint action of G, the canonical symplectic form w_2 defines a 2-form $\overline{w}_2$ on the quotient manifold $J^{-1}(\mu)/G$ by

$$w_2(X_p, Y_p) = \overline{w}_2(\overline{X}_p, \overline{Y}_p),$$

where $\overline{X}_p$, $\overline{Y}_p$ are the respective classes of X_p and Y_p in $T_pJ^{-1}(\mu)/T_p(G \cdot p)$. The definition of $\overline{w}_2$ makes sense, since $T_pJ^{-1}(\mu)$ and $T_p(G \cdot p)$ are orthogonal complements in $T_p(T^*Q)$, as it is not difficult to see.

The geometric frame of the Hamilton-Jacobi theory on the n-dimensional configuration space Q is the phase space of momenta T^*Q and its canonical exact symplectic structure w_2. Thus, given a Hamiltonian function $H \in \mathscr{C}^\infty(T^*Q)$, there exists a vector field X_H provided by the dynamical equation

$$i_{X_H}w_2 = dH \tag{2.3}$$

whose integral curves are the trajectories of the system (v.g.r. see [6], [12] or [9]). In the classics formulation, the Hamilton-Jacobi problem consists in finding a function $S(t,q)$ that satisfies the partial differential equation

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0.$$

If we write $S(t,q) = W - tE$ for a constant E, then the function W satisfies

$$H\left(t, q, \frac{\partial W}{\partial q}\right) = 0.$$

In geometric terms, this equation means $(dW)^*H = E$, where dW is understood as a section of T^*Q. In a more adequate context, as a closed form is locally exact, we seek for closed 1-forms

$\alpha : Q \to T^*Q$ such that

$$H\big|_{\mathrm{Im}\,a} = E.$$

Since the closed character of α implies that $\mathrm{Im}\,\alpha$ to be a Lagrangian submanifold of T^*Q, our aim is to find Lagrangian submanifolds $L \subset T^*Q$ and vector fields $Z_H \in \mathfrak{X}(T^*Q)$ whose integral curves contained in L be the trajectories of the system.

The essence of the geometrical character is gathered in the following central result.

Theorem 2.1 (Hamilton-Jacobi). *Let us consider the dynamical equation*

$$i_{Z_H} w_2 = dH. \tag{2.4}$$

The following assertions are equivalent for a Lagrangian submanifold $L \subset T^*Q$

(i) *The vector field Z_H is tangent to L.*
(ii) *$H|_L$ is constant.*

In any of these cases, we say that L is a solution of (2.4).

Proof. If $Z_H \in \mathfrak{X}(L)$ then the equation (2.4) can be restricted to L, and then (ii) trivially follows as $w_2\big|_L = 0$. Conversely, if $H\big|_L$ is constant, then (2.4) says that $Z_H \in (\mathfrak{X}(L))^{\perp}$. But for a Lagrangian manifold $(\mathfrak{X}(L))^{\perp} = \mathfrak{X}(L)$, which completes the proof of the theorem. □

Some other proofs of this key fact can be consulted in [8], [6] or even in [9] for an interesting proof based on $\mathbb{R}$-actions on T^*Q.

Now let (q_i) the coordinates in M and (q_i, p_i) the induced coordinates on T^*M. If $\alpha = \sum_{i=1}^n \alpha_i dq_i$ is a closed 1-form, since the functions

$$f_i = \alpha_i - p_i, \ \ 1 \le i \le n$$

generate the Lagrangian submanifold $\mathrm{Im}\,\alpha$ in T^*Q, the Hamiltonian vector fields

$$X_{f_i}, \ \ 1 \le i \le n$$

span, by Proposition 2.1, the orthogonal complement $(T_x(\mathrm{Im}\,\alpha))^{\perp}$ $(x \in \mathrm{Im}\,\alpha)$. Hence a necessary and sufficient condition for $\mathrm{Im}\,\alpha$ (or α) to be a solution of the Hamilton-Jacobi equation is that

$$w_x(X_{f_i}, X_H) = 0, \ \ \forall x \in L, \ \ 1 \le i \le n,$$

fact that we state as follows.

Proposition 2.2. *The closed* 1*-form $\alpha = \sum_{i=1}^n \alpha_i dq_i$ on Q is a solution of the Hamilton-Jacobi equation* (2.4) *if and only if*

$$X_{H,x}(\alpha_i - p_i) = 0, \ \ \forall x \in L, \ \ 1 \le i \le n.$$

3. Involutive systems

Let us consider the system of Hamilton-Jacobi equations

$$\begin{cases} i_{X_{H_1}} w_2 &= dH_1 \\ & \dots \\ i_{X_{H_k}} w_2 &= dH_k \end{cases} \tag{3.1}$$

where the functions H_i are pairwise in involution, that is, we have

$$[H_i, H_j] = 0, \ \ 1 \le i, j \le k$$

and whose differentials are linearly independent in every point of T^*Q. In this case, we say the system (3.1) is involutive.

In the framework of Hamilton-Jacobi theory, the question which this definition implies is to find Lagrangian submanifolds L of T^*Q invariant under the flows of the vector fields X_{H_i}, $1 \le i \le k$.

Here we will give a local solution using an easy argument based on a classical result of Jacobi-Lie in relation to the extension of a set of functions, pairwise in involution and functionally independent to a complete set of canonical symplectic coordinates.

In precise terms, we have:

Theorem 3.1. *Let us consider the involutive system* (3.1). *The submanifold M of T^*Q defined by the equations*

$$H_i = 0, \ \ i = 1, \dots, k.$$

is coisotropic and $k \le n$. Let us denote with w_2' and X_{H_i}' the respective restrictions of the symplectic form w_2 and the fields X_{H_i} to M. For each point $p \in M$ there exists a neighborhood U in M in such a way that a family $\{N_{k_j}\}$ of Lagrangian submanifolds contained in U is obtained by equaling to constants $n-k$ first integrals common to X_{H_i}', $1 \le i \le k$, and such that $X_{H_i} \in \mathfrak{X}(N_{k_j})$, $1 \le i \le k$.

Proof. The first claim trivially follows form Proposition 2.1. In this way, the condition $\dim M \ge n$ for a coisotropic submanifold says that $2n - k \ge n$ and hence $k \le n$. From the Carathéodory-Jacobi-Lie theorem (v.g.r., see [13]), in a neighborhood V of each point $p \in T^*Q$ there are another $2n-k$ functions

$$\begin{matrix} H_{k+1}, \dots, H_n \\ J_1, \dots, J_n \end{matrix} \tag{3.2}$$

in such a way that $[H_i, H_j] = 0$, $[J_i, J_j] = 0$, $[H_i, J_j] = \delta_{ij}$.

In fact, these functions constitute a set of Darboux coordinates in which the symplectic form w_2 on V is expressed as

$$w_2 = \sum_{j=1}^{n} dH_j \wedge dJ_j.$$

Consequently for the restriction w_2' of w_2 to $U = V \cap M$, we have $w_2' = \sum_{j=1}^{n-k} dH_{k+j}' \wedge dJ_{k+j}'$.

Now by Proposition 2.1, at each point $p \in M$, the tangent vectors $X_{H_i,p}$, $(1 \le i \le k)$ generate the orthogonal complement $(T_pM)^\perp$ of T_pM in $T_p(T^*Q)$. As M is a coisotropic manifold $(T_pM)^\perp \subseteq$

T_pM, with what the tangent vectors $X_{H_i,p}$ form a basis for the space of tangent vectors $X_p \in T_pM$ such that $i_{X_p} w_2' = 0$.

In this way, the condition $i_{X'_{H_i}} w_2' = 0$, means that H_{k+j}, J_{k+j} $(1 \leq j \leq k-n)$ are first integrals of the fields X'_{H_i}.

Thus, the submanifold of M

$$N_{k_j} = \left\{ H'_{k+j} = k_j, \;\; k_j \in \mathbb{R}, \;\; 1 \leq j \leq n-k \right\}$$

is a Lagrangian submanifold in U.

The argument above combined with the Hamilton-Jacobi theory provides $X_{H_i} \in \mathfrak{X}(N_{k_j})$ for $1 \leq i \leq k$, which completes the proof of the theorem. □

4. Reduction and Reconstruction

In this section we address the reduction and reconstruction procedures for an involutive system of Hamilton-Jacobi equations with symmetries. Maybe one of the seeds in the consideration of the subjection of the Hamilton-Jacobi dynamics to the symplectic reduction might be [8, Th. 4.3.5]. Years later, this topic is approached together with the inverse problem of reconstruction in [9]. We tackle here this deep subject in a simpler way, without using the structural results of [9] neither the consideration of magnetic terms added to the canonical symplectic form of T^*G.

Let G be a connected Lie group acting freely and properly on a manifold Q and let us consider its natural lifted action on T^*G, which is also Hamiltonian with respect to the canonical symplectic structure w_2 with momentum mapping $J : T^*G \to \mathscr{G}^*$. Let $\dim Q = n$ and $\dim G = m$.

Proposition 4.1. *Let $H_i : T^*Q \to \mathbb{R}$ $(1 \leq i \leq k)$ smooth functions invariant under the G-action. Let $\mu \in \mathscr{G}^*$ be a fixed point for the coadjoint action of G, and let us consider the symplectic structure $\overline{w}_2$ defined in the quotient by the projection*

$$\pi : J^{-1}(\mu) \to J^{-1}(\mu)/G.$$

Let us assume that the projections $\overline{H}_i$ of the functions H_i $(1 \leq i \leq k)$ to the quotient manifold $J^{-1}(\mu)/G$ are functionally independent. The Hamilton-Jacobi system

$$\begin{cases} i_{X_{H_1}} w_2 \; = \; dH_1 \\ \qquad \cdots \\ i_{X_{H_k}} w_2 \; = \; dH_k \end{cases} \tag{4.1}$$

with $k \leq n-m$, determines a Hamilton-Jacobi system on the reduced symplectic manifold $J^{-1}(\mu)/G$,

$$\begin{cases} i_{\overline{X}_{H_1}} \overline{w}_2 \; = \; d\overline{H}_1 \\ \qquad \cdots \\ i_{\overline{X}_{H_k}} \overline{w}_2 \; = \; d\overline{H}_k \end{cases} \tag{4.2}$$

in such a way that if $L \subset J^{-1}(\mu)$ is a Lagrangian submanifold of T^*Q solution of the system (4.1) *then $\pi(L)$ is a solution of the system* (4.2).

Conversely if $\overline{L}$ is a Lagrangian submanifold of $J^{-1}(\mu)/G$ solution of the system (4.2)*, then $L = \pi^{-1}(\overline{L}) \subset J^{-1}(\mu)$ is a Lagrangian submanifold of T^*Q solution of the system* (4.1).

Proof. First of all, we must show that every vector field X_{H_i} is tangent to $J^{-1}(\mu)$. It suffices to see that if $p \in J^{-1}(\mu)$ then $J_{*,p}(X_H) = 0$. But

$$\langle J_{*,p}(X_H), A \rangle = w_{2,p}(X_H, A^*) = A_p^* H = 0, \ \ \forall A \in \mathscr{G}, \tag{4.3}$$

by the G-invariance of H. Now, the G-invariance of both w_2 and H_i defines X_{H_i} as a G-invariant vector field on $J^{-1}(\mu)$ hence inducing a vector field $\overline{X}_H \in \mathfrak{X}(J^{-1}(\mu)/G)$. In this manner, each of the equations of the system (4.1) provides an equation of the reduced dynamics

$$i_{\overline{X}_{H_i}} \overline{w}_2 = d\overline{H}_i, \ \ 1 \leq i \leq k, \text{ in } J^{-1}(\mu)/G. \tag{4.4}$$

Let $L \subset J^{-1}(\mu)$ a Lagrangian submanifold of T^*Q. A consideration similar to (4.3) proves that L is G-invariant. In fact, it suffices to see that for every fundamental vector field A^* we have $A^* \in \mathfrak{X}(L)$. Let $p \in L$ and $X \in T_p(L)$, then

$$w_{2,p}(X, A^*) = J_{*,p}(X)(A) = 0.$$

Thus, $\overline{L} = \pi(L)$ is a Lagrangian submanifold of $J^{-1}(\mu)/G$: in fact as $\dim J^{-1}(\mu) = 2n - m$ (and hence $\dim J^{-1}(\mu)/G = 2(n-m)$), as $\dim \overline{L} = n - m$ it suffices to see that $\overline{w}_2\big|_{\overline{L}} = 0$, which is guaranteed by the fact

$$\overline{w}_2\big|_{\overline{L}} = w_2\big|_L \tag{4.5}$$

in a self-explanatory notation. Finally $\overline{L}$ is a solution of the system (4.2), since by the G-invariance

$$\overline{H}_i\big|_{\overline{L}} = H_i\big|_L = c_i \text{ for } 1 \leq i \leq k. \tag{4.6}$$

Conversely, if the Lagrangian submanifold $\overline{L}$ of $J^{-1}(\mu)/G$ is a solution of the system (4.2), then the above argument on dimensions and (4.5) say that $L = \pi^{-1}(\overline{L})$ is a Lagrangian submanifold of $J^{-1}(\mu)$ which, again by (4.6), is a solution of the system (4.1). □

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