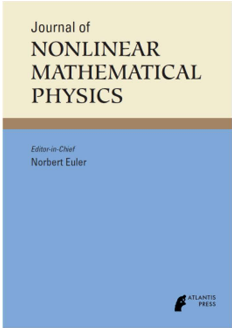

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Variational Operators, Symplectic Operators, and the Cohomology of Scalar Evolution Equations

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For a scalar evolution equation $u_t = K(t,x,u,u_x,\dots,u_{2m+1})$ with $m \geq 1$, the cohomology space $H^{1,2}(\mathscr{R}^\infty)$ is shown to be isomorphic to the space of variational operators and an explicit isomorphism is given. The space of symplectic operators for $u_t = K$ for which the equation is Hamiltonian is also shown to be isomorphic to the space $H^{1,2}(\mathscr{R}^\infty)$ and subsequently can be naturally identified with the space of variational operators. Third order scalar evolution equations admitting a first order symplectic (or variational) operator are characterized. The variational operator (or symplectic) nature of the potential form of a bi-Hamiltonian evolution equation is also presented in order to generate examples of interest.

Keywords: Variational Bicomplex, Cohomology, Scalar Evolution Equation, Symplectic Operator, Hamiltonian Evolution Equation

2010 Mathematics Subject Classification: 58E30,58A20, 37K05, 37K10

1. Introduction

Given a scalar differential equation $\Delta = 0$, the multiplier problem in the calculus of variations consists in determining whether there exists a smooth function m (the multiplier) and a smooth function L (the Lagrangian) such that

$$m \cdot \Delta = \mathbf{E}(L) \tag{1.1}$$

where $\mathbf{E}$ is the Euler-Lagrange operator and $\mathbf{E}(L)$ is the Euler-Lagrange expression for L. The problem of determining whether m and L exists has a long history and is known as the inverse problem in the calculus of variations [4, 7, 9, 11, 12, 14, 19].

The variational bicomplex [2, 3, 20] can be used to provide a solution to the inverse problem in the calculus of variations by utilizing the Helmholtz conditions. The result is that the existence of a solution to the inverse problem in equation (1.1) can be expressed in terms of the existence of special elements in the cohomology space $H^{n-1,2}$ where n is the number of independent variables. In some cases this in turn allows the solution to be expressed directly in terms of the invariants of the equation, see [7, 12].

The main goal of this article is to give a description of the entire cohomology space $H^{1,2}(\mathscr{R}^\infty)$ for scalar evolution equations $u_t = K(t,x,u,u_x,\dots)$ which extends the interpretation of the special elements which control the solution to the inverse problem. The result is a natural generalization of the inverse problem in equation (1.1), we call the variational operator problem, which we now state. Given a differential equation $\Delta = 0$, does there exist a differential operator $\mathcal{E}$ and Lagrangian L such that

$$\mathcal{E}(\Delta) = \mathbf{E}(L)\,. \tag{1.2}$$

A simple example is given by the potential cylindrical KdV equation, $u_t = u_{xxx} + \frac{1}{2}u_x^2 - \frac{u}{2t}$ which admits $\mathcal{E} = tD_x$ as a first order variational operator,

$$tD_x\left(u_t - u_{xxx} - \frac{1}{2}u_x^2 + \frac{u}{2t}\right) = \mathbf{E}\left(-\frac{1}{2}tu_xu_t + \frac{1}{2}tu_xu_{xxx} + \frac{1}{6}tu_x^3\right). \tag{1.3}$$

The variational operator problem in equation (1.2) can be studied for either the case of scalar or systems of ordinary or partial differential equations. Here we restrict our attention to problem (1.2) in the case where Δ is a scalar evolution equation in order to relate this problem to the theory of symplectic and Hamiltonian operators for integrable systems.

In Section 2 we summarize the relevant facts about the variational bicomplex for the case of two independent and one dependent variable. Sections 3 and 4 provide normal forms for the cohomology spaces $H^{r,s}(\mathscr{R}^\infty)$ in the variational bicomplex associated with the equation $\Delta = 0$. These normal forms are then used in Section 5 to show there exists a one to one correspondence between the solution to (1.2) and the cohomology space $H^{1,2}(\mathscr{R}^\infty)$. Even order evolution equations don't admit non-zero variational operators (see Corollary 5.3) but we have the following theorem for odd order equations (the summation convention is assumed).

Theorem 1.1. *Let $\mathcal{E} = r_i(t,x,u,u_x,\ldots)D_x^i$, $i = 0,\ldots,k$ be a k^{th} order differential operator and let the zero set of $\Delta = u_t - K(t,x,u,u_x,\ldots,u_{2m+1})$, $m \geq 1$ define an odd order evolution equation.*

1. *The operator $\mathcal{E}$ is a variational operator for Δ if and only if $\mathcal{E}$ is skew-adjoint and*

$$\omega = dx\wedge\theta^0\wedge\varepsilon - dt\wedge\sum_{j=1}^{2m+1}\left(\sum_{a=1}^{j}(-X)^{a-1}\left(\frac{\partial K}{\partial u_j}\varepsilon\right)\wedge\theta^{j-a}\right) \tag{1.4}$$

   is d_H closed on $\mathscr{R}^\infty$, where $\varepsilon = -\frac{1}{2}r_i\theta^i$ and θ^i are given in equation (2.11).
2. *Let $\mathscr{V}_{op}(\Delta)$ be the vector space of variational operators for Δ. The function $\Phi : \mathscr{V}_{op}(\Delta) \to H^{1,2}(\mathscr{R}^\infty)$ defined from equation* (1.4) *by*

$$\Phi(\mathcal{E}) = [\omega], \tag{1.5}$$

   is an isomorphism.

It follows immediately from Theorem 1.1 that a scalar evolution equation admits a (non-zero) variational operator if and only if $H^{1,2}(\mathscr{R}^\infty) \neq 0$. Consequently the techniques developed for solving the multiplier inverse problem in terms of cohomology [4, 7, 12] can be used to solve the operator problem. The operator $\mathcal{E}$ and the function L in (1.2) are easily determined from the cohomology class $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ (see Theorem 5.3).

The variational operator problem in equation (1.2) is related to the problem of whether a scalar evolution equation can be written in the form of a symplectic Hamiltonian evolution equation [10].

In the time independent case, a scalar evolution $u_t = K(x,u,u_x,\ldots,u_n)$ equation is said to be Hamiltonian with respect to a time independent symplectic operator $\mathcal{S} = s_i(x,u,u_x,\ldots)D_x^i$ if there exists a function H such that,

$$\mathcal{S}(K) = \mathbf{E}(H). \tag{1.6}$$

For a time dependent equation and operator, the symplectic Hamiltonian condition is given in Definition 6.3 (see also Corollary 6.3). Symplectic Hamiltonian evolution equations are reviewed in Section 6 in terms of the variational bicomplex.

Symplectic operators exists on a different space than variational operators but there is a natural identification (see Remark 2.1) between symplectic operators and operators which can be variational operators. With this identification, the variational operator problems and the symplectic operator problem are shown to be the same in Section 7. This leads to the following theorem.

Theorem 1.2. *Let $\mathcal{S} = s_i(t,x,u,u_x,\ldots)D_x^i$ be a differential operator and let $\Delta = u_t - K(t,x,u,u_x,\ldots,u_{2m+1})$. The operator $\mathcal{S}$ is a symplectic operator and $\Delta = 0$ is a symplectic Hamiltonian evolution equation for $\mathcal{S}$ if and only if $\mathcal{S}$ is a variational operator for Δ.*

Theorem 1.2 shows that symplectic operators and variational operators for $u_t = K$ are the essentially the same so that Theorem 1.1 implies the following.

Theorem 1.3. *The function Φ in equation (1.5) defines an isomorphism between the vector space of symplectic operators $\mathcal{S} = \mathcal{E} = r_i(t,x,u,u_x,\ldots)D_x^i$ for which $\Delta = u_t - K$ is Hamiltonian, and the cohomology space $H^{1,2}(\mathscr{R}^\infty)$.*

With Theorem 1.3 in hand, the determination of a symplectic Hamiltonian formulation of $u_t = K$ is resolvable in terms of the cohomology $H^{1,2}(\mathscr{R}^\infty)$ of the differential equation $u_t = K$ and subsequently the invariants of Δ. This characterization of symplectic Hamiltonian evolution equations in terms of $H^{1,2}(\mathscr{R}^\infty)$ allows the techniques in [4, 7, 12] to be used in their study.

A key idea that directly explains the interplay between the symplectic Hamiltonian formulation for an evolution equation and the cohomology $H^{1,2}(\mathscr{R}^\infty)$ is the fact that the equation manifold $\mathscr{R}^\infty$ is canonically diffeomorphic to $\mathbb{R} \times J^\infty(\mathbb{R},\mathbb{R})$. The cohomology of the equation is expressed in terms of the geometric structure that arises from the embedding of the equation into $J^\infty(\mathbb{R}^2,\mathbb{R})$ while the symplectic Hamiltonian formulation of an equation is expressed in terms of the contact structure on $\mathbb{R} \times J^\infty(\mathbb{R},\mathbb{R})$. Theorem 7.1 shows how these are related and this leads to Theorem 1.3. This idea also plays a role in the approach to geometric structures in the article [13].

In Section 8 the case of first order operators for third order equations is examined in detail and the following characterization is found.

Theorem 1.4. *A third order scalar evolution equation $u_t = K(t,x,u,u_x,u_{xx},u_{xxx})$ admits a first order symplectic operator (or variational operator) $\mathcal{E} = 2RD_x + D_xR$ if and only if κ is a trivial conservation law, where*

$$\kappa = \hat{K}_2\,dx + \left(-2K_0 + K_1\hat{K}_2 - \frac{1}{2}\left(X(K_3)\hat{K}_2^2 + K_3\hat{K}_2^3\right) + X\left(K_3X(\hat{K}_2)\right)\right)dt \tag{1.7}$$

and $K_i = \partial_{u_i}K$, $\hat{K}_2 = \frac{2}{3K_3}(K_2 - X(K_3))$, and X is the total x derivative on $\mathscr{R}^\infty$.

Furthermore, when $\kappa = d_H(\log R)$ then $u_t = K$ admits the first order symplectic (or variational) operator $\mathcal{E} = 2RD_x + D_xR$.

In Section 8 we examine the relationship between the Hamiltonian form of an evolution equation and their potential form. In [15] it is shown that the (first order) potential form of a time independent Hamiltonian equation admits a variational operator. We examine this in more detail, as well as the role of bi-Hamiltonian systems as in [18]. In Example 9.1 the Krichever-Novikov equation (or Schwartzian KdV) is shown to be the potential form of the Harry-Dym equation. This demonstrates that the symplectic operators (or variational operators) for the Krichever-Novikov equation ([10]) arise as the lift of the Hamiltonian operators of the Harry-Dym equation as described in Section 8.1.

Theorem 1.4 should be contrasted to the problem of determining a Hamiltonian formulation of a scalar evolution equation in terms of a Hamiltonian operator. An evolution equation $u_t = K$ is Hamiltonian with respect to a Hamiltonian operator $\mathscr{D}$ if there exists a Hamiltonian function H (see [1, 10, 17]) such that

$$u_t = \mathscr{D} \circ \mathbf{E}(H)\,. \tag{1.8}$$

Conditions for the existence of $\mathscr{D}$ and H in equation (1.8) in terms of the invariants of $u_t = K$ is unknown. We illustrate the difference in these problems with the cylindrical KdV and its potential form. The potential form of the cylindrical KdV is easily shown to admit at least two time dependent variational (or symplectic) operators. Section 8.1 then suggests that the cylindrical KdV is a time dependent bi-Hamiltonian system. See Example 9.3 where a bi-Hamiltonian formulation of the cylindrical KdV is proposed ([21] states that no Hamiltonian exists for the cylindrical KdV).

Lastly, in Appendix A we identify the elements of $H^{1,1}(\mathscr{R}^\infty)$, which don't arise as the vertical differential of a conservation law, with a family of variational operators. This is demonstrated in Example 9.2.

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2. Preliminaries

In this section we review some basic facts on the variational bicomplex associated with scalar evolution equations, see [6] for more details.

2.1. *The Variational Bicomplex on* $J^\infty(\mathbb{R}^2,\mathbb{R})$

The t and x total derivative vector fields on $J^\infty(\mathbb{R}^2,\mathbb{R})$ with coordinates $(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots)$ are given by

$$\begin{aligned} D_t &= \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \cdots \\ D_x &= \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \cdots\ . \end{aligned}$$

The contact forms on $J^\infty(\mathbb{R}^2,\mathbb{R})$ are

$$\begin{aligned} \vartheta^0 &= du - u_t dt - u_x dx \\ \vartheta^i &= D_x^i(du - u_t dt - u_x dx) = du_i - u_{t,i}dt - u_{i+1}dx, \quad i \geq 1 \\ \zeta^{a,i} &= D_x^i D_t^a(du - u_t dt - u_x dx) = du_{a,i} - u_{a+1,i}dt - u_{a,i+1}dx, \quad a \geq 1,\ i \geq 0 \end{aligned} \tag{2.1}$$

where $u_i = D_x^i(u)$ and $u_{a,i} = D_x^i D_t^a(u) = u_{tttt\ldots,xxx\ldots}$.

The variational bicomplex on $J^\infty(\mathbb{R}^2,\mathbb{R})$ is denoted by $\Omega^{r,s}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ where $\omega \in \Omega^{r,s}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ is a differential form of degree $r+s$ which is horizontal of degree $r = 0,1,2$ and vertical of degree $s = 0, 1, 2, \dots$ (see Section 2 in [6]). The forms dx and dt are horizontal, while the contact forms are vertical. For example if $\omega \in \Omega^{1,2}(J^\infty(\mathbb{R}^2,\mathbb{R}))$, then ω can be written

$$\begin{aligned}\omega = {} & dx \wedge (A_{ij}\vartheta^i \wedge \vartheta^j + B_{iaj}\vartheta^i \wedge \zeta^{a,j} + C_{aibj}\zeta^{a,i} \wedge \zeta^{b,j}) \\ & + dt \wedge (F_{ij}\vartheta^i \wedge \vartheta^j + G_{iaj}\vartheta^i \wedge \zeta^{a,j} + H_{aibj}\zeta^{a,i} \wedge \zeta^{b,j}),\end{aligned}$$

where $A_{ij}, B_{iaj}, C_{aibj}, F_{ij}, G_{iaj}, H_{aibj} \in C^\infty(J^\infty(\mathbb{R}^2,\mathbb{R}))$. The horizontal and vertical differentials

$$d_H : \Omega^{r,s}(J^\infty(\mathbb{R}^2,\mathbb{R})) \to \Omega^{r+1,s}(J^\infty(\mathbb{R}^2,\mathbb{R})), \quad d_V : \Omega^{r,s}(J^\infty(\mathbb{R}^2,\mathbb{R})) \to \Omega^{r,s+1}(J^\infty(\mathbb{R}^2,\mathbb{R}))$$

are anti-derivations which satisfy

$$\begin{aligned} d_H\omega = {} & dx \wedge D_x(\omega) + dt \wedge D_t(\omega), \qquad & d_V f = \frac{\partial f}{\partial u_i}\vartheta^i + \frac{\partial f}{\partial u_{t,i}}\zeta^{1,i} + \cdots, \\ d_H\vartheta^i = {} & dx \wedge \vartheta^{i+1} + dt \wedge \zeta^{1,i}, & d_V\vartheta^i = 0, \end{aligned}$$

where $f \in C^\infty(J^\infty(\mathbb{R}^2,\mathbb{R}))$, $\omega \in \Omega^{r,s}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ and $D_x(\omega), D_t(\omega)$ are the Lie derivatives. Since $d = d_H + d_V$ this implies,

$$d_H^2 = 0, \quad d_V^2 = 0, \quad \text{and} \quad d_H d_V + d_V d_H = 0.$$

The integration by parts operator $I : \Omega^{2,s}(J^\infty(\mathbb{R}^2,\mathbb{R})) \to \Omega^{2,s}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ is defined by

$$I(\omega) = \frac{1}{s}\vartheta^0 \wedge \sum_{a=0,i=0}^{\infty} (-1)^{i+a} D_x^i D_t^a (\partial_{u_{a,i}} \lrcorner\, \omega), \tag{2.2}$$

and it has the following properties [2], [3],

$$I^2 = I, \quad \omega = I(\omega) + d_H\eta, \qquad \text{for some } \eta \in \Omega^{1,s}(J^\infty(\mathbb{R}^2,\mathbb{R})). \tag{2.3}$$

If we let $J : \Omega^{2,s}(J^\infty(\mathbb{R}^2,\mathbb{R}) \to \Omega^{1,s}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ be

$$J(\kappa) = \sum_{a,i=0}^{\infty} (-1)^{i+a} D_x^i D_t^a (\partial_{u_{a,i}} \lrcorner\, \kappa) \tag{2.4}$$

then $I(\kappa) = \frac{1}{s}\vartheta^0 \wedge J(\kappa)$. Both J and I satisfy,

$$\mathrm{Ker} J = \mathrm{Ker} I = \mathrm{Im}\, d_H\ . \tag{2.5}$$

The operator J is the interior Euler operator, see page 292 in [6] or page 43 in [3].

Let $\mathcal{E} = r_{ia}D_x^i D_t^a$ be a total differential operator. The formal adjoint $\mathcal{E}^*$ is the total differential operator characterized as follows. For any $\rho \in \Omega^{0,s}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ and $\omega \in \Omega^{0,s'}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ there exists $\zeta \in \Omega^{1,s+s'}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ depending on ρ and ω such that

$$(\rho \wedge \mathcal{E}(\omega) - \mathcal{E}^*(\rho) \wedge \omega) \wedge dt \wedge dx = d_H\zeta. \tag{2.6}$$

This leads to

$$\mathcal{E}^*(\alpha) = (-1)^{i+a} D_x^i D_t^a (r_{ia}\alpha)\ , \quad \alpha \in \Omega^{r,s}(J^\infty(\mathbb{R}^2,\mathbb{R})).$$

It follows from (2.6) that the formal adjoint satisfies $(\mathcal{E}^*)^* = \mathcal{E}$.

Let Δ be a smooth function on $J^\infty(\mathbb{R}^2,\mathbb{R})$. The Fréchet derivative of Δ [17] is the total differential operator $\mathbf{F}_\Delta$ satisfying $d_V\Delta = \mathbf{F}_\Delta(\vartheta^0)$. If

$$\Delta = u_t - K(t,x,u,u_x,\dots,u_n),$$

then

$$d_V\Delta = \vartheta_t - K_i\vartheta^i \qquad \text{where} \quad K_i = \frac{\partial}{\partial u_i}K(t,x,u,u_x,\dots,u_n), \quad i=0,\dots,n. \tag{2.7}$$

The Fréchet derivative of Δ is determined from equation (2.7) to be the total differential operator

$$\mathbf{F}_\Delta = D_t - \sum_{i=0}^{n} K_i D_x^i\,. \tag{2.8}$$

The adjoint of the operator in (2.8) is,

$$\mathbf{F}^*_\Delta(\rho) = -D_t(\rho) - \sum_{i=0}^{n}(-D_x)^i(K_i\rho), \quad \rho \in \Omega^*(J^\infty(\mathbb{R}^2,\mathbb{R}).$$

2.2. *The Variational Bicomplex on $\mathscr{R}^\infty$ and $H^{r,s}(\mathscr{R}^\infty)$*

An n^{th} order scalar evolution equation is given by $u_t = K(t,x,u,u_x,\dots,u_n)$ with $K \in C^\infty(J^\infty(\mathbb{R}^2,\mathbb{R}))$ and $K_n = \partial_{u_n}K$ nowhere vanishing. Let $\Delta = u_t - K(t,x,u,u_x,\dots,u_n)$ and let $\mathscr{R}^\infty$ be the infinite dimensional manifold which is the zero set of the prolongation of $\Delta = 0$ in $J^\infty(\mathbb{R}^2,\mathbb{R})$. With coordinates $(t,x,u,u_x,u_{xx},\dots)$ on $\mathscr{R}^\infty$ the embedding $\iota : \mathscr{R}^\infty \to J^\infty(\mathbb{R}^2,\mathbb{R})$ is given by

$$\iota = [t=t, x=x, u=u, u_t=K, u_x=u_x, u_{tt}=T(K), u_{tx}=X(K), u_{xx}=u_{xx},\dots], \tag{2.9}$$

where the vector fields T and X are the restriction of D_t and D_x to $\mathscr{R}^\infty$ given by,

$$\begin{aligned} X &= \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + \cdots \\ T &= \partial_t + K\partial_u + X(K)\partial_{u_x} + \cdots \end{aligned} \tag{2.10}$$

and these satisfy $[X,T]=0$. The Pfaffian system $\mathscr{I} = \{\theta^i\}_{i\geq 0}$ on $\mathscr{R}^\infty$ is generated by the pullback of the 1-forms ϑ^i in equation (2.1)

$$\theta^i = \iota^*\vartheta^i = du_i - X^i(K)dt - u_{i+1}dx. \tag{2.11}$$

The forms

$$\{dt,\ dx, \theta^i = du^i - X^i(K)dt - u_{i+1}dx\} \quad i = 0,\,1,\dots \tag{2.12}$$

form a coframe on $\mathscr{R}^\infty$, and give rise to a vertical and horizontal splitting in the complex of differential forms leading to the bicomplex $\Omega^{r,s}(\mathscr{R}^\infty)$, $r = 0,\,1,\,2$ and $s = 0,\,1,\dots$. For example if $\omega \in \Omega^{1,2}(\mathscr{R}^\infty)$ then

$$\omega = dx \wedge (a_{ij}\theta^i \wedge \theta^j) + dt \wedge (b_{ij}\theta^i \wedge \theta^j)$$

where $a_{ij}, b_{ij} \in C^\infty(\mathscr{R}^\infty)$. The bicomplex $\Omega^{r,s}(\mathscr{R}^\infty)$ is the pullback of the unconstrained bicomplex $\Omega^{r,s}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ by the embedding $\iota : \mathscr{R}^\infty \to J^\infty(\mathbb{R}^2,\mathbb{R})$.

The horizontal exterior derivative $d_H : \Omega^{r,s}(\mathscr{R}^\infty) \to \Omega^{r+1,s}(\mathscr{R}^\infty)$ and vertical exterior derivative $d_V : \Omega^{r,s}(\mathscr{R}^\infty) \to \Omega^{r,s+1}(\mathscr{R}^\infty)$ are anti-derivations computed from the equations,

$$d_H(\omega) = dx \wedge X(\omega) + dt \wedge T(\omega), \quad d_V = d - d_H. \tag{2.13}$$

The horizontal and vertical differentials satisfy

$$d_H^2 = 0 \qquad d_V^2 = 0, \qquad d_H d_V = -d_V d_H. \tag{2.14}$$

The structure equations of $\mathscr{I}$ are computed using (2.11) to be

$$d_H \theta^i = dx \wedge \theta^{i+1} + dt \wedge X^i(d_V K) \quad \text{and} \quad d_V \theta^i = 0. \tag{2.15}$$

Since $d_H^2 = 0$, the complex $d_H : \Omega^{r,s}(\mathscr{R}^\infty) \to \Omega^{r+1,s}(\mathscr{R}^\infty)$ is a differential complex and $H^{r,s}(\mathscr{R}^\infty)$ is defined to be its cohomology,

$$H^{r,s}(\mathscr{R}^\infty) = \frac{\operatorname{Ker}\{d_H : \Omega^{r,s}(\mathscr{R}^\infty) \to \Omega^{r+1,s}(\mathscr{R}^\infty)\}}{\operatorname{Im}\{d_H : \Omega^{r-1,s}(\mathscr{R}^\infty) \to \Omega^{r,s}(\mathscr{R}^\infty)\}}.$$

The conservation laws of Δ are the d_H closed forms in $\Omega^{1,0}(\mathscr{R}^\infty)$ while $H^{1,0}(\mathscr{R}^\infty)$ is the space of equivalence classes of conservation laws modulo the horizontal derivative of a function $d_H f$, $f \in C^\infty(\mathscr{R}^\infty)$.

The vertical complex $d_V : \Omega^{r,s}(\mathscr{R}^\infty) \to \Omega^{r,s+1}(\mathscr{R}^\infty)$ is a differential complex whose cohomology is trivial [3], [6]. Specifically, d_V is the ordinary exterior derivative in the variables u_i, and the DeRham homotopy formula (in u_i variables with parameter) applies. The property $d_H d_V = -d_V d_H$ make $d_V : H^{r,s}(\mathscr{R}^\infty) \to H^{r,s+1}(\mathscr{R}^\infty)$ a co-chain map up to sign, see Appendix A.

Remark 2.1. Every function of the form $Q(t,x,u,u_x,u_{xx},\ldots,u_k)$ on $J^\infty(\mathbb{R}^2,\mathbb{R})$ factors through $\pi : J^\infty(\mathbb{R}^2,\mathbb{R}) \to \mathscr{R}^\infty$, $\pi(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots) = (t,x,u,u_x,u_{xx},\ldots)$, where π is a left inverse of ι in equation (2.9). Therefore by an abuse of notation, we view a function of the form $Q(t,x,u,u_x,u_{xx},\ldots,u_k)$ either on $J^\infty(\mathbb{R}^2,\mathbb{R})$ or $\mathscr{R}^\infty$ where the context will determine which. For example,

$$\mathbf{F}_Q = Q_i D_x^i$$

is a differential operator on $J^\infty(\mathbb{R}^2,\mathbb{R})$ while

$$\mathbf{L}_Q = Q_i X^i$$

is a differential operator on $\mathscr{R}^\infty$ which satisfies $\pi_*(\mathbf{F}_Q) = \mathbf{L}_Q$. Given a differential operator $\bar{\mathcal{E}} = r_i(t,x,u,u_x,\ldots)X^i$ on $\mathscr{R}^\infty$ the differential operator $\mathcal{E} = r_i D_x^i$ satisfies $\pi_*\mathcal{E} = \bar{\mathcal{E}}$ and we'll call $\mathcal{E}$ the (canonical) lift. The formal adjoint of $\bar{\mathcal{E}}$ acting on a form ω is $(-X_i)^i(r_i\omega)$. The operator $\bar{\mathcal{E}}$ is skew-adjoint if and only if $\mathcal{E}$ is skew-adjoint.

3. Normal Forms for $H^{1,s}(\mathscr{R}^\infty)$ and Characteristic Forms

The universal linearization (see [6]) of $\Delta = u_t - K(t,x,u,u_x,\dots,u_n)$ on $\mathscr{R}^\infty$ is the differential operator (on $\mathscr{R}^\infty$),

$$\mathbf{L}_\Delta = T - \sum_{i=0}^{n} K_i X^i \tag{3.1}$$

where $K_i = \partial_{u_i} K$, and the vector fields T and X are defined in equation (2.10). The operator $\mathbf{L}_\Delta$ is the restriction of the Fréchet derivative of Δ to $\mathscr{R}^\infty$. The adjoint of $\mathbf{L}_\Delta$ is the differential operator defined by

$$\mathbf{L}^*_\Delta(\rho) = -T(\rho) - \sum_{i=0}^{n} (-X)^i (K_i \rho), \qquad \rho \in \Omega^*(\mathscr{R}^\infty).$$

This next theorem provides a normal form for a representative of the cohomology classes in $H^{1,s}(\mathscr{R}^\infty)$ and is analogous to Theorem 5.1 in [6].

Theorem 3.1. *Let $\Delta = u_t - K(t,x,u,u_x,\dots,u_n)$ define an n^{th} order evolution equation $\Delta = 0$ and let $H^{r,s}(\mathscr{R}^\infty)$ be its cohomology with $s \geq 1$. For any $[\omega] \in H^{1,s}(\mathscr{R}^\infty)$ there exists a representative,*

$$\omega = dx \wedge \theta^0 \wedge \rho - dt \wedge \beta, \tag{3.2}$$

where $\rho \in \Omega^{0,s-1}(\mathscr{R}^\infty)$, $\beta \in \Omega^{0,s}(\mathscr{R}^\infty)$ and $\mathbf{L}^*_\Delta(\rho) = 0$.

Proof. The proof follows Theorem 5.1 of [6]. Choose $\tilde{\omega}_0 \in \Omega^{1,s}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ such that

$$\iota^*(\tilde{\omega}_0) \in [\omega]. \tag{3.3}$$

where $\iota : \mathscr{R}^\infty \to J^\infty(\mathbb{R}^2,\mathbb{R})$ is given in equation (2.9). Since $\iota^*(d_H\tilde{\omega}_0) = 0$, it there exists $\tilde{\zeta}_{ab} \in \Omega^{0,s}(J^\infty(\mathbb{R}^2,\mathbb{R})), \tilde{\mu}_{ab} \in \Omega^{0,s-1}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ such that (see Lemma 5.2 in [6])

$$d_H\tilde{\omega}_0 = dt \wedge dx \wedge (D_t^a D_x^b(\Delta)\tilde{\zeta}_{ab} + D_t^a D_x^b(d_V\Delta) \wedge \tilde{\mu}_{ab}), \tag{3.4}$$

Applying the identical integration by parts argument on page 292 [6] to (3.4), implies there exists $\tilde{\zeta} \in \Omega^{0,s}(J^\infty(\mathbb{R}^2,\mathbb{R})), \tilde{\rho} \in \Omega^{0,s-1}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ and $\tilde{\omega} \in \Omega^{1,s}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ such that $\tilde{\omega} = \tilde{\omega}_0 + \tilde{d_H}\tilde{\eta}$ and $\iota^*\tilde{\eta} = 0$ (hence $\iota^*\tilde{\omega} = \omega$) and where

$$d_H\tilde{\omega} = dt \wedge dx \wedge (\Delta\tilde{\zeta} + d_V\Delta \wedge \tilde{\rho}). \tag{3.5}$$

We now apply $\iota^* \circ J$ to equation (3.5), where J is defined in equation (2.4). For the first term in right hand side of equation (3.5) we find

$$\iota^* \circ J(\Delta\tilde{\zeta}) = \iota^*\left(\Delta\partial_u \lrcorner (\tilde{\zeta}) - D_t(\Delta\partial_{u_t} \lrcorner \tilde{\zeta}) - D_x(\Delta\partial_{u_x} \lrcorner \tilde{\zeta}) + \cdots\right) = 0 \tag{3.6}$$

since each term contains a total derivative of Δ, and these vanish under pullback to $\mathscr{R}^\infty$.

We now apply $\iota^* \circ J$ to the second term in the right hand side of (3.5),

$$\begin{aligned}\iota^* J(d_V\Delta \wedge \tilde{\rho}) &= \iota^* \left(\sum_{a=0,i=0}^{\infty} (-1)^{i+a} D_x^i D_t^a ((\partial_{u_{a,i}} \lrcorner\, d_V\Delta)\tilde{\rho} - (\partial_{u_{a,i}} \lrcorner\, \tilde{\rho}) d_V\Delta) \right) \\ &= \iota^* \left(\sum_{a=0,i=0}^{\infty} (-1)^{i+a} D_x^i D_t^a ((\partial_{u_{a,i}} \lrcorner\, d_V\Delta)\tilde{\rho}) \right)\end{aligned} \tag{3.7}$$

because $\iota^*(D_t^a D_x^i d_V\Delta) = 0$. Now

$$\partial_{u_{1,0}} \lrcorner\, d_V\Delta = 1, \quad \partial_{u_{0,i}} \lrcorner\, d_V\Delta = -K_i$$

with all other $\partial_{u_{a,i}} \lrcorner\, d_V\Delta = 0$, so that equation (3.7) becomes,

$$\iota^* J(d_V\Delta \wedge \tilde{\rho}) = \iota^*(-D_t(\tilde{\rho}) + \sum_{i=0}^{n} (-1)^i D_x^i(-K_i\tilde{\rho})) = \mathbf{L}_\Delta^*(\iota^*\tilde{\rho}). \tag{3.8}$$

By equation (2.5) $J(d_H\tilde{\omega}) = 0$, so that applying $\iota^* \circ J$ to equation (3.5) implies $\iota^* J(d_V\Delta \wedge \tilde{\rho}) = 0$, and so equation (3.8) gives $\mathbf{L}_\Delta^*(\iota^*\tilde{\rho}) = 0$.

We now turn to showing that equation (3.2) holds using the horizontal homotopy operator (equations (5.15), (5.16) and below (5.16) in [6]), see also proposition 4.12 page 117 of [3] or equation (5.133) in [17]. Using the notation $h_H^{r,s}$ from [3], this operator satisfies

$$\tilde{\omega} = h_H^{2,s}(d_H\tilde{\omega}) + d_H(h_H^{1,s}\tilde{\omega}), \quad \tilde{\omega} \in \Omega^{1,s}(J^\infty(\mathbb{R}^2, \mathbb{R})).$$

Applying the pull back by ι to this formula gives the representative for $[\omega]$,

$$\omega = \iota^* h_H^{2,s}(d_H\tilde{\omega}) \tag{3.9}$$

with $d_H\tilde{\omega}$ in (3.5).

To utilize the formula in [3] for $h_H^{2,s}$ let $(x^1 = t, x^2 = x)$ and so for example $\vartheta^{1122} = D_x D_x D_t D_t \vartheta^0$ and let k be the max of $|I|$ (number of derivatives) of ϑ^I terms in $(\vartheta_t - K_i\vartheta^i) \wedge \rho$. Then by definition 4.13 on page 117 in [3] (or 5.134 in [17])

$$\begin{aligned}h_H^{2,s}(d_H\tilde{\omega}) &= \frac{1}{s} \sum_{|I|=0}^{k-1} D_I \left(\vartheta^0 \wedge J^{Ij}((d_H\tilde{\omega})_j) \right) \\ &= \sum_{|I|=0}^{k-1} D_I \left(\vartheta^0 \wedge (-1)^j d\hat{x}_j \wedge J^{Ij}(\Delta\tilde{\zeta} + d_V\Delta \wedge \tilde{\rho}) \right)\end{aligned} \tag{3.10}$$

where $(d_H\tilde{\omega})_j = D_{x^j} \lrcorner\, d_H\tilde{\omega} = (-1)^{j+1} d\hat{x}_j \wedge (\Delta\tilde{\zeta} + d_V\Delta \wedge \tilde{\rho})$, $(\hat{x}_1 = x, \hat{x}_2 = t)$, $I = (i_1, \ldots, i_l)$, $|I| = l$, and

$$J^{Ij}(\Delta\tilde{\zeta} + d_V\Delta \wedge \tilde{\rho}) = \sum_{|L|=0}^{k-|I|-1} \binom{|I|+|L|+1}{|L|} (-D)_L \left(\partial_{u_{(jIL)}} \lrcorner\, (\Delta\tilde{\zeta} + d_V\Delta \wedge \tilde{\rho}) \right). \tag{3.11}$$

Applying ι^* to equation (3.10) we have the $\iota^* h_H^{2,s}\left(dt \wedge dx \wedge (\Delta\tilde{\zeta})\right) = 0$ because all terms in (3.11) on $\Delta\tilde{\zeta}$ involve total derivatives of Δ. Therefore using equation (3.10), equation (3.9) becomes,

$$
\begin{aligned}
\omega = \frac{1}{s}\iota^*\Bigg(& dx \wedge \sum_{|I|=0}^{k-1} D_I \left[\vartheta^0 \wedge \sum_{|L|=0}^{k-|I|-1} \binom{|I|+|L|+1}{|L|} (-D)_L((\partial_{u_{(1IL)}} \lrcorner\, d_V\Delta)\tilde{\rho}) \right] \\
& -dt \wedge \sum_{|I|=0}^{k-1} D_I \left[\vartheta^0 \wedge \sum_{|L|=0}^{k-|I|-1} \binom{|I|+|L|+1}{|L|} (-D)_L((\partial_{u_{(2IL)}} \lrcorner\, d_V\Delta)\tilde{\rho}) \right] \Bigg)
\end{aligned}
\tag{3.12}
$$

Consider the first term in equation (3.12). The only non-zero interior product is (with $u_{(1)} = u_t$, $u_{(2)} = u_x$ etc.)

$$\partial_{u_{(1)}} \lrcorner\, d_V\Delta = 1,$$

since Δ does not depend on derivatives such as u_{tx}. Therefore the only non-zero terms have $|I| = 0$, $|L| = 0$ in the first term of (3.12) giving,

$$dx \wedge \sum_{|I|=0}^{k-1} D_I \left[\vartheta^0 \wedge \sum_{|L|=0}^{k-|I|-1} \binom{|I|+|L|+1}{|L|} (-D)_L((\partial_{u_{(1IL)}} \lrcorner\, d_V\Delta)\tilde{\rho}) \right] = dx \wedge \vartheta^0 \wedge \tilde{\rho} \tag{3.13}$$

Combining equation (3.13) with (3.12) we have

$$\omega = \frac{1}{s}\iota^*\left(dx \wedge \vartheta^0 \wedge \tilde{\rho} - dt \wedge \sum_{|I|=0}^{k-1} D_I \left[\vartheta^0 \wedge \sum_{|L|=0}^{k-|I|-1} \binom{|I|+|L|+1}{|L|} (-D)_L((\partial_{u_{(2IL)}} \lrcorner\, d_V\Delta)\tilde{\rho} \right] \right) \tag{3.14}$$

which produces equation (3.2) with $\rho = \frac{1}{s}\iota^*\tilde{\rho}$. Equations (3.14) and (3.8) shows that $\rho = \frac{1}{s}\iota^*\tilde{\rho}$ satisfies $\mathbf{L}^*_\Delta(\rho) = 0$. $\square$

If $s = 1$ in Theorem 3.1, then $\rho \in C^\infty(\mathscr{R}^\infty)$ is the characteristic function for the cohomology class $[\omega] \in H^{1,1}(\mathscr{R}^\infty)$, see Theorem 3.3 and Theorem A.1. In general ρ in equation (3.2) is called a characteristic form for $[\omega]$ see [6]. The form β in (3.2) is given in terms of ρ by formula (3.14) which is simplified in Corollary 3.2 for $H^{1,1}(\mathscr{R}^\infty)$ and $H^{1,2}(\mathscr{R}^\infty)$. The term $dx \wedge \theta^0 \wedge \rho$ in equation (3.2) generalizes the conserved density of a conservation law, and plays a critical role in Section 7.

Theorem 3.2. *Let $u_t = K(t, x, u, u_x, \ldots, u_n)$ be an n^{th} order evolution equation where $n \geq 2$. The cohomology satisfies $H^{1,s}(\mathscr{R}^\infty) = 0$ for all $s \geq 3$.*

This is essentially Theorem 1 in [13] and we give a different proof.

Proof. Suppose ω is a representative for an element of $H^{1,s+1}(\mathscr{R}^\infty)$, $(s \geq 2)$, in the form (3.2), where $\rho \in \Omega^{0,s}(\mathscr{R}^\infty)$ is given by

$$\rho = A_{i_1 \ldots i_s} \theta^{i_1} \wedge \cdots \wedge \theta^{i_s}. \tag{3.15}$$

and satisfies $\mathbf{L}^*_\Delta(\rho) = 0$.

Suppose for ρ in (3.15) that the highest form order is (no sum) $A_{m_1 \ldots m_s} \theta^{m_1} \wedge \theta^{m_2} \wedge \cdots \wedge \theta^{m_s}$ where we use lexicographic order so that max of $(m_1 > m_2 > \cdots > m_s)$ determines the highest order. We first claim that in $\mathbf{L}^*_\Delta(\rho)$ the coefficient of $\theta^{m_1+n}\wedge\theta^{m_2}\wedge\cdots\wedge\theta^{m_s}$ is

$$[\mathbf{L}^*_\Delta(\rho)]_{m_1+n,m_2,\dots,m_s} = (-1-(-1)^n)K_nA_{m_1\dots m_s}, \tag{3.16}$$

and that the coefficient of $\theta^{m_1+n-1}\wedge\theta^{m_1}\wedge\cdots\wedge\theta^{m_s}$ when $m_1>m_2+1$ is

$$[\mathbf{L}^*_\Delta(\rho)]_{m_1+n-1,m_2+1,\dots,m_s} = (-1)^n nK_nA_{m_1\dots m_s} \tag{3.17}$$

and when $m_1=m_2+1$ and n is odd, the coefficient of $\theta^{m_1+n-1}\wedge\theta^{m_1}\wedge\cdots\wedge\theta^{m_s}$

$$[\mathbf{L}^*_\Delta(\rho)]_{m_1+n-1,m_1,\dots,m_s} = -(-1)^n nK_nA_{m_1\dots m_s}. \tag{3.18}$$

Therefore $\mathbf{L}^*_\Delta(\rho)=0$ implies $A_{m_1\dots m_s}=0$, $\rho=0$ and $\omega=dt\wedge\beta$. The condition $d_H\omega=0$ gives $X(\beta)=0$. This implies $\beta=0$ since $\beta\in\Omega^{0,s+1}(\mathscr{R}^\infty)$ and so $\omega=0$.

We compute $\mathbf{L}^*_\Delta(\rho)=-T(\rho)-(-X)^i(K_i\rho)$ to find equations (3.16), (3.17), (3.18),

$$\begin{aligned}\mathbf{L}^*_\Delta(\rho) = &-T(A_{i_1\dots i_s})\theta^i_1\cdots\wedge\theta^{i_s} - A_{i_1\dots i_s}T(\theta^{i_1})\cdots\wedge\theta^{i_s} - A_{i_1\dots i_s}\theta^{i_1}\wedge T(\theta^{i_2})\cdots\wedge\theta^{i_s}+\cdots\\ &-(-1)^nX^n(K_nA_{i_1\dots i_s}\theta^{i_1}\wedge\cdots\wedge\theta^{i_s}) - (-1)^{n-1}X^{n-1}(K_{n-1}A_{i_1\dots i_s}\theta^{i_1}\wedge\cdots\wedge\theta^{i_s})\dots\end{aligned} \tag{3.19}$$

where from equation (2.15) we have $T(\theta^i)=K_n\theta^{i+n}+\text{lower order}$. Consider also the highest order terms in expanding $X^n(K_nA_{i_1\dots i_s}\theta^{i_1}\wedge\cdots\wedge\theta^{i_s})$ (no sum),

$$\begin{aligned}&X^n(K_nA_{m_1\dots m_s}\theta^{m_1}\wedge\theta^{m_2}\cdots\wedge\theta^{m_s})\\ &= X^n(K_nA_{m_1\dots m_s})\theta^{m_1}\wedge\theta^{m_2}\cdots\wedge\theta^{m_s}\\ &\quad + K_nA_{m_1\dots m_s}\theta^{m_1+n}\wedge\theta^{m_2}\cdots\wedge\theta^{m_s} + nK_nA_{m_1\dots m_s}\theta^{m_1+n-1}\wedge\theta^{m_2+1}\cdots\wedge\theta^{m_s}+\cdots\\ &\quad + K_nA_{m_1\dots m_s}\theta^{m_1}\wedge\theta^{m_2+n}\cdots\wedge\theta^{m_s}+\cdots\\ &= K_nA_{m_1\dots m_s}\theta^{m_1+n}\wedge\theta^{m_2}\cdots\wedge\theta^{m_s} + nK_nA_{m_1\dots m_s}\theta^{m_1+n-1}\wedge\theta^{m_2+1}\cdots\wedge\theta^{m_s}\\ &\quad - K_nA_{m_1\dots m_s}\theta^{m_2+n}\wedge\theta^{m_1}\cdots\wedge\theta^{m_s} + \text{ lower order}\dots\end{aligned} \tag{3.20}$$

The coefficient of $\theta^{m_1+n}\wedge\theta^{m_2}\wedge\cdots\wedge\theta^{m_s}$ (which is the highest order) occurring in equation (3.19) comes from the second term on the right hand side in (3.19) and the first term on the last right hand side in equation (3.20) to give (3.16).

We consider the next highest order term in (3.19). From equation (3.19), the only possible term that can contain $\theta^{m_1+n-1}\wedge\theta^{m_2+1}\wedge\theta^{m_3}\cdots\theta^{m_s}$ when $m_1>m_2+1$ is from second term on the last right hand side of equation (3.20). Therefore 3.20 produces (3.17).

In the case when $m_1=m_2+1$ we have from the second term in right side of (3.19) at highest order giving (no sum)

$$\begin{aligned}-A_{m_1\dots m_s}\theta^{m_1}\wedge T(\theta^{m_2})\cdots\wedge\theta^{m_s} &= -A_{m_1\dots m_s}\theta^{m_1}\wedge(K_n\theta^{m_2+n})\cdots\wedge\theta^{m_s}+\text{lower order},\\ &= K_nA_{m_1\dots m_s}\theta^{m_1-1+n}\wedge\theta^{m_1}\cdots\wedge\theta^{m_s}+\text{lower order}.\end{aligned} \tag{3.21}$$

The second and third term on the last right hand side in equation (3.20) are (no sum)

$$nK_nA_{m_1\dots m_s}\theta^{m_1+n-1}\wedge\theta^{m_2+1}\cdots\wedge\cdots\theta^{m_s} - K_nA_{m_1\dots m_s}\theta^{m_1+n-1}\wedge\theta^{m_2+1}\cdots\wedge\theta^{m_s}. \tag{3.22}$$

Using $m_1=m_2+1$, equations (3.21), (3.22) and that n is odd in equation (3.19), gives equation (3.18). □

We also have as a corollary of Theorem 3.1.

Corollary 3.1. *If $u_t = K(t,x,u,\ldots,u_{2m})$, $m \geq 1$ is an even order evolution equation, then $H^{1,2}(\mathscr{R}^\infty) = 0$.*

Proof. We show that the only $\rho \in \Omega^{0,1}(\mathscr{R}^\infty)$ satisfying $\mathbf{L}^*_\Delta(\rho) = 0$ is $\rho = 0$. Suppose $\rho = r_i\theta^i$, $i = 0,\ldots,k$ then by direct computation and the fact $2m > 0$

$$\mathbf{L}^*_\Delta(\rho) = -T(\rho) - (-X)^i(K_i\rho) \equiv -2\,r_k K_{2m}\theta^{2m+k} \qquad \mod \theta^0,\ldots,\theta^{2m+k-1}$$

which is non-zero unless $r_k = 0$. Therefore $\rho = 0$. □

This next theorem is a partial converse to Theorem 3.1 which will be used in Corollary 3.2 below to provide a formula for β in equation (3.2).

Theorem 3.3. *Let $\Delta = u_t - K(t,x,u,u_x,\ldots,u_n)$ define an n^{th} order evolution equation $\Delta = 0$. Let $\rho \in \Omega^{0,s-1}(\mathscr{R}^\infty)$ $(s = 1,2)$ satisfies $\theta^0 \wedge \mathbf{L}^*_\Delta(\rho) = 0$, then*

$$\omega = dx \wedge \theta^0 \wedge \rho - dt \wedge \boldsymbol{\beta}(\rho), \quad \boldsymbol{\beta}(\rho) = \sum_{i=1}^{n}\left(\sum_{a=1}^{i}(-X)^{a-1}(K_i\rho) \wedge \theta^{i-a}\right), \tag{3.23}$$

satisfies $d_H\omega = 0$.

Proof. First suppose $\rho \in \Omega^{0,s-1}(\mathscr{R}^\infty)$ and satisfies $\theta^0 \wedge \mathbf{L}^*_\Delta(\rho) = 0$, and let $\omega \in \Omega^{1,s}(\mathscr{R}^\infty)$ be as in equation (3.23). We compute $d_H\omega$,

$$d_H\omega = dt \wedge dx \wedge [T(\theta^0 \wedge \rho) + X(\boldsymbol{\beta}(\rho))]. \tag{3.24}$$

To compute $X(\boldsymbol{\beta}(\rho))$ we need the telescoping identity,

$$X\left(\sum_{a=1}^{i}(-X)^{a-1}(K_i\rho) \wedge \theta^{i-a}\right) = -(-X)^i(K_i\rho) \wedge \theta^0 + K_i\rho \wedge \theta^i \qquad (\text{ no sum on } i). \tag{3.25}$$

Using equation (3.25) in the formula for $\boldsymbol{\beta}(\rho)$ in equation (3.23) gives

$$\begin{aligned} X(\boldsymbol{\beta}(\rho)) &= \sum_{i=1}^{n} X\left[\left(\sum_{a=1}^{i}(-X)^{a-1}(K_i\rho) \wedge \theta^{i-a}\right)\right], \\ &= \sum_{i=1}^{n} -(-X)^i(K_i\rho) \wedge \theta^0 + K_i\rho \wedge \theta^i. \end{aligned} \tag{3.26}$$

We then use $T(\theta^0 \wedge \rho) = T(\theta^0) \wedge \rho + \theta^0 \wedge T(\rho)$ so that together with equation (3.26), equation (3.24) becomes (adding and subtracting $K_0\theta^0 \wedge \rho$)

$$d_H\omega = dt \wedge dx \wedge (T(\theta^0) - \sum_{i=0}^{n} K_i\theta^i) \wedge \rho + dt \wedge dx \wedge \theta^0 \wedge \left(T(\rho) + \sum_{i=0}^{n}(-X)^i(K_i\rho)\right) \tag{3.27}$$

Now by equation (2.7), $\iota^* d_V(-\Delta) = -T(\theta^0) + \sum_{i=0}^{n} K_i\theta^i = 0$, and so equation (3.27) becomes,

$$d_H\omega = dt \wedge dx \wedge \theta^0 \wedge \left(T(\rho) + \sum_{i=0}^{n}(-X)^i(K_i\rho)\right) = -dt \wedge dx \wedge \theta^0 \wedge \mathbf{L}^*_\Delta(\rho) = 0.$$

□

Corollary 3.2. *For any* $[\omega] \in H^{1,s}(\mathscr{R}^\infty)$*,* $s = 1,2$ *the representative,* $\omega \in \Omega^{1,s}(\mathscr{R}^\infty)$*,* $s = 1,2$ *in equation* (3.2) *is given by*

$$\omega = dx \wedge \theta^0 \wedge \rho - dt \wedge \boldsymbol{\beta}(\rho), \quad \boldsymbol{\beta}(\rho) = \sum_{i=1}^{n} \left(\sum_{a=1}^{i} (-X)^{a-1}(K_i \rho) \wedge \theta^{i-a} \right), \tag{3.28}$$

where $\rho \in \Omega^{1,s-1}(\mathscr{R}^\infty)$*. If* $s = 1$*, then the representative* (3.28) *is unique.*

Proof. Starting with the representative in equation (3.2) of Theorem 3.1 we have $\omega = dx \wedge \theta^0 \wedge \rho - dt \wedge \beta$ where $\mathbf{L}^*(\rho) = 0$. Let $\hat{\omega}$ be the form in (3.28) using this ρ which satisfies $\mathbf{L}^*_\Delta(\rho) = 0$ and so by Theorem 3.3, $d_H \hat{\omega} = 0$. The form $\omega' = \omega - \hat{\omega}$ is then d_H closed, leading to

$$0 = d_H \omega' = d_H(dt \wedge (\beta - \boldsymbol{\beta}(\rho)) = dx \wedge dt \wedge X(\beta - \boldsymbol{\beta}(\rho)). \tag{3.29}$$

This implies $X(\beta - \boldsymbol{\beta}(\rho)) = 0$, where $\beta - \boldsymbol{\beta}(\rho) \in \Omega^{0,s}(\mathscr{R}^\infty), s = 1,2$. However, the only contact form satisfying this condition is the zero form. So $\beta = \boldsymbol{\beta}(\rho)$. This proves equation (3.28).

For the final statement in the theorem, suppose $\omega_a = dx \wedge \theta^0 \cdot Q_a - dt \wedge \beta_a$, $a = 1,2$ where $Q_a \in C^\infty(\mathscr{R}^\infty)$ and $\beta_a \in \Omega^{0,1}(\mathscr{R}^\infty)$ satisfy $[\omega_1] = [\omega_2] \in H^{1,1}(\mathscr{R}^\infty)$. This implies there exists $\xi = g_j \theta^j$ such that $\omega_1 - \omega_2 = d_H \xi$ so that

$$dx \wedge \theta^0 (Q_1 - Q_2) = dx \wedge X(g_j \theta^j). \tag{3.30}$$

Since $X(\theta^i) = \theta^{i+1}$, equation (3.30) can only be satisfied when $Q_1 = Q_2$ and $g_j = 0$. Therefore $\omega_1 = \omega_2$ and the form ω in equation (3.28) when $s = 1$ is unique. □

The form ω in equation (3.28) can be derived by a rather lengthy calculation from the second term in equation (3.14).

A form $\rho \in \Omega^{0,1}(\mathscr{R}^\infty)$ can be written $\rho = r_i X^i(\theta^0)$. We define the adjoint of ρ by $\rho^* = (-X)^i(r_i \theta^0)$ while $(\rho^*)^* = \rho$ because the operator $r_i X^i$ has this property, see Remark 2.1.

Theorem 3.4. *Suppose* $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ *admits a representative*

$$\omega = dx \wedge \theta^0 \wedge \varepsilon - dt \wedge \beta, \qquad \varepsilon \in \Omega^{0,1}(\mathscr{R}^\infty), \quad \beta \in \Omega^{0,2}(\mathscr{R}^\infty)$$

where $\varepsilon^* = -\varepsilon$*. Then* $\mathbf{L}^*_\Delta(\varepsilon) = 0$ *and*

$$\beta = \boldsymbol{\beta}(\varepsilon) = \sum_{i=1}^{n} \left(\sum_{a=1}^{i} (-X)^{a-1}(K_i \varepsilon) \wedge \theta^{i-a} \right).$$

Proof. Write $\beta = B_{ab}\theta^a \wedge \theta^b$, and choose in equation (3.3), $\tilde{\omega}_0 = dx \wedge \vartheta^0 \wedge \tilde{\varepsilon}_0 - dt \wedge \tilde{\beta}$, where $\tilde{\varepsilon}_0 = r_i \vartheta^i, \tilde{\beta} = B_{ab}\vartheta^a \wedge \vartheta^b$ (see Remark 2.1). Then there exists $s_{ab} \in C^\infty(J^\infty(\mathbb{R}^2,\mathbb{R}))$ such that,

$$\begin{aligned}
d_H(\tilde{\omega}_0) &= dt \wedge dx \wedge (\vartheta_t \wedge \tilde{\varepsilon}_0 + \vartheta_0 \wedge D_t(\tilde{\varepsilon}_0) + D_x(\tilde{\beta})) \\
&= dt \wedge dx \wedge ((d_V \Delta + K_m \vartheta^m) \wedge \tilde{\varepsilon}_0 + \vartheta_0 \wedge D_t(\tilde{\varepsilon}_0) + D_x(\tilde{\beta})) \\
&= dt \wedge dx \wedge ((d_V \Delta + K_m \vartheta^m) \wedge \tilde{\varepsilon}_0 + \vartheta_0 \wedge (r_{i,t}\vartheta^i + r_i \vartheta^i_t) + D_x(\tilde{\beta})) \\
&= dt \wedge dx \wedge (d_V \Delta \wedge \tilde{\varepsilon}_0 + \vartheta^0 \wedge (r_i D^i_x(d_V \Delta)) + s_{ab}\vartheta^a \wedge \vartheta^b)
\end{aligned} \tag{3.31}$$

since $D_t(\vartheta^0) = d_V\Delta + K_m\vartheta^m$. Now using equations (3.25) and (3.26) with $X = D_x$, $\rho = \vartheta^0$, $K_i = r_i$, and $\theta^0 = d_V\alpha$ while adding and subtracting $r_0\vartheta^0 \wedge d_V\Delta$ we have

$$dt \wedge dx \wedge \vartheta^0 \wedge (r_i D_x^i(d_V\Delta)) = dt \wedge dx \wedge (-D_x)^i(r_i\vartheta^0) \wedge d_V\Delta - d_H\tilde{\eta} \tag{3.32}$$

where

$$\tilde{\eta}_0 = dt \wedge \sum_{i=1}^{k}\sum_{j=1}^{i}(-D_x)^{j-1}(r_i\vartheta^0) \wedge D_x^{i-j}(d_V\Delta) \tag{3.33}$$

and $\tilde{\eta}_0$ satisfies $\iota^*\tilde{\eta}_0 = 0$. Since $\varepsilon^* = -\varepsilon$, we have $\tilde{\varepsilon}_0^* = -\tilde{\varepsilon}_0$, and combining this with equation (3.31) and (3.32) we have

$$d_H(\tilde{\omega}_0) = dt \wedge dx \wedge (d_V\Delta \wedge 2\tilde{\varepsilon}_0 + s_{ab}\vartheta^a \wedge \vartheta^b) - d_H\tilde{\eta}_0 \tag{3.34}$$

Therefore comparing equations (3.34) with equation (3.5) we have $\tilde{\rho} = 2\tilde{\varepsilon}_0$. By equations (3.14) in the proof of Theorem 3.1, $\rho = \frac{1}{2}\iota^*\tilde{\rho} = \varepsilon$ satisfies $\mathbf{L}_\Delta^*(\varepsilon) = 0$. Finally Theorem 3.3 implies $\beta = \boldsymbol{\beta}(\varepsilon)$. □

Corollary 3.3. *Let $\varepsilon \in \Omega^{0,1}(\mathscr{R}^\infty)$ satisfy $\varepsilon^* = -\varepsilon$. Then $\theta^0 \wedge \mathbf{L}_\Delta^*(\varepsilon) = 0$ if and only if $\mathbf{L}_\Delta^*(\varepsilon) = 0$.*

Proof. Let ε be as stated and satisfy $\theta^0 \wedge \mathbf{L}^*(\varepsilon) = 0$. The form ω with $\rho = \varepsilon$ in equation (3.23) of Theorem 3.3 satisfies $d_H\omega = 0$. By Theorem 3.4 $\mathbf{L}_\Delta^*(\varepsilon) = 0$. The if part of the statement is trivial. □

4. A Canonical Form for $H^{1,2}(\mathscr{R}^\infty)$ and the Snake Lemma

We now refine Theorem 3.1 to produce a canonical form for elements of $H^{1,2}(\mathscr{R}^\infty)$ by determining a unique representative for any $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$.

Theorem 4.1. *Let $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$. There exists a unique representative for $[\omega]$ of the form*

$$\omega = dx \wedge \theta^0 \wedge \varepsilon - dt \wedge \boldsymbol{\beta}(\varepsilon)\,, \quad \boldsymbol{\beta}(\varepsilon) = \sum_{i=1}^{n}\left(\sum_{a=1}^{i}(-X)^{a-1}(K_i\varepsilon) \wedge \theta^{i-a}\right), \tag{4.1}$$

where $\varepsilon \in \Omega^{0,1}(\mathscr{R}^\infty)$, $\varepsilon^* = -\varepsilon$, and $\mathbf{L}_\Delta^*(\varepsilon) = 0$.

Proof. We begin by utilizing equation (3.26) and make the substitution $\rho = \theta^0$, $K_i = r_i$ giving the identity,

$$X\left(\sum_{i=1}^{k}\left(\sum_{a=1}^{i}(-X)^{a-1}(r_i\theta^0) \wedge \theta^{i-a}\right)\right) = \sum_{i=1}^{k}\left(-(-X)^i(r_i\theta^0) \wedge \theta^0 + r_i\theta^0 \wedge \theta^i\right). \tag{4.2}$$

If we now write $\rho = \sum_{i=1}^{k} r_i\theta^i$ and let

$$\eta = \sum_{i=1}^{k}\left(\sum_{a=1}^{i}(-X)^{a-1}(r_i\theta^0) \wedge \theta^{i-a}\right), \tag{4.3}$$

the identity (4.2) gives

$$X(\eta) = \theta^0 \wedge (\rho^* + \rho) \tag{4.4}$$

Suppose now $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ with representative $\omega = dx \wedge \theta^0 \wedge \rho - dt \wedge \boldsymbol{\beta}(\rho)$ with $\rho = r_i \theta^i$ from Theorem 3.2. Let $\hat{\omega} = \omega - \frac{1}{2} d_H(\eta)$ where η is given in equation (4.3), so that $[\hat{\omega}] = [\omega]$. We then use equation (4.4) to replace $X(\eta)$ in the following,

$$\begin{aligned}
\hat{\omega} &= dx \wedge \theta^0 \wedge \rho - dt \wedge \boldsymbol{\beta}(\rho) - \frac{1}{2} dx \wedge X(\eta) - \frac{1}{2} dt \wedge T(\eta) \\
&= dx \wedge (\theta^0 \wedge \rho - \frac{1}{2} X(\eta)) - dt \wedge (\boldsymbol{\beta}(\rho) + \frac{1}{2} T(\eta)) \\
&= dx \wedge \left(\theta^0 \wedge \rho - \frac{1}{2}(\theta^0 \wedge \rho + \theta^0 \wedge \rho^*) \right) - dt \wedge (\boldsymbol{\beta}(\rho) + \frac{1}{2} T(\eta)) \\
&= dx \wedge \theta^0 \wedge \frac{1}{2}(\rho - \rho^*) - dt \wedge (\boldsymbol{\beta}(\rho) + \frac{1}{2} T(\eta)).
\end{aligned} \tag{4.5}$$

The representative $\hat{\omega}$ in (4.5) satisfies the skew-adjoint condition in the theorem with

$$\varepsilon = \frac{1}{2}(\rho - \rho^*),$$

while Theorem 3.4 shows $\mathbf{L}^*_\Delta(\varepsilon) = 0$ and $\boldsymbol{\beta}(\rho) + \frac{1}{2} T(\eta) = \boldsymbol{\beta}(\varepsilon)$.

We now show the representative (4.1) unique. Suppose that

$$\omega_\alpha = dx \wedge \theta^0 \wedge \varepsilon_\alpha - dt \wedge \check{\boldsymbol{\beta}}(\varepsilon_a), \qquad \alpha = 1, 2 \tag{4.6}$$

where $\varepsilon^*_\alpha = -\varepsilon_\alpha$ and $[\omega_1] = [\omega_2]$. This implies there exists $\xi = \xi_{ab} \theta^a \wedge \theta^b \in \Omega^{0,2}(\mathscr{R}^\infty)$ such that

$$dx \wedge \theta^0 \wedge (\varepsilon_1 - \varepsilon_2) = dx \wedge X(\xi_{ab} \theta^a \wedge \theta^b). \tag{4.7}$$

Now let $\tilde{\varepsilon}_\alpha = r_{i,\alpha} \vartheta^i \in \Omega^{0,1}(J^\infty(\mathbb{R}^2, \mathbb{R}))$ and $\tilde{\xi} = \xi_{ab} \vartheta^a \wedge \vartheta^b \in \Omega^{0,2}(J^\infty(\mathbb{R}^2, \mathbb{R}))$ so that $\iota^* \tilde{\varepsilon}_\alpha = \varepsilon_\alpha$, $\iota^* \tilde{\xi} = \xi$. Equation (4.7) implies

$$dt \wedge dx \wedge \vartheta^0 \wedge (\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2) = dt \wedge [D_x(\tilde{\xi})] = -d_H(dt \wedge \tilde{\xi}). \tag{4.8}$$

Applying the integration by parts operator I (using (2.5)) to equation (4.8) and that $\tilde{\varepsilon}^*_\alpha = -\tilde{\varepsilon}_\alpha$ gives

$$dt \wedge dx \wedge \theta^0 \wedge (\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2) = 0.$$

Since $\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2$ is skew-adjoint, this implies $\tilde{\varepsilon}_1 = \tilde{\varepsilon}_2$ and hence $\varepsilon_1 = \varepsilon_2$. Therefore $\boldsymbol{\beta}(\varepsilon_1) = \boldsymbol{\beta}(\varepsilon_2)$ and $\omega_1 = \omega_2$. □

Corollary 4.1. *If* $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ *with representative* $\omega = dx \wedge \theta^0 \wedge \rho - dt \wedge \boldsymbol{\beta}(\rho)$ *then the unique representative in Theorem* 4.1 *has*

$$\varepsilon = \frac{1}{2}(\rho - \rho^*). \tag{4.9}$$

The second part of Corollary 3.3 provides an isomorphism between $H^{1,1}(\mathscr{R}^\infty)$ and the solution space $\mathbf{L}^*_\Delta(Q) = 0$, $Q \in C^\infty(\mathscr{R}^\infty)$. We now extend this to $H^{1,2}(\mathscr{R}^\infty)$.

Corollary 4.2. *Let* $S = \{\varepsilon \in \Omega^{0,1}(\mathscr{R}^\infty) \mid \varepsilon^* = -\varepsilon \text{ , and } \theta^0 \wedge \mathbf{L}^*(\varepsilon) = 0\}$. *The linear map* $\chi : S \to H^{1,2}(\mathscr{R}^\infty)$ *given by* $\chi(\varepsilon) = [dx \wedge \theta^0 \wedge \varepsilon - dt \wedge \boldsymbol{\beta}(\varepsilon)]$ *where* $\boldsymbol{\beta}(\varepsilon)$ *is given in equation* (4.1)*, is an isomorphism.*

Proof. Given ε satisfying the conditions of the corollary, Theorem 3.3 shows $\chi(\varepsilon) \in H^{1,2}(\mathscr{R}^\infty)$. Theorem 4.1 shows directly that χ is onto, while the uniqueness of the representative in Theorem 4.1 shows that χ is one-to-one. □

See Section 8.1 for an application of Corollary 4.2.

We now refine Theorem 3.1 and provide a third (non-unique) normal form.

Theorem 4.2. *Given* $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$*, there exists a representative* ω *such that*

$$\omega = dx \wedge \theta^0 \wedge d_V Q - dt \wedge d_V \gamma = d_V \left(dx \wedge \theta^0 \cdot Q - dt \wedge \gamma \right) \tag{4.10}$$

where Q *is a smooth function on* $\mathscr{R}^\infty$ *and* $\gamma \in \Omega^{0,1}(\mathscr{R}^\infty)$.

Proof. We start with equation (3.2) in Theorem 3.1 where a representative for $[\omega]$ can be written

$$\omega = dx \wedge \theta \wedge \rho - dt \wedge \beta,$$

where w.l.o.g. $\rho = r_a \theta^a$, $a = 1, \ldots, m$. Now $d_V \omega \in H^{1,3}(\mathscr{R}^\infty)$, and so by Theorem 3.2 there exists $\xi \in \Omega^{0,3}(\mathscr{R}^\infty)$ such that

$$d_V \omega = d_H \xi.$$

Writing $\xi = A_{ijk} \theta^i \wedge \theta^j \wedge \theta^k$, this gives

$$dx \wedge X(A_{ijk} \theta^i \wedge \theta^j \wedge \theta^k) = dx \wedge \theta^0 \wedge (d_V \rho). \tag{4.11}$$

We now show ξ has the form

$$\xi = A_i \theta^i \wedge \theta^1 \wedge \theta^0, \quad i = 0, \ldots, m-1 \tag{4.12}$$

Suppose there is a term in ξ with $\theta^{M_1} \wedge \theta^{M_2} \wedge \theta^{M_3}$, $1 \le M_1 < M_2 < M_3$, and assume we have the one with the highest M_3. On the left side of (4.11) there will be

$$dx \wedge X(\theta^{M_1} \wedge \theta^{M_2} \wedge \theta^{M_3})$$

which contains $dx \wedge \theta^{M_1} \wedge \theta^{M_2} \wedge \theta^{M_3+1}$, which can't occur on the right side since there is no θ^0. Suppose now that there are terms in ξ of the form $\theta^0 \wedge \theta^{M_2} \wedge \theta^{M_3}$ with $1 < M_2 < M_3$. Consider the maximal M_3, and again

$$dx \wedge X(\theta^0 \wedge \theta^{M_2} \wedge \theta^{M_3}),$$

will contain a term $dx \wedge \theta^0 \wedge \theta^{M_2} \wedge \theta^{M_3+1}$ which can't occur on the right hand side of equation (4.11). This shows equation (4.12).

Now

$$d_H d_V \xi = 0$$

but since $\xi \in \Omega^{0,3}(\mathscr{R}^\infty)$, this implies $d_V \xi = 0$. We apply vertical exactness and let $\zeta \in \Omega^{0,2}(\mathscr{R}^\infty)$ be such that $d_V \zeta = \xi$. By the vertical homotopy on ξ we may assume $\zeta = A\theta^0 \wedge \theta^1$. Finally, we let

$$\tilde{\omega} = \omega + d_H \zeta = dx \wedge \theta^0 \wedge \tilde{\rho} - dt \wedge \tilde{\beta}, \tag{4.13}$$

where $\tilde{\rho} = \rho + X(A\theta^1 \wedge \theta^0)$ and $\tilde{\beta} = \beta - T(\zeta)$. Therefore,

$$d_V \tilde{\omega} = d_V \omega + d_V d_H \zeta = d_V \tilde{\omega} - d_H d_V \zeta = d_V \tilde{\omega} - d_H \xi = 0.$$

This proves there is a representative $\tilde{\omega}$ for $[\omega]$ with $d_V \tilde{\omega} = 0$.

Again we use d_V exactness to find $\eta \in \Omega^{1,1}(\mathscr{R}^\infty)$ such that,

$$\tilde{\omega} = d_V \eta \tag{4.14}$$

where

$$\eta = dx \wedge \alpha - dt \wedge \gamma, \qquad \alpha, \gamma \in \Omega^{0,1}(\mathscr{R}^\infty). \tag{4.15}$$

Writing $\alpha = a_j \theta^j$, $j = 0, \ldots, m$, equation (4.14) and (4.15) give

$$\theta^0 \wedge \tilde{\rho} = -d_V(a_j \theta^j).$$

We now modify η in equation (4.15) and the representative $\tilde{\omega}$ for $[\omega]$ in equation (4.13) by

$$\hat{\eta} = \eta - d_H(a_m \theta^{m-1}), \qquad \hat{\omega} = \tilde{\omega} + d_H d_V(a_m \theta^{m-1}), \quad \text{no summation}$$

so that $\hat{\omega} = d_V \hat{\eta}$. In particular we note

$$\hat{\eta} = dx \wedge (\hat{a}_j \theta^j) - dt \wedge \hat{\gamma} \qquad j = 0, \ldots m-1.$$

Continuing by induction, there exists a representative $\bar{\omega}$ for $[\omega]$ and an $\bar{\eta} \in \Omega^{1,1}(\mathscr{R}^\infty)$, where $\bar{\omega} = d_V \bar{\eta}$ and

$$\bar{\eta} = dx \wedge \theta^0 \cdot Q - dt \wedge \bar{\gamma} \tag{4.16}$$

where Q is a smooth function on $\mathscr{R}^\infty$. Therefore

$$\bar{\omega} = dx \wedge \theta^0 \wedge d_V Q - dt \wedge d_V \bar{\gamma}.$$

□

Combining Theorem 4.2 and Corollary 4.1 gives the following.

Corollary 4.3. *If* $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ *with representative* $\omega = d_V(dx \wedge \theta^0 \cdot Q - dt \wedge \gamma)$ *from Theorem* 4.2 *then the unique representative in Theorem* 4.1 *is determined by*

$$\varepsilon = \frac{1}{2}\left(\mathbf{L}_Q - \mathbf{L}_Q^*\right)\theta^0 \tag{4.17}$$

where $\mathbf{L}_Q = Q_i X^i$.

The snake lemma from the variational bicomplex is the following.

Lemma 4.1. *Let* $\omega \in \Omega^{1,2}(\mathscr{R}^\infty)$ *satisfy* $d_H\omega = 0$, $d_V\omega = 0$. *Let* $\eta \in \Omega^{1,1}(\mathscr{R}^\infty)$ *such that* $d_V\eta = \omega$. *Then there exists* $\lambda = L dt \wedge dx \in \Omega^{2,0}(\mathscr{R}^\infty)$ *such that* $d_H\eta = d_V\lambda$.

Proof. We have

$$d_V d_H \eta = -d_H d_V \eta = -d_H \omega = 0,$$

and the vertical exactness of the variational bicomplex implies the lemma. □

The relationship between ω, λ and η in Lemma 4.1 is represented by the diagram,

$$\begin{tikzcd}
0 & \\
\omega \arrow[u, "d_V"] & \\
\eta \arrow[u, "d_V"] \arrow[r, "d_H"'] & d_H\eta \\
& \lambda \arrow[u, "d_V"]
\end{tikzcd}$$

Corollary 4.4. *Let* $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ *and let* ω *be a* d_V *closed representative as in equation* (4.10), *and let* $\lambda \in \Omega^{2,0}(\mathscr{R}^\infty)$ *be as in Lemma* 4.1, *so that* $d_H\eta = d_V\lambda$. *The linear map* $\Lambda : H^{1,2}(\mathscr{R}^\infty) \to H^{2,0}(\mathscr{R}^\infty)$ *given by*

$$\Lambda([\omega]) = [\lambda] \tag{4.18}$$

is well defined.

Proof. Suppose $\omega_a \in \Omega^{1,2}(\mathscr{R}^\infty)$, $a = 1,2$ where $[\omega_1] = [\omega_2]$ and that $\omega_a = d_V\eta_a$, $a = 1,2$ are d_V closed representatives. Let $\lambda_a \in \Omega^{2,0}(\mathscr{R}^\infty)$ satisfy $d_H\eta_a = d_V\lambda_a$, $a = 1,2$. To demonstrate $[\lambda_1] = [\lambda_2] \in H^{2,0}\mathscr{R}^\infty$ we show there exists $\kappa \in \Omega^{1,0}(\mathscr{R}^\infty)$ such that $\lambda_1 - \lambda_2 = d_H\kappa$.

Since $[\omega_1] = [\omega_2]$, there exists $\xi \in \Omega^{0,2}(\mathscr{R}^\infty)$ such that,

$$d_V(\eta_1 - \eta_2) = d_H\xi\,. \tag{4.19}$$

Taking d_V of equation (4.19) gives $d_V d_H \xi = -d_H d_V \xi = 0$, which implies $d_V\xi = 0$ since $H^{0,2}(\mathscr{R}^\infty) = 0$ (or $d_H\mu = 0$, $\mu \in \Omega^{0,s}(\mathscr{R}^\infty)$ implies $\mu = 0$). Therefore there exists $\phi \in \Omega^{0,1}(\mathscr{R}^\infty)$ such that $\xi = d_V\phi$. Substituting $\xi = d_V\phi$ in equation (4.19) gives

$$d_V(\eta_1 - \eta_2 + d_H\phi) = 0. \tag{4.20}$$

The vertical exactness of the variational bicomplex applied to equation (4.20) implies that there exists $\kappa \in \Omega^{1,1}(\mathscr{R}^\infty)$ such that

$$\eta_1 - \eta_2 + d_H\phi = d_V\kappa\,. \tag{4.21}$$

Taking d_H of equation (4.21) and using $d_H\eta_a = d_V\lambda_a$ gives

$$d_V\lambda_1 - d_V\lambda_2 = d_H d_V \kappa$$

so that

$$d_V(\lambda_1 - \lambda_2 + d_H\kappa) = 0. \tag{4.22}$$

Again by vertical exactness of the augmented variational bicomplex applied to equation (4.22), there exists $\mu \in \Omega^{2,0}(\mathbb{R})$ such that

$$\lambda_1 - \lambda_2 = -d_H\kappa + \mu \tag{4.23}$$

where $\mu \in \Omega^2(\mathbb{R}^2)$. The deRham cohomology of $\mathbb{R}^2$ is trivial so $\mu = d\alpha = d_H\alpha, \alpha \in \Omega^1(\mathbb{R}^2)$. Therefore equation (4.23) becomes

$$\lambda_1 - \lambda_2 = d_H(-\kappa + \alpha)$$

and $[\lambda_1] = [\lambda_2] \in H^{2,0}(\mathscr{R}^\infty)$. □

The relevance of the kernel of Λ is given in Theorem A.2.

5. Variational Operators and $H^{1,2}(\mathscr{R}^\infty)$

A scalar evolution equation defined by the zero set of $\Delta = u_t - K(t,x,u,u_x,\ldots,u_n)$ is said to admit a variational operator of order k if there exists a differential operator

$$\mathcal{E} = \sum_{i=0}^{k} r_i(t,x,u,u_x,u_{xx},\ldots)D_x^i$$

with r_k nowhere vanishing, and a function $L \in C^\infty(\mathbb{R}^2,\mathbb{R})$ such that,

$$\mathcal{E}(\Delta) = \mathbf{E}\left(L(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots)\right) \tag{5.1}$$

where $\mathbf{E}(L)$ is the Euler-Lagrange expression of $L \in C^\infty(J^\infty(\mathbb{R}^2,\mathbb{R}))$. Theorem 2.6 in [7] relates the existence of a multiplier (or zero order operator) to the cohomology of Δ. In this section we will prove a generalization of this result and ultimately prove Theorem 1.1. We start with the following.

Theorem 5.1. *Let $\mathcal{E} = r_iD_x^i$ be a variational operator for $\Delta = u_t - K(t,x,u,u_x,\ldots,u_n)$ with Lagrangian L satisfying* (5.1). *Then there exists $\eta \in \Omega^{1,1}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ such that*

$$d_V(L dt \wedge dx) = dt \wedge dx \wedge \vartheta^0 \cdot \mathcal{E}(\Delta) + d_H\eta. \tag{5.2}$$

With $\iota : \mathscr{R}^\infty \to J^\infty(\mathbb{R}^2,\mathbb{R})$ in equation (2.9) *let,*

$$\omega = d_V(\,\iota^*\eta\,) \in \Omega^{1,2}(\mathscr{R}^\infty).$$

Then $d_H\omega = 0$.

Proof. Suppose $\mathcal{E}$ and L are given satisfying (5.1), then using the standard formula in the calculus of variations (for example equation (3.2) in [2]), we have on account of (5.1)

$$\begin{aligned} d_V(L dt \wedge dx) &= dt \wedge dx \wedge \vartheta^0 \cdot \mathbf{E}(L) + d_H\eta \\ &= dt \wedge dx \wedge \vartheta^0 \cdot \mathcal{E}(\Delta) + d_H\eta \end{aligned} \tag{5.3}$$

which shows (5.2).

By applying ι^* to equation (5.2) we have

$$d_V \iota^* \lambda = d_H \iota^* \eta. \tag{5.4}$$

Letting $\omega = d_V \iota^* \eta$, we compute $d_H \omega$ using equation (5.4) and get

$$d_H \omega = d_H d_V \iota^* \eta = -d_V d_H \iota^* \eta = -d_V(\iota^* d_V \lambda)) = -d_V^2(\iota^* \lambda) = 0.$$

Therefore $d_H \omega = 0$. □

A formula for ω in terms of $\mathcal{E}$ in Theorem 5.1 is given in Theorem 5.2 below. Before giving Theorem 5.2 we note the following property of variational operators for evolutions equations.

Lemma 5.1. *If $\Delta = u_t - K(t,x,u,u_x,\ldots,u_n)$ admits the k^{th} order variational operator $\mathcal{E} = r_i(t,x,u,u_x,\ldots)D_x^i$, then $\mathcal{E}$ is skew-adjoint.*

Proof. Suppose $\mathcal{E}(u_t - K) = \mathbf{E}(L)$ then applying $I \circ d_V$ to equation (5.2) using $d_V^2 = 0$ along with the property (2.5) for I, we have

$$I \circ d_V \left(dt \wedge dx \wedge \vartheta^0 \cdot \mathcal{E}(\Delta) \right) = 0. \tag{5.5}$$

With $\Delta = u_t - K$ let

$$\kappa = d_V\left(\mathcal{E}(\Delta)\right) = r_i D_x^i D_t(\vartheta^0) + u_{t,i} d_V r_i - d_V(r_i D_x^i(K)). \tag{5.6}$$

so that condition (5.5) gives

$$2I(dt \wedge dx \wedge \vartheta^0 \wedge \kappa) = dt \wedge dx \wedge \vartheta^0 \wedge \left(\kappa - \sum_{(a,j)\neq(0,0)}^{\infty} (-1)^{j+a} D_x^j D_t^a (\vartheta^0 \cdot \partial_{u_{a,j}} \lrcorner\, \kappa) \right) \tag{5.7}$$

In the term $\partial_{u_{a,j}} \lrcorner\, \kappa$ where κ is given in equation (5.6) we note that $\partial_{u_{a,j}}(r_j) = 0$, $\partial_{u_{a,j}}(K) = 0$, $a \geq 1$, $j \geq 0$. Therefore the only possible non-zero terms in the summation term in equation (5.7) with $\partial_{u_{a,j}} \lrcorner$ with $a \geq 1$, $j \geq 0$ satisfy

$$\begin{aligned}
-D_t(\partial_{u_t} \lrcorner\, \kappa) &= -D_t(r_0 \vartheta^0) && \equiv -r_0 \vartheta_t && \bmod \{\vartheta^j\}_{j\geq 0} \\
D_x D_t(\partial_{u_{t,1}} \lrcorner\, \kappa) &= D_x D_t(r_1 \vartheta^0) && \equiv D_x(r_1 \theta_t) && \bmod \{\vartheta^j\}_{j\geq 0} \\
-(-1)^k D_x^k D_t(\partial_{u_{t,1}} \lrcorner\, \kappa) &= -(-1)^k D_x^k D_t(r_k \vartheta^0) && \equiv (-1)^k D_x^k(r_k \theta_t) && \bmod \{\vartheta^j\}_{j\geq 0}.
\end{aligned} \tag{5.8}$$

Writing the condition $I(dt \wedge dx \wedge \vartheta^0 \wedge \kappa) \mod \{\vartheta^j\}_{j\geq 0}$ using equation (5.7) and (5.8) gives

$$\begin{aligned}
2I(dt \wedge dx \wedge \vartheta^0 \wedge \kappa) &\equiv dt \wedge dx \wedge \vartheta^0 \wedge \left(r_i D_x^i \vartheta_t + \sum_{i=0}^{k} (-D_x)^i (r_i \vartheta_t) \right) && \bmod \{\vartheta^j\}_{j\geq 0} \\
&\equiv dt \wedge dx \wedge \vartheta^0 \wedge \left(\mathcal{E}(\vartheta_t) + \mathcal{E}^*(\vartheta_t) \right) && \bmod \{\vartheta^i\}_{i\geq 0}.
\end{aligned} \tag{5.9}$$

In order for the right side of equation (5.9) to be zero we must have $\mathcal{E}^* = -\mathcal{E}$. □

Theorem 5.2. *Let $\mathcal{E} = r_i(t,x,u,u_x,\dots)D_x^i,\ i = 0,\dots,k$ be a k^{th} order variational operator for $\Delta = u_t - K(t,x,u,u_x,\dots,u_n)$ and let $[d_V \iota^* \eta] \in H^{1,2}(\mathscr{R}^\infty)$ from Theorem 5.1. Then the unique representative for $[d_V \iota^* \eta]$ in Theorem 4.1 is*

$$\omega = dx \wedge \theta \wedge \varepsilon - dt \wedge \boldsymbol{\beta}(\varepsilon), \quad \varepsilon = -\frac{1}{2}\iota^* \mathcal{E}(\vartheta^0) = -\frac{1}{2} r_i \theta^i. \tag{5.10}$$

Proof. Let $\tilde{\omega}_0 = d_V \tilde{\eta}_0$ where $\tilde{\eta}_0$ satisfies equation (5.3). We have from equations (5.3) and (3.32)

$$\begin{aligned} d_H \tilde{\omega}_0 &= -d_V d_H(\tilde{\eta}_0) = dt \wedge dx \wedge d_V(\vartheta^0 \cdot \mathcal{E}(\Delta)) \\ &= dt \wedge dx \wedge \left(D_x^i(\Delta) d_V r_i \wedge \vartheta^0 + r_i D_x^i (d_V \Delta) \wedge \vartheta^0 \right) \\ &= dt \wedge dx \wedge \left(D_x^i(\Delta) d_V r_i \wedge \vartheta^0 + d_V \Delta \wedge (-D_x)^i (r_i \vartheta^0) \right) + d_H \tilde{\eta} \end{aligned} \tag{5.11}$$

where $\tilde{\eta}$ is given in equation (3.33) and satisfies $\iota^* \tilde{\eta} = 0$. As remarked in the proof of Theorem 3.1, the term $D_x^i(\Delta) d_V r_i \wedge \vartheta^0$ in (5.11) does not contribute to the form $\tilde{\rho}$ in equation (3.5). Therefore we have from equation (5.11) that $\tilde{\rho}$ in equation (3.5) is,

$$\tilde{\rho} = (-D_x)^i (r_i \vartheta) = \mathcal{E}^*(\vartheta^0).$$

Since $\rho = \frac{1}{2}\iota^* \tilde{\rho}$ and $\mathcal{E}$ is skew-adjoint we get equation (5.10). □

We now come to the last main theorem in this section which proves the converse to Theorem 5.1. The proof is again a generalization of the argument given in Theorem 2.6 of [7] for the multiplier problem.

Theorem 5.3. *Let $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ with representative ω as in equation (4.10) of Theorem 4.2,*

$$\omega = d_V \eta, \quad \textit{where} \qquad \eta = dx \wedge \theta \cdot Q - dt \wedge \gamma. \tag{5.12}$$

Let $\lambda = L dt \wedge dx$ satisfying $d_H \eta = d_V \lambda$ from Lemma 4.1. Then $\mathcal{E} = \mathbf{F}_Q^* - \mathbf{F}_Q$ is a variational operator and,

$$\mathcal{E}(\Delta) = (\mathbf{F}_Q^* - \mathbf{F}_Q)(\Delta) = \mathbf{E}(Q\Delta + L). \tag{5.13}$$

The proof requires considerable care whether we are working on $\mathscr{R}^\infty$ or $J^\infty(\mathbb{R}^2,\mathbb{R})$, see Remark 2.1.

Proof. We start by writing $\gamma = g_j \theta^j$ in equation (5.12) and define the form $\tilde{\eta} \in \Omega^{1,1}(J^\infty(\mathbb{R}^2,\mathbb{R}))$ given by

$$\tilde{\eta} = dx \wedge \vartheta^0 \cdot Q - dt \wedge [g_j \vartheta^j] \tag{5.14}$$

which satisfies $\iota^* \tilde{\eta} = \eta$ in equation (5.12), and where the forms ϑ^j are defined in (2.1). Now define the vector fields on $J^\infty(\mathbb{R}^2,\mathbb{R})$,

$$\begin{aligned} \tilde{T} &= \partial_t + K\partial_u + D_x(K)\partial_{u_x} + D_x^2(K)\partial_{u_{xx}} + \dots, \quad & V &= \Delta \partial_u + D_x(\Delta)\partial_{u_x} + \dots, \\ \tilde{X} &= \partial_x + u_x \partial_u + u_{xx}\partial_{u_x} + D_x(K)\partial_{u_t} + \dots, & W &= D_x(\Delta)\partial_{u_t} + D_x^2(\Delta)\partial_{u_{tx}} + \dots \end{aligned} \tag{5.15}$$

so that $D_t = \tilde{T} + V, D_x = \tilde{X} + W$. Then with $\tilde{\eta}$ in equation (5.14) we have,

$$\begin{aligned} d_H \tilde{\eta} &= dt \wedge dx \wedge [D_t(\vartheta^0 \cdot Q) + D_x(g_j \vartheta^j)] \\ &= dt \wedge dx \wedge [\tilde{T}(\vartheta^0 \cdot Q) + \tilde{X}(g_j \vartheta^j)] + dt \wedge dx \wedge [V(\vartheta^0 \cdot Q) + W(g_j \vartheta^j)]. \end{aligned} \tag{5.16}$$

Since $g_j = g_j(t,x,u,u_x,\ldots)$ then $W(g_j) = 0$, while $dt\wedge dx\wedge D_x(\vartheta^j) = dt\wedge dx\wedge\tilde{X}(\vartheta^j)$ and $V(\vartheta^0) = d_V\Delta$. Equation (5.16) then can be written,

$$d_H\tilde{\eta} = dt\wedge dx\wedge[\tilde{T}(Q\vartheta^0)+\tilde{X}(g_j\vartheta^j)] + dt\wedge dx\wedge[\vartheta^0\cdot V(Q)+d_V\Delta\cdot Q]. \tag{5.17}$$

The condition $d_H\eta = d_V\lambda$ (on $\mathscr{R}^\infty$) is

$$dt\wedge dx\wedge[T(\theta\cdot Q)-X(g_j\theta^j)] = dt\wedge dx\wedge\theta^a\cdot L_a. \tag{5.18}$$

Now $\pi^* dt\wedge dx\wedge\theta^j = dt\wedge dx\wedge\vartheta^j$ (see Remark 2.1), and using the vector fields in (5.15) we have

$$\begin{aligned}\pi^*(d_V\lambda) &= d_V(\pi^*\lambda) = d_V(Ldt\wedge dx),\\ \pi^*(dt\wedge dx\wedge[T(Q\theta^0)] &= dt\wedge dx\wedge[\tilde{T}(Q\vartheta^0)]\\ \pi^*(dt\wedge dx\wedge[X(g_j\theta^j)] &= dt\wedge dx\wedge[\tilde{X}(g_j\vartheta^j)]\end{aligned} \tag{5.19}$$

Therefore applying π^* to (5.18) and using (5.19) we have

$$dt\wedge dx\wedge[\tilde{T}(Q\vartheta^0)+\tilde{X}(g_j\vartheta^j)] = d_V(Ldt\wedge dx), \tag{5.20}$$

The first variational formula for $d_V(Ldt\wedge dx)$ on $J^\infty(\mathbb{R}^2,\mathbb{R})$ applied to the right side of (5.20) gives

$$dt\wedge dx\wedge[\tilde{T}(Q\vartheta^0)+\tilde{X}(g_j\vartheta^j)] = dt\wedge dx\wedge\vartheta^0\cdot\mathbf{E}(L)+d_H\tilde{\zeta}_1 \tag{5.21}$$

where $\tilde{\zeta}\in\Omega^{1,1}(J^\infty(\mathbb{R}^2,\mathbb{R}))$. Inserting equation (5.21) into (5.17) we have,

$$d_H\tilde{\eta} = dt\wedge dx\wedge\vartheta^0\cdot\mathbf{E}(L)+dt\wedge dx\wedge[\vartheta^0\cdot V(Q)+d_V\Delta\cdot Q]+d_H\tilde{\zeta}_1 \tag{5.22}$$

The terms $d_V\Delta\cdot Q$ in equation (5.22) can be written as

$$dt\wedge dx\wedge[Qd_V\Delta] = dt\wedge dx\wedge[d_V(Q\Delta)-\Delta d_V Q] \tag{5.23}$$

We now apply the integration by parts operator (see equation 2.8 in [2]) and use the first variational formula for $d_V(Q\Delta dt\wedge dx)$, in equation (5.23) and get

$$\begin{aligned}dt\wedge dx\wedge[Qd_V\Delta] &= dt\wedge dx\wedge\vartheta^0[\mathbf{E}(Q\Delta)-(-D_x)^i(Q_{,i}\Delta)]+d_H\tilde{\zeta}_2\\ &= dt\wedge dx\wedge\vartheta^0[\mathbf{E}(Q\Delta)-\mathbf{F}^*_Q(\Delta)]+d_H\tilde{\zeta}_2\end{aligned} \tag{5.24}$$

Next we expand the term $V(Q)$ in equation (5.22) using V in (5.15) and

$$V(Q) = D_x^i(\Delta)Q_{,i} = \mathbf{F}_Q(\Delta) \tag{5.25}$$

Inserting (5.24), and (5.25) into (5.22) and letting $\tilde{\zeta} = \tilde{\zeta}_1+\tilde{\zeta}_2$ gives,

$$d_H\tilde{\eta} = dt\wedge dx\wedge\vartheta^0[\mathbf{E}(L)+\mathbf{F}_Q(\Delta)-\mathbf{F}^*_Q(\Delta)+\mathbf{E}(Q\Delta)]+d_H\tilde{\zeta}.$$

This implies $d_H(\tilde{\eta}-\tilde{\zeta})$ is a source-form. This is only possible if $d_H(\tilde{\eta}-\tilde{\zeta}) = 0$, and so

$$(\mathbf{F}^*_Q-\mathbf{F}_Q)(\Delta) = \mathbf{E}(Q\Delta+L),$$

which is equation (5.13) as required. □

Remark 5.1. In general three applications of the vertical homotopy operator are required to determine $\lambda \in \Omega^{2,0}(\mathscr{R}^\infty)$ from $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$. The first is to find a representative $\omega \in H^{1,2}(\mathscr{R}^\infty)$ with $d_V\omega = 0$ (Theorem 4.2). The second is to find η such that $d_V\eta = \omega$, and the third is to find λ such that $d_V\lambda = d_H\eta$.

We now have the following corollaries.

Corollary 5.1. *Let $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ with unique representative $\omega = dx\wedge\theta^0\wedge\varepsilon - dt\wedge\boldsymbol{\beta}(\varepsilon)$, $\varepsilon = r_i\theta^i$ as in Theorem 4.1. Then Δ admits $\mathcal{E} = -2r_iD_x^i$ as a variational operator.*

Proof. Starting with equation (5.12), Corollary 4.3 implies

$$\varepsilon = \frac{1}{2}(\mathbf{L}_Q - \mathbf{L}_Q^*)\theta^0 = r_i\theta^i. \tag{5.26}$$

Equation (5.26) together with the fact $\mathbf{F}_Q = Q_iD_x^i$ gives $\varepsilon = \iota^*\frac{1}{2}(\mathbf{F}_Q - \mathbf{F}_Q^*)(\vartheta^0) = r^i\theta^i$, we have $\mathcal{E} = \mathbf{F}_Q^* - \mathbf{F}_Q = -2r_iD_x^i$ is a variational operator by Theorem 5.3. □

Corollary 5.2. *Let $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ with $\omega = dx\wedge\theta^0\wedge(r_i\theta^i) - dt\wedge\boldsymbol{\beta}(\rho)$ as in Theorem 3.1. Then Δ admits*

$$\mathcal{E} = (r_iD_x^i)^* - r_iD_x^i \tag{5.27}$$

as a variational operator.

Proof. By Corollary 4.1 the unique representative $\hat{\omega} = dx\wedge\theta^0\wedge\varepsilon - dt\wedge\boldsymbol{\beta}(\varepsilon)$ has $\varepsilon = \frac{1}{2}(\rho - \rho^*) = \frac{1}{2}(r_i\theta^i - (-X^i)(r_i\theta^0))$. Therefore by Corollary 5.1, $\mathcal{E}$ in equation (5.27) is a variational operator. □

Corollary 5.3. *If an even order evolution equation $u_t = K(t,x,u,\ldots,u_{2m})$ $m \geq 1$ admits a variational operator $\mathcal{E}$, then $\mathcal{E} = 0$.*

Proof. Let $\omega \in \Omega^{2,1}(\mathscr{R}^\infty)$ be the d_H closed form from Theorem 5.2. On the other hand using the representative for $[\omega]$ from Theorem 3.1 combined with Corollaries 3.1 and 5.2 implies $\omega = 0$. Hence $\mathcal{E} = 0$. □

Finally we may also restate Theorem 5.3 without reference to the equation manifold $\mathscr{R}^\infty$ as follows.

Corollary 5.4. *The operator $\mathcal{E} = r_i(t,x,u,u_x,\ldots)D_x^i$, $i = 0,\ldots,k$ is a variational operator for $u_t = K$ if and only if there exists $Q(t,x,u,u_x,u_{xx},\ldots)$ and $L(t,x,u,u_x,u_{xx},\ldots)$ such that*

$$\mathcal{E} = \mathbf{F}_Q^* - \mathbf{F}_Q \quad \text{and} \quad \mathcal{E}(u_t - K) = \mathbf{E}\left(Q(u_t - K) + L\right). \tag{5.28}$$

Lastly we combine the results of Theorems 5.1 and 5.3 to prove Theorem 1.1.

Proof. (Theorem 1.1) We define a linear transformation $\hat{\Phi} : H^{1,2}(\mathscr{R}^\infty) \to \mathscr{V}_{op}(\Delta)$ by using the unique representative in Theorem 4.1 to be

$$\widehat{\Phi}([\omega]) = \widehat{\Phi}\left([dx\wedge\theta^0\wedge\varepsilon - dt\wedge\boldsymbol{\beta}(\varepsilon)]\right) = -2r_iD_x^i \tag{5.29}$$

where $\varepsilon = r_i\theta^i$ and is skew-adjoint. Then Corollary 5.1 shows $\Phi([\omega]) \in \mathscr{V}_{op}(\Delta)$.

We check $\widehat{\Phi} = \Phi^{-1}$. With $\mathcal{E} = -2r_i D_x^i$ a variational operator, let $\varepsilon = r_i\theta^i$. We have from equation (1.5) (or Theorem 5.2) and equation (5.29),

$$\widehat{\Phi} \circ \Phi(\mathcal{E}) = \widehat{\Phi}\left([dx \wedge \theta^0 \wedge \varepsilon - dt \wedge \boldsymbol{\beta}(\varepsilon)]\right) = \mathcal{E}$$

and

$$\Phi \circ \widehat{\Phi}\left([dx \wedge \theta^0 \wedge \varepsilon - dt \wedge \boldsymbol{\beta}(\varepsilon)]\right) = \Phi(-2r_i D_x^i) = [dx \wedge \theta^0 \wedge \varepsilon - dt \wedge \boldsymbol{\beta}(\varepsilon)].$$

Therefore Φ in equation (1.5) is invertible with $\widehat{\Phi}$ in equation (5.29) as the inverse. $\square$

6. Functional 2-Forms, Symplectic Forms and Hamiltonian Vector Fields

In this section we review the space of functional forms on $J^\infty(\mathbb{R},\mathbb{R})$ as in [3], [2] and relate these to symplectic forms, symplectic operators and Hamiltonian vector fields.

6.1. *Functional Forms*

On the space $J^\infty(\mathbb{R},\mathbb{R})$ with coordinates $(x,u,u_x,\ldots,u_i,\ldots)$ the contact forms are $\theta^i = du_i - u_{i+1}dx$ and $D_x = \partial_x + u_x\partial_u + \ldots u_{i+1}\partial_{u_i} + \ldots$ is the total x derivative operator. Again $\Omega^{r,s}(J^\infty(\mathbb{R},\mathbb{R}))$ denotes the $r = 0,1$ horizontal, $s \geq 0$ vertical forms on $J^\infty(\mathbb{R},\mathbb{R})$. The horizontal and vertical differentials $d_H : \Omega^{r,s}(J^\infty(\mathbb{R},\mathbb{R})) \to \Omega^{r+1,s}(J^\infty(\mathbb{R},\mathbb{R}))$, $d_V : \Omega^{r,s}(J^\infty(\mathbb{R},\mathbb{R})) \to \Omega^{r,s+1}(J^\infty(\mathbb{R},\mathbb{R}))$, are anti-derivations which satisfy

$$d_H\omega = dx \wedge D_x(\omega)dx, \quad d_V f = \frac{\partial f}{\partial u_i}\theta^i = f_i\theta^i, \quad d_H\theta^i = dx \wedge \theta^{i+1}, \quad d_V\theta^i = 0,$$

where $f \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$, $\omega \in \Omega^{r,s}(J^\infty(\mathbb{R},\mathbb{R}))$ and $D_x(\omega)$ is the Lie derivative. Since $d = d_H + d_V$ this implies,

$$d_H^2 = 0, \quad d_V^2 = 0, \quad \text{and} \quad d_H d_V + d_V d_H = 0.$$

The integration by parts operator $I : \Omega^{1,s}(J^\infty(\mathbb{R},\mathbb{R})) \to \Omega^{1,s}(J^\infty(\mathbb{R},\mathbb{R}))$ is

$$I(\Sigma) = \frac{1}{s}\theta^0 \wedge \sum_{i=0}^{\infty}(-1)^i(D_x)^i(\partial_{u_i} \lrcorner \Sigma), \quad \Sigma \in \Omega^{1,s}(J^\infty(\mathbb{R},\mathbb{R})) \tag{6.1}$$

and I satisfies the same properties as in (2.3),

$$\Sigma = I(\Sigma) + d_H\eta, \qquad I^2 = I, \qquad \operatorname{Ker} I = \text{Image } d_H. \tag{6.2}$$

The space of functional s forms ($s \geq 1$) on $J^\infty(\mathbb{R},\mathbb{R})$, $\mathscr{F}^s(J^\infty(\mathbb{R},\mathbb{R})) \subset \Omega^{1,s}(J^\infty(\mathbb{R},\mathbb{R}))$, is defined to be the image of $\Omega^{1,s}(E)$ under I,

$$\mathscr{F}^s(J^\infty(\mathbb{R},\mathbb{R})) = I(\Omega^{1,s}(J^\infty(\mathbb{R},\mathbb{R}))). \tag{6.3}$$

Equation (6.1) applied to Definition 6.3 shows that if $\Sigma \in \mathscr{F}^s(E)$ then there exists $\alpha \in \Omega^{0,s-1}(J^\infty(\mathbb{R},\mathbb{R}))$ such that

$$\Sigma = dx \wedge \theta^0 \wedge \alpha. \tag{6.4}$$

However, not every differential form $\Sigma \in \Omega^{1,s}(J^\infty(\mathbb{R},\mathbb{R}))$ written in the form (6.4) is in the space $\mathscr{F}^s(J^\infty(\mathbb{R},\mathbb{R}))$. In the case of $\mathscr{F}^2(J^\infty(\mathbb{R},\mathbb{R}))$ the following is easy to show using the definition of I in (6.1), see also Proposition 3.6 and 3.7 in [3].

Lemma 6.1. *Let $\Sigma \in \mathscr{F}^2(J^\infty(\mathbb{R},\mathbb{R}))$, then there exists a unique skew-adjoint differential operator, $\mathcal{S} = s_i D_x^i$ such that,*

$$\Sigma = dx \wedge \theta^0 \wedge \mathcal{S}(\theta^0). \tag{6.5}$$

The differential $\delta_V : \mathscr{F}^s(J^\infty(\mathbb{R},\mathbb{R})) \to \mathscr{F}^{s+1}(J^\infty(\mathbb{R},\mathbb{R}))$ is defined by

$$\delta_V = I \circ d_V : \mathscr{F}^s(J^\infty(\mathbb{R},\mathbb{R})) \to \mathscr{F}^{s+1}(J^\infty(\mathbb{R},\mathbb{R})), \qquad s = 0, \ldots$$

where we let $\mathscr{F}^0(J^\infty(\mathbb{R},\mathbb{R})) = \Omega^{1,0}(J^\infty(\mathbb{R},\mathbb{R}))$. This leads to the differential complex

$$C^\infty(J^\infty(\mathbb{R},\mathbb{R})) \xrightarrow{d_H} \mathscr{F}^0(J^\infty(\mathbb{R},\mathbb{R})) \xrightarrow{\delta_V} \mathscr{F}^1(J^\infty(\mathbb{R},\mathbb{R})) \xrightarrow{\delta_V} \mathscr{F}^2(J^\infty(\mathbb{R},\mathbb{R})) \ \ldots, \tag{6.6}$$

which is exact and is known as the Euler complex, see Theorem 2.7 [2].

6.2. *Symplectic Forms, Symplectic Operators, and Hamiltonian Vector Fields*

Let Γ be the Lie algebra of prolonged evolutionary vector fields on $J^\infty(\mathbb{R},\mathbb{R})$. We begin by recalling the appropriate definitions (see Section 2.5 [10]).

Definition 6.1. An element $\Sigma \in \mathscr{F}^2(J^\infty(\mathbb{R},\mathbb{R}))$ is a **symplectic form** on Γ if $\Sigma \neq 0$ and $\delta_V(\Sigma) = 0$. A skew-adjoint differential operator $\mathcal{S} = s_i D_x^i$ is symplectic if $dx \wedge \theta^0 \wedge \mathcal{S}(\theta^0)$ is a symplectic form.

Definition 6.1 combined with Lemma 6.1 shows there is a one-to-one correspondence between symplectic forms and symplectic operators. We now define Hamiltonian vector fields.

Definition 6.2. Let Σ be a symplectic form. A vector field $Y \in \Gamma$ is Hamiltonian if

$$\delta_V \circ I(Y \lrcorner \Sigma) = 0. \tag{6.7}$$

The vector field Y is a degenerate direction if $I(Y \lrcorner \Sigma) = 0$.

Definition 6.2 is equivalent to Σ being invariant under the flow of Y on $\mathscr{F}^2(J^\infty(\mathbb{R},\mathbb{R}))$ as shown in the following theorem.

Theorem 6.1. *Let Σ be a symplectic form. An evolutionary vector field $Y \in \Gamma$ is Hamiltonian with respect to Σ if and only if*

$$\mathscr{L}_Y^\natural \Sigma = I \circ \pi^{1,2} \circ \mathscr{L}_Y \Sigma = 0, \tag{6.8}$$

where $\mathscr{L}^\natural = I \circ \pi^{1,2} \circ \mathscr{L}$ is the projected Lie derivative on $\mathscr{F}^2(J^\infty(\mathbb{R},\mathbb{R}))$, see Theorem 3.21 in [3].

Proof. Using Lemma 3.24 in [3] and the fact that $\delta_V \Sigma = 0$, we have

$$\mathscr{L}_Y^\natural \Sigma = I \circ d_V(Y \lrcorner \Sigma) + I(Y \lrcorner \delta_V \Sigma) = I \circ d_V(Y \lrcorner \Sigma). \tag{6.9}$$

By the first property in equation (6.2), $I \circ d_V \circ I = I \circ d_V$, so conditions (6.7) and (6.8) are equivalent through equation (6.9). □

We now write out definition 6.2 in a more familiar form. The exactness of the Euler complex and the condition $\delta_V \circ I(Y \lrcorner \Sigma) = 0$ implies there exists $\lambda = 2Hdx \in \mathscr{F}^0(J^\infty(\mathbb{R},\mathbb{R}))$ such that

$$I(Y \lrcorner \Sigma) = \delta_V \lambda = dx \wedge \theta^0 \cdot \mathbf{E}(2H). \tag{6.10}$$

Writing $Y = \mathrm{pr}(K\partial_u)$ and $\Sigma = dx \wedge \theta^0 \wedge \mathcal{S}(\theta^0)$ where $\mathcal{S} = s_i D_x^i$ is a skew-adjoint differential operator, the left side of equation (6.10) is then

$$\begin{aligned} I(Y \lrcorner \Sigma) &= I(dx \wedge \left(s_i D_x^i(K)\theta^0 - K s_i \theta^i\right)) \\ &= dx \wedge \theta^0 \left(s_i D_x^i(K) - (-D_x)^i(K s_i)\right) \\ &= dx \wedge \theta^0 \cdot 2\mathcal{S}(K). \end{aligned} \tag{6.11}$$

Using this computation in (6.10) shows that condition (6.7) (or (6.8)) is then equivalent to the following.

Corollary 6.1. *Let Σ be a symplectic form with corresponding symplectic operator $\mathcal{S}$. The evolutionary vector field $Y = \mathrm{pr}(K\partial_u) \in \Gamma$ is Hamiltonian if and only if there exists $H \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$ such that*

$$\mathcal{S}(K) = \mathbf{E}(H). \tag{6.12}$$

Corollary 6.1 just shows that Definition 6.2 agrees with the standard symplectic Hamiltonian formulation for time independent evolution equations [10].

6.2.1. *Symplectic Potential*

If Σ is a symplectic form, the exactness of the δ_V complex implies there exists $\psi \in \mathscr{F}^1(J^\infty(\mathbb{R},\mathbb{R}))$ such that $\Sigma = \delta_V(\psi)$. The functional form ψ is a **symplectic potential** for Σ.

Lemma 6.2. *Let $\Sigma \in \mathscr{F}^2(J^\infty(\mathbb{R},\mathbb{R}))$ be symplectic (and so δ_V closed), then there exists a smooth function $P \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$ such that*

$$\Sigma = dx \wedge \theta^0 \wedge \mathcal{S}(\theta^0), \quad \text{where} \quad \mathcal{S} = \frac{1}{2}(\mathbf{F}_P - \mathbf{F}_P^*) \tag{6.13}$$

where $\mathbf{F}_P = P_i D_x^i$ is the Fréchet derivative of P.

Proof. A symplectic potential $\psi \in \mathscr{F}^1(J^\infty(\mathbb{R},\mathbb{R}))$ for Σ can be written using (6.4) as

$$\psi = dx \wedge \theta^0 \cdot P, \quad P \in C^\infty(J^\infty(\mathbb{R},\mathbb{R})). \tag{6.14}$$

Writing $\Sigma = \delta_V \psi$ and using equation (6.14) produces (6.13). □

The Hamiltonian condition on $Y \in \Gamma$ in terms of a symplectic potential ψ is the following.

Lemma 6.3. *The evolutionary vector field $Y \in \Gamma$ is Hamiltonian for the symplectic form $\Sigma = \delta_V \psi$ if and only if there exists $\lambda \in \mathscr{F}^0(J^\infty(\mathbb{R},\mathbb{R}))$ such that*

$$\mathscr{L}_Y^\natural \psi = \delta_V \lambda. \tag{6.15}$$

Proof. Using the exactness of the δ_V complex we show $\delta_V \mathscr{L}_Y^\natural \psi = 0$ which is equivalent to equation (6.15). By Theorem 6.1, Y is Hamiltonian if and only if

$$0 = \mathscr{L}_Y^\natural \delta_V \psi = \delta_V \mathscr{L}_Y^\natural \psi \tag{6.16}$$

where we have used $\mathscr{L}_Y^\natural \circ \delta_V = \delta_V \circ \mathscr{L}_Y^\natural$ (Lemma 3.24 [3]). This proves the lemma. □

Using either Lemma 6.3 or equations (6.13) and (6.1) we have the following simple corollary.

Corollary 6.2. *Let Σ be a symplectic form with symplectic potential $\psi = dx \wedge \theta^0 \cdot P$. The evolutionary vector field $V = \mathrm{pr}(K\partial_u) \in \Gamma$ is Hamiltonian if and only if there exists $H \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$ such that*

$$\frac{1}{2}(\mathbf{F}_P - \mathbf{F}_P^*)(K) = \mathcal{S}(K) = \mathbf{E}(H) \tag{6.17}$$

where $\mathbf{F}_P$ is the Fréchet-derivative of P on $J^\infty(\mathbb{R},\mathbb{R})$.

A straight forward computation writing $\Sigma = \delta_V \psi$ classifies the first order symplectic operators, see also Theorem 6.2 in [10]

Lemma 6.4. *An element $\Sigma \in \mathscr{F}^2(J^\infty(\mathbb{R},\mathbb{R}))$ of the first order form,*

$$\Sigma = dx \wedge \theta^0 \wedge \theta^1 \cdot A \quad A \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$$

is symplectic, if and only if there exists $P(x,u,u_x,u_{xx}) \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$ depending on up to the second order derivative, such that

$$A = \frac{\partial P}{\partial u_x} - D_x\left(\frac{\partial P}{\partial u_{xx}}\right). \tag{6.18}$$

6.3. *Time Dependent Systems*

Most of the definitions and results from Sections 6.1 and 6.2 extend immediately to the case of time dependent systems. Let $E = \mathbb{R} \times J^\infty(\mathbb{R},\mathbb{R})$, and label the extra $\mathbb{R}$ with the parameter t. The contact forms are

$$\theta_E^i = du_i - u_{i+1}dx \tag{6.19}$$

and we let $\Omega_{\mathrm{t_{sb}}}^{r,s}(E)$ be the bicomplex of t semi-basic forms on E,

$$\Omega_{\mathrm{t_{sb}}}^{r,s}(E) = \{\ \omega \in \Omega^{r,s}(E) \mid \partial_t \lrcorner\, \omega = 0\ , \quad r = 0,1\,;\ s = 0\ldots\ \}.$$

A generic form $\omega \in \Omega_{\mathrm{t_{sb}}}^{1,2}(E)$ is given by

$$\omega = dx \wedge (\xi_{ij}\theta_E^i \wedge \theta_E^j), \qquad \xi_{ij} \in C^\infty(E).$$

The anti-derivations $d_H^E : \Omega^{r,s}_{\mathrm{t_{sb}}}(E) \to \Omega^{r+1,s}_{\mathrm{t_{sb}}}(E)$ and $d_V^E : \Omega^{r,s}_{\mathrm{t_{sb}}}(E) \to \Omega^{r,s+1}_{\mathrm{t_{sb}}}(E)$ are determined by

$$d_H^E(\omega) = dx \wedge D_x(\omega)dx, \quad d_V^E(f) = f_i \theta_E^i, \quad d_V^E \theta_E^i = 0, \tag{6.20}$$

and satisfy $(d_H^E)^2 = 0, (d_V^E)^2 = 0, d_H^E d_V^E + d_V^E d_H^E = 0$. However $d \neq d_H^E + d_V^E$. The integration by parts operator I induces a map $I_E : \Omega^{r,s}_{\mathrm{t_{sb}}}(E) \to \Omega^{r,s}_{\mathrm{t_{sb}}}(E)$ having the formula (6.1) and properties (6.2). We let

$$\mathscr{F}^s_{\mathrm{t_{sb}}}(E) = I_E\left(\Omega^{1,s}_{\mathrm{t_{sb}}}(E)\right). \tag{6.21}$$

The mapping $\delta_V^E = I_E \circ d_V^E$ gives rise to the exact sequence as in (6.6). A form $\Sigma \in \mathscr{F}^2_{\mathrm{t_{sb}}}(E)$ is symplectic if $\delta_V^E \Sigma = 0$ and Lemma 6.2 becomes the following.

Lemma 6.5. *An element $\Sigma = dx \wedge \theta_E^0 \wedge \mathcal{S}(\theta_E^0) \in \mathscr{F}^2_{\mathrm{t_{sb}}}(E)$ where $\mathcal{S} = s_i D_x^i$, and $s_i \in C^\infty(E)$ is symplectic if and only if there exists $P \in C^\infty(E)$ such that*

$$\mathcal{S} = \frac{1}{2}(\mathbf{L}_P - \mathbf{L}_P^*) \tag{6.22}$$

where $\mathbf{L}_P = P_i D_x^i$.

We use Theorem 6.1 to define Hamiltonian vector fields in this case.

Definition 6.3. An evolutionary vector field $Y = pr(K\partial_u)$ where $K \in C^\infty(E)$ is Hamiltonian with respect to the symplectic form $\Sigma \in \mathscr{F}^2_{\mathrm{t_{sb}}}(E)$ if

$$\mathscr{L}_T^\natural \Sigma = I_E \circ \pi^{1,2} \circ \mathscr{L}_T \Sigma = 0 \tag{6.23}$$

where $T = \partial_t + Y$ and $\mathscr{L}_T^\natural = I_E \circ \pi^{1,2} \circ \mathscr{L}_T$ is the projected Lie derivative.

Note that T in Definition 6.3 agrees with T in equation (2.10). We can also write condition (6.23) as follows.

Lemma 6.6. *An evolutionary vector field $Y = pr(K\partial_u)$ is a Hamiltonian vector field for the symplectic form $\Sigma = dx \wedge \theta_E^0 \wedge (s_i \theta_E^i)$ if and only if there exists $\xi \in \Omega^{0,2}(E)$ such that*

$$\pi^{1,2} \circ \mathscr{L}_T(dx \wedge \theta_E^0 \wedge (s_i \theta_E^i)) = dx \wedge D_x(\xi) = d_H^E \xi. \tag{6.24}$$

Proof. We have kernel $I_E = $ Image d_H^E, therefore equation (6.23) can be written as equation (6.24). □

A formula for ξ in equation (6.24) in terms of $\rho = s_i \theta_E^i$ is given in the proof of Theorem 7.1.

The analogue to Lemma 6.3 also holds in this case where Y is replaced by T. In order to prove this we now show the commutation formula in equation (6.16) holds where Y is replaced by T.

Lemma 6.7. *If $\psi = dx \wedge \theta_E^0 \cdot P$ where $P \in C^\infty(E)$, then $\mathscr{L}_T^\natural \delta_V^E \psi = \delta_V^E \mathscr{L}_T^\natural \psi$.*

Proof. Since $T = \partial_t + Y$ and $\mathscr{L}_Y^\natural \delta_V^E \psi = \delta_V^E \mathscr{L}_Y^\natural \psi$ (Lemma 3.24 [3]), we need to check

$$\mathscr{L}_{\partial_t}^\natural \delta_V^E \psi = \delta_V^E \mathscr{L}_{\partial_t}^\natural \psi$$

We write out both side of this equation. The left side is

$$I_E \circ \pi^{1,2}\left(\frac{1}{2} dx \wedge \theta_E^0 \wedge [P_{i,t}\theta_E^i - (-D_x)^i(P_{i,t}\theta_E^0)]\right) = \frac{1}{2} dx \wedge \theta_E^0 \wedge [P_{i,t}\theta_E^i - (-D_x)^i(P_{i,t}\theta_E^0)]. \quad (6.25)$$

The right side is

$$\delta_V(dx \wedge \theta_E^0 \cdot P_t) = \frac{1}{2} dx \wedge \theta_E^0 \wedge \left[P_{t,i}\theta_E^i - (-D_x)^i(P_{t,i}\theta_E^0)\right]. \quad (6.26)$$

Since the mixed partials are equal $P_{t,i} = P_{i,t}$, equations (6.25) and (6.26) are equal, which proves the lemma. □

Lemma 6.8. *The evolutionary vector field $Y \in \Gamma$ is Hamiltonian for the symplectic form $\Sigma = \delta_V^E \psi$ if and only if there exists $\lambda \in \mathscr{F}^0(E)$ such that*

$$\mathscr{L}_T^\natural \psi = \delta_V^E \lambda. \quad (6.27)$$

Proof. Using the exactness of the δ_V^E complex we show $\delta_V^E \mathscr{L}_T^\natural \psi = 0$ which is equivalent to equation (6.27). By Definition 6.3, Y is Hamiltonian if and only if

$$0 = \mathscr{L}_T^\natural \Sigma = \mathscr{L}_T^\natural \delta_V^E \psi = \delta_V^E \mathscr{L}_T^\natural \psi$$

where we have used Lemma 6.7. This proves the lemma. □

Using Lemma 6.8 we have the following corollary which is the t-dependent version of Corollary 6.2.

Corollary 6.3. *Let $\Sigma = dx \wedge \theta_E^0 \wedge \mathbb{S}(\theta_E^0)$ be a symplectic form with symplectic potential $\psi = dx \wedge \theta^0 \cdot P$. The evolutionary vector field $Y = \mathrm{pr}(K\partial_u) \in \Gamma$ is Hamiltonian if and only if there exists $H \in C^\infty(E)$ such that*

$$\frac{1}{2}(P_t + (\mathbf{L}_P - \mathbf{L}_P^*)(K)) = \frac{1}{2} P_t + \mathbb{S}(K) = \mathbf{E}(H). \quad (6.28)$$

Proof. We just need to compute

$$\mathscr{L}_T^\natural(dx \wedge \theta_E^0 \cdot P) = dx \wedge \theta_E^0 \cdot P_t + dx \wedge \theta_E^0 \cdot (\mathbf{L}_P K - \mathbf{L}_P^* K) = dx \wedge \theta_E^0 \cdot (P_t + \mathbf{L}_P K - \mathbf{L}_P^* K)$$

Using this computation in equation (6.27) with $\lambda = 2H dx$ gives equation (6.28). □

The function H in (6.28) is the Hamiltonian.

Remark 6.1. A symplectic form Σ is t-invariant if $\mathscr{L}_{\partial_t}\Sigma = 0$. In this case Σ determines a well defined symplectic form $\bar{\Sigma}$ on the quotient of E by the flow of ∂_t, $q : E \to E/\partial_t = J^\infty(\mathbb{R}, \mathbb{R})$ such that $q^*\bar{\Sigma} = \Sigma$. Definition 6.3 where Y and Σ are t-invariant implies Definition 6.2 for $\bar{\Sigma}$ and $\bar{Y} = q_* Y$ and equation (6.28) becomes equation (6.17).

7. Variational and Symplectic Operator Equivalence

A time independent evolution equation $u_t = K(x,u,u_x,\ldots,u_n)$ is a symplectic Hamiltonian evolution equation [10] if there exists a symplectic operator $\mathcal{S}$ and a function H called the Hamiltonian such that equation (1.6) holds. With this definition, the determination of the possible symplectic Hamiltonian evolution equations is typically approached in two ways. The first way consists of determining the possible symplectic operators of a certain order [10]. Then for a given class of symplectic operators $\mathcal{S}$, determine K which satisfy equation (1.6). The second approach starts with a given K and then determines if there exists a symplectic operator $\mathcal{S}$ such that equation (1.6) holds.

By comparison Theorem 1.3, whose proof is given in this section, combines these two questions and resolves the characterization of symplectic Hamiltonian evolution equations by the invariants $H^{1,2}(\mathscr{R}^\infty)$. This can simultaneously solve the existence of $\mathcal{S}$ and the existence of the Hamiltonian function H in equation (1.6) as done for the special case given in Theorem 1.4.

7.1. $H^{1,2}(\mathscr{R}^\infty)$ *and Symplectic Hamiltonian Evolution Equations*

Given a scalar evolution equation $u_t = K(t,x,u,u_x,\ldots)$, we identify the manifolds $\mathscr{R}^\infty$ and $E = \mathbb{R}\times J^\infty(\mathbb{R},\mathbb{R})$ by identifying their coordinates which in turn induces an identification of smooth functions. Define the projection map $\Pi : T^*(\mathscr{R}^\infty) \to T^*(E)$ by

$$\Pi(\theta^i) = \theta^i_E, \quad \Pi(dx) = dx, \quad \Pi(dt) = 0, \tag{7.1}$$

where θ^i are given in equation (2.11) and θ^i_E are in equation (6.19). Also denote by Π the induced projection map $\Pi : \Omega^{r,s}(\mathscr{R}^\infty) \to \Omega^{r,s}_{\mathrm{tsb}}(E)$ where for example

$$\Pi\left(dx\wedge\theta^0\wedge(s_i\theta^i) + dt\wedge\beta\right) = dx\wedge\theta^0_E\wedge(s_i\theta^i_E). \tag{7.2}$$

Lemma 7.1. *The map $\Pi : \Omega^{r,s}(\mathscr{R}^\infty) \to \Omega^{r,s}_{\mathrm{tsb}}(E)$ is a bicomplex co-chain map,*

$$\Pi(d_H\omega) = d^E_H\,\Pi(\omega), \qquad \Pi(d_V\omega) = d^E_V\,\Pi(\omega). \tag{7.3}$$

Proof. Equation (7.3) follows for the case $\omega = \theta^i$ directly from equations (2.15) and (6.20), and generically from the anti-derivation property of the operators. □

Lemma 7.2. *The function $\Pi : \Omega^{1,2}(\mathscr{R}^\infty) \to \Omega^{1,2}_{\mathrm{tsb}}(E)$ induces a well defined injective linear map $\widehat{\Pi} : H^{1,2}(\mathscr{R}^\infty) \to \operatorname{Ker}\delta^E_V \subset \mathscr{F}^2_{\mathrm{tsb}}(E)$ defined by*

$$\widehat{\Pi}([\omega]) = I_E\circ\Pi(\omega) \tag{7.4}$$

where ω is a representative of $[\omega]$.

Proof. To show $\widehat{\Pi}$ is well defined, suppose $\omega' = \omega + d_H\xi$. Then by equation (7.3) and property 3 in equation (6.2) applied to I_E gives

$$I_E\circ\Pi(\omega') = I_E\circ(\Pi(\omega)+\Pi(d_H\xi)) = I_E\circ(\Pi(\omega)+d^E_H(\Pi(\xi)) = I_E\circ\Pi(\omega).$$

Therefore $\widehat{\Pi}$ is well defined.

We now show $\widehat{\Pi}([\omega])$ is δ_E^V closed. We use equation (7.3) and compute

$$I_E \circ d_V^E \circ I_E \circ \Pi(\omega) = I_E \circ d_V^E \circ \Pi(\omega) = I_E \circ \Pi(d_V \omega). \tag{7.5}$$

Since $d_V \omega \in H^{1,3}(\mathscr{R}^\infty)$, Theorem 3.2 implies there exists $\xi \in \Omega^{0,3}(\mathscr{R}^\infty)$ such that $d_V \omega = d_H \xi$ so equation (7.5) becomes

$$I_E \circ d_V^E \circ I_E \circ \Pi(\omega) = I_E \circ \Pi(d_H \xi) = I_E \circ d_H^E \circ \Pi(\xi) = 0.$$

Therefore $\widehat{\Pi}([\omega])$ is δ_V closed.

We now show $\widehat{\Pi}$ is injective. Let $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ and let $\omega = dx \wedge \theta^0 \wedge \varepsilon - dt \wedge \boldsymbol{\beta}(\varepsilon)$ be the unique representative from Theorem 4.1, where $\varepsilon = r_i \theta^i$ and $\varepsilon^* = -\varepsilon$. Then $\widehat{\Pi}([\omega]) = dx \wedge \theta_E^0 \wedge (s_i \theta_E^i)$, and $(s_i \theta_E^i)^* = -(s_i \theta_E^i)$ since $X = D_x$. If $\widehat{\Pi}([\omega]) = 0$, then $s_i \theta_E^i = 0$ and $\omega = 0$. This shows $\widehat{\Pi}([\omega]) \in \mathscr{F}^2_{\mathrm{tsb}}(E)$ and that $\widehat{\Pi}$ is injective. □

In particular we have

Corollary 7.1. *If $[\omega] \neq 0$ then $\widehat{\Pi}([\omega]) \in \mathscr{F}^2_{\mathrm{tsb}}(E)$ is a symplectic form.*

We now set out to prove the fact that $\widehat{\Pi}$ in Lemma 7.2 is in fact a bijection which will imply Theorem 1.2 in the Introduction. We will use the following Lemma.

Lemma 7.3. *Let $s_i, \xi_{ij} \in C^\infty(\mathscr{R}^\infty)$ then*

$$\begin{aligned} 1] \quad & dt \wedge \left(\pi^{1,2} \circ \mathscr{L}_T(dx \wedge \theta_E^0 \wedge s_i \theta_E^i)\right) = dt \wedge \mathscr{L}_T(dx \wedge \theta^0 \wedge s_i \theta^i), \\ 2] \quad & dt \wedge dx \wedge D_x(\xi_{ij} \theta_E^i \wedge \theta_E^j) = dt \wedge dx \wedge X(\xi_{ij} \theta^i \wedge \theta^j)\,. \end{aligned} \tag{7.6}$$

Proof. Since $dt \wedge \theta_E^i = dt \wedge \theta^i$ and $X = D_x$ these identities follow. □

We now have the main theorem.

Theorem 7.1. *Let $\mathcal{S} = s_i D_x^i$ be a skew-adjoint differential operator on E. The form $\Sigma = dx \wedge \theta_E^0 \wedge (s_i \theta_E^i)$ is symplectic, and $Y = pr(K\partial_u)$ is a Hamiltonian vector-field for Σ if and only if*

$$\omega = dx \wedge \theta^0 \wedge \varepsilon - dt \wedge \boldsymbol{\beta}(\varepsilon) \tag{7.7}$$

satisfies $d_H \omega = 0$, where $\varepsilon = \mathcal{S}(\theta^0)$ and $\boldsymbol{\beta}(\varepsilon)$ is given in equation (4.1).

Proof. Supposed Σ is symplectic and Y is Hamiltonian, then Lemma 6.6 produces $\xi = \xi_{ab} \theta_E^a \wedge \theta_E^b$ satisfying equation (6.24). Let

$$\omega = dx \wedge \theta^0 \wedge (s_i \theta^i) + dt \wedge (\xi_{ab} \theta^a \wedge \theta^b). \tag{7.8}$$

Equations (7.6) and (6.24) give

$$\begin{aligned} d_H \omega &= dt \wedge T(dx \wedge \theta^0 \wedge (s_i \theta^i)) + dx \wedge X(dt \wedge \xi_{ab} \theta^a \wedge \theta^b) \\ &= dt \wedge \mathscr{L}_T(dx \wedge \theta_E^0 \wedge (s_i \theta_E^i) + dx \wedge dt \wedge D_x(\xi_{ab} \theta_E^a \wedge \theta_E^b) \\ &= dt \wedge \left(\pi^{1,2} \circ \mathscr{L}_T(dx \wedge \theta_E^0 \wedge (s_i \theta_E^i)) - dx \wedge D_x(\xi_{ab} \theta_E^a \wedge \theta_E^b)\right) = 0. \end{aligned}$$

Therefore $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$. Now Theorem 3.4 implies $\xi_{ab} \theta^a \wedge \theta^b = -\boldsymbol{\beta}(s_i \theta^i)$ so that ω in equation (7.8) and equation (7.7) are the same.

Suppose now that ω in equation (7.7) is d_H closed. By Lemma 7.2, $\Sigma = \widehat{\Pi}(\omega)$ is a symplectic form. So we need only show that Y is Hamiltonian. Again we refer to Lemma 6.6 and show the existence of $\xi = \xi_{ab}\theta_E^a \wedge \theta_E^b$ in equation (6.24).

Writing $\boldsymbol{\beta}(s_i\theta^i) = B_{ab}\theta^a \wedge \theta^b$ and using equations (7.6) we have

$$\begin{aligned} d_H\omega &= dt \wedge T(dx \wedge \theta^0 \wedge (s_i\theta^i)) - dx \wedge X(dt \wedge B_{ab}\theta^a \wedge \theta^b) \\ &= dt \wedge \left(\pi^{1,2} \circ \mathscr{L}_T(dx \wedge \theta_E^0 \wedge (s_i\theta_E^i)) + dx \wedge D_x(B_{ab}\theta_E^a \wedge \theta_E^b)\right). \end{aligned} \tag{7.9}$$

This will vanish if and only if

$$\pi^{1,2} \circ \mathscr{L}_T(dx \wedge \theta_E^0 \wedge (s_i\theta_E^i)) + dx \wedge D_x(B_{ab}\theta_E^a \wedge \theta_E^b) = 0 \tag{7.10}$$

because this term is t semi-basic. Equation (7.10) produces $\xi = -\Pi(\boldsymbol{\beta}(s_i\theta^i)) = B_{ab}\theta_E^a \wedge \theta_E^b$ in equation (6.24) and therefore Y is a Hamiltonian vector field for Σ. □

We now summarize the results of Lemma 7.2 and Theorem 7.1 by the following corollary.

Corollary 7.2. *Let $u_t = K$ be an evolution equation, and let $Y = pr(K\partial_u)$ be the corresponding evolutionary vector field on E and let $\mathscr{Z}_Y(E) \subset \mathscr{F}^2_{\mathrm{tsb}}(E)$ be the subset of symplectic forms for which Y is a Hamiltonian vector field. Define the function $\Psi : \mathscr{Z}_Y(E) \to H^{1,2}(\mathscr{R}^\infty)$ given by*

$$\Psi(dx \wedge \theta_E^0 \wedge \varepsilon_E) = [dx \wedge \theta^0 \wedge \varepsilon - dt \wedge \boldsymbol{\beta}(\varepsilon)], \tag{7.11}$$

where $dx \wedge \theta_E^0 \wedge \varepsilon_E \in \mathscr{Z}_Y(E)$ with $\varepsilon_E = \mathcal{S}(\theta_E^0)$ and $\mathcal{S} = s_iD_x^i$ is the corresponding symplectic operator, and $\varepsilon = \mathcal{S}(\theta^0) = s_iX^i(\theta^0)$. The function $\Psi : \mathscr{Z}_Y(E) \to H^{1,2}(\mathscr{R}^\infty)$ is an isomorphism and $\hat{\Psi} = \Psi^{-1}$ where $\hat{\Psi}$ is defined in equation (7.4).

With Theorem 1.1 and Corollary 7.2 in hand the proof of Theorem 1.2 and 1.3 are now easily given.

Proof. (Theorems 1.2 and 1.3) We start with Theorem 1.2 and suppose that $\mathcal{S} = s_iD_x^i \in \mathscr{Z}_Y(E)$ is a symplectic operator for the scalar evolution equation $\Delta = u_t - K$ and that $Y = \mathrm{pr}(K\partial_u)$ is a Hamiltonian vector field for Σ. Then with Ψ from equation (7.11) in Corollary 7.2 and Φ in equation (1.5) in Theorem 1.1 we have,

$$\Phi^{-1} \circ \Psi(\mathcal{S}) = -\frac{1}{2}s_iD_x^i$$

is a variational operator and so $\mathcal{S}$ is a variational operator for Δ (by the abuse of notation in Remark 2.1). The fact that $\Phi^{-1} \circ \Psi$ is an isomorphism then proves Theorem 1.2.

To prove Theorem 1.3 we identify as above, a symplectic operator $\mathcal{S}$ on E as an operator on $J^\infty(\mathbb{R}^2, \mathbb{R})$ (see Remark 2.1). The function Φ in equation (1.5) defines an isomorphism between symplectic operators for Δ and $H^{1,2}(\mathscr{R}^\infty)$. This proves Theorem 1.3. □

As our final Lemma we show for completeness how formula (5.13) can be determined from the symplectic potential.

Lemma 7.4. *Let $\mathcal{S} = s_iD_x^i$ be a symplectic operator and let $\psi = dx \wedge \theta_E^0 \cdot P \in C^\infty(E)$ be a symplectic potential. The unique representative for $\Psi(\mathcal{S}) \in H^{1,2}(\mathscr{R}^\infty)$ in Theorem* 5.2 *has $\varepsilon = \mathcal{S}(\theta^0)$. Furthermore there exists a representative ω for $\Psi(\mathcal{S})$ where ω in equation* (5.12) *can be written $\omega = d_V\eta$*

where

$$\eta = dx \wedge \theta^0 \cdot P - dt \wedge \gamma. \tag{7.12}$$

Proof. By equation (7.11) of Corollary 7.2 we have the unique representative as stated in the lemma.

We prove the second part of the lemma by first using Theorem 4.2 to construct a representative ω_0 for $\Psi(\mathcal{S})$ such that $\omega_0 = d_V\eta_0$ with $\eta_0 = dx\wedge\theta^0\cdot Q - dt\wedge\gamma_0$. By equation (4.17) of Corollary 4.3 and equation (6.22) for the operator in the form $\varepsilon = \mathcal{S}(\theta^0)$ gives

$$\varepsilon = \frac{1}{2}(\mathbf{L}_Q - \mathbf{L}_Q^*)\theta^0 = \frac{1}{2}(\mathbf{L}_P - \mathbf{L}_P^*)\theta^0. \tag{7.13}$$

Lemma 6.5 and equation (7.13) show $\psi_0 = dx\wedge\theta_E^0\cdot Q$ is a symplectic potential for $\mathcal{S}$ and that $\delta_V^E\psi_0 = \delta_V^E\psi$. Therefore using equation (6.2) (for I_E) and the exactness of the d_V^E complex,

$$\psi = \psi_0 + d_V^E(Adx) + d_H^E\xi \tag{7.14}$$

for some $A\in C^\infty(E)$ and $\xi\in\Omega^{0,1}(E)$. We then let

$$\eta = \eta_0 + d_V(Adx) + d_H\xi, \quad \text{and} \quad \omega = \omega_0 - d_H d_V\xi, \tag{7.15}$$

where we are computing d_H and d_V on $\mathscr{R}^\infty$. Note that by equation (7.3) and (7.14) we have $\Pi(\eta) = \psi$ so that η has the form in equation (7.12). We then compute using equation (7.15) and $\omega_0 = d_V\eta_0$ that

$$d_V\eta = d_V(\eta_0 + d_H\xi) = \omega_0 + d_V d_H\xi = \omega,$$

which proves the lemma. □

7.2. *Time Independent Operators*

Equation (1.6) defines when the time independent evolution equation $u_t = K(x,u,u_x,\ldots)$ is a Hamiltonian evolution equation with symplectic operator $\mathcal{S}$. This is precisely the same definition that the ordinary differential equation $K(x,u,u_x,\ldots)=0$ admits $\mathcal{S}$ as a variational operator. The following simple lemma is the key to decoupling the variational operator problem for time independent scalar evolution equations.

Lemma 7.5. *Let $\mathcal{S} = s_i D_x^i$ be a time independent symplectic operator with symplectic potential $P\in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$ (equation (6.13) in Lemma 6.2). Then*

$$\mathcal{S}(u_t) = \mathbf{E}\left(-\frac{1}{2}Pu_t\right). \tag{7.16}$$

Proof. By the product formula in the calculus of variations (equation 5.80 in [17]) the left side of equation (7.16) is

$$
\begin{aligned}
\mathbf{E}\left(-\frac{1}{2}Pu_t\right) &= -\frac{1}{2}\left(\mathbf{F}_P^* u_t + \mathbf{F}_{u_t}^* P\right) \\
&= -\frac{1}{2}\left(\mathbf{F}_P^* u_t - D_t P\right) \\
&= -\frac{1}{2}\left(\mathbf{F}_P^* u_t - P_i D_x^i u_t\right) \\
&= -\frac{1}{2}\left(\mathbf{F}_P^* u_t - \mathbf{F}_P u_t\right).
\end{aligned}
\tag{7.17}
$$

Equation (7.17) together with the fact from equation (6.13) that $2\mathcal{S}(u_t) = \mathbf{F}_P u_t - \mathbf{F}_P^* u_t$ show that the two sides of equation (7.16) agree. □

We then have the following.

Theorem 7.2. *Let $\mathcal{S}$ be a t-independent symplectic operator. The following are equivalent,*

(1) $u_t = K(x, u, u_x, \ldots, u_{2m+1})$, $m \geq 1$, *satisfies* $\mathcal{S}(K) = \mathbf{E}(H)$.
(2) $\mathcal{S}$ *is a symplectic variational operator for the ODE* $K = 0$,
(3) $\mathcal{S}$ *is a variational operator for* $u_t = K$ (*see Remark* 2.1).

This converts the symplectic Hamiltonian question for the evolution equation into a variational operator problem for the ODE $K = 0$.

Proof. Suppose $u_t = K(x, u, \ldots, u_{2m+1})$ is Hamiltonian for the t-independent symplectic operator $\mathcal{S}$, so that $\mathcal{S}(K) = \mathbf{E}(H)$ on $J^\infty(\mathbb{R}, \mathbb{R})$. By definition $\mathcal{S}$ is a variational operator for the ODE $K = 0$. So (1) and (2) are trivially equivalent.

We show (1) implies (3). Suppose that $\mathcal{S}(K) = \mathbf{E}(H)$. Using equation (7.17) in Lemma 7.5 we have

$$
\mathcal{S}(u_t - K) = \mathcal{S}(u_t) - \mathbf{E}(H) = \mathbf{E}\left(-\frac{1}{2}Pu_t - H\right),
$$

where P is a symplectic potential for $\mathcal{S}$ (see equation (6.13)). Therefore $\mathcal{S}$ is a variational operator for $u_t - K$.

Finally we show (3) implies (1). Lemma 7.4 which allows Q in equation (5.13) to be replaced with the symplectic potential P so that hypothesis (3) implies,

$$
(\mathbf{F}_P^* - \mathbf{F}_P)(u_t - K) = \mathbf{E}(\Delta P + L) = \mathbf{E}(Pu_t - PK + L). \tag{7.18}
$$

Substituting from equation (7.17) into equation (7.18) we get

$$
(\mathbf{F}_P^* - \mathbf{F}_P)(-K) = \mathbf{E}(-PK + L).
$$

Therefore

$$
\mathcal{S}(K) = \mathbf{E}(2(L - PK)),
$$

and $u_t = K$ is a time independent Hamiltonian evolution equation for the symplectic operator $\mathcal{S}$.

□

It is worth noting that if $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ then $[\partial_t \lrcorner\, \omega]_{K=0} \in H^{1,2}(K=0)$ [5]. That is, in the time independent case, the form β in Lemma 3.2 (restricted to the ODE $K=0$) defines an $H^{0,2}$ cohomology class for the ODE $K=0$.

8. First Order Operators and Hamiltonian Evolution Equations

8.1. *First Order Operators for Third Order Equations*

For a third order evolution equation

$$u_t = K(t,x,u,u_x,u_{xx},u_{xxx}) \tag{8.1}$$

we write the conditions for when $\mathcal{E} = 2RD_x + X(R)$, $R \in C^\infty(\mathscr{R}^\infty)$ is a variational operator. This will prove Theorem 1.4 in the Introduction.

Proof. (Theorem 1.4) By Theorem 1.1 and Corollary 4.2 the skew-adjoint operator $\mathcal{E} = 2RD_x + X(R)$ is a variational operator for (8.1) if and only if the skew-adjoint form $\varepsilon = -R\theta^1 - \frac{1}{2}R_0\theta^0$ is a solution to

$$-\mathbf{L}^*_\Delta(\varepsilon)\wedge\theta^0 = \big(T(\varepsilon) - X^3(K_3\varepsilon) + X^2(K_2\varepsilon) - X(K_1\varepsilon) + K_0\varepsilon\big)\wedge\theta^0 = 0 \tag{8.2}$$

where $K_i = \partial_{u_i}K$. Using $T(\theta^0) = d_V K = K_i\theta^i$ and $T(\theta^1) = X(d_V K) = X(K_i\theta^i)$ we have

$$T(\varepsilon) = -T(R)\theta^1 - \frac{1}{2}T(X(R)))\theta^0 - RX(K_i\theta^i) - \frac{1}{2}X(R)K_i\theta^i. \tag{8.3}$$

The highest possible $\theta^i\wedge\theta^0$ term in equation (8.2) using (8.3) is θ^4. We find from equation (8.2)

$$[\theta^4\wedge\theta^0] = -RK_3 + RK_3 = 0.$$

While for $\theta^3\wedge\theta^0$, $\theta^2\wedge\theta^0$ and $\theta^1\wedge\theta^0$ we have from equations (8.3) and (8.2),

$$\begin{aligned}
[\theta^3\wedge\theta^0] =& 2X(K_3)R - 2K_2R + 3K_3X(R),\\
[\theta^2\wedge\theta^0] =& -3X(K_2R) + 3X^2(RK_3) + \frac{3}{2}X(K_3X(R)),\\
[\theta^1\wedge\theta^0] =& -T(R) - 2K_0R + K_1X(R) + \frac{3}{2}X^2(K_3X(R))\\
&+ X^3(K_3R) - X^2(K_2R) - X(K_2X(R)).
\end{aligned} \tag{8.4}$$

For the coefficient of $\theta^3\wedge\theta^0$ to be zero we have from equation (8.4),

$$X(R) = \frac{2}{3K_3}(K_2 - X(K_3))R = \hat{K}_2R \tag{8.5}$$

where $\hat{K}_2 = \frac{2}{3K_3}(K_2 - X(K_3))$. The coefficient of $\theta^2\wedge\theta^0$ in equation (8.4) is zero on account of (8.5). For the coefficient of $\theta^1\wedge\theta^0$ in (8.4) to be zero gives

$$T(R) = -2K_0R + K_1X(R) + \frac{3}{2}X^2(K_3X(R)) + X^3(K_3R) - X^2(K_2R) - X(K_2X(R)). \tag{8.6}$$

Simplifying equation (8.6) using equation (8.5) we get

$$T(R) = \left(-2K_0 + K_1\hat{K}_2 - \frac{1}{2}\left(X(K_3)\hat{K}_2^2 + K_3\hat{K}_2^3\right) + X(K_3X(\hat{K}_2))\right)R \tag{8.7}$$

It follows that a non-vanishing R (which we may assume to be positive) satisfying equations (8.5) and (8.7), which is necessary and sufficient for the existence of a first order variational operator for $\Delta = u_t - K$ in equation (8.1), is equivalent to $A = X(\log R)$ and $B = T(\log R)$ satisfying the conditions in Theorem 1.4. This proves Theorem 1.4. □

The form κ in equation (1.7) for the KdV equation $u_t = u_{xxx} + uu_x$ satisfies

$$\kappa = -u_x dt, \quad d_H \kappa = -u_{xx} dx \wedge dt.$$

Therefore according to Theorem 1.4 there is no first order formulation for the KdV equation as a symplectic Hamiltonian evolution equation.

8.2. *First Order Hamiltonians Operators and their Potential Form*

Let $v_t = \mathscr{D} \circ \mathbf{E}(H(x, v, v_x, \ldots))$ be a time independent Hamiltonian evolution equation where $\mathscr{D}$ is a first order Hamiltonian operator. According to [16] or [1] we may choose coordinates (using a contact transformation) such that $\mathscr{D} = D_x$. The following is Theorem 1 in [15] in the context of scalar evolution equations.

Lemma 8.1. *The potential form of the Hamiltonian evolution equation,*

$$v_t = D_x \mathbf{E}(H_1(x, v, v_x, \ldots)). \tag{8.8}$$

is given by the equation

$$u_t = \mathbf{E}(H_1)|_{v=u_x}. \tag{8.9}$$

Equation (8.9) *admits* $\mathcal{E} = D_x$ *as a first order variational operator, and satisfies*

$$D_x \left(u_t - \mathbf{E}(H_1)|_{v=u_x} \right) = \mathbf{E}(-\frac{1}{2} u_x u_t + H_1|_{v=u_x}). \tag{8.10}$$

There is an abuse of notation in this lemma where D_x is used as the total x derivative operator in either variable u or v depending on context.

Proof. Starting with equation (8.8), let $v = u_x$ so that (8.8) becomes

$$u_{tx} = (D_x \mathbf{E}(H_1))|_{v=u_x} = D_x \left(\mathbf{E}(H_1)|_{v=u_x} \right). \tag{8.11}$$

Integrating equation (8.11) with respect to x gives the potential form (8.9).

To prove equation (8.10) holds, we simply need the change of variables formula, see exercise 5.49 in [17],

$$\mathbf{E}\left(H_1|_{v=u_x}\right) = (D_x)^* \left(\mathbf{E}(H_1|_{v=u_x}) \right) = -D_x \left(\mathbf{E}(H_1|_{v=u_x}) \right). \tag{8.12}$$

Equation (8.12) together with the simple fact $-2\mathbf{E}(u_t u_x) = u_{tx}$ proves equation (8.10). □

The second term in the right hand side of equation (8.10) is just the pullback of the Hamiltonian function in (8.8). We also note the following simple corollary.

Corollary 8.1. *Every Hamiltonian evolution equation* $v_t = \mathscr{D}(\mathbf{E}(H_1(x, v, v_x, \ldots)))$ *with first order Hamiltonian operator* $\mathscr{D}$ *is the symmetry reduction of an equation* $u_t = K(x, u, u_x, \ldots)$*, of the same order, which admits an invariant first order variational operator.*

8.3. *Bi-Hamiltonian Evolution Equations with a First Order Hamiltonian Operator*

We now present sufficient conditions when the potential form of a compatible bi-Hamiltonian system admits another variational operator.

Theorem 8.1. *Let $v_t = K(x,v,v_x,\ldots) = D_x(\mathbf{E}(H_1(x,v,v_x,\ldots)))$ be a Hamiltonian evolution equation with potential form*

$$u_t = \mathbf{E}(H_1)|_{v=u_x}. \tag{8.13}$$

Let $\mathscr{D}_0$ be second time independent Hamiltonian operator with Hamiltonian $H_0(x,v,v_x,\ldots)$ satisfying,

$$v_t = D_x(\mathbf{E}(H_1)) = \mathscr{D}_0(\mathbf{E}(H_0)).$$

Assume $\mathscr{D}_0$ also satisfies the compatibility condition (equation 7.29 in [17])

$$\mathscr{D}_0(\mathbf{E}(H_1)) = D_x\mathbf{E}(H_2). \tag{8.14}$$

Then the right hand side of the potential form satisfies

$$\mathcal{E}(\mathbf{E}(H_1)|_{v=u_x}) = -\mathbf{E}(H_2|_{v=u_x}) \tag{8.15}$$

where [a], $\mathcal{E} = \mathscr{D}_0|_{v=u_x}$. *Furthermore if $\mathcal{E} = \mathscr{D}_0|_{v=u_x}$ is symplectic, then $\mathcal{E}$ is a variational operator for the evolution equation* (8.13) *and*

$$\mathcal{E}(u_t - K) = \mathbf{E}(Qu_t + H_2|_{v=u_x}) \tag{8.16}$$

where Q is defined in equation (5.28) *where $\mathcal{E} = F_Q^* - F_Q$.*

Proof. First we apply $\mathcal{E} = \mathscr{D}_0|_{v=u_x}$ to the right hand side of equation (8.13), and use condition (8.14) to get

$$\begin{aligned}\mathcal{E}(\mathbf{E}(H_1)|_{v=u_x}) &= (\mathscr{D}_0(\mathbf{E}(H_1)))|_{v=u_x} \\ &= (D_x(\mathbf{E}(H_2)))|_{v=u_x} \\ &= -\mathbf{E}(H_2|_{v=u_x}).\end{aligned} \tag{8.17}$$

Again the last line follows from the change of variables formula in the calculus variations (exercise 5.49 in [17]). This verifies equation (8.15). Then by part (1) of Theorem 7.2 equation (8.15) shows that $\mathcal{E}$ is a variational operator for equation (8.13). If Q is the function from equation (5.28) we then have $\mathcal{E}(u_t) = (F_Q^* - F_Q)(u_t) = E(Qu_t)$ and equation (8.17) that

$$\mathcal{E}(u_t - \mathbf{E}(H_1)|_{v=u_x}) = \mathbf{E}(Qu_t + H_2|_{v=u_x}). \tag{8.18}$$

□

[a] $\mathscr{D}_0$ is the push-forward of $\mathcal{E}$ by the quotient map $\mathbf{q} : (t,x,u,u_x,\ldots) \to (t,x,v,v_x,\ldots)$.

Theorem 8.1 makes the hypothesis that $\mathcal{E} = \mathscr{D}_0|_{v=u_x}$ is a symplectic operator. This holds in the case of the Hamiltonian operators given by Theorem 5.3 in [10],

$$\mathscr{D}_0 = h(v)\left(\sqrt{c_1 + c_2\int_0^v \frac{1}{h(y)}dy}D_x \circ \sqrt{c_1 + c_2\int_0^v \frac{1}{h(y)}dy} + D_x^3\right)\circ h(v) \tag{8.19}$$

satisfy the compatibility conditions with D_x in Corollary 3.2 of [8] when $h(v) = (k_1 v + k_2)^{-1}$. This gives

$$\mathcal{E} = \mathscr{D}_0|_{v=u_x} = \frac{1}{k_1 u_x + k_2}\left(\sqrt{c_1 + \frac{1}{2}k_1 u_x^2 + k_2 u_x}D_x \circ \sqrt{c_1 + \frac{1}{2}k_1 u_x^2 + k_2 u_x} + D_x^3\right)\circ \frac{1}{k_1 u_x + k_2} \tag{8.20}$$

which are symplectic [10].

9. Examples

Example 9.1. The Harry-Dym equation $z_t = z^3 z_{xxx}$ [10,21] is a compatible bi-Hamiltonian system,

$$z_t = \widehat{\mathscr{D}}_1\mathbf{E}(\widehat{H}_1) = \widehat{\mathscr{D}}_0\mathbf{E}(\widehat{H}_0) \tag{9.1}$$

where

$$\widehat{\mathscr{D}}_1 = z^2 \circ D_x \circ z^2\ ,\ \widehat{H}_1 = -\frac{1}{2}\frac{z_x^2}{z}\ ,\ \text{and}\quad \widehat{\mathscr{D}}_0 = z^3 \circ D_x^3 \circ z^3\ ,\ \widehat{H}_0 = -\frac{1}{z}. \tag{9.2}$$

The change of variable $z = v^{-1}$ maps the Hamiltonian operator $\widehat{\mathscr{D}}_1$ to canonical form [1,16], and the Hamilonian operators and the associated Hamiltonians in equation (9.2) become

$$\mathscr{D}_1 = D_x\ ,\ H_1 = -\frac{1}{2}\frac{v_x^2}{v^3},\quad \text{and}\quad \mathscr{D}_0 = v^{-1}D_x^3 \circ v^{-1}\ ,\ H_0 = -v. \tag{9.3}$$

The Harry-Dym equation (9.1) in these coordinates is then,

$$v_t = D_x\mathbf{E}(H_1) = \mathscr{D}_0(\mathbf{E}(H_0)) = \frac{v_{xxx}}{v^3} - 6\frac{v_x v_{xx}}{v^4} + \frac{6v_x^3}{v^5}. \tag{9.4}$$

The potential form of equation (9.4) is found by letting $v = w_x$ and integrating to get (see also (8.9))

$$w_t = E(H_1)|_{v=w_x} = \frac{w_{xxx}}{w_x^3} - \frac{3w_{xx}^2}{2w_x^4}. \tag{9.5}$$

Equation (8.10) of Lemma 8.1 as it applies to the potential Harry-Dym equation (9.5) produces the following variational operator equation for D_x,

$$D_x\left(w_t - \frac{w_{xxx}}{w_x^3} + \frac{3w_{xx}^2}{2w_x^4}\right) = \mathbf{E}\left(-\frac{1}{2}w_x w_t - \frac{1}{2}\frac{w_{xx}^2}{w_x^3}\right).$$

We now apply Theorem 8.1 to obtain a second variational operator. The compatibility condition in equation (8.14) is satisfied with the operators from equation (9.3) with

$$\mathscr{D}_0(\mathbf{E}(H_1)) = D_x\mathbf{E}(H_2),\quad \text{where}\quad H_2 = \frac{1}{2}\frac{v_{xx}^2}{v^5} - \frac{15}{8}\frac{v_x^4}{z^7}. \tag{9.6}$$

The operator $\mathscr{D}_0$ in equation (9.3) is of the form (8.19) so that by equation (8.20)

$$\mathcal{E} = \mathscr{D}_0|_{v=w_x} = w_x^{-1} D_x^3 \circ w_x^{-1} \tag{9.7}$$

is a symplectic or variational operator. Since the compatibility condition in equation (8.14) is satisfied and $\mathcal{E}$ is a symplectic operator Theorem 8.1 applies. Therefore the operator $\mathcal{E}$ in equation (9.7) is a variational operator for the potential Harry-Dym equation in (9.5). The function Q in equation (5.28) is easily determined for $\mathcal{E}$ (using the fact that $-2\mathcal{E}$ is a symplectic operator) to be

$$Q = \frac{w_{xx}^2 - w_x w_{xxx}}{2w_x^3}. \tag{9.8}$$

Equation (8.16) with Q in equation (9.8) and H_2 in equation (9.6) (with $v = w_x$) gives the variational operator equation for the potential Harry-Dym equation (9.5),

$$\mathcal{E}\left(w_t - \frac{w_{xxx}}{w_x^3} + \frac{3w_{xx}^2}{2w_x^4}\right) = \mathbf{E}\left(\frac{w_{xx}^2 - w_x w_{xxx}}{2w_x^3} w_t + \frac{1}{2}\frac{w_{xxx}^2}{w_x^5} - \frac{15}{8}\frac{w_{xx}^4}{w_x^7}\right).$$

If we return to the original coordinates for the Harry-Dym equation and make the change of variable given by $x = u, w = x, w_x = u_x^{-1}, \ldots$ to the potential form in equation (9.5) we get the Krichever-Novikov equation (or Schwarzian KdV), pg. 120 in [10],

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x}. \tag{9.9}$$

In particular the Krichever-Novikov in equation (9.9) is the potential form of the Harry-Dym equation (9.1). These different coordinate representations of the Harry-Dym equation and the Krichever-Novikov equation is summarized by the diagram,

$$\begin{tikzcd}[column sep=huge, row sep=huge]
u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} \arrow[r, dash, "(x = w, w = x, w_x = u_x^{-1})"] \arrow[d, "(x = u, z = u_x, z_x = u_{xx}u_x^{-1})"] & w_t = \frac{w_{xxx}}{w_x^3} - \frac{3w_{xx}^2}{2w_x^4} \arrow[d, "(x = x, v = w_x)"] \\
z_t = z^3 z_{xxx} \arrow[r, dash, "(x = x, v = z^{-1})"'] & v_t = -v^{-1}D_x^3(v^{-1})
\end{tikzcd} \tag{9.10}$$

The variational or symplectic operators for the Krichever-Novikov equation are obtained by applying the change of variables $x = u, w = x, w_x = u_x^{-1}, \ldots$ to D_x and equation (9.7) giving the well known symplectic or variational operators for the Krichever-Novikov equation [10],

$$\mathcal{E}_1 = u_x^{-1} D_x \circ u_x^{-1} = \frac{1}{u_x^2}D_x - \frac{u_{xx}}{u_x^3}, \quad \mathcal{E}_0 = \frac{1}{u_x^2}D_x^3 - 3\frac{u_{xx}}{u_x^3}D_x^2 + \left(3\frac{u_{xx}^2}{u_x^4} - \frac{u_{xxx}}{u_x^3}\right)D_x. \tag{9.11}$$

With quotient map $\mathbf{q}(t, x, u, u_x, u_{xx}, \ldots) = (t = t, x = u, z = u_x, z_x = u_{xx}u_x^{-1}, \ldots)$, the operators from (9.11) project $\mathbf{q}_* \mathcal{E}_i = \widehat{\mathscr{D}}_i$ to the Hamiltonian operators in equation (9.2).

We now compute the explicit unique representative for the $H^{1,2}(\mathscr{R}^\infty)$ cohomology class for the Krichever-Novikov equation (9.9) corresponding to the first operator in (9.11) (Theorem 4.1). This is computed using formula (1.4) in Theorem 1.1 to be,

$$\omega_1 = -\frac{1}{2u_x^2}dx\wedge\theta^0\wedge\theta^1 + dt\wedge\left[\theta^0\wedge\left(\frac{4u_{xxx}u_x-3u_{xx}^2}{4u_x^4}\theta^1+\frac{u_{xx}}{2u_x^3}\theta^2-\frac{1}{2u_x^2}\theta^3\right)+\frac{1}{u_x^2}\theta^1\wedge\theta^2\right]. \tag{9.12}$$

We have $d_V\omega_1 = 0$ and for the forms η and λ in Theorem 5.3 we may choose

$$\begin{aligned}\eta_1 &= \frac{1}{2u_x}dx\wedge\theta^0 + dt\wedge\left(\frac{u_{xx}^2-2u_{xxx}u_x}{4u_x^3}\theta^0+\frac{u_{xx}}{2u_x^2}\theta^1+\frac{1}{2u_x}\theta^2\right)\\ \lambda_1 &= -\frac{3u_{xx}^2}{4u_x^2}dt\wedge dx.\end{aligned} \tag{9.13}$$

Likewise formula (1.4) for the second operator in (9.11) gives the unique cohomology representative (Theorem 4.1),

$$\begin{aligned}\hat{\omega}_0 &= dx\wedge\theta^0\wedge\left(\frac{u_xu_{xxx}-3u_{xx}^2}{2u_x^4}\theta^1+\frac{3u_{xx}}{2u_x^3}\theta^2-\frac{1}{2u_x^2}\theta^3\right)-\frac{1}{2u_x^2}dt\wedge\theta^2\wedge\theta^3\\ &\quad + dt\wedge\theta^1\wedge\left(\frac{1}{2u_x^2}\theta^4-\frac{u_{xx}}{u_x^3}\theta^3-\frac{5u_xu_{xxx}-6u_{xx}^2}{2u_x^4}\theta^2\right)-\frac{1}{2u_x^2}dt\wedge\theta^0\wedge\theta^5\\ &\quad + \frac{2u_x^3u_{xxxxx}-18u_x^2u_{xx}u_{xxxx}-12u_x^2u_{xxx}^2+69u_xu_{xx}^2u_{xxx}-39u_{xx}^4}{4u_x^6}dt\wedge\theta^0\wedge\theta^1\\ &\quad + dt\wedge\theta^0\left(\frac{10u_x^2u_{xxxx}-48u_xu_{xx}u_{xxx}+39u_{xx}^3}{4u_x^5}\theta^2+\frac{3(4u_xu_{xxx}-7u_{xx}^2)}{4u_x^4}\theta^3+\frac{2u_{xx}}{u_x^3}\theta^4\right).\end{aligned} \tag{9.14}$$

In this case $d_V\hat{\omega}_0 \neq 0$, but $[\hat{\omega}_0] = [\omega_0]$ where

$$\omega_0 = \hat{\omega}_0 + d_H\left(\frac{u_{xx}}{2u_x^3}\theta^0\wedge\theta^1\right) \tag{9.15}$$

and $d_V\omega_0 = 0$. Furthermore with ω_0 in equation (9.15) the forms η and λ in Theorem 5.3 can be chosen to be

$$\begin{aligned}\eta_0 &= \frac{2u_{xx}^2-u_xu_{xxx}}{2u_x^3}dx\wedge\theta^0+\frac{2u_{xx}^2-u_xu_{xxx}}{2u_x^3}dt\wedge\theta^2+\frac{u_{xxxx}u_x^2-3u_xu_{xx}u_{xxx}+u_{xx}^3}{2u_x^4}dt\wedge\theta^1\\ &\quad -\frac{2u_x^3u_{xxxxx}-10u_x^2u_{xx}u_{xxxx}-6u_x^2u_{xxx}^2+27u_xu_{xx}^2u_{xxx}-12u_{xx}^4}{4u_x^5}dt\wedge\theta^0\\ \lambda_0 &= -\frac{u_{xx}^4}{8u_x^4}dt\wedge dx.\end{aligned} \tag{9.16}$$

For λ_i in equations (9.13) and (9.16), it is difficult to determine whether $[\lambda_i]\in H^{2,0}(\mathscr{R}^\infty)$ is trivial or not (see Theorem A.2). However, it is possible but not easy to show $\lambda_i \neq d\kappa_i$ where κ_i is t-invariant by using the infinite sequence of conservation laws [10] for the Krichever-Novikov (Schwarzian KdV) equation (9.9). The forms λ_i define a non-trivial cohomology class in the t-invariant variational bi-complex for (9.9).

Example 9.2. The Harry Dym equation can be written in the form

$$v_t = D_x^3\left(\frac{1}{\sqrt{v}}\right) = \mathscr{D}_i\left(\mathbf{E}(H_i)\right), \qquad i = 0, 1 \tag{9.17}$$

where the Hamilonian operators and their Hamiltonians are

$$\mathscr{D}_0 = D_x^3\ ,\ H_0 = 2\sqrt{v} \quad \text{and} \quad \mathscr{D}_1 = 2vD_x + v_x\ ,\ H_1 = \frac{1}{8}v^{-\frac{5}{2}}v_x^2.$$

Equation (9.17) is obtained from equation (9.4) by substituting $v = -2^{\frac{1}{3}}\sqrt{\hat{v}}$.

Another potential form (or integrable extension) for the Harry-Dym equation (9.17) can be obtained by letting $v = u_{xxx}$ in equation (9.17) so that

$$u_{txxx} = \left(D_x^3\mathbf{E}(H_0)\right)|_{v=u_{xxx}},$$

which after integrating three times gives,

$$u_t = \mathbf{E}(H_0)|_{v=u_{xxx}} = \sqrt{\frac{1}{u_{xxx}}}. \tag{9.18}$$

We show that D_x^3 is a variational operator. First using the change of variables formula in the calculus of variation for $v = u_{xxx}$ (exercise 5.49 [17]) we have

$$\left(-D_x^3\mathbf{E}(H_0)\right)\Big|_{v=u_{xxx}} = \mathbf{E}\left(H_0|_{v=u_{xxx}}\right). \tag{9.19}$$

The operator D_x^3 is symplectic which together with equation (9.19) shows that D_x^3 is a variational operator for equation (9.18) and giving,

$$D_x^3\left(u_t - \mathbf{E}(H_0)|_{v=u_{xxx}}\right) = u_{txxx} + \mathbf{E}\left(H_0|_{v=u_{xxx}}\right) = \mathbf{E}\left(-\frac{1}{2}u_t u_{xxx} + 2\sqrt{u_{xxx}}\right).$$

In equation (8.14) compatibility was used to show the second Hamiltonian operator for a bi-Hamiltonian equation became a variational operator for the potential form. In order to use a similar argument in this case we need to show $\mathscr{D}_1\mathbf{E}(H_0) = \mathscr{D}_0\mathbf{E}(H_{-1})$. We find

$$\mathscr{D}_1\left(\mathbf{E}(H_0)\right) = (2vD_x + v_x)\left(\frac{1}{\sqrt{v}}\right) = 0 = \mathscr{D}_0(0). \tag{9.20}$$

In analogy to equation (8.14), this gives rise with $H_{-1} = 0$ to the variational operator

$$\mathcal{E} = \mathscr{D}_1|_{v=u_{xxx}} = 2u_{xxx}D_x + u_{xxxx}. \tag{9.21}$$

Using the fact that operator $\mathcal{E}$ in equation (9.21) is a symplectic operator, the compatibility condition (9.20) gives

$$\begin{aligned}\mathcal{E}\left(u_t - \mathbf{E}(H_0)|_{v=u_{xxx}}\right) &= 2u_{xxx}u_{tx} + u_{xxxx}u_t - \mathcal{E}\left(\mathbf{E}(H_0)\right)|_{v=u_{xxx}} \\ &= \mathbf{E}\left(\frac{1}{2}u_{xx}^2 u_t\right).\end{aligned} \tag{9.22}$$

Equation (9.22) shows directly that $\mathcal{E}$ in (9.21) is a variational operator for equation (9.18).

It is worth noting that $\mathcal{E}(K) = 0$ in this example and that $[\omega] = d_V[\eta]$ where $[\eta] \in H^{1,1}(\mathscr{R}^\infty)$. The representative

$$\omega = dx \wedge \theta^0 \wedge \varepsilon - dt \wedge \boldsymbol{\beta}(\varepsilon) + d_H(\theta^0 \wedge \theta^1 \cdot u_{xx}) + \frac{1}{3} d_H \circ d_V(uu_{xx}\theta^1 - (u_x u_{xx} + uu_{xxx})\theta^0)$$

with $\varepsilon = -\frac{1}{2}\mathcal{E}(\theta^0) = -u_{xxx}\theta^1 - \frac{1}{2}u_{xxxx}\theta^0$ satisfies $\omega = d_V\eta$ where

$$\eta = dx \wedge \theta^0 \cdot \left(-\frac{2}{3}u_x u_{xxx} - \frac{1}{3}uu_{xxxx}\right) - dt \wedge \boldsymbol{\beta}(-\frac{2}{3}u_x u_{xxx} - \frac{1}{3}uu_{xxxx}).$$

Since $d_H\eta = 0$, $[\eta] \in H^{1,1}(\mathscr{R}^\infty)$. This also produces an example where

$$Q = -\frac{2}{3}u_x u_{xxx} - \frac{1}{3}uu_{xxxx}$$

satisfies $\mathbf{L}^*_\Delta(Q) = 0$, as well as equation (A.6). By Theorem A.1, Corollary A.1 or Corollary A.3, Q is not the characteristic of a classical conservation law

Example 9.3. The cylindrical KdV equation is (see [21])

$$v_t = v_{xxx} + vv_x - \frac{v}{2t} \tag{9.23}$$

while it's potential form is

$$u_t = u_{xxx} + \frac{1}{2}u_x^2 - \frac{u}{2t}. \tag{9.24}$$

The form κ in equation (1.7) in Theorem 1.4 is $\kappa = t^{-1}dt = d_H(\log t)$ and so equation (9.24) admits $\mathcal{E}_1 = tD_x$ as a variational operator. In equation (5.13) we have $Q_1 = -\frac{1}{2}tu_x$ leading to

$$\mathcal{E}_1\left(u_t - u_{xxx} - \frac{1}{2}u_x^2 + \frac{u}{2t}\right) = \mathbf{E}\left(-\frac{1}{2}tu_x\left(u_t - u_{xxx} - \frac{1}{2}u_x^2 + \frac{u}{2t}\right) - \frac{1}{12}tu_x^3\right).$$

Note that the Lagrangian on the right side of this equation differs from that in equation (1.3) by a total divergence.

By solving the equation $\theta^0 \wedge \mathbf{L}^*_\Delta(\varepsilon) = 0$ from (4.1) for third order forms ε we find that equation (9.24) admits a third order symplectic or variational operator,

$$\mathcal{E}_0 = t^2 D_x^3 + \frac{1}{3}(2t^2 u_x + tx)D_x + \frac{1}{6}(2t^2 u_{xx} + t).$$

For $\mathcal{E}_0$ we have

$$Q_0 = -\frac{1}{6}\left(t^2 u_x^2 + txu_x + 3u_{xxx}t^2\right)$$

in equation (5.13) leading to

$$\mathcal{E}_0\left(u_t - u_{xxx} - \frac{1}{2}u_x^2 + \frac{u}{2t}\right) = \mathbf{E}\left(Q_0\left(u_t - u_{xxx} - \frac{1}{2}u_x^2 + \frac{u}{2t}\right) - \frac{1}{72}\left(t^2 u_x^4 + 2txu_x^3\right)\right)$$

If we now compute the reduction of the potential cylindrical KdV by substituting $w = \sqrt{t}\,u_x$ into the x-derivative of equation (9.24) we get

$$w_t = w_{xxx} + \frac{1}{\sqrt{t}}ww_x = \mathscr{D}_1(\mathbf{E}(H_1)) = \mathscr{D}_0(\mathbf{E}(H_0)) \tag{9.25}$$

where

$$\mathscr{D}_1 = D_x,\ H_1 = \frac{1}{2}w_x^2 + \frac{1}{6\sqrt{t}}w^3, \quad \mathscr{D}_0 = D_x^3 + \frac{2w}{3\sqrt{t}}D_x + \frac{w_x}{3\sqrt{t}}, \quad H_0 = \frac{1}{2}w^2. \tag{9.26}$$

Equation (9.25) can of course be obtained from the cylindrical KdV equation (9.23) by the change of variables $w = \sqrt{t}\,v$. It is unclear (to the authors) if the cylindrical KdV in equation (9.24) is a bi-Hamiltonian evolution equation for which $\mathscr{D}_1$ and $\mathscr{D}_0$ in equation (9.26) are Hamiltonian operators. Reference [21] states there are no Hamiltonians for the cylindrical KdV. It is straightforward to work out the symplectic or variational operators for the potential cylindrical KdV in these new variables from equation (9.25) by following Theorem 8.1.

More generally any evolution equation of the form

$$u_t = u_{xxx} + a(t)u_x^2. \tag{9.27}$$

admits D_x as a first order variational operator. We find after a long computation that equation (9.27) admits a third order variational operator in the case where $a(t)\dot{a}(t) \neq 0$ only when

$$a(t) = \pm\frac{1}{\sqrt{c_1 t + c_2}}. \tag{9.28}$$

For the $+$ sign in equation (9.28), the change of variables $t = c_1^{-1}(\hat{t} - c_2),\ \ x = c_1^{-\frac{1}{3}}\hat{x},\ u = \frac{1}{2}c_1^{\frac{1}{3}}\hat{u}$, takes equation (9.27) with $a(t)$ in (9.28) to the potential form of the cylindrical KdV obtained from equation (9.25). The same result holds in the other cases with slightly different changes of variable.

10. Conclusions

The determination of a variational or symplectic operator for a scalar evolution equation has been shown to be equivalent to the non-vanishing of a cohomology class in $H^{1,2}(\mathscr{R}^\infty)$. The arguments used to prove this clearly extend to other types of differential equations including systems. For example Theorem 5.1 holds independently of Δ being a evolution equation and so the variational operators for Δ always determine an element of the cohomology $H^{n-1,2}(\mathscr{R}^\infty)$ as in Theorem 5.1.

There remain many open theoretical questions such as how the compatibility condition for symplectic operators appears in the cohomology. Another interesting problem is to determine under what conditions the symmetry reduction of a variational operator equation is a Hamiltonian system (the converse of Lemma 8.1).

Many difficult computational questions have also not been resolved. We were unable to compute the dimension of $H^{1,2}(\mathscr{R}^\infty)$ in our examples. Preliminary computations using equation $\theta^0 \wedge \mathbf{L}^*_\Delta(\rho) = 0$ from Theorem 3.1 suggests that $\dim H^{1,2}(\mathscr{R}^\infty) = 2$ for the Krichever-Novikov equation in Example 1. However we were not able to give a full proof of this fact. We have also not explored in any detail the obvious generalization of Noether's Theorem which arises from the existence of a variational operator or equivalently by utilizing a non-trivial element of $H^{1,2}(\mathscr{R}^\infty)$. This would provide an alternate derivation for identifying symmetries and conservation laws for symplectic Hamiltonian systems, see Theorem 7.15 in [17] and [13].

A. The Vertical Differential

The vertical differential induces a mapping $d_V : H^{r,s}(\mathscr{R}^\infty) \to H^{r,s+1}(\mathscr{R}^\infty)$ defined by $d_V[\omega] = [d_V\omega]$. Let $u_t = K$ be a scalar evolution equation with equation manifold $\mathscr{R}^\infty$. We now examine when $[\omega] \in \mathrm{Image}\, d_V$.

Theorem A.1. *Let $[\zeta] \in H^{1,1}(\mathscr{R}^\infty)$. There exists $[\kappa] \in H^{1,0}(\mathscr{R}^\infty)$ such that $[\zeta] = d_V[\kappa]$ if and only if $\delta_V \circ \Pi(\zeta) = 0$ where $\Pi : \Omega^{1,1}(\mathscr{R}^\infty) \to \Omega^{1,1}_{\mathrm{t_{sb}}}(E)$ is the induced map from equation* (7.1) *and ζ is any representative of $[\zeta]$.*

This answers the question of when $[\zeta]$ is the image of a classical conservation law $[\kappa]$. To relate Theorem A.1 to the theory of characteristics for a conservation law, suppose $[\zeta] \in H^{1,1}(\mathscr{R}^\infty)$ with (unique) canonical representative given in Theorem 3.3 by

$$\zeta = dx \wedge \theta^0 \cdot Q - dt \wedge \boldsymbol{\beta}(Q)$$

where the function Q satisfies $\mathbf{L}^*_\Delta(Q) = 0$. Theorem A.1 states that the function Q is the characteristic of a classical conservation law for Δ if and only if $Q = E(L)$. The test for this condition is the Helmholtz condition $\delta^E_V(dx \wedge \theta^0_E \cdot Q) = 0$.

Proof. Suppose $[\zeta] \in H^{1,1}(\mathscr{R}^\infty)$ where $\zeta = dx \wedge (a_i\theta^i) - dt \wedge \beta$ is a representative, then

$$I_E \circ d^E_V \circ \Pi(dx \wedge (a_i\theta^i) - dt \wedge \beta) = I_E \circ d^E_V \left(dx \wedge (a_i\theta^i_E)\right) = 0.$$

This implies by equation (6.2),

$$dx \wedge (a_i\theta^i_E) = d^E_V(gdx) + d^E_H(m_i\theta^i_E). \tag{A.1}$$

Let $\mu = m_i\theta^i \in \Omega^{1,0}(\mathscr{R}^\infty)$ and

$$\hat{\zeta} = \zeta - d_H\mu. \tag{A.2}$$

so that $[\hat{\zeta}] = [\zeta]$. Now by equation (7.3), the definition of Π in equation (7.1), and equation (A.1),

$$\Pi(\hat{\zeta}) = \Pi(\zeta) - d^E_H \circ \Pi(\mu) = dx \wedge (a_i\theta^i_E) - d^E_H(m_i\theta^i_E) = d^E_V(gdx) = g_i\theta^i_E.$$

Therefore there exists $\hat{\beta} \in \Omega^{0,1}(\mathscr{R}^\infty)$ such that,

$$\hat{\zeta} = g_i\theta^i \wedge dx - dt \wedge \hat{\beta} = d_V(gdx) - dt \wedge \hat{\beta}. \tag{A.3}$$

Now $d_H d_V \hat{\zeta} = -d_V d_H \zeta = 0$ and therefore from equation (A.3),

$$d_H d_V \hat{\zeta} = dt \wedge dx \wedge X(d_V\beta) = 0. \tag{A.4}$$

However $d_V\beta \in \Omega^{0,2}(\mathscr{R}^\infty)$ and the only way the contact two form $d_V\beta$ satisfies equation (A.4) is if $d_V\beta = 0$. This implies from equation (A.3) that $d_V\hat{\eta} = 0$.

Using the vertical exactness of $\Omega^{1,1}(\mathscr{R}^\infty)$ we conclude there exists $\kappa \in \Omega^{1,0}(\mathscr{R}^\infty)$ such that $\hat{\zeta} = d_V\kappa$. Now

$$d_V d_H \kappa = -d_H d_V \kappa = -d_H \hat{\zeta} = 0.$$

Again by vertical exactness of the (augmented) variational bicomplex for $d_V : \Omega^{2,0}(\mathscr{R}^\infty) \to \Omega^{2,1}(\mathscr{R}^\infty)$ applied to $d_H\kappa$ we have,

$$d_H\kappa = a(t,x)dt \wedge dx.$$

Since $\mathbb{R}^2$ is simply connected we may write

$$d_H\kappa = a(t,x)dt \wedge dx = d(g(t,x)dx + h(t,x)dt). \tag{A.5}$$

Finally let

$$\hat{\kappa} = \kappa - g(t,x)dx - h(t,x)dt,$$

so that $d_V\hat{\kappa} = d_V\kappa = \hat{\zeta}$, and equation (A.5) gives

$$d_H\hat{\kappa} = d_H\kappa - d(g(t,x)dx + h(t,x)dt) = 0.$$

Therefore $[\zeta] = d_V[\hat{\kappa}]$ and $[\hat{\kappa}] \in H^{1,0}(\mathscr{R}^\infty)$. $\square$

Corollary A.1. *Let $[\zeta] \in H^{1,1}(\mathscr{R}^\infty)$ with canonical representative given by*

$$\zeta = dx \wedge \theta^0 \cdot Q - dt \wedge \beta$$

where $\mathbf{L}^*_\Delta(Q) = 0$ (see Theorem 3.1). Then $[\zeta] = d_V[\kappa]$ where $[\kappa] \in H^{1,0}(\mathscr{R}^\infty)$ if and only if the function Q is in the image of the Euler operator. That is if and only if there exists $A(t,x,u,u_x,u_{xx},\ldots) \in C^\infty(E)$ such that $Q = \mathbf{E}(A)$.

Corollary A.2. *If $u_t = K(t,x,u,\ldots,u_{2m})$, $m \geq 1$ is even order, then every solution Q to $\mathbf{L}^*_\Delta(Q) = 0$ is the characteristic of a conservation law.*

As is well known, the characteristic of a conservation law is a solution to $\mathbf{L}^*_\Delta(Q) = 0$ but the converse is not necessarily true. Corollary A.1 identifies the characteristics which come from conservation laws. See Example 9.2 for a solution to $\mathbf{L}^*_\Delta(Q) = 0$ which is not the characteristic of a conservation law.

We now examine the case of $H^{1,2}(\mathscr{R}^\infty)$.

Theorem A.2. *Let $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$. Then $[\omega] = d_V[\eta]$ where $[\eta] \in H^{1,1}(\mathscr{R}^\infty)$ if and only if $[\omega] \in$* Ker Λ *where $\Lambda : H^{1,2}(\mathscr{R}^\infty) \to H^{2,0}(\mathscr{R}^\infty)$ is defined in equation* (4.18).

Proof. Let $[\omega] \in H^{1,2}(\mathscr{R}^\infty)$ with representative ω satisfying $\omega = d_V\eta$ and λ be as in Lemma 4.1. That is

$$[\omega] = [d_V\eta] \qquad d_H\eta = d_V\lambda.$$

Suppose now that $\lambda = d_H\kappa$ so that $[\omega] \in \operatorname{Ker}\Lambda$. Let $\hat{\eta} = \eta + d_V\kappa$. Then

$$[d_V\hat{\eta}] = [d_V\eta], \quad d_H\hat{\eta} = d_H\eta + d_Hd_V\kappa = d_V\lambda - d_Vd_H\kappa = 0.$$

Therefore $[\omega] = d_V[\hat{\eta}]$ where $[\hat{\eta}] \in H^{1,1}(\mathscr{R}^\infty)$. This proves sufficiency of the condition.

Suppose now that $[\omega] = d_V[\eta]$ where $[\eta] \in H^{1,1}(\mathscr{R}^\infty)$. Let ω be the representative such that $\omega = d_V\eta$. By hypothesis $d_H\eta = 0$ and so for λ in Lemma 4.1 we have

$$d_H\eta = d_V\lambda = 0.$$

The same argument as in the second part of the proof of Corollary 4.4 implies that there exists $\kappa \in \Omega^{1,0}(\mathbb{R}^2)$ such that $\lambda = d_H\kappa$. Therefore $\Lambda([\omega]) = [\lambda] = [d_H\kappa] = 0$. $\square$

Theorem A.2 is demonstrated in Example 9.2. As a simple corollary to Theorem A.2 we can identify the elements of $H^{1,1}(\mathscr{R}^\infty)$ which are not the image of a conservation law as follows.

Corollary A.3. *The map $d_V : H^{1,1}(\mathscr{R}^\infty)/d_V(H^{1,0}(\mathscr{R}^\infty)) \to \operatorname{Ker}\Lambda$ is an isomorphism. Moreover, we can identify $\eta \in H^{1,1}(\mathscr{R}^\infty)/d_V(H^{1,0}(\mathscr{R}^\infty))$ with the space of functions $Q \in C^\infty(\mathscr{R}^\infty)$ such that*

$$(\mathbf{F}^*_Q - \mathbf{F}_Q)(u_t - K) = \mathbf{E}(Q(u_t - K)) \qquad \delta^E_V(dx \wedge \theta_E \cdot Q) \neq 0. \tag{A.6}$$

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