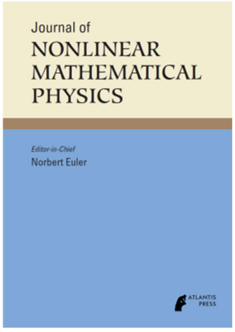

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On μ-symmetries, μ-reductions, and μ-conservation laws of Gardner equation

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In this study, we represent an application of the geometrical characterization of μ-prolongations of vector fields to the nonlinear partial differential Gardner equation with variable coefficients. First, μ-symmetries and the corresponding μ-symmetry classification are investigated and then μ-reduction forms of the equations are obtained. Furthermore, μ-invariant solutions are determined and μ-conservation laws of Gardner equation are studied.

Keywords: μ-symmetries, μ-conservation laws, μ-reductions, classification, Gardner equation.

2000 Mathematics Subject Classification: 35J05, 35KO5, 54H15

1. Introduction

Symmetry group analysis is one of the most efficient methods in the analysis of differential equations and there are many studies in the literature dealing with the analysis of differential equations based on Lie symmetries, reductions and invariant solutions [1, 4, 17–20, 22, 25–27]. In addition to application of Lie point symmetries to differential equations, λ-symmetry approach, which is first studied by Muriel and Romero [14, 15], plays important role for analyzing the analytical implicit solutions of ordinary differential equations (ODEs). In the literature, in order to obtain the extension of λ-symmetries for the case of partial differential equations (PDEs), it can be characterized by λ-prolongation in $J^{(n)}M$, where (M, π, B) is the space B of independent variables seen as a bundle over the space B. Once this characterization is obtained, it is extended to the ordinary differential equations for the case $B = \mathbb{R}$ and to the partial differential equations for the case $B = \mathbb{R}^p$, and then to the case of systems of PDEs.

Cicogna, Gaeta and Morando [4–6] improved the λ-symmetry approach for PDEs. These symmetries are called μ-symmetries and the corresponding conservation laws for μ-symmetries are called μ-conservation laws. In this approach, for p independent variables $x = (x^1, \ldots, x^p)$ and q dependent variables $u = (u^1, \ldots, u^q)$, μ function is a horizontal one-form $\mu = \lambda_i dx^i$ on the first-order jet space $(J^{(1)}M, \pi, M)$, where μ is a compatible, that is, $D_i\lambda_j - D_j\lambda_i = 0$. If μ-prolongation is applied to differential equations then the determining equations are obtained for μ-symmetries

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and by solving these determining equations, the infinitesimals functions ξ, τ, φ and λ_i are determined. The application of μ-symmetry approach not only to ODEs but also to PDEs can be seen as the new application area of the symmetry group theory to differential equations.

In the literature, the Gardner equation was initially discussed by Miura in 1968 [13]. It is known that the Gardner equation is an extension of KdV equation and it introduces the identical features of the classical KdV equation, but it expands its order of availability to a larger interval of the parameters of the interior wave motion for a certain surrounding. The event of shallow-water wave is governed by Gardner equation, which is the KdV equation with dual power nonlinearity. Experimental investigations have shown that Gardner equation models deep ocean waves rather than shallow-water waves that is used by KdV equation. It is widely used in various areas of physics, such that plasma physics, fluid dynamics, quantum field theory, and it is a useful model for the description of a great variety of wave phenomena in plasma and solid state [2,3]. The μ-symmetries and μ-conservations laws of the extended KdV equation are investigated in the study [9].

In the studies [11] and [12], Johnpillai and Khalique investigated the optimal system of one-dimensional subalgebras of the Lie symmetry algebras of the classes

$$u_t + uu_x + A_1(t)u_{xxx} + A_4(t)u = 0, \tag{1.1}$$

and

$$u_t + u^2u_x + A_1(t)u_{xxx} + A_4(t)u = 0, \tag{1.2}$$

where $A_1(t)$ and $A_4(t)$ represent the dispersion term and the linear damping term, respectively and $u(x,t)$ denotes the amplitude of the relevant wave mode, which is a function of two independent variables x and t indicating the space variable in the direction of wave propagation and time parameter, respectively. Later, the same authors constructed conservation laws for Eq. (1.1) for some special forms of the functions $A_1(t)$ and $A_4(t)$. Additionally, Vaneeva [28] examined the variable-coefficient Gardner equation

$$u_t + A_2(t)uu_x + A_3(t)u^2u_x + A_1(t)u_{xxx} = 0, \tag{1.3}$$

where $A_2(t)$ and $A_3(t)$ are smooth functions satisfying the condition $A_1(t).A_3(t) \neq 0$. Bruzon and Rosa studied [23] the Gardner equation of the form

$$u_t + A_2(t)u^nu_x + A_3(t)u^{2n}u_x + A_1(t)u_{xxx} + A_4(t)u = 0, \tag{1.4}$$

where n is an arbitrary positive integer. In addition, the Lie point symmetries and the corresponding conservation laws of the Gardner equation with variable coefficients for n =1 are investigated in the study [24].

We propose herein a generalized variable-coefficient Gardner equation of the form

$$u_t + A_2(t)u^mu_x + A_3(t)u^{2n}u_x + A_1(t)u_{xxx} + A_4(t)u = 0, \tag{1.5}$$

where m is an arbitrary positive integer. It is known that PDEs with two independent variables can be reduced to ODEs by using Lie point symmetry groups. If one obtains some solutions for the differential equations under the symmetries that leave invariant the equation itself then these solutions are called similarity solutions, which are invariant under the Lie group of transformations. In the literature, there are some studies in which exact solutions of PDEs are obtained from the similarity reductions, for example [20, 21]. For the cases of μ-symmetries, μ-reductions and μ-invariant solutions, the concept of the infinitesimal symmetry generator and the standard conservation laws should be redefined by taking account of μ-infinitesimal functions and λ_i functions [7, 8]. One of the aims of the present study is also to classify the μ-symmetries of the Gardner equation and to obtain μ-invariant solutions of the Gardner equation by using μ-symmetries and to determine μ-conservation laws.

This study is organized as follows. In section 2, we present the fundamental definitions about μ-prolongation and μ-symmetries. In section 3, we discuss some different forms of the Gardner equation and the corresponding determining equations. This section also includes different cases corresponding to different choices of coefficients. Furthermore, μ-symmetries for each different case are presented. Section 4 presents similarity reductions and some invariant solutions of a generalized variable coefficient Gardner equation. In the section 5, we examine μ-conservation laws of the Gardner equation. The last section summarizes some important results in the study.

2. Preliminaries

In this section, we introduce the concept of μ-symmetries for PDEs. Let us consider a space $M = B \times U$ with coordinates $x \in B \simeq \mathbb{R}^p$ and $u \in \mathbb{R}^q$, x is an independent and u is a dependent variable. M is the total space of a linear bundle (M, π, B) over the base space B. The bundle M can be prolonged to the k-jet bundle $(J^{(k)}M, \pi_k, B)$, with $J^{(0)}M \equiv M$, the total space of the jet bundle is also called the jet space. Now we suppose any bundle P, the set of sections of this bundle is denoted by $\Gamma(P)$ and the set of vector fields in P by $\chi(P)$. And $\mu = \lambda_i dx^i$ be horizontal one-form on first-order jet space $(J^{(1)}M, \pi, M)$ and compatible with contact structure ε on $J^{(k)}M$ for $k \geq 2$, i.e. $d\mu \in J(\varepsilon)$, where $J(\varepsilon)$ is Cartan ideal generated by contact structure ε and $\lambda_i : J^{(1)}M \longrightarrow \mathbb{R}$. μ-symmetries should satisfy the compatibility condition, that is $D_i\lambda_j - D_j\lambda_i = 0$, where D_i is total derivative of x^i. The horizontal one-form $\mu \in \Lambda^1(J^1M)$ on the vector bundle $(M, \pi, \mathbb{R})$ is the one-form $\mu = \lambda(x, u, u_x)dx$, where $\lambda(x, u, u_x) : J^{(1)}M \longrightarrow \mathbb{R}$ is smooth real function.

Definition 2.1 ([16]). Let us consider x-derivatives of function u in $J^{(k)}M$. Then one can introduce a canonical contact conjecture in the jet space $J^{(k)}M$, that is, the module generated by the set of canonical contact one-forms $\upsilon^{a_j} := du^a_j - u^a_{j,m}dx^m$.

Definition 2.2 ([7]). Let Y be a vector field on J^kM. The vector field Y preserves the contact structure if $L_Y : \varepsilon \longrightarrow \varepsilon$.

Definition 2.3 ([7]). A vector field $X \in \chi(M)$ can be written as

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \varphi^j(x,u)\frac{\partial}{\partial x^i}. \tag{2.1}$$

This can be extended to a vector field $X(k)$ in $J^{(k)}M$ by requiring that it preserves the contact structure. Thus, the vector field in $J^{(k)}M$ can be written

$$Y = X + \sum_{n=1}^{k} \Psi_n \frac{\partial}{\partial_{u_n}}. \tag{2.2}$$

Definition 2.4 ([7]). Let $X = \xi\partial_x + \eta\partial_t + \varphi\partial_u$ be a vector field on M and $Y = X + \sum_{n=1}^{k} \Psi_n(\partial/\partial_{u_n})$ be a vector field on J^kM. Let $\lambda : J^1M \longrightarrow \mathbb{R}$ be a smooth function. We say that Y is the λ-prolongation of X if its coefficients satisfy the λ-prolongation formula

$$\Psi_{n+1} = ((D_x + \lambda)\Psi_n) - u_{n+1}((D_x + \lambda)\xi), \tag{2.3}$$

for all $n = 0, \ldots, k-1$.

Definition 2.5 ([7,8]). Let Δ be a k-th order ODE for $u = u(x)$, $u \in U = \mathbb{R}$, and let $(M = U \times B, \pi, B)$ be the corresponding variables bundle. Let the vector field Y in J^kM be the λ-prolongation of the Lie-point vector field X in M. Then we say that X is a λ-symmetry of Δ if and only if Y is tangent to the solution manifold S_Δ, that is, iff there is a smooth function Φ on J^kM such that $Y(\Delta) = \Phi\Delta$.

We take $\lambda : J^1M \longrightarrow \mathbb{R}$, and then the λ-prolongation of a Lie-point vector field in M is a vector field in each J^nM. We can consider $\lambda : J^rM \longrightarrow \mathbb{R}$, obtaining obvious generalizations of the results. In this case the λ-prolongations of X would be generalized vector fields in each J^nM with $n > 0$ even if X is a Lie-point vector field. The same procedure can be applied to the μ-prolongation.

Proposition 2.1. *The vector field $Y \in \chi(J^kM)$, projecting to a vector field $X \in \chi(M)$ if and only if it preserves the contact structure in J^kM.*

Theorem 2.1 ([7,8]). *Let Δ be a scalar PDEs of order k for $u = u(x^1, \ldots, x^p)$. Let $X = \xi\partial_x + \eta\partial_t + \varphi\partial_u$ be a vector field on M, with characteristic $Q = \varphi - \xi u_x - \tau u_t$ and let Y be the μ-prolongation of order k of X. If X is a μ-symmetry for Δ, then $Y : \varphi_X \longrightarrow T\varphi_X$, where $\varphi_X \subset J^kM$ is the solution manifold for the system Δ_X made of Δ and $E_J := D_JQ = 0$ for all J with $|J| = 0, 1, \ldots, k-1$.*

3. μ-symmetries of Gardner equation

In this section, we study μ-symmetry properties of the Gardner equations with variable coefficients. To deal with μ-symmetry analysis, the Gardner equation can be considered in different forms including a general form. For this purpose, we first, deal with the generalized variable-coefficient Gardner equation of the form

$$u_t + A_2(t)u^m u_x + A_3(t)u^{2n}u_x + A_1(t)u_{xxx} + A_4(t)u = 0, \tag{3.1}$$

where n and m are non-zero integers. To determine μ-symmetries, first μ-prolongation of the vector field Y is applied to Gardner equation and then the determining equations are obtained. In order to determine μ-symmetries of PDEs, one can use the similar method as in the case for λ-symmetries of ODEs. Let us say X be a vector field on M and Y be μ-prolongation of $\mu = \lambda_i dx^i$. When the μ-prolongation of the vector field Y is applied to the differential Eq. (3.1) and then the determining equations can be obtained for μ-symmetries and the infinitesimal functions ξ, τ, φ, and λ_i. In addition, $X = \xi\partial_x + \eta\partial_t + \varphi\partial_u$ is a vector field and $\mu = \lambda_1 dx + \lambda_2 dt$ is a horizontal one-form. For λ_1 and λ_2 functions, the necessary condition to be satisfied is called the *compatibility condition*, which is represented by the formula $D_t\lambda_1 = D_x\lambda_2$. Thus, μ-prolongation of infinitesimal generator X (2.1) is written

$$Y = X + \psi^x\partial_{u_x} + \psi^t\partial_{u_t} + \psi^{xx}\partial_{u_{xx}} + \cdots + \psi^{ttt}\partial_{u_{ttt}}, \tag{3.2}$$

where

$$\begin{aligned}
\psi^x &= (D_x+\lambda_1)\varphi - u_x(D_x+\lambda_1)\xi - u_t(D_x+\lambda_1)\eta,\\
\psi^t &= (D_t+\lambda_2)\varphi - u_x(D_t+\lambda_2)\xi - u_t(D_t+\lambda_2)\eta,\\
\psi^{xx} &= (D_x+\lambda_1)\psi^x - u_{xx}(D_x+\lambda_1)\xi - u_{xt}(D_x+\lambda_1)\eta,\\
\psi^{xt} &= (D_t+\lambda_2)\psi^x - u_{xx}(D_x+\lambda_2)\xi - u_{xt}(D_x+\lambda_2)\eta,\\
\psi^{tt} &= (D_t+\lambda_2)\psi^t - u_{tx}(D_t+\lambda_2)\xi - u_{tt}(D_t+\lambda_2)\eta,\\
\psi^{xxx} &= (D_x+\lambda_1)\psi^{xx} - u_{xxx}(D_x+\lambda_1)\xi - u_{xxt}(D_x+\lambda_1)\eta,\\
\psi^{xxt} &= (D_t+\lambda_2)\psi^{xx} - u_{xxx}(D_t+\lambda_2)\xi - u_{xxt}(D_t+\lambda_2)\eta,\\
\psi^{xtt} &= (D_t+\lambda_2)\psi^{xt} - u_{xtx}(D_t+\lambda_2)\xi - u_{xtt}(D_t+\lambda_2)\eta,\\
\psi^{ttt} &= (D_t+\lambda_2)\psi^{tt} - u_{ttx}(D_t+\lambda_2)\xi - u_{ttt}(D_t+\lambda_2)\eta.
\end{aligned} \tag{3.3}$$

Now if we suppose $\mu = \lambda_i dx^i$ is horizontal 1-form and $V = \exp(\int \mu)X$ is exponential vector field, where X is a vector field given by formula (2.1) then the characteristic function is defined as $Q = \varphi - u_i\xi^i$. Thus, one can write

$$V = \exp\Big(\int \lambda_1 dx + \lambda_2 dt\Big)X, \qquad Q = \varphi - \xi u_x - \eta u_t. \tag{3.4}$$

If the μ-prolongation Y is acted on the Eq. (3.1) and by substituting $(-A_2(t)u^m u_x - A_3(t)u^{2n}u_x - A_1(t)u_{xxx} - A_4(t)u)$ for u_t, we can use the property of functional independence of the variables $u(x,t)$ and its derivatives and rearrange all polynomials, and separate independent functions, and equate the coefficients of various powers of the variables to zero. As a result, the following over-determined system of partial differential equations called *determining equations* are obtained in terms of $\eta(x,t,u)$, $\xi(x,t,u)$, $\varphi(x,t,u)$, $\lambda_1(x,t,u)$, and $\lambda_2(x,t,u)$ functions

$$\begin{gathered}
3A_1^2(t)\eta_u = 0, \qquad A_1(t)\xi_{uuu} = 0, \qquad \varphi_{uu} = 0,\\
A_1(t)(-3\xi_u + A_1(t)(3\eta_x(\lambda_1)_u + \eta(3\lambda_1(\lambda_1)_u + 2(\lambda_1)_{ux}))) = 0,\\
4A_1(t)A_3(t)(\eta\lambda_1 + \eta_x) = 0, \qquad 6n(\lambda_1)_u + u(\lambda_1)_{uu} = 0,\\
2u^{2n}A_3(t)(9n\eta_x + \eta(9n\lambda_1 + 2u(\lambda_1)_u) + u^m A_2(t)(9m\eta_x + \eta(9m\lambda_1 + 4u(\lambda_1)_u)\\
+u(4\xi(\lambda_1)_u + 9\lambda_1\xi_u - 3\varphi_{uu} + 9\xi_{xu}))) = 0,\\
6n(2n-1)\eta_x + \eta(6n(2n-1)\lambda_1 + u(6n(\lambda_1)_u) + u(\lambda_1)_{uu}) = 0,\\
3(m-1)m\eta_x + \eta(3(m-1)m\lambda_1 + u(3m\lambda_1)_u + u(\lambda_1)_{uu}) = 0,\\
2uA_1(t)A_3(t)\eta^3(2nA_1(t)A_3(t)A_2'(t)\eta^2 + A_2(t)(-mA_1(t)A_3'(t)\eta^2 = 0,\\
\eta A_3'(t) + 2u^m A_2(t)A_3(t)(\eta\lambda_1 + \eta_x + u^{2n}A_3^2(t)(\eta\lambda_1 + \eta_x) + A_3(t)(\eta\lambda_2 - \lambda_1\xi + \eta_t - \xi_x\\
+A_1(t)(3\lambda_1^2\eta_x + 3\lambda_1\eta_{xx} + \eta_{xxx} + 3\eta_x(\lambda_1)_u u_x + 3\eta_x(\lambda_1)_x + \eta(\lambda_1^3 + 4(\lambda_1)_u u_{xx} + (\lambda_1)_{uu}u_x^2\\
+3\lambda_1(u_x(\lambda_1)_u + (\lambda_1)_x) + 2u_x(\lambda_1)_{xu} + (\lambda_1)_{xx})))) = 0,
\end{gathered} \tag{3.5}$$

$$\eta A_1'(t)+A_1(t)(\eta\lambda_2-3\lambda_1\xi+\eta_t+4u^m A_2(t)((\eta\lambda_1+\eta_x)+4u^{2n}A_3(t)(\eta\lambda_1+\eta_x)-3\xi_u u_x-3\xi_x)$$
$$+A_1^2(t)(3\lambda_1^2\eta_x+3\lambda_1\eta_{xx}+\eta_{xxx}+3\eta_x(\lambda_1)_u u_x+3\eta_x(\lambda_1)_x+\eta(\lambda_1^3+(\lambda_1)_u u_{xx}+(\lambda_1)_{uu}u_x^2$$
$$+3\lambda_1(u_x(\lambda_1)_u+(\lambda_1)_x)+2u_x(\lambda_1)_{xu}+(\lambda_1)_{xx}))=0,$$
$$4(m-5n)nA_1^2(t)A_3^2(t)A_2'(t)-4nA_1(t)A_2(t)(-3mA_1(t)A_2'(t)A_3'(t)+(m-2n)A_3(t)(2A_1'(t)A_2'(t)$$
$$+A_1(t)A_2''(t)))+A_2^2(t)(m(4n-5m)A_1^2(t)A_3'^2(t)+(m-2n)^2A_3^2(t)(A_1'^2(t)-2A_1(t)A_1''(t))$$
$$+2m(m-2n)A_1(t)A_3(t)(2A_1'(t)A_3'(t)+A_1(t)A_3''(t))=0,$$
$$2(m-5n)A_1(t)A_3^2(t)A_2'(t)+A_2(t)A_3(t)((2n-m)A_3(t)A_1'(t)A_2'(t)+A_1(t)(3(m+2n)A_2'(t)A_3'(t)$$
$$+2(m-2n)A_3(t)(3nA_4(t)A_2'(t)-A_2''(t)+A_2^2(t)((4n-5m)A_1(t)A_3'^2(t)+(m-2n)^2A_3^2(t)(A_4(t)A_1'(t)$$
$$+2A_1(t)A_4'(t))-(m-2n)A_3(t)(-A_1'(t)A_3'(t)+A_1(t)(3mA_4(t)A_3'(t)-2A_3''(t)))))=0.$$

The standard but somewhat troublesome calculations for above determining equations yield unknown functions as infinitesimal functions, namely, η, ξ, φ of the infinitesimal generator (2.1) of the form

$$\xi(x,t,u)=F(x,t),$$
$$\eta(x,t,u)=-\left(\sqrt{A_1(t)}\left(c_5\exp\left(\int\frac{c_4+mA_4(t)(2mc_2-4nc_2-3mt)}{2mc_2-4nc_2-3mt}dt\right)\right)^{\frac{3n}{m-2n}}(2mc_2-4nc_2-3mt)F(x,t)\right)/$$
$$\left(-2^{\frac{m}{m-2n}}c_1(2c_2(m-2n)-3mt)\left(2^{-\frac{4n}{3m}}c_3\left(c_5\exp\left(\int\frac{c_4+mA_4(t)(2mc_2-4nc_2-3mt)}{2mc_2-4nc_2-3mt}dt\right)\right)^{\frac{2n}{m}}\right.\right.$$
$$\left.\left.\left(\frac{2mc_2-4nc_2-3mt}{m-2n}\right)^{\frac{4n-2m}{3m}}\right)^{\frac{3m}{2(m-2n)}}+\sqrt{A_1(t)}mx\left(c_5\exp\left(\int\frac{c_4+mA_4(t)(2mc_2-4nc_2-3mt)}{2mc_2-4nc_2-3mt}dt\right)\right)^{\frac{3n}{m-2n}}\right),\tag{3.6}$$

$$\varphi(x,t,u)=\left(\sqrt{A_1(t)}\left(c_5\exp\left(\int\frac{c_4+mA_4(t)(2mc_2-4nc_2-3mt)}{2mc_2-4nc_2-3mt}dt\right)\right)^{\frac{3n}{m-2n}}u(c_4-2m+mA_4(t)\right.$$
$$\left.(2mc_2-4nc_2-3mt))F(x,t)\right)/\left(m(-2^{\frac{m}{m-2n}}c_1(2c_2(m-2n)-3mt)\right.$$
$$\left(2^{-\frac{4n}{3m}}c_3\left(c_5\exp\left(\int\frac{c_4+mA_4(t)(2mc_2-4nc_2-3mt)}{2mc_2-4nc_2-3mt}dt\right)\right)^{\frac{2n}{m}}\right.$$
$$\left.\left(\frac{2mc_2-4nc_2-3mt}{m-2n}\right)^{\frac{4n-2m}{3m}}\right)^{\frac{3m}{2(m-2n)}}+\sqrt{A_1(t)}mx\left(c_5\exp\left(\int\frac{c_4+mA_4(t)(2mc_2-4nc_2-3mt)}{2mc_2-4nc_2-3mt}dt\right)\right)^{\frac{3n}{m-2n}}\Bigg)\Bigg),$$

and the corresponding λ_1 and λ_2 functions, which satisfy the compatibility condition $D_t\lambda_1 = D_x\lambda_2$, are

$$\begin{aligned}\lambda_1(x,t,u) = &-\Big(2\sqrt{A_1(t)}c_1(m-2n)A_2(t)^{1-\frac{3n}{m-2n}}A_3(t)^{1+\frac{3m}{2(m-2n)}}F_x(x,t)\\ &-4nA_1(t)A_3(t)A_2'(t)(F(x,t)-xF_x(x,t))+2mA_1(t)A_2(t)A_3'(t)(F(x,t)-xF_x(x,t)\Big)\\ &/\Big(2F(x,t)\Big(\sqrt{A_1(t)}c_1(m-2n)A_2(t)^{1-\frac{3n}{m-2n}}A_3(t)^{1+\frac{3m}{2(m-2n)}}\\ &+2nxA_1(t)A_3(t)A_2'(t)-mxA_1(t)A_2(t)A_3'(t)\Big)\Big)\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\lambda_2(x,t,u) = &\Big(8A_1(t)^2(m-5n)nxA_2(t)^{\frac{3n}{(m-2n)}}A_3(t)^2F(x,t)A_2'(t)^2\\ &+4c_1A_1(t)^{\frac{3}{2}}(m-2n)^2A_2(t)^2A_3(t)^{2+\frac{3m}{2(m-2n)}}F_t(x,t)+4nxA_1(t)A_2(t)^{\frac{m+n}{m-2n}}A_3(t)\\ &\Big(6mA_1(t)A_2'(t)A_3'(t)F(x,t)-2A_1(t)(m-2n)A_3(t)\Big(F(x,t)A_2''(t)-A_2'(t)F_t(x,t)))\\ &+2mxA_1(t)A_2(t)^{2+\frac{3n}{m-2n}}\Big(-A_1(t)(5m-4n)A_3'(t)^2F(x,t)\\ &-2A_1(t)(m-2n)A_3(t)(F(x,t)A_3''(t)-A_3'(t)F_t(x,t))\Big)\\ &/\Big(4(m-2n)A_1(t)A_2(t)A_3(t)F(x,t)\Big(-\sqrt{A_1(t)}c_1(m-2n)A_2(t)A_3(t)^{1+\frac{3m}{2(m-2n)}}\\ &-2nxA_1(t)A_3(t)A_2'(t)A_2(t)^{\frac{3n}{m-2n}}+mxA_1(t)A_3'(t)A_2(t)^{\frac{m+n}{m-2n}}\Big)\Big),\end{aligned} \tag{3.8}$$

where $m \neq 2n$, $F(x,t)$ is an arbitrary function and c_1, c_2, c_3, c_4 and c_5 are arbitrary constants. In addition, we have following differential relations among the coefficients $A_1(t)$, $A_2(t)$, $A_3(t)$ and $A_4(t)$ for the μ-invariance condition of the Eq. (3.1), which are obtained from solutions of determining equations,

$$\begin{aligned}&4(m-5n)nA_1^2(t)A_3^2(t)A_2'(t)-4nA_1(t)A_2(t)(-3mA_1(t)A_2'(t)A_3'(t)+(m-2n)A_3(t)(2A_1'(t)A_2'(t)\\ &+A_1(t)A_2''(t)))+A_2^2(t)(m(4n-5m)A_1^2(t)A_3'^2(t)+(m-2n)^2A_3^2(t)(A_1'^2(t)-2A_1(t)A_1''(t))\\ &+2m(m-2n)A_1(t)A_3(t)(2A_1'(t)A_3'(t)+A_1(t)A_3''(t))=0,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&2(m-5n)A_1(t)A_3^2(t)A_2'(t)+A_2(t)A_3(t)((2n-m)A_3(t)A_1'(t)A_2'(t)+A_1(t)(3(m+2n)A_2'(t)A_3'(t)\\ &+2(m-2n)A_3(t)(3nA_4(t)A_2'(t)-A_2''(t)+A_2^2(t)((4n-5m)A_1(t)A_3'^2(t)+(m-2n)^2A_3^2(t)(A_4(t)A_1'(t)\\ &+2A_1(t)A_4'(t))-(m-2n)A_3(t)(-A_1'(t)A_3'(t)+A_1(t)(3mA_4(t)A_3'(t)-2A_3''(t)))))=0.\end{aligned} \tag{3.10}$$

Therefore, it is possible to carry out a one-type of μ-symmetry classification from Eq. (3.9) and Eq. (3.10) based on the relations between coefficients $A_1(t)$, $A_2(t)$, $A_3(t)$ and $A_4(t)$ of the Gardner equation. In addition, the other type of classification is possible with respect to the relations between parameters m and n in (3.9) and (3.10). From these two equations, it is clear that five different cases can be considered to examine μ-symmetries for different choice of integer parameters m and n, which are $m \neq n$, $m = n$, $m = -2n$, $m = 5n$, and $m = (4/5)n$ since Eq. (3.9) and Eq. (3.10) have different forms and different solutions for $A_1(t)$, $A_2(t)$, $A_3(t)$, and $A_4(t)$ for each case. In this study, we only examine the μ-symmetry classification with respect to relations between the parameters m and n. We consider each case in order below.

CASE I: $m \neq n$. From the mathematical point of view, it is not possible to determine the general solutions of $A_1(t)$, $A_2(t)$, $A_3(t)$ and $A_4(t)$ from Eqs. (3.9) and (3.10). For this reason, we consider some special chooses of the functions $A_1(t)$, $A_2(t)$, $A_3(t)$ and $A_4(t)$. In order to obtain a specific solution for (3.9) and (3.10), for example, lets choose $A_1(t) = A_1$, where A_1 is a constant. Under this consideration, Eq. (3.9) becomes

$$\begin{aligned} &A_1^2(m(4n-5m)A_2(t)^2A_3'(t)^2 + 4nA_3(t)^2\left((m-5n)A_2'(t)^2 - (m-2n)A_2(t)A_2''(t)\right) \\ &\quad +2mA_2(t)A_3(t)(6nA_2'(t)A_3'(t) + (m-2n)A_2(t)A_3''(t))) = 0, \end{aligned} \tag{3.11}$$

and Eq. (3.10) is written in the form

$$\begin{aligned} &A_1(4(m-5n)A_3(t)^2A_2'(t)^2 + 2A_2(t)A_3(t)(3(m+2n)A_2'(t)A_3'(t) \\ &+2(m-2n)A_3(t)\left(3nA_4(t)A_2'(t) - A_2''(t)\right)) + 2A_2(t)^2(-5m+4n)A_3'(t)^2 \\ &+2(m-2n)^2A_3(t)^2A_4'(t) - (m-2n)A_3(t)\left(3mA_4(t)A_3'(t) - 2A_3''(t)\right) = 0. \end{aligned} \tag{3.12}$$

It is clear that from Eq. (3.11), first, the solution of function $A_3(t)$ can be determined and then the function $A_2(t)$ is found from the equation (3.12) as below

$$\begin{aligned} A_3(t) &= c_3A_2(t)^{\frac{2n}{m}}\left(c_2 - \frac{3mt}{2m-4n}\right)^{\frac{4n-2m}{3m}}, \\ A_2(t) &= c_5\exp\left(\int \frac{c_4 + mA_4(t)(2mc_2 - 4nc_2 - 3mt)}{2mc_2 - 4nc_2 - 3mt}\,dt\right), \end{aligned} \tag{3.13}$$

where c_2, c_3, c_4 and c_5 are constants. From equations (3.6) and (3.13), the corresponding μ-infinitesimals are written in the following form

$$\xi(x,t,u) = F(x,t),$$

$$\eta(x,t,u) = -\Big(\sqrt{A_1}\left(c_5\exp\left(\int \frac{c_4 + mA_4(t)(2mc_2 - 4nc_2 - 3mt)}{2mc_2 - 4nc_2 - 3mt}dt\right)\right)^{\frac{3n}{m-2n}}(2mc_2 - 4nc_2 - 3mt)F(x,t)\Big)\Big/$$

$$\Big(-2^{\frac{m}{m-2n}}c_1(2c_2(m-2n) - 3mt)\Big(2^{-\frac{4n}{3m}}c_3\left(c_5\exp\left(\int \frac{c_4 + mA_4(t)(2mc_2 - 4nc_2 - 3mt)}{2mc_2 - 4nc_2 - 3mt}dt\right)\right)^{\frac{2n}{m}}$$

$$\left(\frac{2mc_2-4nc_2-3mt}{m-2n}\right)^{\frac{4n-2m}{3m}}\Bigg)^{\frac{3m}{2(m-2n)}}+\sqrt{A_1}mx\left(c_5\exp\left(\int\frac{c_4+mA_4(t)(2mc_2-4nc_2-3mt)}{2mc_2-4nc_2-3mt}dt\right)\right)^{\frac{3n}{m-2n}}\Bigg)\Bigg),$$

$$\varphi(x,t,u)=\Bigg(\sqrt{A_1}\left(c_5\exp\left(\int\frac{c_4+mA_4(t)(2mc_2-4nc_2-3mt)}{2mc_2-4nc_2-3mt}dt\right)\right)^{\frac{3n}{m-2n}}u(c_4-2m+mA_4(t)$$

$$(2mc_2-4nc_2-3mt)F(x,t)\Bigg)\Bigg/\Bigg(m(-2^{\frac{m}{m-2n}}c_1(2c_2(m-2n)-3mt) \tag{3.14}$$

$$\Bigg(2^{-\frac{4n}{3m}}c_3\left(c_5\exp\left(\int\frac{c_4+mA_4(t)(2mc_2-4nc_2-3mt)}{2mc_2-4nc_2-3mt}dt\right)\right)^{\frac{2n}{m}}$$

$$\left(\frac{2mc_2-4nc_2-3mt}{m-2n}\right)^{\frac{4n-2m}{3m}}\Bigg)^{\frac{3m}{2(m-2n)}}$$

$$+\sqrt{A_1}mx\left(c_5\exp\left(\int\frac{c_4+mA_4(t)(2mc_2-4nc_2-3mt)}{2mc_2-4nc_2-3mt}dt\right)\right)^{\frac{3n}{m-2n}}\Bigg)\Bigg),$$

where c_1 is an arbitrary constant.

CASE II: $m=n$. To provide specific solutions for the last two equations, we now choose $A_1(t)=A_1t^k$, where A_1 and k are constants. Under this consideration, Eq. (3.9) gets

$$-n^2t^{-2+2k}A_1^2(16t^2A_3(t)^2A_2'(t)^2-4tA_2(t)A_3(t)\left(3tA_2'(t)A_3'(t)+A_3(t)\left(2kA_2'(t)+tA_2''(t)\right)\right)$$

$$A_2(t)^2\left((-2+k)kA_3(t)^2+t^2A_3'(t)^2+2tA_3(t)\left(2kA_3'(t)+tA_3''(t)\right)\right)=0. \tag{3.15}$$

Similarly, the equation (3.10) becomes

$$2nt^{k-1}A_1(A_2(t)A_3(t)\left(9tA_2'(t)A_3'(t)+A_3(t)\left((k-6ntA_4(t))A_2'(t)+2tA_2''(t)\right)\right)$$

$$-2nt^{-1+k}A_18tA_3(t)^2A_2'(t)^2-A_2(t)^2\left(-tA_3'(t)^2+nA_3(t)^2\left(kA_4(t)+2tA_4'(t)\right)\right)$$

$$-A_2(t)^2A_3(t)\left((k-3ntA_4(t))A_3'(t)+2tA_3''(t)\right)=0. \tag{3.16}$$

The solution of (3) is written

$$A_3(t)=c_3A_2(t)^{\frac{2n}{m}}\left(c_2-\frac{3mt}{2m-4n}\right)^{\frac{4n-2m}{3m}}, \tag{3.17}$$

where c_2 and c_3 are arbitrary constants. Combining this solution (3.17) with the solution (3), the corresponding μ-infinitesimals are determined as below

$$\eta(x,t,u)=\frac{3A_1(c_2+t^{k+1})F(x,t)}{3c_1c_3^3t^{k/2}\sqrt{A_1t^k}+A_1(k+1)t^kx},\quad \xi(x,t,u)=F(x,t),$$

$$\varphi(x,t,u)=\frac{A_1F(x,t)u\Big((3c_2k+(k-2)t^{k+1})A_3(t)-3t(c_2+t^{k+1})A_3'(t)\Big)}{2nt\sqrt{A_1t^k}A_3(t)(3c_1c_3^3t^{k/2}+(k+1)x\sqrt{A_1t^k})}, \tag{3.18}$$

where c_1 is a constant.

CASE III: $m=-2n$. To consider Eqs. (3.9) and (3.10) together, the function $A_1(t)$ can be considered in the same form with the previous case as $A_1(t)=A_1t^k$. Hence, Eq. (3.9) gets

$$-4n^2t^{-2+2k}A_1^2(7t^2A_3(t)^2A_2'(t)^2-2tA_2(t)A_3(t)\left(-3tA_2'(t)A_3'(t)+2A_3(t)\left(2kA_2'(t)+tA_2''(t)\right)\right)$$
$$A_2(t)^2\left(4(-2+k)kA_3(t)^2+7t^2A_3'(t)^2-4tA_3(t)\left(2kA_3'(t)+tA_3''(t)\right)\right)=0, \qquad (3.19)$$

and Eq. (3.10) reads

$$4nt^{-1+k}A_1(-7tA_3(t)^2A_2'(t)^2+2A_2(t)A_3(t)^2\left((k-6ntA_4(t))A_2'(t)+2tA_2''(t)\right) \qquad (3.20)$$
$$A_2(t)^2\left(7tA_3'(t)^2+8nA_3(t)^2\left(kA_4(t)+2tA_4'(t)\right)-2A_3(t)\left((k+6ntA_4(t))A_3'(t)+2tA_3''(t)\right)\right)=0.$$

From the solution of Eq. (3.19), the function $A_2(t)$

$$A_2(t)=\frac{c_3t^{2k}}{A_3(t)(t^{k+1}+c_2)^{\frac{4}{3}}}, \qquad (3.21)$$

is found, where c_2 and c_3 are arbitrary constants. If one substitutes the function $A_2(t)$ into Eq. (3.20) and the function $A_4(t)$ can be obtained from the solution of Eq. (3.20) in the form

$$A_4(t)=\frac{t^k}{c_2+t^{1+k}}\Big(c_4+\int 1/(2nA_3^2(t))t^{-2-k}(c_2k(1+k)A_3^2(t)-t^2(c_2+t^{1+k})(A_3')^2(t)$$
$$+tA_3(t)(\Big(t^{1+k}-c_2k)A_3'(t)+t(c_2+t^{1+k})A_3''(t)\Big)dt\Big), \qquad (3.22)$$

where c_4 is an arbitrary constant. For this case, from solutions of equations (3.19), (3.20) and (3.6), μ-infinitesimal functions

$$\xi(x,t,u)=F(x,t),$$
$$\eta(x,t,u)=\frac{3A_3(t)\sqrt{A_1t^k}(c_2+t^{1+k})(c_1+t^{1+k})^{4/3}F(x,t)}{(1+k)t^kx(c_1+t^{1+k})^{4/3}A_3(t)\sqrt{A_1t^k}+3c_1(c_2+t^{1+k})(A_3(t)c_2t^{2k})^{3/4}}, \qquad (3.23)$$
$$\varphi(x,t,u)=-\frac{\sqrt{A_1t^k}(c_1+t^{1+k})F(x,t)(-(3c_2k+(k-2)t^{1+k})A_3(t)+3t(c_2+t^{1+k})A_3'(t)}{2ntA_3(t)\Big((1+k)t^kx\sqrt{A_1t^k}+3c_1(c_2+t^{1+k})(c_2t^{2k})^{3/4}\Big)},$$

are determined, where c_1 is a constant.

CASE IV: $m=5n$. For the function $A_1(t)=A_1t^k$ and the Eqs. (3.9) and (3.10) are converted to the following differential equations

$$3n^2t^{-2+2k}A_1^2A_2(t)-4tA_3(t)\left(-5tA_2'(t)A_3'(t)+A_3(t)\left(2kA_2'(t)+tA_2''(t)\right)\right)$$
$$+A_2(t)\left(-3(-2+k)kA_3(t)^2-35t^2A_3'(t)^2+10tA_3(t)\left(2kA_3'(t)+tA_3''(t)\right)\right)=0, \qquad (3.24)$$

and

$$6nt^{k-1}A_1A_2(t)(A_3(t)\left(7tA_2'(t)A_3'(t)-A_3(t)\left((k-6ntA_4(t))A_2'(t)+2tA_2''(t)\right)\right) \qquad (3.25)$$
$$A_2(t)\left(-7tA_3'(t)^2+3nA_3(t)^2\left(kA_4(t)+2tA_4'(t)\right)+A_3(t)\left((k-15ntA_4(t))A_3'(t)+2tA_3''(t)\right)\right)=0.$$

Thus, the solution of Eq. (3.24) yields

$$A_2(t) = \frac{t^{-3k/2}A_3(t)^{5/2}((1+k)c_2+t^{1+k}c_3)}{1+k}, \tag{3.26}$$

where c_2 and c_3 are arbitrary constants. From solution of Eq. (3.25), the function $A_4(t)$

$$A_4(t) = \frac{2nc_4t^k + t^k\int\left(t^{-2-k}(c_2k(1+k)^2A_3(t)^2 - t^2(c_2+c_2k+c_3t^{1+k})A_3'(t)^2/A_3(t)^2\right)dt}{2n(c_2+c_2k+c_3t^{1+k})}$$
$$+\frac{t^k\int\left(tA_3(t)((-c_2k(1+k)+c_3t^{1+k})A_3'(t)+t(c_2+c_2k+c_3t^{1+k})A_3''(t)))/A_3(t)^2\right)dt}{2n(c_2+c_2k+c_3t^{1+k})}, \tag{3.27}$$

is determined. Substituting Eqs. (3.26) and (3.27) into (3.6) leads to

$$\eta(x,t,u) = \frac{3A_1(c_2+c_2k+c_3t^{1+k})F(x,t)}{\sqrt{A_1t^k}(1+k)(3c_1t^{k/2}+c_3x\sqrt{A_1t^k})}, \quad \xi(x,t,u) = F(x,t),$$
$$\varphi(x,t,u) = \frac{\sqrt{A_1t^k}(c_2+c_2k+c_3t^{1+k})u(x,t)F(x,t)\left((3c_1k(1+k)+c_2(k-2)t^{k+1})A_3(t)\right)}{2ntA_3(t)(3c_1(1+k)t^{3k/2}(c_1+c_1k+c_2t^{k+1})+\sqrt{A_1t^k}(c_2+c_2k+c_3t^{1+k})c_2(1+k)xt^k)}$$
$$-\frac{\sqrt{A_1t^k}(c_2+c_2k+c_3t^{1+k})u(x,t)F(x,t)\left(3t(c_1+c_1k+c_2t^{1+k})A_3'(t)\right)}{2ntA_3(t)(3c_1(1+k)t^{3k/2}(c_1+c_1k+c_2t^{k+1})+\sqrt{A_1t^k}(c_2+c_2k+c_3t^{1+k})c_2(1+k)xt^k)}, \tag{3.28}$$

where c_1 and c_4 are constants.

CASE V: $m = (4/5)n$. For the same function $A_1(t)$ considered above, Eq. (3.9) becomes

$$-\frac{12}{25}n^2t^{2k-2}A_1^2A_3(t)\Big(35t^2A_3(t)A_2'(t)^2 - 10tA_2(t)\left(2tA_2'(t)A_3'(t)+A_3(t)\left(2kA_2'(t)+tA_2''(t)\right)\right)$$
$$+A_2(t)^2\left(3(-2+k)kA_3(t)+4t\left(2kA_3'(t)+tA_3''(t)\right)\right)\Big) = 0, \tag{3.29}$$

and the Eq. (3.10) is written

$$\frac{12}{25}nt^{-1+k}A_1A_3(t)(-35tA_3(t)A_2'(t)^2+5A_2(t)(7tA_2'(t)A_3'(t)+A_3(t)((k-6ntA_4(t))A_2'(t)+2tA_2''(t)))$$
$$+A_2(t)^2\left((-5k+12ntA_4(t))A_3'(t)+6nA_3(t)\left(kA_4(t)+2tA_4'(t)\right)-10tA_3''(t)\right)) = 0. \tag{3.30}$$

Additionally, the solution of Eq. (3.29) leads to

$$A_2(t) = \frac{c_3t^{3k/5}A_3(t)^{2/5}}{(t^{1+k}+c_2)^{2/5}}, \tag{3.31}$$

where c_2 and c_3 are constants. By substituting $A_2(t)$ into (3.30), the function $A_4(t)$

$$A_4(t) = \frac{c_4t^k + t^k\int\left(t^{-2-k}(c_2k(1+k)^2A_3(t)^2 - t^2(c_2+c_2k+c_3t^{1+k})A_3'(t)^2/2nA_3(t)^2\right)dt}{c_2+t^{1+k}}$$
$$+\frac{t^k\int\left(tA_3(t)((-c_2k(1+k)+c_3t^{1+k})A_3'(t)+t(c_2+c_2k+c_3t^{1+k})A_3''(t)))/2nA_3(t)^2\right)dt}{c_2+t^{1+k}}, \tag{3.32}$$

is determined. Finally, the corresponding μ-infinitesimal functions are

$$\eta(x,t,u) = \frac{3\sqrt{t}(c_2+t^{1+k})A_3(t)F(x,t)}{A_3(t)t^{3k/2}((1+k)x+3c_1c_3^{5/2}}, \quad \xi(x,t,u) = F(x,t), \tag{3.33}$$

$$\varphi(x,t,u) = \frac{t^{k/2}u(x,t)A_3(t)F(x,t)\Big((3c_2k+(k-2)t^{k+1})A_3(t)-3t(c_2+t^{1+k})A_3'(t)\Big)}{2ntx(1+k)t^{3k/2}A_3(t)^2+3c_1c_3^{5/2}t^{3k/2}A_3(t)},$$

where c_1 and c_4 are constants.

In addition to these cases analyzed above, it is possible to investigate the other different forms of Gardner equations, which are represented in the introduction section. Firstly, we examine Eq. (1.1) ([11]-[12]) of the form

$$u_t + uu_x + A_1(t)u_{xxx} + A_4(t)u = 0.$$

For Eq. (1.1), by using the same form for $A_1(t)$ function, the infinitesimal functions

$$\eta(x,t,u) = \frac{(c_1-3tA_1)F(x,t)}{c_2-x}, \quad \xi(x,t,u) = F(x,t), \quad \varphi(x,t,u) = \frac{2uF(x,t)}{(c_2-x)}, \tag{3.34}$$

are found, where c_1 and c_2 are constants. Similarly, for Eq. (1.2) of the form

$$u_t + u^2u_x + A_1(t)u_{xxx} + A_4(t)u = 0,$$

the μ-infinitesimals read

$$\eta(x,t,u) = \frac{(3t-3t\log(t))F(x,t)}{c_1-x}, \quad \xi(x,t,u) = F(x,t), \quad \varphi(x,t,u) = \frac{(3\log(t)-1)uF(x,t)}{2c_1-x},$$

where c_1 is a constant. Finally, we deal with Eq. (1.3)

$$u_t + A_2(t)uu_x + A_3(t)u^2u_x + A_1(t)u_{xxx} = 0,$$

and by considering the function of form $A_1(t) = 1/(a+bt)$, where a and b are arbitrary constants, the functions $A_2(t)$ and $A_3(t)$

$$A_2(t) = \frac{c_5(3\log(a+bt)-bc_1)^{-\frac{2b+b^2c_1+c_3}{6b}}}{a+bt}, \quad A_3(t) = \frac{c_4(3\log(a+bt)-bc_1)^{-\frac{b^2c_1+c_3}{3b}}}{a+bt}, \tag{3.35}$$

are determined, where c_3, c_4, and c_5 are constants. Thus, the μ-infinitesimal functions for the equation become

$$\eta(x,t,u) = \frac{(bc_1-3\log(a+bt))(a+bt)F(x,t)}{b(c_2-x)}, \quad \xi(x,t,u) = F(x,t), \tag{3.36}$$

$$\varphi(x,t,u) = \frac{(c_3+c_1b^2-2b)uF(x,t)}{2b(c_2-x)},$$

where c_2 is a constant.

4. μ-reduction forms and μ-invariant solutions of Gardner equation

In this section, we consider μ-reduced forms and corresponding μ-invariant solutions of Gardner equation by using the μ-symmetries, which are determined in the previous section. In the classical Lie symmetry group theory, the reduced forms of the original differential equations can be obtained by using their symmetry groups. In this manner, one can transform a PDE to a ODE after reduction if the original equation has two independent variables. In the case of μ-symmetries of PDEs the procedure is similar to the case for the Lie point symmetries of differential equations. To obtain the μ-reduced forms, we should write the characteristic equation based on the symmetry groups of the equation for different cases. As the first application to determine μ-reduced forms of the equation, we consider the case $m = n$. If the functions $A_1(t)$ and $A_3(t)$ are chosen as constants and then the functions $A_2(t)$ and $A_4(t)$ are determined from the related relations given in the previous section. As a result, for these infinitesimals for the case $m = n$, the characteristic equation

$$\frac{dx}{F(x,t)} = \frac{(c_1c_2^3 - \sqrt{A_1}x)dt}{\sqrt{A_1}(c_1 - 3t)F(x,t)} = \frac{(c_1c_2^3 - \sqrt{A_1}x)du}{\sqrt{A_1}uF(x,t)}, \tag{4.1}$$

is written and then the similarity variable (invariant), which is the new independent variable for the reduced forms of the original equation is obtained by the integration of the first two terms of the characteristic equation

$$\zeta = \frac{c_1c_2^3 - \sqrt{A_1}x}{(c_1 - 3t)^{1/3}}, \tag{4.2}$$

and the other invariant is obtained by the integration of equations of the other terms in the characteristic equation. Thus, the corresponding similarity variable becomes

$$u(x,t) = \frac{\tilde{u}(\zeta)}{(c_1 - 3t)^{1/3}}. \tag{4.3}$$

Hereby, if we substitute similarity forms into the original equation, the μ-reduced equation, for example $n = 1$,

$$(1 + c_3)\tilde{u}(\zeta) + (\zeta + \tilde{u}(\zeta)(c_2 + \tilde{u}(\zeta)))\tilde{u}'(\zeta) - A_1^{5/2}\tilde{u}'''(\zeta) = 0, \tag{4.4}$$

is found. Here, it is possible to determine other forms of similarity variables and invariants for other cases of the different choices of m and n.

As the second case, we take $m = 5n$, $A_1(t)$ and $A_3(t)$ are considered as constants and thus, $A_2(t)$ and $A_4(t)$ are determined and the infinitesimals are found. For this reason, the characteristic equation becomes

$$\frac{dx}{F(x,t)} = \frac{(3c_1 + c_3x)dt}{3(c_2 + c_3t)F(x,t)} = -\frac{(3c_1n + c_3nx)du}{c_3uF(x,t)}, \tag{4.5}$$

and the first invariant is

$$\zeta = \frac{(3c_1 + c_3x)^3}{c_2 + c_3t}, \tag{4.6}$$

for $n = 1$, the other invariant is

$$u(x,t) = \frac{\tilde{u}(\zeta)}{3c_1 + c_3x}, \tag{4.7}$$

and the corresponding μ-reduced equation reads

$$c_4\zeta^2 - 6c_3^3\zeta - c_3\zeta\tilde{u}^2(\zeta) - c_3\tilde{u}^5(\zeta) = 0. \tag{4.8}$$

As the third case, we consider $m = 4/5n$, $A_3(t)$ is a constant and $k = -1$, that is $A_1(t) = A_1/t$ and the functions $A_2(t)$ and $A_4(t)$ are determined according to these choices. Hence, one can write characteristic equation for the infinitesimal functions as below

$$\frac{dx}{F(x,t)} = \frac{c_1c_3^{5/2}dt}{(1+c_2)F(x,t)} = \frac{2nc_1c_3^{5/2}}{u(1+c_2)F(x,t)}, \tag{4.9}$$

which yields the corresponding invariants

$$\zeta = x - \frac{c_1c_3^{5/2}\ln(t+c_2t)}{(1+c_2)} \quad \text{and} \quad u(x,t) = \frac{t}{\tilde{u}(\zeta)}. \tag{4.10}$$

For $n = 1/2$, the μ-reduced equation form is

$$\begin{aligned}&\tilde{u}^3(\zeta) + c_4\tilde{u}^3(\zeta) - \tilde{u}(\zeta)\tilde{u}'(\zeta) - \tilde{u}^{8/5}(\zeta)\tilde{u}'(\zeta) + \tilde{u}^2(\zeta)\tilde{u}'(\zeta)\\ &+A_1\left(6\tilde{u}(\zeta)\tilde{u}'(\zeta)\tilde{u}''(\zeta) - 6\tilde{u}'^3(\zeta) - \tilde{u}^2\tilde{u}'''(\zeta)\right) = 0.\end{aligned} \tag{4.11}$$

In addition to the previous cases, it is possible to consider μ-reduced equations for other forms of the Gardner equation. For example, let's consider Eq. (1.1) and then for the special choices of the function $A_1(t)$, the function $A_4(t)$ is determined and the characteristic equation becomes

$$\frac{dx}{F(x,t)} = \frac{A_1(c_2-x)dt}{(c_1-3A_1t)F(x,t)} = \frac{(c_2-x)du}{2uF(x,t)}, \tag{4.12}$$

giving the related invariants such that

$$\zeta = \frac{(c_1-3A_1t)^{1/3}}{c_2-x} \quad \text{and} \quad u(x,t) = \frac{\tilde{u}(\zeta)}{(c_2-x)^2}. \tag{4.13}$$

The associated μ-reduced equation for Eq. (1.1) is

$$24A_1\zeta + c_3 + 2\tilde{u}(\zeta) = 0. \tag{4.14}$$

Solving μ-reduced equation and substituting into (4.13) lead to the invariant solution of the form

$$u(x,t) = \frac{24A_1^2t + c_3(x-c_2) - 8c_1A_1}{2(c_2-x)^3}. \tag{4.15}$$

We now examine μ-reduced equation for the Gardner equation (1.2). For the special choice of $A_1(t)$, the function $A_4(t)$ is determined and the characteristic equation is written

$$\frac{dx}{F(x,t)} = \frac{(c_2-x)dt}{(c_1-3t)F(x,t)} = \frac{du}{uF(x,t)}. \tag{4.16}$$

The corresponding invariants

$$\zeta = \frac{(c_1-3t)^{1/3}}{c_2-x} \quad \text{and} \quad u(x,t) = \frac{\tilde{u}(\zeta)}{(c_2-x)}, \tag{4.17}$$

are defined and thus the μ-reduced equation for differential equation (1.2) becomes

$$6\zeta^3 + c_3 + \zeta^3 \tilde{u}^2(\zeta) = 0. \tag{4.18}$$

The solution of μ-reduced equation yields the μ-invariant solution of the form

$$u(x,t) = \frac{\sqrt{c_3(x-c_2)^3 - 6(c_1 - 3t)}}{(c_2 - x)(c_1 - 3t)^{1/2}}. \tag{4.19}$$

Finally, for the μ-reduced equation of the Gardner equation given by Eq. (1.3), we write the characteristic equation

$$\frac{dx}{F(x,t)} = \frac{b(c_2 - x)dt}{(a+bt)(bc_1 - 3\ln(a+bt))F(x,t)} = \frac{2b(c_2 - x)du}{(2b - c_1 b^2 - c_3)uF(x,t)}, \tag{4.20}$$

thus, the similarity variables are

$$\zeta = \frac{c_2 - x}{(2tc_3 - 3\ln(2a+t))^{1/3}} \quad \text{and} \quad u(x,t) = \tilde{u}(2c_2 - x), \tag{4.21}$$

and the μ-reduced equation for Eq. (1.3) is

$$\tilde{u}'(\zeta)(\tilde{u}'(\zeta) + \zeta^4(2 + c_3 - 3\ln(2\exp^{2/3 - 1/3\zeta^3}))) = 0, \tag{4.22}$$

and solving μ-reduced equation, the μ-invariant solution

$$u(x,t) = c_4(2c_2 - x), \tag{4.23}$$

is determined.

Remark 4.1. It is important to mention that μ-invariant solutions obtained in this section are both solutions of the μ-reduced equations and solutions of the original differential equations.

5. Lagrangians and μ-conservation laws

In this section, we deal with the Lagrangian functions of the Gardner equation and the corresponding μ-conservation laws. It is known that μ-symmetry is defined in the following form

$$\mu = \lambda_i dx^i, \tag{5.1}$$

where the functions λ_i are square matrices and it is called M-functions of the dependent variable u and independent variable x. S_Δ is the solution manifold in J^rM for given equation Δ of differential equation of order r. Thus, X is μ-symmetry of the equation Δ if $Y = X_M^r : S_\Delta \to TS_\Delta$.

For a function M and one can have the relation $M : J^rM \to \mathbb{R}$. Under this relation, X is μ-symmetry of the equation Δ for the function H that H is μ-invariant under X if $Y[H] = 0$ and it is mentioned as the invariance of the level manifolds of H under Y. We now consider the vector fields X having $\xi^i = 0$ that is $Q^j = \varphi^j$ thus, it is represented by a standard formulation of the classical Noether's theorem. It is clear that these assumptions simplify the procedure. For the applicability of Noether's theorem, we need some form of variational structure in the systems. The construction of the Euler-Lagrange equations characterizes the minimizers of a variational problem.

Definition 5.1 ([10, 16]). Let $\Omega \subset X$ be an open and *variational problem* that consists of finding maximum and minimum values of

$$\mathfrak{L}[\mathfrak{u}] = \int_{\Omega} L(x, u^{(n)}) dx. \tag{5.2}$$

The integrand $L(x, u^{(n)})$ called the *Lagrangian* of the variational problem $\mathfrak{L}$, is depend on x, u and derivatives of u.

Definition 5.2 ([16]). We suppose that $f(x)$ is real-valued function and $\nabla f(x)$ is the gradient of $f(x)$. The change of gradient can be seen by

$$\langle \nabla f(x), y \rangle = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(x+\varepsilon y), \tag{5.3}$$

where $\langle x, y \rangle$ is the usual inner product. For functional $\mathfrak{L}[\mathfrak{u}]$, the gradient is called *variational derivative* of $\mathfrak{L}$.

Proposition 5.1. *Let $\mathfrak{L}[\mathfrak{u}]$ be a variational problem. The variational derivative of $\mathfrak{L}$ is*

$$\delta \mathfrak{L}[\mathfrak{u}] = (\delta_1 \mathfrak{L}[\mathfrak{u}], \ldots, \delta_q \mathfrak{L}[\mathfrak{u}]). \tag{5.4}$$

If u is an extremum of $\mathfrak{L}[\mathfrak{u}]$, then

$$\delta \mathfrak{L}[\mathfrak{u}] = 0. \tag{5.5}$$

Theorem 5.1. *If $\mathfrak{L}[\mathfrak{u}] = \int_{\Omega} L(x, u^{(n)}) dx$ is variational problem, then it must be a solution of the Euler-Lagrange equations*

$$E(L) = 0. \tag{5.6}$$

Definition 5.3 ([16]). Let

$$D = \sum_j P_j[u] D_j, \qquad P_j \in \mathscr{A} \tag{5.7}$$

is a differential operator, its adjoint is the differential operator D^* is

$$D^* = \sum_j P_j[u](-D)_j. \tag{5.8}$$

The differential operator $D : \mathscr{A}^k \to \mathscr{A}^\ell$ has adjoint $D^* : \mathscr{A}^\ell \to \mathscr{A}^k$, the adjoint of the transposed entries of D. An operator D is *self-adjoint* if $D^* = D$ and it is *skew-adjoint* if $D^* = -D$.

Note that, if $P \in \mathscr{A}^\ell$, its *Frechet derivative* has adjoint $D_p^* : \mathscr{A}^\ell \to \mathscr{A}^q$.

Theorem 5.2. *Let Δ is the Euler-Lagrange expression for some variational problem $\mathfrak{L} = \int L dx$, i.e. $\Delta = E(L)$, if and only if the Frechet derivative D_Δ is self-adjoint: $D_\Delta^* = D_\Delta$. The function Lagrangian for Δ can be found using the formula $L[u] = \int_0^1 u.\Delta[\lambda u] d\lambda$.*

Definition 5.4. A standard conservation law is

$$D_i P^i = 0, \tag{5.9}$$

where P^i is a p-dimensional vector and we know that $\mu = \lambda_i dx^i$ is a horizontal one-form and it satisfies the compatibility condition i.e. $D_i\lambda_j = D_j\lambda_i$. Now, using these property we can define *μ-conservation law* for this μ-symmetry [4] as

$$(D_i + \lambda_i)P^i = 0, \tag{5.10}$$

where P^i is a M-vector and it is called *μ-conserved vector.*

As a result, one can obtain μ-conservation law using the following Theorem:

Theorem 5.3 ([8]). *Let X be a vector field and $\mathscr{L} = L(x, u^{(n)})$ be the n-th order Lagrangian. X is a μ-symmetry for $\mathscr{L}$ if and only if the equation has the M-vector P^i and it satisfies the μ-conservation law $(D_i + \lambda_i)P^i = 0$.*

Using this Theorem, the M-vector P^i can be found using Lagrangian. Firstly, we suppose $\mathscr{L}(x,t,u,u_x,u_t)$ is first order Lagrangian, the vector field $X = \varphi(\partial/\partial u)$ is a μ-symmetry for $\mathscr{L}$. Then, we can define the M-vector $P^i = \varphi(\partial\mathscr{L}/\partial u_i)$. For second-order Lagrangian $\mathscr{L}$ and the vector field $X = \varphi(\partial/\partial u)$ is a μ-symmetry for $\mathscr{L}$ and then, the M-vector

$$P^i = \varphi\frac{\partial\mathscr{L}}{u_i} + ((D_j + \lambda_j)\varphi)\frac{\partial\mathscr{L}}{u_{ij}} - \varphi D_j\frac{\partial\mathscr{L}}{u_{ij}}, \tag{5.11}$$

is a μ-conserved vector.

5.1. *Lagrangian of Gardner equation*

Now, we consider the Gardner equation given by (1.3)

$$\Delta : u_t + A_2(t)uu_x + A_3(t)u^2u_x + A_1(t)u_{xxx} = 0,$$

where $A_1(t)$, $A_2(t)$ and $A_3(t)$ are smooth functions satisfying the condition $A_1.A_3 \neq 0$. It is a fact that the Gardner equation does not have standard Lagrangian function itself and the equation has Lagrangian if and only if its Frechet derivative is self-adjoint. However, the Frechet derivative of Gardner equation is not self-adjoint since it is of odd order. To prove this fact, let us take the following Frechet derivative of Δ

$$D_\Delta = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} = \Big((u_t+\varepsilon D_t)+A_1(t)(u_{xxx}+\varepsilon D_x^3)+A_2(t)(u+\varepsilon)(u_x+\varepsilon D_x)+A_3(t)(u^2+\varepsilon)(u_x+\varepsilon D_x)\Big).$$

The simplification gives

$$D_\Delta = D_t + A_1(t)D_x^3 + A_2(t)uD_x + A_2(t)u_x + A_3(t)u^2D_x + A_3(t)u_x, \tag{5.12}$$

and then, the adjoint operator becomes

$$D_\Delta^* = (-D_t) + A_1(t)(-D_x)^3 + A_2(t)u(-D_x) + A_2(t)u_x + A_3(t)u^2(-D_x) + A_3(t)u_x. \tag{5.13}$$

From the equations (5.12) and (5.13), one can say that it is not a variational problem since it is not self-adjoint, that is $D_\Delta^* \neq D_\Delta$. Therefore, the potential form of Gardner equation having a standart Lagrangian should be considered. For this purpose, one can consider the differential substitution $u = v_x$ to Eq. (1.3) to write

$$\Delta_v : v_{xt} + A_1(t)v_{xxxx} + A_2(t)v_x v_{xx} + A_3(t)v_x^2 v_{xx} = 0. \tag{5.14}$$

This equation is called "the Gardner equation in potential form" having the Frechet derivative of Δ_v of the form

$$D_{\Delta_v} = D_{xt} + A_1(t)D_x^4 + (A_2(t)v_x + A_3(t)v_x^2)D_x^2 + (A_2(t)v_{xx} + 2A_3(t)v_x v_{xx})D_x. \tag{5.15}$$

Thus, one can see easily that the Gardner equation in potential form is self-adjoint, that is $D_\Delta^* = D_\Delta$. By the Theorem 5.2, it has a Lagrangian of the form

$$L[v] = \int_0^1 v.\Delta_v[\lambda v]d\lambda = \frac{1}{2}vv_{xt} - \frac{1}{2}A_1(t)vv_{xxxx} + \frac{1}{3}A_2(t)vv_x v_{xx} + \frac{1}{4}A_3(t)vv_x^2 v_{xx}. \tag{5.16}$$

By reducing the order of Lagrangian, one can write

$$L[v] = -\frac{1}{2}v_x v_t + \frac{1}{2}A_1(t)v_{xx}^2 - \frac{1}{6}A_2(t)v_x^3 - \frac{1}{12}A_3(t)v_x^4 + \mathrm{Div}\, P. \tag{5.17}$$

Hence, the Lagrangian of equation Δ_v becomes

$$\mathscr{L} = -\frac{1}{2}v_x v_t + \frac{1}{2}A_1(t)v_{xx}^2 - \frac{1}{6}A_2(t)v_x^3 - \frac{1}{12}A_3(t)v_x^4. \tag{5.18}$$

5.2. *μ-conservation laws of Gardner equation*

In order to analyze μ-conservation laws, let us consider the vector field in the form

$$Y = \varphi\partial_v + \psi^x\partial_{v_x} + \psi^t\partial_{v_t} + \psi^{xx}\partial_{v_{xx}} + \psi^{xt}\partial_{v_{xt}} + \psi^{tt}\partial_{v_{tt}}, \tag{5.19}$$

where

$$\psi^x = (D_x+\lambda_1)\varphi,\ \ \psi^t = (D_t+\lambda_2)\varphi,\ \ \psi^{xx} = (D_x+\lambda_1)\psi^x,\ \ \psi^{xt} = (D_t+\lambda_2)\psi^x,\ \ \psi^{tt} = (D_t+\lambda_2)\psi^t.$$

In order to determine the μ-conservation forms, we deal with the application of μ-prolongation of the vector field Y (5.19) to the Lagrangian of the Gardner equation in potential form (5.18) by considering following two different cases.

Case I: In the first case, the function λ_i is considered not only functions of x, t, v and but also functions of the derivatives of v, i.e the function λ_i can be written as $\lambda_i = \lambda_i(x,t,v,v_x,v_t)$, which is the most general case. In this case, the coefficients of the μ-prolongation of the vector field Y should be reformulated by considering the dependence of λ_i functions with respect to the derivatives of the dependent function. Hence, if the μ-prolongation of the vector field Y acts on Eq. (5.18) by substituting $v_t = \big(A_1(t)v_{xx}^2 - A_2(t)v_x^3/3 - A_3(t)v_x^4/6\big)/v_x$ and after straight forward but rigorous calculations one can get the following determining equation

$$-\frac{1}{2}v_x\Big(\lambda_2(x,t,v,v_x,v_t)\varphi+(A_1(t)v_{xx}^2-A_2(t)v_x^3/3-A_3(t)v_x^4/6)\varphi_u/v_x+\varphi_t\Big)$$
$$-\Big(\frac{1}{2}A_2(t)v_x^2+\frac{1}{3}A_3(t)v_x^3-(A_1(t)v_{xx}^2-A_2(t)v_x^3/3-A_3(t)v_x^4/6)/2v_x\Big)(\lambda_1(x,t,v,v_x,v_t)\varphi+v_x\varphi_u+\varphi_x)$$
$$+A_1(t)v_{xx}(v_{xx}\varphi_u+\lambda_1(x,t,v,v_x,v_t)(v_x\varphi_u+\varphi_x)+(\lambda_1(x,t,v,v_x,v_t)(\lambda_1(x,t,v,v_x,v_t)\varphi+v_x\varphi_u+\varphi_x)$$
$$+v_x\varphi_{xu}+v_x(v_x\varphi_{uu}+\varphi_{xu})+\varphi_{xx}+\varphi(v_{xt}(\lambda_1(x,t,v,v_x,v_t))_{v_t}+v_{xx}(\lambda_1(x,t,v,v_x,v_t))_{v_x}$$
$$+v_x(\lambda_1(x,t,v,v_x,v_t))_{v_x}+(\lambda_1(x,t,v,v_x,v_t))_x))=0. \tag{5.20}$$

In this equation, one can consider λ_i functions as linear in terms of v_x and v_t since the functions λ_i depend on derivatives of v function, which satisfies the compatibility condition $D_t\lambda_1=D_x\lambda_2$. Then, the solution of the determining equation is

$$\varphi=C(x)\exp\Big(-v(x,t)-\int(h(t)+G'(t)+K_t(x,t))dt\Big), \tag{5.21}$$

$$\lambda_1=v_x-\frac{C'(x)}{C(x)}+K_x(x,t),\quad \lambda_2=v_t+h(t)+G'(t)+K_t(x,t), \tag{5.22}$$

where $K(x,t)$, $C(x)$, $h(t)$ and $G(t)$ are arbitrary functions.

Therefore, the Theorem 5.3 yields μ-conservation vectors such that

$$P^1=-\frac{C(x)}{6}\exp\Big(-v(x,t)-\int(h(t)+G'(t)+K_t(x,t))dt\Big)$$
$$\Big(3v_t+6A_1(t)v_{xxx}+3A_2(t)v_x^2+2A_3(t)v_x^3\Big)K(x,t),$$
$$P^2=-\frac{C(x)}{2}\exp\Big(-v(x,t)-\int(h(t)+G'(t)+K_t(x,t))dt\Big)v_x, \tag{5.23}$$

where $v(x,t)=\int u(x,t)dx$. By considering Noether Theorem for μ-symmetries, one can write μ-conservation law for the Gardner equation in the following potential form

$$(D_i+\lambda^i)P^i=(D_x+\lambda^1)P^1+(D_t+\lambda^2)P^2$$
$$=C(x)\exp\Big(-v-\int(h(t)+G'(t)+K_t(x,t))dt\Big)\Big(v_{xt}+A_1(t)v_{xxxx}+A_2(t)v_xv_{xx}+A_3(t)v_x^2v_{xx}\Big)$$
$$=QE(\mathscr{L}). \tag{5.24}$$

In order to write μ-conservation law for the Gardner equation in its original form (1.3) (not potential), first, the equation is written in following form

$$D_x(v_t+A_1(t)v_{xxx}+\frac{1}{2}A_2(t)v_x^2+\frac{1}{3}A_3(t)v_x^3)=0, \tag{5.25}$$

and then substituting u in v_x gives

$$(v_t+A_1(t)u_{xx}+\frac{1}{2}A_2(t)u^2+\frac{1}{3}A_3(t)u^3)=R(t), \tag{5.26}$$

where $R(t)$ is an arbitrary function. Hence, one can compute μ-conserved vectors in the forms

$$P^1 = -\frac{C(x)}{12}\exp\left(-v(x,t) - \int (h(t) + G'(t) + K_t(x,t))dt\right)\left(6R(t) + 6A_1(t)u_{xx} + 3A_3(t)u^2 + 2A_4(t)u^3\right),$$
$$P^2 = -\frac{C(x)}{2}\exp\left(-v(x,t) - \int (h(t) + G'(t) + K_t(x,t))dt\right)u. \tag{5.27}$$

As a consequence, the corresponding μ-conservation law of the Gardner equation in the original form is written as

$$\begin{aligned}(D_x + \lambda^1)P^1 + (D_t + \lambda^2)P^2 \\ = -\frac{C(x)}{2}\exp\left(-v(x,t) - \int (h(t) + G'(t) + K_t(x,t))dt\right) \\ \left(u_t + A_3(t)uu_x + A_4(t)u^2u_x + A_1(t)u_{xxx}\right) = Q\Delta.\end{aligned} \tag{5.28}$$

Case II: As a second case, we consider the case $\lambda_i = \lambda_i(x,t,v)$ and then the solution of determining equation (5.20) yields

$$\varphi = S(x,t), \tag{5.29}$$

where $S(x,t)$ is an arbitrary function and it can be shown that it satisfies the condition $\mathscr{L}[v] = 0$. Firstly, the corresponding λ_1 and λ_2 functions

$$\lambda_1 = -\frac{S_x(x,t)}{S(x,t)}, \quad \lambda_2 = -\frac{S_t(x,t)}{S(x,t)}, \tag{5.30}$$

are determined. It is clear that λ_1 and λ_2 satisfy the compatibility condition. Then, μ-conservation vectors P^i

$$P^1 = -\frac{1}{6}(3v_t + 6A_1(t)v_{xxx} + A_2(t)3v_x^2 + 2A_3(t)v_x^3)S(x,t), \quad P^2 = -\frac{v_x}{2}S(x,t), \tag{5.31}$$

are obtained and the corresponding μ-conservation law for the potential form

$$\begin{aligned}(D_i + \lambda^i) = (D_x + \lambda^1)P^1 + (D_t + \lambda^2)P^2 \\ = S(x,t)(v_{xt} + A_1(t)v_{xxxx} + A_2(t)v_xv_{xx} + A_3(t)v_x^2v_{xx})),\end{aligned} \tag{5.32}$$

is found. Similar to the first case, μ-conservation vectors for the original Gardner equation is determined in the following form

$$P^1 = -\frac{1}{12}\left(6A_1(t)u_{xx} + 3A_2(t)u^2 + 2A_3(t)u^3 + 6H(t)\right)S(x,t), \quad P^2 = -\frac{u}{2}S(x,t), \tag{5.33}$$

where $H(t)$ is an arbitrary function. Finally, μ-conservation law of the original Gardner equation is of the form

$$(D_x + \lambda^1)P^1 + (D_t + \lambda^2)P^2 = S(x,t)(u_t + A_2(t)uu_x + A_3(t)u^2u_x + A_1(t)u_{xxx}) = Q\Delta. \tag{5.34}$$

Remark 5.1. It is also important to express that the μ-conservation laws of the Gardner equations with variable coefficients obtained in this section are new. In addition, the corresponding conservation laws of the Gardner equations with variable coefficients are obtained in two different forms by using μ-symmetry approach for the first time in the literature.

6. Concluding Remarks

In this study the concept of μ-symmetry is represented to analyze μ-symmetry classification, μ-reductions, μ-invariant solutions, and μ-conservations laws of nonlinear partial differential Gardner equation with variable coefficients. The important object in this method is a horizontal one-form $\mu = \lambda_i(x,u,u_x)dx_i$, which must satisfy compatibility conditions. Firstly, some important properties of μ-symmetries are presented. Then we show that the method of μ-reduction can also be interpreted in terms of the formulation of the Noether theorem when μ-symmetries are considered to find the invariant solutions of partial differential equations, which are called μ-invariant solutions. In this view, we investigate μ-symmetries for different cases of equation parameters, namely m and n, μ-reduction forms and some invariant solutions of the Gardner equations.

Furthermore, we analyze μ-conservation laws of Gardner equation. It is known that Gardner equation does not have standard Lagrangian function itself. We know that the equation has Lagrangian if and only if its Frechet derivative is self-adjoint. In addition, the Frechet derivative of the equation is not self-adjoint since it is of odd order. However, the equation in potential form is self-adjoint that is $D^*_\Delta = D_\Delta$ that this means it has a Lagrangian.

The μ-conservations laws of the Gardner equation in potential form are considered in two different cases. Firstly the new conservation law is determined by considering the dependence of λ_i functions with respect to the derivatives of the dependent function that is $\lambda_i = \lambda_i(x,t,v,v_x,v_t)$, which is the most general case. For this case, μ-conservations laws are determined not in the potential form but also in the form of original variables. As a second case, the function λ_i is considered only functions of x, t, and v variables. In this case, we also obtain new conservation laws of the equation both in the potential form and in the form of the original variables. We show that the both conservation laws corresponding to these two cases have different forms.

References

[1] G. Bluman, Temuerchaolu, and S.C. Anco, New conservation laws obtained directly from symmetry action on a known conservation law, *J. Math. Anal. Appl.*, **322**, (2006), 233–250.

[2] H. Bulut and S. Demiray, New exact solutions for generalized Gardner equation, *Kuwait J. Sci.*, **44** (1), (2017), 1–8.

[3] H. Bulut, Classification of exact solutions for generalized form of $K(m,n)$ equation. *Abstr. Appl. Anal.*, (2013), Article ID:742643, 11 pages.

[4] G. Cicogna and G. Gaeta, Noether theorem for μ-symmetries, *J. Phys. A: Math. Theor.,* **40**, (2007), 11899–11921.

[5] G. Cicogna, G. Gaeta and P. Morando, On the relation between standard and μ-symmetries for PDEs, *J. Phys. A*, **37**, (2004), 9467–9486.

[6] G. Gaeta and P. Morando, On the geometry of lambda-symmetries and PDEs reduction, *J. Phys. A*, **37**, (2004), 6955–6975.

[7] G. Gaeta and P. Morando, PDESs reduction and λ-symmetries, *Note Mat.*, **23** (2), (2004/2005), 33–73.

[8] G. Gaeta, Lambda and mu-symmetries, *in Symmetry and Perturbation Theory (SPT2004)* (G. Gaeta *et al.* eds.), World Scientific, Singapore (2005).

[9] Kh. Goodarzi and M. Nadjafikhah, μ-symmetry and μ-conservation law for the extended mKDV equation, *J. Nonlinear Math. Phys.*, **21:3**, (2014), 371–381.

[10] N.H. Ibragimov and T. Kolsrud, Lagrangian approach to evolution equations: symmetries and conservation laws, *Nonlinear Dyn.*, **36**, (2004), 29–40.

[11] A.G. Johnpillai, C.M. Khalique, Gruop analysis of KdV equation with time dependent coefficients, *Appl. Math. Comput.*, **216**, (2010), 3761–3771.

[12] A.G. Johnpillai, C.M. Khalique, Conservation laws of KdV equation with time dependent coefficients, *Commun Nonlinear Sci. Numer. Simulat.*, **16**, (2011), 3081–3089.

[13] R.M. Miura, C.S. Gardner and M.D. Kruskal, Korteweg-devries equation and generalizations II. Existence of conservation laws and constants of motion. *J. Math. Phys.*, **9**, (1968), 1204–1209.

[14] C. Muriel and J. L. Romero, New methods of reduction for ordinary differential equation, *IMA J. Appl. Math.*, **66**, (2001), 111–125.

[15] C. Muriel and J. L. Romero, $\mathscr{C}^{\infty}$-symmetries and reduction of equation without Lie point symmetries, *J. Lie Theory*, **13**, (2003), 67–188.

[16] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, (1986).

[17] Ö. Orhan and T. Özer, New conservation forms and Lie algebras of Ermakov-Pinney equation, *AIMS Discrete Continuous Dyn. Syst. Ser. S*, **11** (4), (2018), 735–746.

[18] Ö. Orhan and T. Özer, Analysis of Lienard II-type oscillator equation by symmetry-transformation methods, *Adv. Differ. Equ.*, **259**, (2016).

[19] Ö. Orhan and T. Özer, Linearization properties, first integrals, nonlocal transformations for the heat transfer equation, *Int. J. Mod. Phys. B*, **30**, (2016), 1640024.

[20] T. Özer, On symmetry group properties and general similarity forms of the Benney equations in the Lagrangian variables, *J. Comput. Appl. Math.*, **169** (2), (2004), 297–313.

[21] T. Özer, Symmetry group analysis and similarity solutions of variant nonlinear long-wave equations, *Chaos Solitons Fractals*, **33**, (2008), 722–730.

[22] E. Pucci and G. Saccomandi, On the reduction methods for ordinary differential equations, *J. Phys. A*, **35**, (2002), 6145–6155.

[23] R. De La Rosa and M.S. Bruzon, Differential Invariants of a generelized variable coefficient Gardner equation, *AIMS Discrete Continuous Dyn. Sys.t Ser. S*, **11** (4), (2018), 747–757.

[24] M. Rosa, M.S. Bruzon, and M.L. Gandarias, Lie symmetries and conservation laws of a Fisher equation with nonlinear convection term, *Discrete Continuous Dyn. Syst. Ser. S*, **8** (6), (2015), 1331–1339.

[25] M. Torrisi and R. Tracina, Equivalence transformations for systems of first order quasilinear partial differential equations, *Modern Group Analysis VI: Developments in Theory, Computation and Application*, (1996).

[26] M. Torrisi and R. Tracina, Equivalence transformations and symmetries for a heat conduction model, *Int. J. Nonlinear Mech.*, **33** (3), (1998), 473–487.

[27] R. Tracina, Invariants of a family of nonlinear wave equations, *Commun. Nonlinear. Sci. Numer. Simulat.*, **9**, (2004), 127–133.

[28] O. Vaneeva, O. Kuriksha, C. Sophocleous, Enhanced group classification of Gardner equations with time-dependent coefficients, *Commun. Nonlinear. Sci. Numer. Simulat.*, **22**, (2015), 1243–1251.