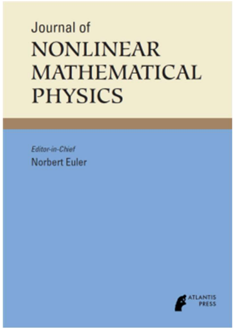

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: Kostyantyn Zheltukhin, Natalya Zheltukhina (2020) On the discretization of Darboux Integrable Systems, Journal of Nonlinear Mathematical Physics 27:4, 616–632, DOI: https://doi.org/10.1080/14029251.2020.1819597

To link to this article: https://doi.org/10.1080/14029251.2020.1819597

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 27, No. 4 (2020) 616–632

On the discretization of Darboux Integrable Systems

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Received 2 July 2019

Accepted 6 January 2020

We study the discretization of Darboux integrable systems. The discretization is done using x-, y-integrals of the considered continuous systems. New examples of semi-discrete Darboux integrable systems are obtained.

Keywords: semi-discrete system; Darboux integrability; x-integral; discretization.

2000 Mathematics Subject Classification: 35Q51,35L70,37K60, 37K10.

1. Introduction

The classification problem of Darboux integrable equations has attracted a considerable interest in the recent time, see the survey paper [1] and references there in. There are many classification results in the continuous case. The case of semi-discrete and discrete equations is not that well studied. Discrete models play a big role in many areas of physics and discretization of existing integrable continuous models is an important problem. There is a currently discussed conjecture saying that for each continuous Darboux integrable system it is possible to find a semi-discrete Darboux integrable system that admits the same set of x-integrals. To better understand properties of semi-discrete and discrete Darboux integrable systems it is important to have enough examples of such systems. We can test the conjecture and obtain new semi-discrete Darboux integrable systems, corresponding to given continuous ones, following an approach proposed by Habibullin *et al.*, see [2]. In this case we take a Darboux integrable continuous equation and look for a semi-discrete equation admitting the same integrals. The method was successfully applied to many Darboux integrable continuous equations, see [2]–[4]. In almost all considered cases such semi-discrete equations exist and they are Darboux integrable.

In the present paper we apply this method of discretization to Darboux integrable systems to obtain new Darboux integrable semi-discrete systems. Let us give necessary definitions and formulate the main results of our work.

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Co-published by Atlantis Press and Taylor & Francis

Consider a hyperbolic continuous system

$$p_{xy} = \mathscr{A}(p, p_x, p_y) \qquad \left(p^i_{xy} = \mathscr{A}^i(p^1 \dots p^N, p^1_x \dots p^N_x, p^1_y \dots p^N_y) \quad i = 1, \dots, N\right), \tag{1.1}$$

where $p^i(x,y)$, $i = 1, \dots, N$, are functions of continuous variables $x, y \in \mathbb{R}$. We say that a function $F(x, y, p, p_y, p_{yy}, \dots)$ is an x-integral of the system (1.1) if

$$D_x F(x, y, p, p_y, p_{yy}, \dots) = 0 \qquad \text{on all the solutions of the system (1.1).}$$

The operator D_x represents the total derivative with respect to x. The y-integral of the system (1.1) is defined in a similar way. The system (1.1) is called Darboux integrable if it admits N functionally independent non-trivial x-integrals and N functionally independent non-trivial y-integrals.

Consider a hyperbolic semi-discrete system

$$q_{x1} = \mathscr{B}(q, q_x, q_1), \qquad \left(q^i_{x1} = \mathscr{B}^i(q^1 \cdots q^N, q^1_x \cdots q^N_x, q^1_1 \cdots q^N_1), \quad i = 1, \dots, N\right), \tag{1.2}$$

where $q^i(x,n)$, $i = 1, \dots, N$, are functions of a continuous variable $x \in \mathbb{R}$ and a discrete variable $n \in \mathbb{N}$. Note that we use notation $q_1(x,n) = Dq(x,n) = q(x, n+1)$ and $q_k(x,n) = D^k q(x,n) = q(x, n+k)$, where D is the shift operator. To state the Darboux integrability of a semi-discrete system we need to define x- and n-integrals for such systems, see [5]. An x-integral is defined in the same way as in continuous case and a function $I(x, n, q, q_x, q_{xx}, \dots)$ is an n-integral of system (1.2) if

$$DI(x, n, q, q_x, q_{xx}, \dots) = I(x, n, q, q_x, q_{xx}, \dots) \qquad \text{on all the solutions of the system (1.2).}$$

The system (1.2) is called Darboux integrable if it admits N functionally independent non-trivial x-integrals and N functionally independent non-trivial n-integrals.

To find new Darboux integrable semi-discrete systems we applied the discretization method proposed in [2] to one of the continuous systems derived by Zhiber, Kostrigina in [6] and continuous systems derived by Shabat, Yamilov in [7]. In [6] the authors considered the classification problem for Darboux integrable continuous systems that admit the x- and y-integrals of the first and second order. In [7] the authors considered the exponential type system

$$\mu^i_{xy} = e^{\sum a_{ij}\mu^j}, \qquad i, j = 1, 2, \dots, N.$$

It was shown that such a system is Darboux integrable if and only if the matrix $A = (a_{ij})$ is a Cartan matrix of a semi-simple Lie algebra. Such systems are closely related to the classical Toda field theories, see [8]–[10] and references there in. In this case we obtain the Darboux integrable semi-discrete systems that were already described in [11].

First we consider the following system (see [6])

$$\begin{cases} u_{xy} = \dfrac{u_x u_y}{u+v+c} + \left(\dfrac{1}{u+v+c} + \dfrac{1}{u+v-c}\right) u_x v_y \\ v_{xy} = \dfrac{v_x v_y}{u+v-c} + \left(\dfrac{1}{u+v+c} + \dfrac{1}{u+v-c}\right) u_x v_y, \end{cases} \tag{1.3}$$

where c is an arbitrary constant. This system is Darboux integrable and admits the following y-integrals

$$I_1 = 2v - \frac{v_x(u+v+c)}{u_x} + 2c \ln \frac{u_x}{u+v+c} \tag{1.4}$$

and

$$I_2 = \frac{u_{xx}}{u_x} - \frac{2u_x + v_x}{u + v + c}. \tag{1.5}$$

The x- integrals have the same form in u, v, u_y, v_y, $\dots$ variables.

Now we look for semi-discrete systems admitting these functions as n-integrals. The obtained results are given in Theorems 1.1 and 1.2 below.

Theorem 1.1. *The system*

$$\begin{cases} u_{1x} = f(x,n,u,v,u_1,v_1,u_x,v_x) \\ v_{1x} = g(x,n,u,v,u_1,v_1,u_x,v_x) \end{cases} \tag{1.6}$$

possessing n-integrals (1.4) *and* (1.5)*, where c is a function of n satisfying* $c(n) \neq c(n+1)$ *for all* $n \in \mathbb{Z}$*, has the form*

$$\begin{cases} u_{1x} = \dfrac{(u_1 + v_1 + c_1)u_x}{u + v + c} \\ v_{1x} = \dfrac{2(v_1 - v)u_x}{u + v + c} + \dfrac{2(c_1 - c)u_x}{u + v + c} \ln \dfrac{u_x}{u + v + c} + v_x . \end{cases} \tag{1.7}$$

Moreover, the system above also possesses x-integrals

$$F_1 = \frac{(c - c_1)(v_2 - v) - (c - c_2)(v_1 - v)}{(c - c_2)(v_3 - v) - (c - c_3)(v_2 - v)} \tag{1.8}$$

and

$$F_2 = \frac{(c_1 - c_2)u + (c_2 - c)u_1 + (c - c_1)u_2}{\sqrt{(c_1 - c_2)v + (c_2 - c)v_1 + (c - c_1)v_2}} - \sqrt{(c_1 - c_2)v + (c_2 - c)v_1 + (c - c_1)v_2}\,. \tag{1.9}$$

Hence, semi-discrete system (1.7) *is Darboux integrable.*

Theorem 1.2. *The system* (1.6) *possessing n-integrals* (1.4) *and* (1.5)*, where c is a constant, is either*

$$\begin{cases} u_{1x} = \dfrac{(u_1 + v_1 + c)u_x}{u + v + c} \\ v_{1x} = \dfrac{2(v_1 - v)u_x}{u + v + c} + v_x \end{cases} \tag{1.10}$$

with x-integrals $F_1 = \dfrac{v_1 - v}{v_2 - v_1}$ *and* $F_2 = \dfrac{u_2 - u + v - v_2}{\sqrt{v_1 - v}}$*, or*

$$\begin{cases} u_{1x} = \dfrac{(u_1 + v_1 + c)Bu_x}{u + v + c} \\ v_{1x} = \dfrac{2B(v_1 - v + c\ln B)}{u + v + c} u_x + Bv_x\,, \end{cases} \tag{1.11}$$

where B is defined by equality $H(K_1,K_2) = 0$ *with*

$$K_1 = \frac{v_1 - vB + B(1 - B)u + c\ln B}{(B - 1)^2} + c\ln(B - 1) - c\ln B$$

and

$$K_2 = \frac{u_1 + cB - c - c\ln B}{B-1} + \frac{B^2 v - Bv_1 - cB\ln B}{(B-1)^2} + c\ln(B-1) - c\ln B,$$

and H being any smooth function.

Remark 1.1. We considered some special cases of the system (1.11) and get Darboux integrable systems.

(I) System (1.11) with $B = \dfrac{u - v + (-1)^n\sqrt{(u-v)^2 + 4uv_1}}{2u}$ is Darboux Integrable. (The expression for B is found from $K_1 = 0$, with $c = 0$.)

(II) System (1.11) with $B = \dfrac{v_1 - u_1 + (-1)^n\sqrt{(v_1-u_1)^2 + 4u_1 v}}{2v}$ is Darboux Integrable. (The expression for B is found from $K_2 = 0$, with $c = 0$.)

Remark 1.2. Expansion of the function $B(u,v,v_1)$, given implicitly by $(B-1)^2 K_1 = 0$, into a series of the form

$$B(u,v,v_1) = \sum_{n=0}^{\infty} a_n (v_1 - v)^n, \tag{1.12}$$

where coefficients a_n depend on variables u and v, yields $a_0 = 1$ and $a_1 = \dfrac{1}{u+v-c}$. So B can be written as

$$B(u,v,v_1) = 1 + \frac{1}{u+v-c}(v_1 - v) + \sum_{n=2}^{\infty} a_n (v_1 - v)^n. \tag{1.13}$$

By letting $u_1 = u + \varepsilon u_y$ and $v = v + \varepsilon v_y$ and taking $\varepsilon \to 0$ one can see that the system (1.11) has a continuum limit (1.3).

Let us discuss the exponential type systems. We consider the discretization of such systems corresponding to 2×2 matrices, namely,

$$\begin{aligned} \mu_{xy} &= e^{2\mu - \nu}, \\ \nu_{xy} &= e^{-c\mu - 2\nu}, \end{aligned} \tag{1.14}$$

where $c = 1, 2, 3$. The obtained results are given in Theorem 1.3 below. The discretization of such systems was also considered in [11], where the form of the corresponding semi-discrete system was directly postulated and then the Darboux integrability proved. In our approach we do not make any specific assumptions about the form of the corresponding semi-discrete system. Note that the integrals corresponding to Darboux integrable exponential systems are given in the statement of Theorem 1.3.

Theorem 1.3.

(1) *The system*

$$\begin{cases} u_{1x} = \widetilde{f}(u, v, u_1, v_1, u_x, v_x) \\ v_{1x} = \widetilde{g}(u, v, u_1, v_1, u_x, v_x), \end{cases} \tag{1.15}$$

possessing n-integrals

$$I_1 = u_{xx} + v_{xx} - u_x^2 + u_x v_x - v_x^2 \tag{1.16}$$

and

$$I_1^* = u_{xxx} + u_x(v_{xx} - 2u_{xx}) + u_x^2 v_x - u_x v_x^2 \tag{1.17}$$

has the form

$$\begin{cases} u_{1x} = u_x + Ae^{u_1 + u - v_1} \\ v_{1x} = v_x + Be^{-u + v + v_1}, \end{cases} \tag{1.18}$$

or

$$\begin{cases} u_{1x} = u_x + Ae^{u_1 + u - v} \\ v_{1x} = v_x + Be^{-u_1 + v + v_1}, \end{cases} \tag{1.19}$$

where A and B are arbitrary constants.

(2) *The system* (1.15) *possessing n-integrals*

$$I_2 = 2u_{xx} + v_{xx} - 2u_x^2 + 2u_x v_x - v_x^2 \tag{1.20}$$

and

$$\begin{aligned} I_2^* = u_{xxxx} + u_x(v_{xxx} - 2u_{xxx}) + u_{xx}(4u_x v_x - 2u_x^2 - v_x^2) \\ + u_{xx}(v_{xx} - u_{xx}) + v_{xx}u_x(u_x - 2v_x) + u_x^4 + u_x^2 v_x^2 - 2u_x^3 v_x \end{aligned} \tag{1.21}$$

has the form

$$\begin{cases} u_{1x} = u_x + Ae^{u + u_1 - v_1} \\ v_{1x} = v_x + Be^{-2u + v + v_1}, \end{cases} \tag{1.22}$$

where A and B are arbitrary constants.

(3) *The system* (1.15) *possessing n-integrals*

$$I_3 = u_{xx} + \frac{1}{3}v_{xx} - u_x^2 + u_x v_x - \frac{1}{3}v_x^2 \tag{1.23}$$

and

$$\begin{gathered}
I_3^* = u_{(6)} - 2u_{(5)}u_x + v_{(5)}u_x + u_{(4)}(32(u_x)^2 - 30u_xv_x + 11(v_x)^2 - 40u_{xx} - 11v_{xx}) \\
+v_{(4)}(14(u_x)^2 - 15u_xv_x + (13/3)(v_x)^2 - 10u_{xx} - (13/3)v_{xx}) + 19(u_{(3)})^2 + (13/6)(v_{(3)})^2 + 16u_{(3)}v_{(3)} \\
+u_{(3)}(-36u_{xx}u_x + 18u_{xx}v_x + 80v_{xx}u_x - 45v_{xx}v_x) + v_{(3)}(-52u_{xx}u_x + 33u_{xx}v_x - 5v_{xx}u_x) \\
+u_{(3)}(-64(u_x)^3 + 102(u_x)^2v_x - 62u_x(v_x)^2 + 13(v_x)^3) + v_{(3)}(32(u_x)^3 - 58(u_x)^2v_x \\
+38u_x(v_x)^2 - (26/3)(v_x)^3) + 66(u_{xx})^3 + (26/3)(v_{xx})^3 - 35(u_{xx})^2(v_{xx}) - 5u_{xx}(v_{xx})^2 \\
+(u_{xx})^2(30(u_x)^2 - 18u_xv_x - (11/2)(v_x)^2) + u_{xx}v_{xx}(-34(u_x)^2 + 32u_xv_x - 2(v_x)^2) - 2(v_{xx})^2u_xv_x \\
+u_{xx}(6(u_x)^4 - 24(u_x)^3v_x + 25(u_x)^2(v_x)^2 - 9u_x(v_x)^3 + (v_x)^4) + v_{xx}(-(u_x)^4 + 8(u_x)^3v_x - 8(u_x)^2(v_x)^2 \\
+2u_x(v_x)^3) + (-2(u_x)^6 + 6(u_x)^5v_x - (13/2)(u_x)^4(v_x)^2 + 3(u_x)^3(v_x)^3 - (1/2)(u_x)^2(v_x)^4)
\end{gathered} \tag{1.24}$$

has the form

$$\begin{cases} u_{1x} = u_x + Ae^{u+u_1-v_1} \\ v_{1x} = v_x + Be^{-3u+v+v_1}, \end{cases} \tag{1.25}$$

where A and B are arbitrary constants.

Remark 1.3. We note that while considering systems with integrals (1.20) and (1.21) we also obtain two degenerate systems

$$\begin{cases} u_{1x} = u_x \\ v_{1x} = v_x + Be^{-(2+c)u+cu_1+v+v_1}, \end{cases} \tag{1.26}$$

and

$$\begin{cases} u_{1x} = u_x + Ae^{u+u_1+2cv-(2c+1)v_1} \\ v_{1x} = v_x, \end{cases} \tag{1.27}$$

where A, B and c are arbitrary constants, which are equivalent to a Darboux integrable equation.

Remark 1.4. By letting $u=\mu^1$, $u_1=\mu^1+\varepsilon\mu_y^1$, $v=\mu^2$, $v_1=\mu^2+\varepsilon\mu_y^2$ and $A=\varepsilon$, $B=\varepsilon$ in equations (1.18), (1.22), (1.25) and taking $\varepsilon\to 0$ one can see that the considered systems have corresponding continuum limit given by (1.14).

2. Proof of Theorems 1.1 and 1.2

Let us find a semi-discrete system (1.6) possessing n-integrals (1.4) and (1.5), where c is an arbitrary constant, possibly dependent on n. Let $Dc = c_1$. It follows from $DI_2 = I_2$ that

$$\frac{u_{1xx}}{u_{1x}} - \frac{2u_{1x}+v_{1x}}{u_1+v_1+c_1} = \frac{u_{xx}}{u_x} - \frac{2u_x+v_x}{u+v+c},$$

that is

$$\frac{f_x+f_u u_x+f_v v_x+f_{u_1} f+f_{v_1} g+f_{u_x} u_{xx}+f_{v_x} v_{xx}}{f}-\frac{2f+g}{u_1+v_1+c_1}=\frac{u_{xx}}{u_x}-\frac{2u_x+v_x}{u+v+c}. \tag{2.1}$$

Compare the coefficients by v_{xx} and u_{xx}, we get $f_{v_x}=0$ and $\frac{f_{u_x}}{f}=\frac{1}{u_x}$. Hence

$$f(x,n,u,v,u_1,v_1,u_x,v_x)=A(x,n,u,v,u_1,v_1)u_x. \tag{2.2}$$

It follows from $DI_1=I_1$ that

$$2v_1-\frac{(u_1+v_1+c_1)g}{f}+2c_1\ln\frac{f}{u_1+v_1+c_1}=2v-\frac{v_x(u+v+c)}{u_x}+2c\ln\frac{u_x}{u+v+c}. \tag{2.3}$$

Using (2.2) we obtain

$$2v_1-\frac{(u_1+v_1+c_1)g}{Au_x}+2c_1\ln\frac{Au_x}{u_1+v_1+c_1}=2v-\frac{v_x(u+v+c)}{u_x}+2c\ln\frac{u_x}{u+v+c}$$

and find g as

$$g=\left(\frac{2(v_1-v)A}{(u_1+v_1+c_1)}+\frac{2Ac_1}{(u_1+v_1+c_1)}\ln\frac{(u+v+c)A}{(u_1+v_1+c_1)}\right)u_x+\frac{2(c_1-c)A}{(u_1+v_1+c_1)}u_x\ln\frac{u_x}{u+v+c}+\frac{(u+v+c)A}{(u_1+v_1+c_1)}v_x. \tag{2.4}$$

Substituting the expressions (2.2) and (2.4) into equality (2.1) and comparing coefficients by u_x, v_x, $u_x\ln\frac{u_x}{u+v+c}$ and free term we get the following equalities

$$\frac{A_x}{A}=0 \tag{2.5}$$

$$\frac{2(c_1-c)A_{v_1}}{(u_1+v_1+c_1)}-\frac{2(c_1-c)A}{(u_1+v_1+c_1)^2}=0 \tag{2.6}$$

$$\frac{A_u}{A}+A_{u_1}+\left(\frac{A_{v_1}}{A}-\frac{1}{(u_1+v_1+c_1)}\right)\left(\frac{2(v_1-v)A}{(u_1+v_1+c_1)}+\frac{2c_1A}{(u_1+v_1+c_1)}\ln\frac{(u+v+c)A}{(u_1+v_1+c_1)}\right)-\frac{2A}{(u_1+v_1+c_1)}+\frac{2}{(u+v+c)}=0 \tag{2.7}$$

$$\frac{A_v}{A}+\frac{(u+v+c)A_{v_1}}{(u_1+v_1+c_1)}-\frac{(u+v+c)A}{(u_1+v_1+c_1)^2}+\frac{1}{(u+v+c)}=0. \tag{2.8}$$

We have two possibilities: $c_1\neq c$ and $c_1=c$.

2.1. *c depends on n*

First we consider the case $c_1 \neq c$, that is c depends on n and satisfies $c(n) \neq c(n+1)$ for all n. Then equations (2.6)-(2.8) are transformed into

$$\frac{A_{v_1}}{A} - \frac{1}{(u_1+v_1+c_1)} = 0 \tag{2.9}$$

$$\frac{A_u}{A} + A_{u_1} - \frac{2A}{(u_1+v_1+c_1)} + \frac{2}{(u+v+c)} = 0 \tag{2.10}$$

$$\frac{A_v}{A} + \frac{1}{(u+v+c)} = 0\,. \tag{2.11}$$

Equations (2.9) and (2.11) imply that

$$A = \frac{(u_1+v_1+c_1)}{(u+v+c)} M(n,u,u_1). \tag{2.12}$$

Substituting the above A into (2.10) we get that M satisfies

$$(u+v+c)\frac{M_u}{M} + (u_1+v_1+c_1)M_{u_1} + (1-M) = 0. \tag{2.13}$$

Differentiating equation (2.13) with respect to v and v_1 we get that $M_u = 0$ and $M_{u_1} = 0$ respectively. Thus, equation (2.13) implies that $M = 1$. So in the case $c_1 \neq c$ we arrive to the system of equations (1.7). We note that the system (1.7) is Darboux integrable. It admits two n-integrals (1.4) and (1.5) and two x-integrals (1.8) and (1.9). The x-integrals can be found by considering the characteristic x-ring for system (1.7).

2.2. *c does not depend on n*

Now we consider the case $c = c_1$, that is c is a constant independent of n. Then we have equations (2.7) and (2.8). Introducing new variable $B = \frac{(u+v+c)}{(u_1+v_1+c)}A$ we can rewrite the equations as

$$\frac{B_u}{B} + \frac{(u_1+v_1+c)}{(u+v+c)}B_{u_1} + 2\frac{(v_1 - v + c\ln B)}{(u+v+c)}B_{v_1} + \frac{1-B}{(u+v+c)} = 0 \tag{2.14}$$

$$\frac{B_v}{B} + B_{v_1} = 0\,. \tag{2.15}$$

The set of solutions of the above system is not empty, for example it admits a solution $B = 1$. Setting $B = 1$ we arrive to the system of equations (1.10). We note that the system (1.10) is Darboux integrable. It admits two n-integrals (1.4) and (1.5) and two x-integrals

$$F_1 = \frac{v_1 - v}{v_2 - v_1}, \qquad F_2 = \frac{u_2 - u + v - v_2}{\sqrt{v_1 - v}}.$$

The x-integrals are calculated by considering the characteristic x-ring for system (1.10).

Now let us consider case when $B \neq 1$ identically. For function $W = W(u, v, u_1, v_1, B)$ equations (2.14) and (2.15) become

$$\frac{W_u}{B} + \frac{(u_1+v_1+c)}{(u+v+c)}W_{u_1} + 2\frac{(v_1-v+c\ln B)}{(u+v+c)}W_{v_1} + \frac{B-1}{(u+v+c)}W_B = 0 \tag{2.16}$$

$$\frac{W_v}{B} + W_{v_1} = 0\,. \tag{2.17}$$

After the change of variables $\widetilde{v} = v + c$, $\widetilde{v_1} = v_1 + c - (v+c)B$, $\widetilde{u} = u$, $\widetilde{u_1} = u_1$, $\widetilde{B} = B$ equations (2.17) and (2.16) become $W_{\widetilde{v}} = 0$ and

$$\frac{\widetilde{u}+\widetilde{v}}{\widetilde{B}}W_{\widetilde{u}} + (\widetilde{u_1}+\widetilde{v_1}+\widetilde{v}\widetilde{B})W_{\widetilde{u_1}} + (2\widetilde{v_1}+2c\ln\widetilde{B}+\widetilde{v}(\widetilde{B}-1))W_{\widetilde{v_1}} + (\widetilde{B}-1)W_{\widetilde{B}} = 0.$$

We differentiate the last equality with respect to $\widetilde{v}$, use $W_{\widetilde{v}} = 0$, and find that W satisfies the following equations

$$\frac{W_{\widetilde{u}}}{\widetilde{B}} + \widetilde{B}W_{\widetilde{u_1}} + (\widetilde{B}-1)W_{\widetilde{v_1}} = 0$$
$$\frac{\widetilde{u}}{\widetilde{B}}W_{\widetilde{u}} + (\widetilde{u_1}+\widetilde{v_1})W_{\widetilde{u_1}} + (2\widetilde{v_1}+2c\ln\widetilde{B})W_{\widetilde{v_1}} + (\widetilde{B}-1)W_{\widetilde{B}} = 0\,.$$

After doing another change of variables $u_1^* = \widetilde{u_1} - \widetilde{B}^2\widetilde{u}$, $v_1^* = \widetilde{v_1} + \widetilde{B}(1-\widetilde{B})\widetilde{u}$, $u^* = \widetilde{u}$, $B^* = \widetilde{B}$, we obtain that $W_{u^*} = 0$ and

$$(u_1^*+v_1^*)W_{u_1^*} + (2v_1^*+2c\ln B^*)W_{v_1^*} + (B^*-1)W_{B^*} = 0.$$

The first integrals of the last equation are

$$K_1 = \frac{v_1^*}{(B^*-1)^2} + \frac{c\ln B^*}{(B^*-1)^2} - c\ln B^* + c\ln(B^*-1) + \frac{c}{B^*-1}$$

and

$$K_2 = \frac{u_1^*-c-c\ln B^*}{B^*-1} - \frac{B^*v_1^*}{(B^*-1)^2} - \frac{cB^*\ln B^*}{(B^*-1)^2} + c\ln(B^*-1) - c\ln B^*.$$

They can be rewritten in the original variables as

$$K_1 = \frac{v_1 - vB + B(1-B)u + c\ln B}{(B-1)^2} + c\ln(B-1) - c\ln B$$

and

$$K_2 = \frac{u_1 + cB - c - c\ln B}{B-1} + \frac{B^2v - Bv_1 - cB\ln B}{(B-1)^2} + c\ln(B-1) - c\ln B.$$

Therefore, system (1.6) becomes (1.11) due to (2.2) and (2.4).

2.3. *Proof of Remark 1.1*

Function B is any function satisfying the equality $H(K_1,K_2)=0$, where H is any smooth function. (I) By taking function H as $H(K_1,K_2)=K_1$ we obtain one possible function B. It satisfies the equality $-uB^2+(u-v)B+v_1=0$ and can be taken as $B=\dfrac{u-v+(-1)^n\sqrt{(u-v)^2+4uv_1}}{2u}$.
(II) By taking function H as $H(K_1,K_2)=K_2$ we obtain another possible function B. It satisfies the equality $vB^2+(u_1-v_1)B-u_1=0$ and can be taken as $B=\dfrac{v_1-u_1+(-1)^n\sqrt{(v_1-u_1)^2+4u_1v}}{2v}$.

In both cases ((I) and (II)) let us consider the corresponding x-rings. Denote by $X=D_x$, $Y_1=\dfrac{\partial}{\partial u_x}$, $Y_2=\dfrac{\partial}{\partial v_x}$, $E_1=\dfrac{u+v}{B}[Y_1,X]$, $E_2=\dfrac{1}{B}[Y_2,X]$, $E_3=[E_1,E_2]$. Note that $X=u_xE_1+v_xE_2$. We have,

$[E_i,E_j]$	E_1	E_2	E_3
E_1	0	E_3	$\alpha_1E_2+\alpha_2E_3$
E_2	$-E_3$	0	0
E_3	$-(\alpha_1E_2+\alpha_2E_3)$	0	0

where

$$\alpha_1=\frac{2v_1(u-v)+2(uv-v^2+2uv_1)B}{v_1(u-v)+((u-v)^2+2uv_1)B},\qquad \alpha_2=-3+\frac{2}{B}$$

in case (I) and

$$\alpha_1=\frac{2u_1^2+4u_1v-2u_1v_1+2(-(u_1-v_1)^2+vv_1-3vu_1)B}{u_1(v_1-u_1)+((u_1-v_1)^2+2u_1v)B},\qquad \alpha_2=-3+\frac{2}{B}$$

in case (II).

3. Proof of Theorem 1.3

3.1. *Case (1)*

Let us find a system

$$\begin{cases} u_{1x}=\widetilde{f}(x,n,u,v,u_1,v_1,u_x,v_x)\\ v_{1x}=\widetilde{g}(x,n,u,v,u_1,v_1,u_x,v_x)\end{cases}\tag{3.1}$$

possessing n-integrals (1.16) and (1.17). The equality $DI=I$ implies

$$u_{1xx}+v_{1xx}-u_{1x}^2+u_{1x}v_{1x}-v_{1x}^2=u_{xx}+v_{xx}-u_x^2+u_xv_x-v_x^2,\tag{3.2}$$

or the same

$$\begin{aligned}&\widetilde{f}_x+\widetilde{f}_uu_x+\widetilde{f}_vv_x+\widetilde{f}_{u_1}\widetilde{f}+\widetilde{f}_{v_1}\widetilde{g}+\widetilde{f}_{u_x}u_{xx}+\widetilde{f}_{v_x}v_{xx}+\widetilde{g}_x+\widetilde{g}_uu_x+\widetilde{g}_vv_x\\ &\quad+\widetilde{g}_{u_1}\widetilde{f}+\widetilde{g}_{v_1}\widetilde{g}+\widetilde{g}_{u_x}u_{xx}+\widetilde{g}_{v_x}v_{xx}-\widetilde{f}^2+\widetilde{f}\widetilde{g}-\widetilde{g}^2=u_{xx}+v_{xx}-u_x^2+u_xv_x-v_x^2.\end{aligned}\tag{3.3}$$

We consider the coefficients by u_{xx} and v_{xx} in (3.3) to get

$$\widetilde{f}_{u_x}+\widetilde{g}_{u_x}=1\tag{3.4}$$
$$\widetilde{f}_{v_x}+\widetilde{g}_{v_x}=1.\tag{3.5}$$

The equality $DI_1^* = I_1^*$ implies

$$u_{1xxx} + u_{1x}(v_{1xx} - 2u_{1xx}) + u_{1x}^2 v_{1x} - u_{1x} v_{1x}^2 = u_{xxx} + u_x(v_{xx} - 2u_{xx}) + u_x^2 v_x - u_x v_x^2. \tag{3.6}$$

Since $DI_1^* = u_{1xxx} + \cdots = \widetilde{f}_{u_x} u_{xxx} + \ldots$, where the remaining terms do not depend on u_{xxx}, the equality (3.6) implies

$$\widetilde{f}_{u_x} = 1. \tag{3.7}$$

Note that $J = D_x I_1 - I_1^* = v_{xxx} + v_x(u_{xx} - 2v_{xx}) + v_x^2 u_x - u_x^2 v_x$ is an n-integral as well. Since $DJ = J$ and $DJ = v_{1xxx} + \cdots = \widetilde{g}_{v_x} v_{xxx} + \ldots$, where the remaining terms do not depend on v_{xxx}, then

$$\widetilde{g}_{v_x} = 1. \tag{3.8}$$

It follows from equalities (3.4), (3.5), (3.7) and (3.8) that $\widetilde{f}_{v_x} = 0$ and $\widetilde{g}_{u_x} = 0$. Therefore the system (3.1) and equality (3.3) become

$$\begin{cases} u_{1x} = u_x + f(x,n,u,v,u_1,v_1) \\ v_{1x} = v_x + g(x,n,u,v,u_1,v_1) \end{cases} \tag{3.9}$$

and

$$\begin{aligned} f_x + f_u u_x + f_v v_x + f_{u_1}(u_x + f) + f_{v_1}(v_x + g) + g_x + g_u u_x + g_v v_x + g_{u_1}(u_x + f) \\ + g_{v_1}(v_x + g) - 2u_x f - f^2 + u_x g + v_x f + fg - 2v_x g - g^2 = 0. \end{aligned} \tag{3.10}$$

By considering coefficients by u_x, v_x and $u_x^0 v_x^0$ in the last equality, we get

$$(f+g)_u + (f+g)_{u_1} + (f+g) - 3f = 0, \tag{3.11}$$

$$(f+g)_v + (f+g)_{v_1} + (f+g) - 3g = 0, \tag{3.12}$$

$$f(f+g)_{u_1} + g(f+g)_{v_1} + (f+g)_x - (f+g)^2 + 3fg = 0. \tag{3.13}$$

Now let us rewrite inequality (3.6) for the system (3.9)

$$\begin{aligned} D_x\big(f_x + f_u u_x + f_v v_x + f_{u_1}(u_x + f) + f_{v_1}(v_x + g)\big) \\ + (u_x + f)\big(g_x + g_u u_x + g_v v_x + g_{u_1}(u_x + f) + g_{v_1}(v_x + g) + v_{xx}\big) \\ + (u_x + f)\big(-2f_x - 2f_u u_x - 2f_v v_x - 2f_{u_1}(u_x + f) - 2f_{v_1}(v_x + g) - 2u_{xx}\big) \\ + (u_x^2 + 2u_x f + f^2)(v_x + g) - (v_x^2 + 2v_x g + g^2)(u_x + f) = u_x(v_{xx} - 2u_{xx}) + u_x^2 v_x - u_x v_x^2. \end{aligned} \tag{3.14}$$

By comparing the coefficients by u_{xx} and v_{xx} in the last equality, we get

$$\begin{aligned} f_u + f_{u_1} &= 2f \\ f_v + f_{v_1} &= -f. \end{aligned} \tag{3.15}$$

It follows from equality $DJ = J$ that

$$\begin{aligned} D_x\big(g_x + g_u u_x + g_v v_x + g_{u_1}(u_x + f) + g_{v_1}(v_x + g)\big) \\ + (v_x + g)\big(f_x + f_u u_x + f_v v_x + f_{u_1}(u_x + f) + f_{v_1}(v_x + g) + u_{xx}\big) \\ - 2(v_x + g)\big(g_x + g_u u_x + g_v v_x + g_{u_1}(u_x + f) + g_{v_1}(v_x + g) + v_{xx}\big) \\ + (u_x + f)(v_x^2 + 2v_x g + g^2) - (v_x + g)(u_x^2 + 2u_x f + f^2) = v_x(u_{xx} - 2v_{xx}) + v_x^2 u_x - u_x^2 v_x. \end{aligned} \tag{3.16}$$

By comparing the coefficients by u_{xx} and v_{xx} in the last equality, we get

$$\begin{aligned} g_u + g_{u_1} &= -g \\ g_v + g_{v_1} &= 2g\,. \end{aligned} \tag{3.17}$$

Note that the equalities (3.11) and (3.12) follow from equalities (3.15) and (3.17). Let us use equalities (3.15) and (3.17) to rewrite equality (3.14)

$$\begin{aligned} D_x(f_x + 2fu_x - fv_x + f_{u_1}f + f_{v_1}g) + (u_x + f)(g_x + g_{u_1}f + g_{v_1}g + v_{xx} - 4fu_x - 2f_x) \\ + (u_x + f)(2fv_x - 2f_{u_1}f - 2f_{v_1}g - 2u_{xx} + u_xv_x + fv_x + fg - v_x^2 - g^2) \\ = u_x(v_{xx} - 2u_{xx}) + u_x^2v_x - u_xv_x^2\,. \end{aligned}$$

We note that the consideration of the coefficients by u_{xx}, v_{xx}, u_x^2, v_x^2, u_xv_x in the above equality give us equations that follow immediately from (3.15) and (3.17). Considering coefficient by u_x we get

$$\begin{aligned} f_{xu} + f_{xu_1} + 2f_x + 2ff_{u_1} + 2f_{v_1}g + ff_{u_1u} + f_{u_1}f_u + f_{u_1}^2 + gf_{v_1u} \\ + gf_{u_1v_1} + f_{v_1}g_u + f_{v_1}g_{u_1} + f_{u_1u_1}f + g_x + g_{u_1}f + g_{v_1}g - 2f_x - 2f_{u_1}f - 2f_{v_1}g + fg - g^2 - 4f^2 = 0. \end{aligned}$$

Using equations (3.15) and (3.17) we get

$$2f_x + g_x + 4ff_{u_1} + f_{v_1}g + g_{u_1}f + g_{v_1}g + fg - g^2 - 4f^2 = 0\,,$$

or using equation (3.13) ,

$$f_x + 3f(f_{u_1} - f) = 0. \tag{3.18}$$

Considering coefficient by v_x we get

$$\begin{aligned} f_{xv} + f_{xv_1} - f_x - ff_{u_1} - f_{v_1}g + ff_{u_1v} + ff_{u_1v_1} + f_{u_1}f_v + f_{u_1}f_{v_1} \\ + gf_{v_1v} + gf_{v_1v_1} + f_{v_1}g_v + f_{v_1}g_{v_1} + 3f^2 = 0. \end{aligned}$$

Using equations (3.15) and (3.17) we get

$$2f_x + 3f(f_{u_1} - f) = 0. \tag{3.19}$$

It follows from equations (3.18) and (3.19) that $f_x = 0$ and $f(f_{u_1} - f) = 0$. Thus either $f = 0$ or

$$\begin{cases} f = f_{u_1}\,, \\ f = f_u\,. \end{cases} \tag{3.20}$$

Now we consider the coefficient by $u_x^0v_x^0$ in (3.14) we get

$$\begin{aligned} f^2f_{u_1u_1} + fgf_{u_1v_1} + ff_{u_1}^2 + f_{u_1}f_{v_1}g + fgf_{u_1v_1} + g^2f_{v_1v_1} + f_{v_1}g_x + ff_{v_1}g_{u_1} \\ + gf_{v_1}g_{v_1} + fg_x + f^2g_{u_1} + fgg_{v_1} - 2f^2f_{u_1} - 2fgf_{v_1} + f^2g - fg^2 = 0. \end{aligned}$$

First assume that $f \neq 0$ then using (3.20) we can rewrite the above equality as

$$fgf_{v_1} + g^2f_{v_1v_1} + f_{v_1}g_x + f_{v_1}g_{u_1}f + f_{v_1}g_{v_1}g + fg_x + f^2g_{u_1} + fgg_{v_1} + f^2g - fg^2 = 0\,. \tag{3.21}$$

Also we can rewrite equality (3.16), using equations (3.15), (3.17) and (3.13) then considering coefficients by u_x and v_x we obtain

$$2g_x + 3g(g_{v_1} - g) = 0,$$
$$g_x + 3g(g_{v_1} - g) = 0.$$

From above equalities and (3.17) it follows that $g_x = 0$, $g_{v_1} = g$ and $g_v = g$ (we assume that $g \neq 0$). We have

$$\begin{aligned} f_{u_1} = f, \quad f_u = f, \qquad & f_v + f_{v_1} = -f \\ g_{v_1} = g, \quad g_v = g, \qquad & g_u + g_{u_1} = -g \\ & f_{v_1} g + g_{u_1} f = -fg\,. \end{aligned} \tag{3.22}$$

Using (3.22), the equality (3.21) takes form $g_{u_1} f_{v_1}(-g+f) = 0$. This equality implies that under assumptions that $f \neq 0$ and $g \neq 0$ we have three possibilities: (I) $g_{u_1} = 0$, (II) $f_{v_1} = 0$ and (III) $g = f$. Let us consider these possibilities.

Case (I) From $g_{u_1} = 0$, using (3.22), we get that $g_u = -g$, $g_{v_1} = g$, $g_v = g$. Thus $g = Be^{-u+v+v_1}$, where B is a constant. We also get that $f_u = f$, $f_{u_1} = f$, $f_v = 0$ and $f_{v_1} = -f$. Thus $f = Ae^{u_1+u-v_1}$, where A is a constant. So the system (3.9) takes form (1.18).

Case (II) From $f_{v_1} = 0$, using (3.22), we get that $f_u = f$, $f_{u_1} = f$, $f_v = -f$. Thus $f = Ae^{u_1+u-v}$, where A is a constant. We also get that $g_u = 0$, $g_{u_1} = -g$, $g_v = g$ and $g_{v_1} = g$. Thus $g = Be^{-u_1+v_1+v}$, where B is a constant. So the system (3.9) takes form (1.19).

Case (III) From $g = f$, using (3.22), we get that $f = 0$ and $g = 0$. So the system (3.9) takes form

$$\begin{cases} u_{1x} = u_x\,, \\ v_{1x} = v_x\,. \end{cases}$$

3.2. *Case (2)*

Let us find system (1.15) possessing n-integrals (1.20) and (1.21). We compare the coefficients in $DI_2 = I_2$ by u_{xx} and v_{xx} and get

$$\begin{aligned} 2\widetilde{f}_{u_x} + \widetilde{g}_{u_x} &= 2\,, \\ 2\widetilde{f}_{v_x} + \widetilde{g}_{v_x} &= 1\,. \end{aligned} \tag{3.23}$$

We also compare the coefficients in $DI_2^* = I_2^*$ and $D(D_x^2 I_2 - 2I_2^*) = (D_x^2 I_2 - 2I_2^*)$ by u_{xxxx} and v_{xxxx} respectively and get $\widetilde{f}_{u_x} = 1$ and $\widetilde{g}_{v_x} = 1$. It follows from (3.23) that $\widetilde{f}_{v_x} = 0$ and $\widetilde{g}_{u_x} = 0$. Therefore, our system (1.15) becomes

$$\begin{cases} u_{1x} = u_x + f(u, v, u_1, v_1), \\ v_{1x} = v_x + g(u, v, u_1, v_1). \end{cases}$$

We write equality $DI_2 = I_2$ and get

$$\begin{aligned} 2u_{xx} + 2f_u u_x + 2f_v v_x + 2f_{u_1}(u_x + f) + 2f_{v_1}(v_x + g) + v_{xx} + g_u u_x + g_v v_x + g_{u_1}(u_x + f) \\ + g_{v_1}(v_x + g) - 2(u_x + f)^2 + 2(u_x + f)(v_x + g) - (v_x + g)^2 = 2u_{xx} + v_{xx} - 2u_x^2 + 2u_x v_x - v_x^2\,. \end{aligned}$$

By comparing the coefficients by u_x, v_x and $u_x^0 v_x^0$ in the last equality we obtain the system of equations

$$\begin{aligned} 2f_u + f_{u_1} + g_u + g_{u_1} - 4f + 2g &= 0\,, \\ 2f_v + 2f_{v_1} + g_v + g_{v_1} + 2f - 2g &= 0\,, \\ 2ff_{u_1} + 2gf_{v_1} + fg_{u_1} + gg_{v_1} - 2f^2 + 2fg - g^2 &= 0\,. \end{aligned}$$

That suggests the following change of variables

$$u = P,\;\; u_1 - u = Q,\;\; v = S,\;\; v_1 - v = T$$

to be made. In new variables the system (1.15) becomes

$$\begin{cases} Q_x = F(P,Q,S,T)\,, \\ T_x \;= G(P,Q,S,T)\,. \end{cases} \tag{3.24}$$

The comparison of coefficients in $DI_2 = I_2$ by P_x, S_x and $P_x^0 S_x^0$ gives

$$\begin{aligned} -4F + 2G + 2F_P + G_P &= 0\,, \\ 2F - 2G + 2F_S + G_S &= 0\,, \\ -2F^2 + G(-G + 2F_T + G_T) + F(2G + 2F_Q + G_Q) &= 0\,. \end{aligned} \tag{3.25}$$

The coefficients in $DI_2^* = I_2^*$ by S_{xxx} and P_{xxx} are compared and we obtain the following equalities

$$\begin{aligned} F + F_S &= 0\,, \\ -2F + F_P &= 0\,. \end{aligned} \tag{3.26}$$

It follows from (3.25) and (3.26) that $G_S = 2G$, $G_P = -2G$, $F_S = -F$ and $F_P = 2F$. Therefore, system (3.24) can be written as

$$\begin{cases} Q_x = A(Q,T)e^{-S+2P}\,, \\ T_x \;= B(Q,T)e^{2S-2P}\,. \end{cases}$$

We compare the coefficient in $DI_2^* = I_2^*$ by S_{xx} and get

$$3e^{4P-2S}A^2 - 3e^{4P-2S}AA_Q = 0,$$

that is $A = A_Q$. Hence, $A(Q,T) = e^Q\widetilde{A}(T)$. Now we compare the coefficient in $DI_2 = I_2$ by $P_x^0 S_x^0$ and get

$$\widetilde{A} + \widetilde{A}_T = \frac{1}{2}e^{-4P+3S-Q}(B - B_T) - \frac{\widetilde{A}}{2B}B_Q\,. \tag{3.27}$$

Since functions $\widetilde{A}(T)$ and $B(Q,T)$ do not depend on variable P, then it follows from (3.27) that $B = B_T$, that is $B = \widetilde{B}(Q)e^T$. Now (3.27) becomes

$$-2\frac{\widetilde{A} + \widetilde{A}_T}{\widetilde{A}} = \frac{\widetilde{B}_Q}{\widetilde{B}}\,.$$

Note that the right side of the last equality depends on Q only, while the left side depends on T only. Hence, $-2\frac{\widetilde{A}+\widetilde{A}_T}{\widetilde{A}} = c$ and $\frac{\widetilde{B}_Q}{\widetilde{B}} = c$, where c is some constant. One can see that $\widetilde{A} = c_1 e^{-(2c+1)T}$ and

$\widetilde{B} = c_2 e^{cQ}$ and therefore system (3.24) becomes

$$\begin{cases} Q_x = c_1 e^{-S+2P+Q-(2c+1)T}\,, \\ T_x \;= c_2 e^{2S-2P+T+cQ}\,, \end{cases}$$

where c, c_1 and c_2 are some constants. Equality $DI_2 - I_2 = 0$ becomes $-3cc_1c_2e^{s+(c+1)Q-2cT} = 0$, which implies that either $c = 0$, or $c_1 = 0$, or $c_2 = 0$. Note that the $DI_2^* = I_2^*$ is also satisfied if either $c = 0$ or $c_1 = 0$ or $c_2 = 0$. So we have three cases:

- when $c = 0$ the system (1.15) becomes (1.22) with $c_1 = A$ and $c_2 = B$.
- when $c_1 = 0$ the system (1.15) becomes (1.26) with $c_2 = B$.
- when $c_2 = 0$ the system (1.15) becomes (1.27) with $c_1 = A$.

3.3. *Case (3)*

Let us find system (1.15) possessing n-integrals (1.23) and (1.24). We compare the coefficients in $DI_3 = I_3$ by u_{xx} and v_{xx} and get

$$\begin{aligned} \widetilde{f}_{u_x} + \tfrac{1}{3}\widetilde{g}_{u_x} &= 1\,, \\ \widetilde{f}_{v_x} + \tfrac{1}{3}\widetilde{g}_{v_x} &= 1\,. \end{aligned} \tag{3.28}$$

We also compare the coefficients in $DI_3^* = I_3^*$ and $D(D_x^4 I_3 - I_3^*) = (D_x^4 I_3 - I_3^*)$ by $u_{(6)}$ and $v_{(6)}$ respectively and get $\widetilde{f}_{u_x} = 1$ and $\widetilde{g}_{v_x} = 1$. It follows from (3.28) that $\widetilde{f}_{v_x} = 0$ and $\widetilde{g}_{u_x} = 0$. Therefore, our system (1.15) becomes

$$\begin{cases} u_{1x} = u_x + f(u,v,u_1,v_1)\,, \\ v_{1x} = v_x + g(u,v,u_1,v_1)\,. \end{cases}$$

By comparing the coefficients by u_x, v_x and $u_x^0 v_x^0$ in $DI_3 = I_3$ we obtain the system of equations

$$\begin{aligned} f_u + f_{u_1} + \tfrac{1}{3}g_u + \tfrac{1}{3}g_{u_1} - 2f + g &= 0\,, \\ f_v + f_{v_1} + \tfrac{1}{3}g_v + \tfrac{1}{3}g_{v_1} + f - \tfrac{2}{3}g &= 0\,, \\ ff_{u_1} + gf_{v_1} + \tfrac{1}{3}fg_{u_1} + \tfrac{1}{3}gg_{v_1} - f^2 + fg - \tfrac{1}{3}g^2 &= 0\,. \end{aligned}$$

That suggests the following change of variables

$$u = P,\ ,\ u_1 - u = Q,\ \ v = S,\ \ v_1 - v = T$$

to be made. In new variables the system (1.15) becomes

$$\begin{cases} Q_x = F(P,Q,S,T)\,, \\ T_x \;= G(P,Q,S,T)\,. \end{cases} \tag{3.29}$$

The comparison of coefficients in $DI_3 = I_3$ by P_x, S_x and $P_x^0 S_x^0$ gives

$$\begin{aligned} 6F - 3G - 3F_P - G_P &= 0\,, \\ -3F + 2G - 3F_S - G_S &= 0\,, \\ F^2 - FG + \tfrac{1}{3}G^2 - 2GF_T - \tfrac{1}{3}GG_T - FF_Q - \tfrac{1}{3}FG_Q &= 0\,. \end{aligned} \tag{3.30}$$

The comparison of coefficients in $DI_3^* = I_3^*$ by $S_{(5)}$ and $P_{(5)}$ gives

$$\begin{aligned} F + F_S &= 0\,, \\ -2F + F_P &= 0\,. \end{aligned} \tag{3.31}$$

Using equations (3.30) and (3.31) we get $G_S = 2G$, $G_P = -3G$, $F_S = -F$, and $F_P = 2F$. Therefore, system (3.29) can be written as

$$\begin{cases} Q_x = A(Q,T)e^{-S+2P}, \\ T_x \;= B(Q,T)e^{2S-3P}, \end{cases}$$

where A and B are some functions depending on Q and T only. We compare the coefficients in $DI_3 - I_3 = 0$ by $S_x^0 P_x^0$ and the coefficients in $DI_3^* - I_3^* = 0$ by $P_{(4)}$, $S_{(4)}$ and $P_{(3)}P_x$ respectively and get

$$\begin{aligned} a_{11}A_T + a_{12}B_T + a_{13}A_Q + a_{14}B_Q + b_1 &= 0, \\ a_{21}A_T + a_{22}B_T + a_{23}A_Q + a_{24}B_Q + b_2 &= 0, \\ a_{31}A_T + a_{32}B_T + a_{33}A_Q + a_{34}B_Q + b_3 &= 0, \\ a_{41}A_T + a_{42}B_T + a_{43}A_Q + a_{44}B_Q + b_4 &= 0, \end{aligned} \tag{3.32}$$

where

$$\begin{array}{llll} a_{11} = -e^{-P+S}B, & a_{12} = -\frac{1}{3}e^{-6P+4S}B, & a_{13} = -e^{4P-2S}A, & a_{14} = -\frac{1}{3}e^{-P+S}A, \\ a_{21} = -33e^{-P+S}B, & a_{22} = -11e^{-6P+4S}B, & a_{23} = -28e^{4P-2S}A, & a_{24} = -11e^{-P+S}A, \\ a_{31} = -13e^{-P+S}B, & a_{32} = -\frac{13}{3}e^{-6P+4S}B, & a_{33} = -16e^{4P-2S}A, & a_{34} = -\frac{13}{3}e^{-P+S}A, \\ a_{41} = 18e^{-P+S}B, & a_{42} = -79e^{-6P+4S}B, & a_{43} = 328e^{4P-2S}A, & a_{44} = 6e^{-P+S}A, \end{array}$$

and

$$\begin{aligned} b_1 &= e^{4P-2S}A^2 - e^{-P+S}AB + \tfrac{1}{3}e^{-6P+4S}B^2, \\ b_2 &= 28e^{4P-2S}A^2 - 33e^{-P+S}AB + 11e^{-6P+4S}B^2, \\ b_3 &= 16e^{4P-2S}A^2 - 13e^{-P+S}AB + \tfrac{13}{3}e^{-6P+4S}B^2, \\ b_4 &= -328e^{4P-2S}A^2 + 18e^{-P+S}AB + 79e^{-6P+4S}B^2. \end{aligned}$$

We solve the linear system of equations (3.32) with respect to A_T, A_Q, B_T and B_Q and get the following system of differential equations $A_T = -A$, $A_Q = A$, $B_T = B$ and $B_Q = 0$. Thus the system (3.29) is written as

$$\begin{cases} Q_x = c_1 e^{2P+Q-S-T}, \\ T_x \;= c_2 e^{-3P+2S+T}, \end{cases}$$

where c_1 and c_2 are arbitrary constants. It is equivalent to system (1.25) with $A = c_1$ and $B = c_2$.

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