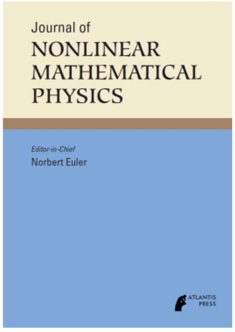

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Study on geometric structures on Lie algebroids with optimal control applications

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We construct $\rho_{\pounds}$-covariant derivatives in $\pi^*\pi$ as the generalization of covariant derivative in $\pi^*\pi$ to $\pounds^{\pi}E$. Moreover, we introduce Berwald and Yano derivatives as two important classes of $\rho_{\pounds}$-covariant derivatives in $\pi^*\pi$ and we study properties of them. Finally, we solve an optimal control problem using some geometric structures and Pontryagin Maximum Principle on Lie algebroids.

Keywords: Berwald and Yano-derivatives; Covariant derivative; Douglas tensor; Lie algebroid; Optimal control.

2010 Mathematics Subject Classification: 17B66, 34A26, 53C05

1. Introduction

The framework of differential geometry is very useful in modelling and understanding of a large class of natural phenomena. The Lie geometric methods are applied successfully in differential equations, optimal control theory or theoretical physics. In most of the cases the study starts with a variational problem formulated for a regular Lagrangian on the tangent bundle TM over the manifold M and very often the whole set of problems is transferred on the dual space T^*M, endowed with a Hamiltonian function, via Legendre transformation. The problem is that in a lot of cases the proposed Lagrangian formalism yields a singular Lagrangian description, which makes the Legendre transformation ill-defined and thus no straightforward Hamiltonian formulation can be related. In the last years the investigations have led to a geometric framework which is covering these phenomena. It is precisely the underlying structure of a Lie algebroid on the phase space which allows a unified treatment. This idea was first introduced by A. Weinstein [24] in order to define a Lagrangian formalism which is very useful for the various types of such systems. One of the motivations for the present work is the study of Lagrangian systems subjected to external holonomic constraints, which come from the theory of optimal control, using the framework of Lie algebroids.

Optimal control problems belong to the class of extremum optimization theory, i.e., minimization or maximization of some functions equipped with some external constraints. This theory

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extends classical variational calculus that is based on control variations of a continuous trajectory. One of the most important methods in the analysis of solutions for the optimal control problems is provided by Pontryagin's Maximum Principle. A curve $c(t) = (x(t), u(t))$ is an optimal trajectory if there exists a lifting of $x(t)$ to the dual space $(x(t), p(t))$ satisfying the Hamilton-Jacobi-Bellman equations. This theory has very important applications in different domains because it provides a pseudo-Hamiltonian formulation of the variational problem in the case in which the standard Legendre transformation is not well defined.

However, finding a complete solution to an optimal control problem, still remains extremely difficult for several reasons. Firstly, we are dealing with the problem of integrating a Hamiltonian system, which is generally difficult to integrate, except for particular dynamics and costs. Secondly, even though all solutions are found, there remains the problem of selecting optimal solutions from them. For these reasons, it is very important to find new methods and new working space that could simplify the study. In this paper we prove that the framework of Lie algebroids is more suitable than the cotangent bundle for the study of driftless control affine systems with holonomic distribution.

Lie algebroids are also important issues in physics and mechanics since the extension of Lagrangian and Hamiltonian systems to their entity [9, 10, 12, 24] and catching the Poisson structure [17]. They are also related to optimization theory [14, 15, 18]. They have such a flexibility that holonomy of orbit foliation carried on them [6]. Thus Lie algebroids are strong assorted structures to assemble the physics and mechanics notions on them. For good details about penetration of Lie algebroids, see [23].

The notion of Lie algebroids was first introduced by Pradines [19]. Research on this field has been continued by mathematicians with various purposes up to now. Lie algebroids are studied pure or in relation with other subjects [1–3, 11, 14–16, 22]. Precisely, a Lie algebroid is a vector bundle with the property that its sections involve a real Lie algebra. Each section is anchored on a vector field, by means of a linear bundle map named as anchor map, which is further supposed to induce a Lie algebra homomorphism. Especially, when the base manifold M is a point, a Lie algebroid reduces to a Lie algebra. The simplest examples of Lie algebroids are the zero bundle over M which is denoted by M and tangent bundle over M with identity as anchor map which is denoted by TM. Then the tangent bundle is a special case of Lie algebroid structure. Therefore, a Lie algebroid is a generalization of a Lie algebra and tangent bundle.

The second motivation of this paper is the study of some geometric structures on Lie algebroids, such as (non)-linear connections, torsion and curvature. These structures can be useful in the study of optimal solutions behavior of control problems. In many situations we cannot find the exact solution of the optimal control problem, but using the geometry of the space, we can find information on their local or global behavior. Indeed, if the geodesics curves in the framework of Lie algebroids, which are also the optimal trajectories of control systems, are situated on a manifold with positive constant curvature, then the geodesics focus and the negative curvature spreads geodesics out, these never intersect again.

The Berwald connection is a well-known concept in Finsler geometry. This connection can be characterized as the unique good vertical connection with a minimal amount of metric compatibility and the most vanishing of the torsion. Also, this connection can be regarded as associated primarily with the geodesic spray of the energy metric, and its metrical properties as consequences of those of the geodesic spray. The construction of the Berwald connection from the geodesic spray is in turn a particular case of a more general construction which associates, in a unique way, a certain linear connection with an arbitrary non-linear connection on the tangent bundle of a differentiable manifold, which is called a Berwald-type connection [5]. Using Berwald-type connections we can construct other linear connections, which are called Yano-type connections. The Yano connection is a special case of a Yano-type connection that is used to construct the Douglas tensor. This tensor always vanishes in affinely connected manifolds (hence, in particular, in Riemannian manifolds). So, it has a typically "non-Riemannian" character. The importance of the Douglas tensor can be readily realized. Those Finsler manifolds which have vanishing projective Weyl tensor and Douglas tensor are just the solutions of the Hilbert's fourth problem [21]. The third aim of the paper is the study of the Berwald-type and Yano-type linear connections, Douglas tensor and their properties. Moreover, the study of the $\rho_{\pounds}$-covariant derivatives in $\pi^*\pi$ (in particular, Berwald and Yano derivatives), theirs torsions and curvatures and the study of their properties are the other aims of the paper.

This paper is organized as follows. In Section 2, we briefly review general definitions on Lie algebroids and we recall some geometric objects on these spaces such as vertical lifts, complete lifts, almost complex structures and sprays. In Section 3, considering two classes of distinguished connections, namely Berwald-type and Yano-type connections, we introduce Douglas tensor of Berwald endomorphism and we obtain some properties of this tensor. In Section 4, we construct $\rho_{\pounds}$-covariant derivatives in $\pi^*\pi$ as the generalization of covariant derivative in $\pi^*\pi$ to $\pounds^{\pi}E$. Moreover, Berwald and Yano derivatives as two important classes of $\rho_{\pounds}$-covariant derivatives in $\pi^*\pi$ are introduced in this section.

In the last section, we present an example from optimal control theory. We prove that the framework of Lie algebroids is more useful than the cotangent space in order to apply the Pontryagin Maximum Principle and find the optimal solution. Also, we calculate in this case some geometric structures as semispray, nonlinear connection, curvature, Berwald connection and Douglas tensor. We prove that the solutions of control system are integral curves of the spray. Finally, we find the optimal solution of the control system, using the framework of a Lie algebroid, which is in this case an integrable distribution of the tangent bundle.

2. Basic concepts on Lie algebroids

Let E be a vector bundle of rank n over a manifold M of dimension m and $\pi : E \to M$ be the vector bundle projection. Denote by $\Gamma(E)$ the $C^\infty(M)$-module of sections of $\pi : E \to M$. A *Lie algebroid structure* $([\cdot,\cdot]_E, \rho)$ on E is a Lie bracket $[\cdot,\cdot]_E$ on the space $\Gamma(E)$ and a bundle map $\rho : E \to TM$, called the anchor map, such that if we also denote by $\rho : \Gamma(E) \to \chi(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map then

$$[X, fY]_E = f[X,Y]_E + \rho(X)(f)Y, \quad \forall X, Y \in \Gamma(E), \ \forall f \in C^\infty(M).$$

Moreover, we have the relations

$$[\rho(X), \rho(Y)] = \rho([X,Y]_E).$$

Then triple $(E, [\cdot,\cdot]_E, \rho)$ is called a *Lie algebroid* over M.

Trivial examples of Lie algebroids are real Lie algebras of finite dimension, the tangent bundle TM of an arbitrary manifold M and an integrable distribution of TM.

If we take local coordinates (x^i) on M and a local basis $\{e_\alpha\}$ of sections of E, then we have the corresponding local coordinates $(\mathbf{x}^i, \mathbf{y}^\alpha)$ on E, where $\mathbf{x}^i = x^i \circ \pi$ and $\mathbf{y}^\alpha(u)$ is the α-th coordinate of $u \in E$ in the given basis. Such coordinates determine local functions ρ^i_α, $L^\gamma_{\alpha\beta}$ on M which contain the local information of the Lie algebroid structure, and accordingly they are called the structure functions of the Lie algebroid. They are given by

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}, \quad [e_\alpha, e_\beta]_E = L^\gamma_{\alpha\beta} e_\gamma,$$

such that

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma L^\gamma_{\alpha\beta}, \quad \sum_{(\alpha,\beta,\gamma)} \left[\rho^i_\alpha \frac{\partial L^\nu_{\beta\gamma}}{\partial x^i} + L^\nu_{\alpha\mu} L^\mu_{\beta\gamma} \right] = 0,$$

which are usually called the structure equations.

For a function f on M, one defines its vertical lift $f^\vee$ on E by $f^\vee(u) = f(\pi(u))$ for $u \in E$. Now, let X be a section of E. Then, we can consider the vertical lift of X as the vector field on E given by $X^\vee(u) = X(\pi(u))^\vee_u$, $u \in E$, where ${}^\vee_u : E_{\pi(u)} \to T_u(E_{\pi(u)})$ is the canonical isomorphism between the vector spaces $E_{\pi(u)}$ and $T_u(E_{\pi(u)})$. Let $\{e_\alpha\}$ be a basis of sections of E, then we have $e^\vee_\alpha = \frac{\partial}{\partial \mathbf{y}^\alpha}$. Also, using it we deduce that if $X = X^\alpha e_\alpha \in \Gamma(E)$, then the vertical lift $X^\vee$ has the local expression $X^\vee = (X^\alpha \circ \pi) \frac{\partial}{\partial \mathbf{y}^\alpha}$.

The complete lift of a smooth function $f \in C^\infty(M)$ into $C^\infty(E)$ is the smooth function

$$f^c : E \longrightarrow \mathbb{R}, \quad f^c(u) = d^E f(u) = \rho(u) f,$$

where d^E is the differential of a function on E (see, [10, 11] for more details). In the local basis we have $f^c|_{\pi^{-1}(U)} = \mathbf{y}^\alpha((\rho^i_\alpha \frac{\partial f}{\partial x^i}) \circ \pi)$.

Also, the complete lift X^c of a section X on E is the unique vector field on E given by [7], [8]:

$$X^c = \left\{ (X^\alpha \rho^i_\alpha) \circ \pi \right\} \frac{\partial}{\partial \mathbf{x}^i} + \mathbf{y}^\beta \left\{ \left(\rho^j_\beta \frac{\partial X^\alpha}{\partial x^j} - X^\gamma L^\alpha_{\gamma\beta} \right) \circ \pi \right\} \frac{\partial}{\partial \mathbf{y}^\alpha}. \tag{2.1}$$

2.1. *The Prolongation of a Lie algebroid*

Let $\pounds^\pi E$ be the subset of $E \times TE$ defined by $\pounds^\pi E = \{(u,z) \in E \times TE \mid \rho(u) = \pi_*(z)\}$ and denote by $\pi_\pounds : \pounds^\pi E \to E$ the mapping given by $\pi_\pounds(u,z) = \pi_E(z)$, where $\pi_E : TE \to E$ is the natural projection. Then $(\pounds^\pi E, \pi_\pounds, E)$ is a vector bundle over E of rank $2n$. Indeed, the total space of the prolongation is the total space of the pull-back of $\pi_* : TE \to TM$ by the anchor map ρ.

We introduce the vertical subbundle

$$V\pounds^\pi E = \operatorname{Ker} \tau_\pounds = \{(u,z) \in \pounds^\pi E \mid \tau_\pounds(u,z) = 0\},$$

where $\tau_\pounds : \pounds^\pi E \to E$ is the projection onto the first factor, i.e., $\tau_\pounds(u,z) = u$. Therefore an element of $V\pounds^\pi E$ is of the form $(0,z) \in E \times TE$ such that $\pi_*(z) = 0$ which is called vertical. Since $\pi_*(z) = 0$ and $\operatorname{Ker} \pi_* = VE$ ($\pi_* : TE \to TM$), then we deduce that if $(0,z)$ is vertical then z is a vertical vector on E.

For a local basis $\{e_\alpha\}$ of sections of E and coordinates $(\mathbf{x}^i, \mathbf{y}^\alpha)$ on E, we have local coordinates $(\mathbf{x}^i, \mathbf{y}^\alpha, k^\alpha, z^\alpha)$ on $\pounds^\pi E$ given as follows. If (u,z) is an element of $\pounds^\pi E$, then by using $\rho(u) = \pi_*(z)$,

z has the form

$$z = ((\rho^i_\alpha u^\alpha)\circ\pi)\frac{\partial}{\partial \mathbf{x}^i}\bigg|_v + z^\alpha\frac{\partial}{\partial \mathbf{y}^\alpha}\bigg|_v, \quad z\in T_vE.$$

The local basis $\{\mathscr{X}_\alpha, \mathscr{V}_\alpha\}$ of sections of $£^\pi E$ associated to the coordinate system is given by

$$\mathscr{X}_\alpha(v) = \left(e_\alpha(\pi(v)), (\rho^i_\alpha\circ\pi)\frac{\partial}{\partial \mathbf{x}^i}\bigg|_v\right), \quad \mathscr{V}_\alpha(v) = \left(0, \frac{\partial}{\partial \mathbf{y}^\alpha}\bigg|_v\right). \tag{2.2}$$

If V is a section of $£^\pi E$ which in coordinates writes

$$V(x,y) = (\mathbf{x}^i, \mathbf{y}^\alpha, Z^\alpha(x,y), V^\alpha(x,y)),$$

then the expression of V in terms of base $\{\mathscr{X}_\alpha, \mathscr{V}_\alpha\}$ is [10]

$$V = Z^\alpha\mathscr{X}_\alpha + V^\alpha\mathscr{V}_\alpha.$$

The vertical lift X^V and the complete lift X^C of a section $X\in\Gamma(E)$ as the sections of $£^\pi E\to E$ are given by

$$X^V(u) = (0, X^\vee(u)), \quad X^C(u) = (X(\pi(u)), X^c(u)), \quad u\in E.$$

It is known that $X^\vee$ and X^C have the following coordinate expressions [13]:

$$X^V = (X^\alpha\circ\pi)\mathscr{V}_\alpha, \quad X^C = (X^\alpha\circ\pi)\mathscr{X}_\alpha + \mathbf{y}^\beta\left[\left(\rho^j_\beta\frac{\partial X^\alpha}{\partial x^j} - X^\gamma L^\alpha_{\gamma\beta}\right)\circ\pi\right]\mathscr{V}_\alpha, \tag{2.3}$$

where $X = X^\alpha e_\alpha\in\Gamma(E)$. In particular we have

$$e^V_\alpha = \mathscr{V}_\alpha. \tag{2.4}$$

Here, we consider the anchor map $\rho_£ : £^\pi E\to TE$ defined by $\rho_£(u,z) = z$ and the bracket $[\cdot,\cdot]_£$ satisfying the relations

$$[X^V, Y^V]_£ = 0, \quad [X^V, Y^C]_£ = [X,Y]^V_E, \quad [X^C, Y^C]_£ = [X,Y]^C_E, \tag{2.5}$$

for $X, Y\in\Gamma(E)$. Then this vector bundle $(£^\pi E, \pi_£, E)$ is a Lie algebroid with structure $([\cdot,\cdot]_£, \rho_£)$.

2.2. *A setting for semispray on $£^\pi E$*

A section of π along smooth map $f : N\to M$ is a smooth map $\sigma : N\to E$ such that $\pi\circ\sigma = f$. The set of sections of π along f will be denoted by $\Gamma_f(\pi)$. Then there is a canonical isomorphism between $\Gamma(f^*\pi)$ and $\Gamma_f(\pi)$ (see [20]). Now we consider the pullback bundle $\pi^*\pi = (\pi^*E, pr_1, E)$

of the vector bundle (E,π,M), where

$$\pi^*E := E\times_M E := \{(u,v)\in E\times E \mid \pi(u)=\pi(v)\},$$

and pr_1 is the projection map onto the first component. The fibres of $\pi^*\pi$ are the n-dimensional real vector spaces $\{u\}\times E_{\pi(u)}\cong E_{\pi(u)}$, and so any section in $\Gamma(\pi^*\pi)$ is of the form

$$\overline{X}: u\in E\to \overline{X}(u)=(u,\underline{X}(u)),$$

where $\underline{X}: E\to E$ is a smooth map such that $\pi\circ\underline{X}=\pi$. In these terms, the map

$$\overline{X}\in\Gamma(\pi^*\pi)\to\underline{X}\in\Gamma_\pi(\pi),$$

is an isomorphism of $C^\infty(E)$-modules. Therefore we have $\Gamma(\pi^*\pi)\cong\Gamma_\pi(\pi)$. In $\Gamma(\pi^*\pi)$, there is a distinguished section

$$\delta: u\in E\to\delta(u)=(u,u)\in\pi^*E, \tag{2.6}$$

that is called *the canonical section along* π. This section corresponds to the identity map 1_E under the isomorphism $\Gamma_\pi(\pi)\cong\Gamma(\pi^*\pi)$. For any section X on E, the map

$$\widehat{X}: E\to\pi^*E,$$

defined by $\widehat{X}(u)=(u,X\circ\pi(u))$ is a section of $\pi^*\pi$, called the lift of X into $\Gamma(\pi^*\pi)$. $\widehat{X}$ may be identified with the map $X\circ\pi: E\to E$. It is easy to see that, $\{\widehat{X}|X\in\Gamma(E)\}$ generates locally the $C^\infty(E)$-module $\Gamma(\pi^*\pi)$.

Here, we consider the exact sequence

$$0\longrightarrow\pi^*(E)\stackrel{\mathbf{i}}{\longrightarrow}\pounds^\pi E\stackrel{\mathbf{j}}{\longrightarrow}\pi^*(E)\longrightarrow 0, \tag{2.7}$$

with $\mathbf{j}(u,z)=(\pi_E(z),Id(u))=(v,u)$, $z\in T_vE$, and $\mathbf{i}(u,v)=(0,v_u^\vee)$. Function $J=\mathbf{i}\circ\mathbf{j}:\pounds^\pi E\to\pounds^\pi E$ is called the *vertical endomorphism* (*almost tangent structure*) of $\pounds^\pi E$. This endomorphism has the locally expression

$$J=\mathscr{V}_\alpha\otimes\mathscr{X}^\alpha, \tag{2.8}$$

and it has the following properties

$$\mathrm{Im}J=\mathrm{Im}\,\mathbf{i}=V\pounds^\pi E,\quad \mathrm{Ker}J=\mathrm{Ker}\,\mathbf{j}=V\pounds^\pi E,\quad J\circ J=0.$$

Let δ be the canonical section along π given by (2.6). Then section C given by

$$C:=\mathbf{i}\circ\delta,$$

is called *Liouville* or *Euler* section. The Liouville section C has the coordinate expression $C=\mathbf{y}^\alpha\mathscr{V}_\alpha$ with respect to $\{\mathscr{X}_\alpha,\mathscr{V}_\alpha\}$. Moreover, we have

$$\text{(i) } [J,C]^{F-N}_\pounds=J,\quad \text{(ii) } [X^V,C]^{F-N}_\pounds=X^V,\quad \text{(iii) } JC=0, \tag{2.9}$$

where $[\cdot,\cdot]^{F-N}_\pounds$ is the generalized Frölicher-Nijenhuis bracket on $\pounds^\pi E$ (see [13], for more details).

A section $\widetilde{X}$ of vector bundle $(\pounds^\pi E,\pi_\pounds,E)$ is said to be homogeneous of degree r, where r is an integer, if $[C,\widetilde{X}]_\pounds=(r-1)\widetilde{X}$. Moreover, $\widetilde{f}\in C^\infty(E)$ is said to be homogeneous of degree r if

$\pounds_C^{\pounds}\widetilde{f}=\rho_{\pounds}(C)(\widetilde{f})=r\widetilde{f}$. It is known that a real valued smooth function $\widetilde{f}$ on E is homogeneous of degree r if and only if $\mathbf{y}^\alpha\frac{\partial\widetilde{f}}{\partial\mathbf{y}^\alpha}=r\widetilde{f}$ [13]. A section S of the vector bundle $(\pounds^\pi E,\pi_{\pounds},E)$ is said to be a semispray if it satisfies the condition $J(S)=C$. Moreover, if S is homogeneous of degree 2, i.e., $[C,S]_{\pounds}=S$, then we call it spray. It is known that $S=A^\alpha\mathscr{X}_\alpha+S^\alpha\mathscr{V}_\alpha$ is a spray on $\pounds^\pi E$ if and only if $A^\alpha=\mathbf{y}^\alpha$ and $2S^\beta=\mathbf{y}^\alpha\frac{\partial S^\beta}{\partial\mathbf{y}^\alpha}$. In [13], the first author proved that if S_1 is a spray on $\pounds^\pi E$ and $\widetilde{f}:E\to\mathbb{R}$ is a homogeneous function of degree 1 on $E-\{0\}$, then $S_2=S_1+\widetilde{f}C$ is a spray on $\pounds^\pi E$. This is said to be *projective change of* S_1 *by* $\widetilde{f}$.

A function $\mathbf{h}:\pounds^\pi E\to\pounds^\pi E$ is called a *horizontal endomorphism* if $\mathbf{h}\circ\mathbf{h}=\mathbf{h}$, $\mathrm{Ker}\,\mathbf{h}=V\pounds^\pi E$ and $\mathbf{h}$ is smooth on $\overset{\circ}{\pounds^\pi E}=\pounds^\pi E-\{0\}$. Also, $\mathbf{v}:=Id-\mathbf{h}$ is called the vertical projector associated to $\mathbf{h}$. It is known that $\mathbf{h}$ has the following locally expression:

$$\mathbf{h}=(\mathscr{X}_\beta+\mathscr{B}^\alpha_\beta\mathscr{V}_\alpha)\otimes\mathscr{X}^\beta.$$

Setting $H\pounds^\pi E:=\mathrm{Im}\,\mathbf{h}$ we have the following splitting for $\pounds^\pi E$:

$$\pounds^\pi E=V\pounds^\pi E\oplus H\pounds^\pi E. \tag{2.10}$$

Moreover, we get

$$\mathrm{Im}\,\mathbf{v}=\mathrm{Ker}\,J=V\pounds^\pi E,\quad \mathrm{Ker}\,\mathbf{h}=\mathrm{Im}\,J=\mathrm{Ker}\,J=\mathrm{Im}\,\mathbf{v}=V\pounds^\pi E,$$

$$\mathbf{h}J=\mathbf{h}\mathbf{v}=J\mathbf{v}=0,\ \ \mathbf{v}\circ\mathbf{v}=\mathbf{v},\ \ \mathbf{v}\mathbf{h}=0,\ \ J\mathbf{h}=J=\mathbf{v}J,$$

and

$$\mathbf{j}\circ\mathbf{h}=\mathbf{j}, \tag{2.11}$$

where $\mathbf{j}:\pounds^\pi E\to E\times_M E$ is the map introduced in (2.7).

Let $\mathbf{h}$ be a horizontal endomorphism on $\pounds^\pi E$. Then $\mathbf{H}=[\mathbf{h},C]^{F-N}_{\pounds}:\pounds^\pi E\to\pounds^\pi E$ is called the *tension* of $\mathbf{h}$. If $\mathbf{H}=0$, then $\mathbf{h}$ is called homogeneous. Using the above definition $\mathbf{H}$ has the coordinate expression

$$\mathbf{H}=(\mathscr{B}^\alpha_\beta-\mathbf{y}^\gamma\frac{\partial\mathscr{B}^\alpha_\beta}{\partial\mathbf{y}^\gamma})\mathscr{V}_\alpha\otimes\mathscr{X}^\beta.$$

It is known that $\mathbf{h}$ is homogeneous if and if $\mathscr{B}^\alpha_\beta=\mathbf{y}^\gamma\frac{\partial\mathscr{B}^\alpha_\beta}{\partial\mathbf{y}^\gamma}$. The *weak torsion* of $\mathbf{h}$ is defined by $t=[J,\mathbf{h}]^{F-N}_{\pounds}\in\Gamma(\pounds^\pi E)$. It is known that, t has the following coordinate expression:

$$t=\frac{1}{2}t^\gamma_{\alpha\beta}\mathscr{X}^\alpha\wedge\mathscr{X}^\beta\otimes\mathscr{V}_\gamma, \tag{2.12}$$

where

$$t^\gamma_{\alpha\beta}:=\frac{\partial\mathscr{B}^\gamma_\beta}{\partial\mathbf{y}^\alpha}-\frac{\partial\mathscr{B}^\gamma_\alpha}{\partial\mathbf{y}^\beta}-(L^\gamma_{\alpha\beta}\circ\pi). \tag{2.13}$$

The curvature of a horizontal endomorphism $\mathbf{h}$ is defined by $\Omega=-N_{\mathbf{h}}$, where $N_{\mathbf{h}}$ is the Nijenhuis tensor of $\mathbf{h}$ given by

$$N_{\mathbf{h}}(\widetilde{X},\widetilde{Y})=[\mathbf{h}\widetilde{X},\mathbf{h}\widetilde{Y}]_{\pounds}-\mathbf{h}[\mathbf{h}\widetilde{X},\widetilde{Y}]_{\pounds}-\mathbf{h}[\widetilde{X},\mathbf{h}\widetilde{Y}]_{\pounds}+\mathbf{h}[\widetilde{X},\widetilde{Y}]_{\pounds},\quad\forall\widetilde{X},\widetilde{Y}\in\Gamma(\pounds^\pi E).$$

For sections $\widetilde{X}$ and $\widetilde{Y}$ of $\pounds^\pi E$ we have

$$\Omega(\widetilde{X},\widetilde{Y}) = \Omega(\mathbf{h}\widetilde{X},\mathbf{h}\widetilde{Y}) = -\mathbf{v}[\mathbf{h}\widetilde{X},\mathbf{h}\widetilde{Y}]_\pounds.$$

Moreover, the curvature Ω has the following coordinate expression:

$$\Omega = -\frac{1}{2}R^\gamma_{\alpha\beta}\mathscr{X}^\alpha\wedge\mathscr{X}^\beta\otimes\mathscr{V}_\gamma, \tag{2.14}$$

where

$$R^\gamma_{\alpha\beta} = (\rho^i_\alpha\circ\pi)\frac{\partial\mathscr{B}^\gamma_\beta}{\partial\mathbf{x}^i} - (\rho^i_\beta\circ\pi)\frac{\partial\mathscr{B}^\gamma_\alpha}{\partial\mathbf{x}^i} + \mathscr{B}^\lambda_\alpha\frac{\partial\mathscr{B}^\gamma_\beta}{\partial\mathbf{y}^\lambda} - \mathscr{B}^\lambda_\beta\frac{\partial\mathscr{B}^\gamma_\alpha}{\partial\mathbf{y}^\lambda} + (L^\lambda_{\beta\alpha}\circ\pi)\mathscr{B}^\gamma_\lambda. \tag{2.15}$$

If S is an arbitrary semispray of $\pounds^\pi E$, then $\overline{S} = \mathbf{h}S$ is also a semispray of $\pounds^\pi E$ which does not depend on the choice of S. $\overline{S}$ is called the *semispray associated* to $\mathbf{h}$. Moreover, if $\mathbf{h}$ is homogeneous, then the semispray associated to $\mathbf{h}$ is a spray [13]. Let S be the semispray associated to $\mathbf{h}$. Then the almost complex structure $F : \pounds^\pi E \to \pounds^\pi E$ given by $F := \mathbf{h}[S,\mathbf{h}]^{F-N}_\pounds - J$ is called *the almost complex structure induced by* $\mathbf{h}$. The following relations hold

$$\text{(i)}\ F\circ J = \mathbf{h},\qquad \text{(ii)}\ F\circ\mathbf{h} = -J,\qquad \text{(iii)}\ J\circ F = \mathbf{v},\qquad \text{(iv)}\ F\circ\mathbf{v} = \mathbf{h}\circ F.$$

Also, F has the following coordinate expression

$$F = -(\mathscr{B}^\gamma_\alpha(\mathscr{X}_\gamma + \mathscr{B}^\beta_\gamma\mathscr{V}_\beta) + \mathscr{V}_\alpha)\otimes\mathscr{X}^\alpha + (\mathscr{X}_\alpha + \mathscr{B}^\beta_\alpha\mathscr{V}_\beta)\otimes\mathscr{V}^\alpha.$$

Considering $\mathscr{H} := F\circ\mathbf{i} : E\times_M E\to\pounds^\pi E$ and $\mathscr{V} := \mathbf{j}\circ F : \pounds^\pi E\to E\times_M E$, the following sequence is a double short exact sequence

$$0 \rightleftarrows \pi^*E \underset{\mathscr{V}}{\overset{\mathbf{i}}{\rightleftarrows}} \pounds^\pi E \underset{\mathscr{H}}{\overset{\mathbf{j}}{\rightleftarrows}} \pi^*E \rightleftarrows 0.$$

Furthermore, we have

$$\text{(i)}\ \mathbf{h} = \mathscr{H}\circ\mathbf{j},\qquad \text{(ii)}\ \mathbf{v} = \mathbf{i}\circ\mathscr{V}.$$

We define the *horizontal endomorphism generated by a semispray* S by $\mathbf{h}_S := \frac{1}{2}(1_{\pounds^\pi E} + [J,S]^{F-N}_\pounds)$. $\mathbf{h}_S$ has the coordinate expression

$$\mathbf{h}_S = (\mathscr{X}_\alpha + \mathscr{B}^\gamma_\alpha\mathscr{V}_\gamma)\otimes\mathscr{X}^\alpha, \tag{2.16}$$

where

$$\mathscr{B}^\gamma_\alpha = \frac{1}{2}\left(\frac{\partial S^\gamma}{\partial\mathbf{y}^\alpha} - \mathbf{y}^\beta(L^\gamma_{\alpha\beta}\circ\pi)\right). \tag{2.17}$$

It is known that the horizontal endomorphism generated by a semispray S is torsion-free (see [13], Th. 4.20). We also have $\mathbf{H}_S = \frac{1}{2}[[C,S]_\pounds - S, J]^{F-N}_\pounds$, where $\mathbf{H}_S$ is the tension of $\mathbf{h}_S$. Moreover, the tension $\mathbf{H}_S$ has the following coordinate expression [13]

$$\mathbf{H}_S = \frac{1}{2}\left(\frac{\partial S^\alpha}{\partial\mathbf{y}^\beta} - \mathbf{y}^\gamma\frac{\partial^2 S^\alpha}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\beta}\right)\mathscr{V}_\alpha\otimes\mathscr{X}^\beta.$$

The horizontal endomorphism generated by an spray is called *Berwald endomorphism* and we denote it by $\mathbf{h}^B_S$.

Lemma 2.1 ([13]). *Let $\mathbf{h}_S$ be the horizontal endomorphism generated by a semispray S. Then the semispray associated by $\mathbf{h}_S$ is $\frac{1}{2}(S+[C,S]_{£})$. Moreover the spray associated by $\mathbf{h}_S$ is S and $\mathbf{h}_S$ is homogeneous.*

Let $\mathbf{h}$ be a horizontal endomorphism on $£^{\pi}E$. We consider the map

$$X \in \Gamma(E) \to X^h := \mathbf{h}X^C \in H£^{\pi}E,$$

and we call it *horizontal lift b* $\mathbf{h}$. If $X = X^{\alpha}e_{\alpha}$, then we have

$$X^h = (X^{\alpha}\circ\pi)(\mathscr{X}_{\alpha} + \mathscr{B}^{\beta}_{\alpha}\mathscr{V}_{\beta}).$$

The following relations hold for any $X, Y \in \Gamma(E)$.

$$\text{(i)}\ JX^h = X^V, \qquad \text{(ii)}\ \mathbf{h}[X^h, Y^h]_{£} = [X,Y]^h_E, \qquad \text{(iii)}\ [X,Y]^V_E = J[X^h,Y^h]_{£}.$$

Setting $\delta_{\alpha} = e^h_{\alpha} = \mathscr{X}_{\alpha} + \mathscr{B}^{\beta}_{\alpha}\mathscr{V}_{\beta}$, it is easy to see that $\{\delta_{\alpha}\}$ generate a basis of $H£^{\pi}E$ and $\{\delta_{\alpha}, \mathscr{V}_{\alpha}\}$ is a local basis of $£^{\pi}E$ adapted to splitting (2.10) which is called adapted basis (see [13], for more details). The dual adapted basis is $\{\mathscr{X}^{\alpha}, \delta\mathscr{V}^{\alpha}\}$, where

$$\delta\mathscr{V}^{\alpha} = \mathscr{V}^{\alpha} - \mathscr{B}^{\alpha}_{\beta}\mathscr{X}^{\beta}.$$

The Lie brackets of the adapted basis $\{\delta_{\alpha}, \mathscr{V}_{\alpha}\}$ are

$$[\delta_{\alpha}, \delta_{\beta}]_{£} = (L^{\gamma}_{\alpha\beta}\circ\pi)\delta_{\gamma} + R^{\gamma}_{\alpha\beta}\mathscr{V}_{\gamma}, \quad [\delta_{\alpha}, \mathscr{V}_{\beta}]_{£} = -\frac{\partial\mathscr{B}^{\gamma}_{\alpha}}{\partial\mathbf{y}^{\beta}}\mathscr{V}_{\gamma}, \quad [\mathscr{V}_{\alpha}, \mathscr{V}_{\beta}]_{£} = 0,$$

where $R^{\gamma}_{\alpha\beta}$ is given by (2.15). It is easy to check that $\mathbf{h}$ and F have the following coordinate expressions with respect to adapted basis

$$\mathbf{h} = \delta_{\alpha}\otimes\mathscr{X}^{\alpha}, \quad F = -\mathscr{V}_{\alpha}\otimes\mathscr{X}^{\alpha} + \delta_{\alpha}\otimes\delta\mathscr{V}^{\alpha}.$$

3. The Douglas tensor of a Berwald endomorphism

In [13], the first author introduced two distinguished connections, namely Berwald-type and Yano-type connections and he studied some properties of them. In this section, using these connections, we introduce the Douglas tensor of the Berwald endomorphism and study properties of it.

A linear connection on a Lie algebroid $(E, [\cdot,\cdot]_E, \rho)$ is a map

$$D : \Gamma(E)\times\Gamma(E) \to \Gamma(E)$$

which satisfies the rules

$$D_{fX+Y}Z = fD_XY + D_YZ,$$
$$D_X(fY+Z) = (\rho(X)f)Y + fD_XY + D_XZ,$$

for any function $f \in C^{\infty}(M)$ and $X, Y, Z \in \Gamma(E)$. Let D be a linear connection on $£^{\pi}E$ and $\mathbf{h}$ be a horizontal endomorphism on $£^{\pi}E$. Then $(D,\mathbf{h})$ is called a distinguished connection (or d-connection) on $£^{\pi}E$, if

i) D is reducible, i.e., $D\mathbf{h} = 0$, which gives us

$$D_{\widetilde{X}}\mathbf{h}\widetilde{Y} = \mathbf{h}D_{\widetilde{X}}\widetilde{Y} \in H£^{\pi}E, \quad D_{\widetilde{X}}\mathbf{v}\widetilde{Y} = \mathbf{v}D_{\widetilde{X}}\widetilde{Y} \in V£^{\pi}E, \quad \forall\widetilde{X}, \widetilde{Y} \in \Gamma(£^{\pi}E),$$

ii) D is almost complex, i.e., $DF = 0$,
where F is the almost complex structure associated to $\mathbf{h}$. It is known that $DJ = 0$ [13]. Moreover, it is shown that a d-connection $(D, \mathbf{h})$ has the following coordinate expressions

$$D_{\delta_\alpha}\delta_\beta = D_{\delta_\alpha}\mathscr{V}_\beta = F^\gamma_{\alpha\beta}\mathscr{V}_\gamma, \quad D_{\mathscr{V}_\alpha}\delta_\beta = D_{\mathscr{V}_\alpha}\mathscr{V}_\beta = C^\gamma_{\alpha\beta}\mathscr{V}_\gamma.$$

Let $(D, \mathbf{h})$ be a d-connection. Then

$$\begin{cases} D^{\mathbf{h}} : \Gamma(\pounds^\pi E) \times \Gamma(\pounds^\pi E) \to \Gamma(\pounds^\pi E) \\ \qquad (\widetilde{X}, \widetilde{Y}) \mapsto D^{\mathbf{h}}_{\widetilde{X}}\widetilde{Y} := D_{\mathbf{h}\widetilde{X}}\widetilde{Y} \end{cases},$$

and

$$\begin{cases} D^{\mathbf{v}} : \Gamma(\pounds^\pi E) \times \Gamma(\pounds^\pi E) \to \Gamma(\pounds^\pi E) \\ \qquad (\widetilde{X}, \widetilde{Y}) \mapsto D^{\mathbf{v}}_{\widetilde{X}}\widetilde{Y} := D_{\mathbf{v}\widetilde{X}}\widetilde{Y} \end{cases},$$

are called $\mathbf{h}$-covariant derivative and $\mathbf{v}$-covariant derivative, respectively. Moreover,

$$\begin{cases} \mathbf{h}^*(DC) : \Gamma(\pounds^\pi E) \to \Gamma(\pounds^\pi E) \\ \quad \widetilde{X} \mapsto DC(\mathbf{h}\widetilde{X}) := D_{\mathbf{h}\widetilde{X}}C \end{cases},$$

and

$$\begin{cases} \mathbf{v}^*(DC) : \Gamma(\pounds^\pi E) \to \Gamma(\pounds^\pi E) \\ \quad \widetilde{X} \mapsto DC(\mathbf{v}\widetilde{X}) := D_{\mathbf{v}\widetilde{X}}C \end{cases},$$

are called $\mathbf{h}$-deflection and $\mathbf{v}$-deflection of $(D, \mathbf{h})$, respectively. It is easy to see that $\mathbf{h}^*(DC)$ and $\mathbf{v}^*(DC)$ have the following coordinate expressions:

$$\mathbf{h}^*(DC) = (\mathscr{B}^\gamma_\alpha + \mathbf{y}^\beta F^\gamma_{\alpha\beta})\mathscr{V}_\gamma \otimes \mathscr{X}^\alpha, \qquad \mathbf{v}^*(DC) = (\delta^\gamma_\alpha + \mathbf{y}^\beta C^\gamma_{\alpha\beta})\mathscr{V}_\gamma \otimes \delta\mathscr{V}^\alpha,$$

where δ^γ_α is the Kronecker symbol. Also, the curvature tensor field K of D is completely determined by the following (see [13], for more details)

$$\begin{aligned} R(\widetilde{X}, \widetilde{Y})\widetilde{Z} &:= K(h\widetilde{X}, h\widetilde{Y})J\widetilde{Z}, \\ P(\widetilde{X}, \widetilde{Y})\widetilde{Z} &:= K(h\widetilde{X}, J\widetilde{Y})J\widetilde{Z}, \\ Q(\widetilde{X}, \widetilde{Y})\widetilde{Z} &:= K(J\widetilde{X}, J\widetilde{Y})J\widetilde{Z}, \end{aligned}$$

where R, P and Q are horizontal, mixed and vertical curvatures, respectively.

For a d-connection $(D, \mathbf{h})$ on $\pounds^\pi E$, the tensor field

$$\begin{cases} P_{ric} : \Gamma(\pounds^\pi E) \times \Gamma(\pounds^\pi E) \to C^\infty(E), \\ (\widetilde{X}, \widetilde{Y}) \to tr[F \circ (\widetilde{Z} \to P(\widetilde{Y}, \widetilde{Z})\widetilde{X})], \end{cases}$$

is called the mixed Ricci tensor of d-connection $(D, \mathbf{h})$, where F is the almost complex structure associated to $\mathbf{h}$. It is known that the mixed Ricci tensor of $(D, \mathbf{h})$ has the coordinate expression $P_{ric} = P_{\alpha\beta}\mathscr{X}^\alpha \otimes \mathscr{X}^\beta$, where $P_{\alpha\beta} = P_{\alpha\beta\gamma}{}^{\beta}$ [13].

Here we present two examples of d-connections on $\pounds^\pi E$. These connections which are called Berwald-type and Yano-type connections have studied by the first author in [13].

Let $\mathbf{h}$ be a horizontal endomorphism on $\pounds^\pi E$. The couple $(\overset{\text{B}}{D}, \mathbf{h})$, where

$$\begin{cases} \overset{\text{B}}{D} : \Gamma(\pounds^\pi E) \times \Gamma(\pounds^\pi E) \to \Gamma(\pounds^\pi E), \\ \qquad (\widetilde{X}, \widetilde{Y}) \mapsto \overset{\text{B}}{D}_{\widetilde{X}} \widetilde{Y}, \end{cases}$$

is given by

$$\overset{\text{B}}{D}_{\widetilde{X}} \widetilde{Y} := \mathbf{h}F[\mathbf{h}\widetilde{X}, J\widetilde{Y}]_\pounds + \mathbf{v}[\mathbf{h}\widetilde{X}, \mathbf{v}\widetilde{Y}]_\pounds + \mathbf{h}[\mathbf{v}\widetilde{X}, \widetilde{Y}]_\pounds + J[\mathbf{v}\widetilde{X}, F\widetilde{Y}]_\pounds,$$

is called the *Berwald-type* connection. If, in particular, $\mathbf{h}^B_S$ is a Berwald endomorphism, then we call $(\overset{\text{B}}{D}, \mathbf{h}^B_S)$ a *Berwald connection*. It is easy to see that

$$\overset{\text{B}}{D}_{\delta_\alpha} \mathcal{V}_\beta = \overset{\text{B}}{D}_{\delta_\alpha} \delta_\beta = -\frac{\partial \mathscr{B}^\gamma_\alpha}{\partial y^\beta} \delta_\gamma, \quad \overset{\text{B}}{D}_{\mathcal{V}_\alpha} \delta_\beta = \overset{\text{B}}{D}_{\mathcal{V}_\alpha} \mathcal{V}_\beta = 0.$$

We have also

$$\begin{aligned} \overset{\text{B}}{R}{}_{\alpha\beta\gamma}^{\ \ \lambda} &= -(\rho^i_\alpha \circ \pi)\frac{\partial^2 \mathscr{B}^\lambda_\beta}{\partial \mathbf{x}^i \partial \mathbf{y}^\gamma} - \mathscr{B}^\mu_\alpha \frac{\partial^2 \mathscr{B}^\lambda_\beta}{\partial \mathbf{y}^\mu \partial \mathbf{y}^\gamma} + (\rho^i_\beta \circ \pi)\frac{\partial^2 \mathscr{B}^\lambda_\alpha}{\partial \mathbf{x}^i \partial \mathbf{y}^\gamma} + \mathscr{B}^\mu_\beta \frac{\partial^2 \mathscr{B}^\lambda_\alpha}{\partial \mathbf{y}^\mu \partial \mathbf{y}^\gamma} \\ &\quad + \frac{\partial \mathscr{B}^\mu_\beta}{\partial \mathbf{y}^\gamma}\frac{\partial \mathscr{B}^\lambda_\alpha}{\partial \mathbf{y}^\mu} - \frac{\partial \mathscr{B}^\mu_\alpha}{\partial \mathbf{y}^\gamma}\frac{\partial \mathscr{B}^\lambda_\beta}{\partial \mathbf{y}^\mu} + (L^\mu_{\alpha\beta} \circ \pi)\frac{\partial \mathscr{B}^\lambda_\mu}{\partial \mathbf{y}^\gamma}, \qquad (3.1) \\ \overset{\text{B}}{P}{}_{\alpha\beta\gamma}^{\ \ \lambda} &= \frac{\partial^2 \mathscr{B}^\lambda_\alpha}{\partial \mathbf{y}^\beta \partial \mathbf{y}^\gamma}, \qquad (3.2) \\ \overset{\text{B}}{S}{}_{\alpha\beta\gamma}^{\ \ \lambda} &= 0, \qquad (3.3) \end{aligned}$$

where $\overset{\text{B}}{R}{}_{\alpha\beta\gamma}^{\ \ \lambda}$, $\overset{\text{B}}{P}{}_{\alpha\beta\gamma}^{\ \ \lambda}$ and $\overset{\text{B}}{S}{}_{\alpha\beta\gamma}^{\ \ \lambda}$ are the coefficients of the horizontal, mixed and vertical curvatures of d-connection $(\overset{\text{B}}{D}, \mathbf{h})$, respectively. Here, let $\mathbf{h}$ be a homogeneous and torsion-free horizontal endomorphism on $\pounds^\pi E$ and $\overset{\text{B}}{P}_{ric}$ be the mixed Ricci tensor of the Berwald-type connection $(\overset{\text{B}}{D}, \mathbf{h})$. We consider the following d-connection

$$\begin{aligned} \overset{\text{Y}}{D}_{\mathbf{v}\widetilde{X}} \mathbf{v}\widetilde{Y} &= \overset{\text{B}}{D}_{\mathbf{v}\widetilde{X}} \mathbf{v}\widetilde{Y}, \\ \overset{\text{Y}}{D}_{\mathbf{h}\widetilde{X}} \mathbf{v}\widetilde{Y} &= \overset{\text{B}}{D}_{\mathbf{h}\widetilde{X}} \mathbf{v}\widetilde{Y} + \frac{1}{n+1} \overset{\text{B}}{P}_{ric} (\widetilde{X}, F\widetilde{Y})C, \\ \overset{\text{Y}}{D}_{\mathbf{v}\widetilde{X}} \mathbf{h}\widetilde{Y} &= \overset{\text{B}}{D}_{\mathbf{v}\widetilde{X}} \mathbf{h}\widetilde{Y}, \\ \overset{\text{Y}}{D}_{\mathbf{h}\widetilde{X}} \mathbf{h}\widetilde{Y} &= \overset{\text{B}}{D}_{\mathbf{h}\widetilde{X}} \mathbf{h}\widetilde{Y} + \frac{1}{n+1} \overset{\text{B}}{P}_{ric} (\widetilde{X}, \widetilde{Y})FC, \end{aligned}$$

where $n = \operatorname{rank} E$. This d-connection is said to be the *Yano-type connection induced by* $\mathbf{h}$. If, in particular, $\mathbf{h}^B_S$ is a Berwald endomorphism, then we call it a *Yano connection*. It is easy to see that the Yano-type connection has the following coordinate expression:

$$\overset{\text{Y}}{D}_{\delta_\alpha}\mathscr{V}_\beta = \overset{\text{Y}}{D}_{\delta_\alpha}\delta_\beta = \left(\frac{1}{n+1}\frac{\partial^2\mathscr{B}^\lambda_\alpha}{\partial y^\lambda\partial y^\beta}y^\gamma - \frac{\partial\mathscr{B}^\gamma_\alpha}{\partial y^\beta}\right)\delta_\gamma, \quad \overset{\text{Y}}{D}_{\mathscr{V}_\alpha}\delta_\beta = \overset{\text{Y}}{D}_{\mathscr{V}_\alpha}\mathscr{V}_\beta = 0.$$

It is known that the coefficients of mixed curvature of Yano-type connection are

$$\overset{\text{Y}}{P}{}^{\lambda}_{\alpha\beta\gamma} = \frac{\partial^2\mathscr{B}^\lambda_\alpha}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma} - \frac{1}{n+1}\left(\frac{\partial^2\mathscr{B}^\mu_\alpha}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\gamma}\delta^\lambda_\beta + \frac{\partial^3\mathscr{B}^\mu_\alpha}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\mu\partial\mathbf{y}^\gamma}\mathbf{y}^\lambda\right). \tag{3.4}$$

Definition 3.1. Let $\boldsymbol{h}^B_S$ be a Berwald endomorphism on the manifold $\pounds^\pi E$. If $(\overset{\text{Y}}{D}, \boldsymbol{h}^B_S)$ is the Yano connection induced by $\boldsymbol{h}^B_S$ and $\overset{\text{Y}}{P}$ is the mixed curvature of $\overset{\text{Y}}{D}$, then the tensor

$$\mathbf{D} = \overset{\text{Y}}{P} - \frac{1}{2}(\overset{\text{Y}}{P}_{ric}\otimes J + J\otimes\overset{\text{Y}}{P}_{ric}),$$

is said to be the Douglas tensor of the Berwald endomorphism.

Using (2.8) and (3.4), the Douglas tensor $\mathbf{D}$ has the following coordinate expression:

$$\mathbf{D} = D^\lambda_{\alpha\beta\gamma}\mathscr{V}_\lambda\otimes\mathscr{X}^\alpha\otimes\mathscr{X}^\beta\otimes\mathscr{X}^\gamma, \tag{3.5}$$

where

$$D^\lambda_{\alpha\beta\gamma} = \frac{\partial^2\mathscr{B}^\lambda_\alpha}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma} - \frac{1}{n+1}\left(\frac{\partial^2\mathscr{B}^\mu_\alpha}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\gamma}\delta^\lambda_\beta + \frac{\partial^3\mathscr{B}^\mu_\beta}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\mu\partial\mathbf{y}^\gamma}y^\lambda + \frac{\partial^2\mathscr{B}^\mu_\alpha}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\beta}\delta^\lambda_\gamma + \frac{\partial^2\mathscr{B}^\mu_\beta}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\gamma}\delta^\lambda_\alpha\right). \tag{3.6}$$

Since the Berwald endomorphism is homogeneous and torsion-free, then from the above equation we deduce $D^\lambda_{\alpha\beta\gamma} = D^\lambda_{\beta\alpha\gamma} = D^\lambda_{\gamma\beta\alpha}$, i.e., $\mathbf{D}$ is symmetric.

Proposition 3.1. *Let* $\boldsymbol{D}$ *be the Douglas tensor of a Berwald endomorphism. Then* $i_S\boldsymbol{D} = 0$ *and* $\boldsymbol{D}_{ric} = 0$.

Proof. Let $\widetilde{X} = \widetilde{X}^\alpha\delta_\alpha + \widetilde{X}^{\overline{\alpha}}\mathscr{V}_\alpha$ and $\widetilde{Y} = \widetilde{Y}^\gamma\delta_\gamma + \widetilde{Y}^{\overline{\gamma}}\mathscr{V}_\gamma$. Since $\mathbf{D}$ is symmetric then using (3.5) we get

$$(i_S\mathbf{D})(\widetilde{X},\widetilde{Y}) = \mathbf{D}(\widetilde{X},S)\widetilde{Y} = \mathbf{y}^\beta\widetilde{X}^\alpha\widetilde{Y}^\gamma D^\lambda_{\alpha\beta\gamma}.$$

Moreover, since $\mathbf{h}^B_S$ is homogeneous, we have $y^\beta\frac{\partial\mathscr{B}^\lambda_\alpha}{\partial y^\beta} = \mathscr{B}^\lambda_\alpha$. Differentiating this with respect to y^γ we obtain

$$y^\beta\frac{\partial^2\mathscr{B}^\lambda_\alpha}{\partial y^\beta\partial y^\gamma} = 0. \tag{3.7}$$

Differentiating (3.7) with respect to y^μ gives us

$$y^\beta\frac{\partial^3\mathscr{B}^\lambda_\alpha}{\partial y^\mu\partial y^\beta\partial y^\gamma} = -\frac{\partial^2\mathscr{B}^\lambda_\alpha}{\partial y^\mu\partial y^\gamma}. \tag{3.8}$$

But using (3.7) and (3.8) we deduce $\mathbf{y}^\beta D^\lambda_{\alpha\beta\gamma} = 0$. Therefore we have $i_S\mathbf{D} = 0$. Now we prove the second part of assertion. It is easy to see that

$$\mathbf{D}_{ric} = D_{\alpha\gamma}\mathscr{X}^\alpha\otimes\mathscr{X}^\gamma,$$

where $D_{\alpha\gamma} = D^\lambda_{\alpha\lambda\gamma}$. But using (3.8) and (3.6) we deduce $D_{\alpha\gamma} = 0$ and consequently $\mathbf{D}_{ric} = 0$. □

Theorem 3.1. *The Douglas tensor of a Berwald endomorphism is invariant under the projective changes of the associated spray.*

Proof. Let $\mathbf{h}_S^B$ be a Berwald endomorphism on $\pounds^\pi E$ with associated spray S and $\mathbf{D}$ be the Douglas tensor of $\mathbf{h}_S^B$. Also, let $\overline{S}$ be the projective change of S by $\widetilde{f}$. Then $\overline{S}$ generates a Berwald endomorphism $\overline{\mathbf{h}}_S^B$. Denote by $\overline{\mathbf{D}}$ the Douglas tensor of $\overline{\mathbf{h}}_S^B$. If $S = \mathbf{y}^\alpha \mathscr{X}_\alpha + S^\alpha \mathscr{V}_\alpha$ and $\overline{S} = \mathbf{y}^\alpha \mathscr{X}_\alpha + \overline{S}^\alpha \mathscr{V}_\alpha$, then $\overline{S} = S + \widetilde{f}C$ gives us

$$\overline{S}^\alpha = S^\alpha + y^\alpha \widetilde{f}. \tag{3.9}$$

From (2.16) and (2.17), $\mathbf{h}_S^B$ and $\overline{\mathbf{h}}_S^B$ have the following coordinate expressions:

$$\mathbf{h}_S^B = (\mathscr{X}_\alpha + \mathscr{B}_\alpha^\gamma \mathscr{V}_\gamma) \otimes \mathscr{X}^\alpha, \quad \overline{\mathbf{h}}_S^B = (\mathscr{X}_\alpha + \overline{\mathscr{B}}_\alpha^\gamma \mathscr{V}_\gamma) \otimes \mathscr{X}^\alpha,$$

where

$$\mathscr{B}_\alpha^\gamma = \frac{1}{2}\left(\frac{\partial S^\gamma}{\partial \mathbf{y}^\alpha} - \mathbf{y}^\beta (L_{\alpha\beta}^\gamma \circ \pi)\right), \quad \overline{\mathscr{B}}_\alpha^\gamma = \frac{1}{2}\left(\frac{\partial \overline{S}^\gamma}{\partial \mathbf{y}^\alpha} - \mathbf{y}^\beta (L_{\alpha\beta}^\gamma \circ \pi)\right). \tag{3.10}$$

Using (3.9) and (3.10) we get

$$\overline{\mathscr{B}}_\alpha^\gamma = \mathscr{B}_\alpha^\gamma + \widetilde{f}_\alpha^\gamma, \tag{3.11}$$

where $\widetilde{f}_\alpha^\gamma = \frac{1}{2}(\widetilde{f}\delta_\alpha^\gamma + \mathbf{y}^\gamma \frac{\partial \widetilde{f}}{\partial \mathbf{y}^\alpha})$. If we denote by $D_{\alpha\beta\gamma}^\lambda$ and $\overline{D}_{\alpha\beta\gamma}^\lambda$ the coefficients of $\mathbf{D}$ and $\overline{\mathbf{D}}$, respectively, then using (3.6) and (3.11) we get

$$\overline{D}_{\alpha\beta\gamma}^\lambda = D_{\alpha\beta\gamma}^\lambda + \frac{\partial \widetilde{f}_\alpha^\lambda}{\partial \mathbf{y}^\beta \partial \mathbf{y}^\gamma} - \frac{1}{n+1}\Bigg(\frac{\partial^2 \widetilde{f}_\alpha^\mu}{\partial \mathbf{y}^\mu \partial \mathbf{y}^\gamma}\delta_\beta^\lambda + \frac{\partial^3 \widetilde{f}_\beta^\mu}{\partial \mathbf{y}^\alpha \partial \mathbf{y}^\mu \partial \mathbf{y}^\gamma}\mathbf{y}^\lambda + \frac{\partial^2 \widetilde{f}_\alpha^\mu}{\partial \mathbf{y}^\mu \partial \mathbf{y}^\beta}\delta_\gamma^\lambda + \frac{\partial^2 \widetilde{f}_\beta^\mu}{\partial \mathbf{y}^\mu \partial \mathbf{y}^\gamma}\delta_\alpha^\lambda\Bigg). \tag{3.12}$$

Since $\widetilde{f}$ is homogeneous of degree 1, then we can obtain

$$\frac{\partial^3 \widetilde{f}}{\partial \mathbf{y}^\beta \partial \mathbf{y}^\gamma \partial \mathbf{y}^\alpha}\mathbf{y}^\beta = -\frac{\partial^2 \widetilde{f}}{\partial \mathbf{y}^\gamma \partial \mathbf{y}^\alpha}.$$

The above equation and direct calculation give us

$$\frac{\partial^2 \widetilde{f}_\alpha^\lambda}{\partial \mathbf{y}^\beta \partial \mathbf{y}^\gamma} = \frac{1}{2}\left(\frac{\partial^2 \widetilde{f}}{\partial \mathbf{y}^\beta \partial \mathbf{y}^\gamma}\delta_\alpha^\lambda + \frac{\partial^2 \widetilde{f}}{\partial \mathbf{y}^\gamma \partial \mathbf{y}^\alpha}\delta_\beta^\lambda + \frac{\partial^2 \widetilde{f}}{\partial \mathbf{y}^\beta \partial \mathbf{y}^\alpha}\delta_\gamma^\lambda + \frac{\partial^3 \widetilde{f}}{\partial \mathbf{y}^\beta \partial \mathbf{y}^\gamma \partial \mathbf{y}^\alpha}\mathbf{y}^\lambda\right), \tag{3.13}$$

$$\frac{\partial^2 \widetilde{f}_\alpha^\mu}{\partial \mathbf{y}^\mu \partial \mathbf{y}^\gamma} = \frac{1}{2}(n+1)\frac{\partial^2 \widetilde{f}}{\partial \mathbf{y}^\gamma \partial \mathbf{y}^\alpha}, \tag{3.14}$$

$$\frac{\partial^2 \widetilde{f}_\alpha^\mu}{\partial \mathbf{y}^\mu \partial \mathbf{y}^\beta} = \frac{1}{2}(n+1)\frac{\partial^2 \widetilde{f}}{\partial \mathbf{y}^\beta \partial \mathbf{y}^\alpha}, \tag{3.15}$$

$$\frac{\partial^2 \widetilde{f}_\beta^\mu}{\partial \mathbf{y}^\mu \partial \mathbf{y}^\gamma} = \frac{1}{2}(n+1)\frac{\partial^2 \widetilde{f}}{\partial \mathbf{y}^\gamma \partial \mathbf{y}^\beta}, \tag{3.16}$$

$$\frac{\partial^3 \widetilde{f}_\beta^\mu}{\partial \mathbf{y}^\alpha \partial \mathbf{y}^\mu \partial \mathbf{y}^\gamma} = \frac{1}{2}(n+1)\frac{\partial^3 \widetilde{f}}{\partial \mathbf{y}^\alpha \partial \mathbf{y}^\gamma \partial \mathbf{y}^\beta}. \tag{3.17}$$

Setting (3.13)–(3.17) in (3.12) we obtain $\overline{D}^{\lambda}_{\alpha\beta\gamma} = D^{\lambda}_{\alpha\beta\gamma}$, i.e., $\overline{\mathbf{D}} = \mathbf{D}$. □

4. $\rho_{\pounds}$-covariant derivatives in $\pi^*\pi$

In this section, we introduce $\rho_{\pounds}$-covariant derivatives in $\pi^*\pi$ and we investigate geometric properties of $\rho_{\pounds}$-covariant derivatives in $\pi^*\pi$ like torsion and partial curvature. Results are in a deep relation with the Berwald derivative.

We can deduce the following double-exact short sequence from the double-exact short sequence (2.7)

$$0 \rightleftarrows \Gamma(\pi^*\pi) \underset{\overline{\mathscr{V}}}{\overset{\bar{\mathbf{i}}}{\rightleftarrows}} \Gamma(\pounds^{\pi}E) \underset{\overline{\mathscr{H}}}{\overset{\bar{\mathbf{j}}}{\rightleftarrows}} \Gamma(\pi^*\pi) \rightleftarrows 0,$$

such that for every $\overline{X} \in \Gamma(\pi^*\pi)$ and $\xi \in \Gamma(\pounds^{\pi}E)$ the following hold

$$\bar{\mathbf{i}}(\overline{X}) := \mathbf{i} \circ \overline{X}, \qquad \bar{\mathbf{j}}(\xi) := \mathbf{j} \circ \xi, \qquad \overline{\mathscr{H}}(\overline{X}) := \mathscr{H} \circ \overline{X}, \qquad \overline{\mathscr{V}}(\xi) := \mathscr{V} \circ \xi.$$

Proposition 4.1. *Let X belongs to $\Gamma(E)$. Then we have the following*

(i) $\bar{\mathbf{i}}(\widehat{X}) = X^V$,
(ii) $\bar{\mathbf{j}}(X^V) = 0$,
(iii) $\bar{\mathbf{j}}(X^C) = \widehat{X}$,
(iv) $\overline{\mathscr{H}}(\widehat{X}) = X^h$,
(v) $\overline{\mathscr{V}}(X^V) = \widehat{X}$,
(vi) $\overline{\mathscr{V}}(X^h) = 0$.

Proof. Let $u \in E$. Then we have

$$\begin{aligned}\bar{\mathbf{i}}(\widehat{X})(u) &= \mathbf{i} \circ \widehat{X} = \mathbf{i}(u, X(\pi(u))) = (0, X(\pi(u))^{\vee}_{u}) \\ &= (0, X^{\vee}(u)) = X^V(u),\end{aligned}$$

that gives us the first one. The second one is obvious. For the third, since $J(X^C) = X^V$, then we have $\mathbf{i} \circ \mathbf{j}(X^C) = X^V = \mathbf{i}(\widehat{X})$. Because $\mathbf{i}$ is injective, $\mathbf{j}(X^C) = \widehat{X}$ and consequently $\bar{\mathbf{j}}(X^C) = \widehat{X}$. For the forth, we can deduce

$$\overline{\mathscr{H}}(\widehat{X}) = \mathscr{H} \circ \widehat{X} = F \circ \mathbf{i} \circ \widehat{X} = F \circ X^V = F \circ J(X^C) = \mathbf{h}(X^C) = X^h.$$

Using (2.11), the fifth equation proves as follows

$$\overline{\mathscr{V}}(X^V) = \mathbf{j} \circ F \circ X^V = \mathbf{j} \circ F \circ J(X^C) = \mathbf{j} \circ \mathbf{h}(X^C) = \mathbf{j} \circ X^C = \widehat{X}.$$

The last one is obvious. □

Remark 4.1. The mapping $\bar{\boldsymbol{i}}$ is an isomorphism between $\Gamma(\pi^*\pi)$ and $\Gamma(V\pounds^{\pi}E)$. Thus every section of $V(\pounds^{\pi}E)$ can be shown like $\bar{\boldsymbol{i}}\overline{X}$ where $\overline{X} \in \Gamma(\pi^*\pi)$. Moreover, since $\bar{\boldsymbol{j}}$ is surjective, then each element of $\Gamma(\pi^*\pi)$ has the format $\bar{\boldsymbol{j}}(\xi)$, where $\xi \in \Gamma(\pounds^{\pi}(E))$.

Definition 4.1. Operator ∇^{ν} with properties

(i) $\nabla^{\nu}_{\overline{X}}\widetilde{f} := \rho_{\pounds}(\bar{\boldsymbol{i}}\overline{X})\widetilde{f}$,
(ii) $\nabla^{\nu}_{\overline{X}}\overline{Y} := \bar{\boldsymbol{j}}[\bar{\boldsymbol{i}}\overline{X}, \overline{\mathscr{H}}\overline{Y}]_{\pounds}$,
(iii) $(\nabla^{\nu}_{\overline{X}}\overline{\alpha})(\overline{Y}) := \rho_{\pounds}(\bar{\boldsymbol{i}}\overline{X})(\overline{\alpha}(\overline{Y})) - \overline{\alpha}(\nabla^{\nu}_{\overline{X}}\overline{Y})$,

is called the *canonical* $\boldsymbol{v}$*-covariant differential*, where $\widetilde{f} \in C^\infty(E)$, $\overline{X}, \overline{Y} \in \Gamma(\pi^*\pi)$, $\overline{\alpha} \in \Omega^1(\pi)$.

Remark 4.2. The second condition of the above definition is independent of choosing $\overline{\mathscr{H}}$. Indeed since $\bar{\boldsymbol{j}}$ is surjective, there is some $\widetilde{Y} \in \Gamma(\pounds^\pi E)$, such that $\overline{Y} = \bar{\boldsymbol{j}}\widetilde{Y}$. Thus

$$\nabla^{\boldsymbol{v}}_{\overline{X}} \bar{\boldsymbol{j}}\widetilde{Y} = \bar{\boldsymbol{j}}\,[\bar{\boldsymbol{i}}\overline{X}, \overline{\mathscr{H}} \circ \bar{\boldsymbol{j}}\widetilde{Y}\,]_\pounds = \bar{\boldsymbol{j}}[\bar{\boldsymbol{i}}\overline{X}, \boldsymbol{h}\widetilde{Y}]_\pounds.$$

But $[\bar{\boldsymbol{i}}\overline{X}, \boldsymbol{v}\widetilde{Y}\,]_\pounds$ is vertical. Therefore

$$\nabla^{\boldsymbol{v}}_{\overline{X}} \bar{\boldsymbol{j}}\widetilde{Y} = \bar{\boldsymbol{j}}\,[\bar{\boldsymbol{i}}\overline{X}, \widetilde{Y}\,]_\pounds.$$

Let $\overline{A} \in \mathscr{T}^k_l(\pi)$. Then we define

$$\begin{aligned}(\nabla^{\mathbf{v}}_{\overline{X}}\overline{A})(\overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_k}, \overline{X_1}, \overline{X_2}, \ldots, \overline{X_l}) := {} & \rho_\pounds(\bar{\mathbf{i}}\overline{X})(\overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_k}, \overline{X_1}, \overline{X_2}, \ldots, \overline{X_l}) \\ & - \sum_{i=1}^k \overline{A}(\overline{\alpha_1}, \ldots, \nabla^{\mathbf{v}}_{\overline{X}}\overline{\alpha_i}, \ldots, \overline{\alpha_k}, \overline{X_1}, \overline{X_2}, \ldots, \overline{X_l}) \\ & - \sum_{i=1}^l \overline{A}(\overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_k}, \overline{X_1}, \ldots, \nabla^{\mathbf{v}}_{\overline{X}}\overline{X_i}, \ldots, \overline{X_l}).\end{aligned}$$

Moreover, for $\overline{A} \in \mathscr{T}^k_l(\pi)$, the tensor field $\nabla^{\mathbf{v}}\overline{A} \in \mathscr{T}^k_{l+1}(\pi)$ is defined by the following rule

$$(\nabla^{\mathbf{v}}\overline{A})(\overline{X}, \overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_k}, \overline{X_1}, \overline{X_2}, \ldots, \overline{X_l}) := (\nabla^{\mathbf{v}}_{\overline{X}}\overline{A})(\overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_k}, \overline{X_1}, \overline{X_2}, \ldots, \overline{X_l}).$$

Definition 4.2. Let $\widetilde{f}$ be a smooth function on E. Then the tensor field

$$\nabla^{\boldsymbol{v}}\nabla^{\boldsymbol{v}}\widetilde{f} := \nabla^{\boldsymbol{v}}(\nabla^{\boldsymbol{v}}\widetilde{f}) \in \mathscr{T}^0_2(\pi),$$

is said to be hessian of $\widetilde{f}$.

Proposition 4.2. *$\widetilde{f} \in C^\infty(E)$ is homogeneous of degree* 1 *if and only if* $\nabla^{\mathbf{v}}_\delta \widetilde{f} = \widetilde{f}$.

Proof. Let $\widetilde{f}$ be a homogeneous function of degree 1 on E. Then we have $\rho_\pounds(C)\widetilde{f} = \widetilde{f}$. Thus

$$\nabla^{\mathbf{v}}_\delta \widetilde{f} = \rho_\pounds(\bar{\mathbf{i}}\delta)\widetilde{f} = \rho_\pounds(\mathbf{i} \circ \delta)\widetilde{f} = \rho_\pounds(C)\widetilde{f} = \widetilde{f}.$$

From the above equation, also we can deduce the converse of assertion. □

Proposition 4.3. *Let X and Y be sections of E and $\widetilde{f} \in C^\infty(E)$. Then*

$$\nabla^{\mathbf{v}}\nabla^{\mathbf{v}}\widetilde{f}(\widehat{X}, \widehat{Y}) = \rho_\pounds(X^V)(\rho_\pounds(Y^V)\widetilde{f}).$$

Moreover, the hessian of $\widetilde{f}$ is symmetric.

Proof. Using the definition of hessian of $\widetilde{f}$, (i) of proposition 4.1 and (iii) of definition 4.1 we get

$$\begin{aligned}\nabla^{\mathbf{v}}\nabla^{\mathbf{v}}\widetilde{f}(\widehat{X},\widehat{Y}) &= (\nabla^{\mathbf{v}}_{\widehat{X}}(\nabla^{\mathbf{v}}\widetilde{f}))(\widehat{Y}) = \rho_{\pounds}(\bar{\mathbf{i}}\widehat{X})((\nabla^{\mathbf{v}}\widetilde{f})(\widehat{Y})) - \nabla^{\mathbf{v}}\widetilde{f}(\nabla_{\widehat{X}}\widehat{Y})\\ &= \rho_{\pounds}(\bar{\mathbf{i}}\widehat{X})(\rho_{\pounds}(\bar{\mathbf{i}}\widehat{Y})\widetilde{f}) - \rho_{\pounds}(\bar{\mathbf{i}}(\nabla_{\widehat{X}}\widehat{Y}))\widetilde{f}\\ &= \rho_{\pounds}(X^V)(\rho_{\pounds}(Y^V)\widetilde{f}) - \rho_{\pounds}(\bar{\mathbf{i}}(\nabla_{\widehat{X}}\widehat{Y}))\widetilde{f}. \end{aligned} \tag{4.1}$$

But using (i), (ii) and (iv) of proposition 4.1 we deduce

$$\nabla^{\mathbf{v}}_{\widehat{X}}\widehat{Y} = \bar{\mathbf{j}}\,[\bar{\mathbf{i}}\widehat{X}, \overline{\mathscr{H}}\widehat{Y}]_{\pounds} = \bar{\mathbf{j}}\,[X^V, Y^h]_{\pounds} = 0,$$

because $[X^V, Y^h]_{\pounds} \in \Gamma(V\pounds^{\pi}E)$. Plugging the above equation into (4.1) implies the first part of assertion. Now, we prove the second part of the assertion. Since $[X^V, Y^V]_{\pounds} = 0$, then using the first part of assertion we get

$$\begin{aligned}\nabla^{\mathbf{v}}\nabla^{\mathbf{v}}\widetilde{f}(\widehat{X},\widehat{Y}) &= \rho_{\pounds}(X^V)(\rho_{\pounds}(Y^V)\widetilde{f}) = \rho_{\pounds}([X^V, Y^V]_{\pounds})(\widetilde{f}) + \rho_{\pounds}(Y^V)(\rho_{\pounds}(X^V)\widetilde{f})\\ &= \rho_{\pounds}(Y^V)(\rho_{\pounds}(X^V)\widetilde{f}) = \nabla^{\mathbf{v}}\nabla^{\mathbf{v}}\widetilde{f}(\widehat{Y},\widehat{X}).\end{aligned}$$

□

Proposition 4.4. *Let $\widetilde{f} \in C^{\infty}(E)$ be a homogeneous function of degree* 1. *Then*

$$\nabla^{\mathbf{v}}_{\delta}(\nabla^{\mathbf{v}}\nabla^{\mathbf{v}}\widetilde{f}) = -\nabla^{\mathbf{v}}\nabla^{\mathbf{v}}\widetilde{f}.$$

Proof. Setting $\overline{A} = \nabla^{\mathbf{v}}\nabla^{\mathbf{v}}\widetilde{f}$, we must show $\nabla^{\mathbf{v}}_{\delta}\overline{A} = -\overline{A}$. Let X and Y be sections of E. Then we have

$$(\nabla^{\mathbf{v}}_{\delta}\overline{A})(\widehat{X},\widehat{Y}) = \rho_{\pounds}(\bar{\mathbf{i}}\delta)\overline{A}(\widehat{X},\widehat{Y}) - \overline{A}(\nabla^{\mathbf{v}}_{\delta}\widehat{X},\widehat{Y}) - \overline{A}(\widehat{X},\nabla^{\mathbf{v}}_{\delta}\widehat{Y}). \tag{4.2}$$

But using (ii) of Definition 4.1, we deduce

$$\nabla^{\mathbf{v}}_{\delta}\widehat{X} = \bar{\mathbf{j}}[\bar{\mathbf{i}}\delta, \overline{\mathscr{H}}\widehat{Y}]_{\pounds} = \bar{\mathbf{j}}[C, Y^h]_{\pounds} = 0,$$

because $[C, Y^h]_{\pounds} \in \Gamma(V\pounds^{\pi}E)$. Similarly we have $\nabla^{\mathbf{v}}_{\delta}\widehat{Y} = 0$. Therefore (4.2) reduces to the following

$$(\nabla^{\mathbf{v}}_{\delta}\overline{A})(\widehat{X},\widehat{Y}) = \rho_{\pounds}(C)\overline{A}(\widehat{X},\widehat{Y}) = \rho_{\pounds}(C)\Big(\rho_{\pounds}(X^V)(\rho_{\pounds}(Y^V)\widetilde{f})\Big). \tag{4.3}$$

In other hand, using (ii) of (2.9) we get

$$\begin{aligned}\overline{A}(\widehat{X},\widehat{Y}) &= \rho_{\pounds}(X^V)(\rho_{\pounds}(Y^V)\widetilde{f}) = \rho_{\pounds}([X^V, C]_{\pounds})(\rho_{\pounds}(Y^V)\widetilde{f})\\ &= [\rho_{\pounds}(X^V), \rho_{\pounds}(C)](\rho_{\pounds}(Y^V)\widetilde{f}) = \rho_{\pounds}(X^V)\Big(\rho_{\pounds}(C)(\rho_{\pounds}(Y^V)\widetilde{f})\Big)\\ &\quad - \rho_{\pounds}(C)\Big(\rho_{\pounds}(X^V)(\rho_{\pounds}(Y^V)\widetilde{f})\Big) = \rho_{\pounds}(X^V)\Big([\rho_{\pounds}(C), \rho_{\pounds}(Y^V)]\widetilde{f}\\ &\quad + \rho_{\pounds}(Y^V)(\rho_{\pounds}(C)\widetilde{f})\Big) - \rho_{\pounds}(C)\Big(\rho_{\pounds}(X^V)(\rho_{\pounds}(Y^V)\widetilde{f})\Big)\\ &= \rho_{\pounds}(X^V)\Big(\rho_{\pounds}[C, Y^V]_{\pounds}\widetilde{f} + \rho_{\pounds}(Y^V)(\rho_L(C)\widetilde{f})\Big)\\ &\quad - \rho_{\pounds}(C)\Big(\rho_{\pounds}(X^V)(\rho_{\pounds}(Y^V)\widetilde{f})\Big).\end{aligned}$$

Since $\widetilde{f}$ is homogeneous of degree 1, then we have $\rho_\pounds(C)\widetilde{f} = \widetilde{f}$. Setting this in the above equation and using (ii) of (2.9) we get

$$\overline{A}(\widehat{X},\widehat{Y}) = -\rho_\pounds(C)\Big(\rho_\pounds(X^V)(\rho_\pounds(Y^V)\widetilde{f})\Big). \tag{4.4}$$

From (4.3) and (4.4) we have the assertion. □

Definition 4.3. Let $\boldsymbol{h}$ be a horizontal endomorphism and $\overline{\mathscr{H}}$ be a horizontal map of π associated to $\boldsymbol{h}$. Operator $\nabla^{\boldsymbol{h}}$ with properties

(i) $\nabla^{\boldsymbol{h}}_{\overline{X}}\widetilde{f} := \rho_\pounds(\overline{\mathscr{H}}\overline{X})\widetilde{f}$,
(ii) $\nabla^{\boldsymbol{h}}_{\overline{X}}\overline{Y} := \overline{\mathscr{V}}[\overline{\mathscr{H}}\overline{X}, \bar{\mathbf{i}}\overline{Y}]_\pounds$,
(iii) $(\nabla^{\boldsymbol{h}}_{\overline{X}}\overline{\alpha})(\overline{Y}) := \rho_\pounds(\overline{\mathscr{H}}\overline{X})(\overline{\alpha}(\overline{Y})) - \overline{\alpha}(\nabla^{\boldsymbol{h}}_{\overline{X}}\overline{Y})$,

is called the canonical $\boldsymbol{h}$-covariant differential, where $\widetilde{f} \in C^\infty(E)$, $\overline{X},\overline{Y} \in \Gamma(\pi^*\pi)$, $\overline{\alpha} \in \Omega^1(\pi)$.

Lemma 4.1. *Let $\boldsymbol{H}$ be the tension of* $\mathbf{h}$ *and $\widetilde{X}$ be a section of $\pounds^\pi E$. Then*

$$(\nabla^{\mathbf{h}}\delta)(\bar{\mathbf{j}}\widetilde{X}) = \overline{\mathscr{V}}\boldsymbol{H}(\widetilde{X}). \tag{4.5}$$

Proof. Using (ii) of the above definition we get

$$(\nabla^{\mathbf{h}}\delta)(\bar{\mathbf{j}}\widetilde{X}) = \nabla^{\mathbf{h}}_{\bar{\mathbf{j}}\widetilde{X}}\delta = \overline{\mathscr{V}}[\overline{\mathscr{H}}\bar{\mathbf{j}}\widetilde{X}, \bar{\mathbf{i}}\delta]_\pounds = \overline{\mathscr{V}}[\mathbf{h}\widetilde{X}, C]_\pounds = \overline{\mathscr{V}}[\mathbf{h}, C]^{F-N}_\pounds(\widetilde{X}) = \overline{\mathscr{V}}\mathbf{H}(\widetilde{X}).$$

□

Since $\bar{\mathbf{i}}\overline{\mathscr{V}} = \mathbf{v}$, (4.5) gives us

$$\bar{\mathbf{i}}(\nabla^{\mathbf{h}}\delta)(\bar{\mathbf{j}}\widetilde{X}) = \mathbf{v}\mathbf{H}(\widetilde{X}) = \mathbf{H}(\widetilde{X}).$$

By reason of the above relation, the (1, 1) tensor field $\overline{\mathbf{H}} = \nabla^{\mathbf{h}}\delta$ is called the tension of the horizontal map $\overline{\mathscr{H}}$. Indeed, we have

$$\overline{\mathbf{H}}(\overline{X}) = \overline{\mathscr{V}}[\overline{\mathscr{H}}\overline{X}, C]_\pounds, \quad \forall \overline{X} \in \Gamma(\pi^*\pi). \tag{4.6}$$

Let $\overline{A} \in \mathscr{T}^k_l(\pi)$. Then we define

$$\begin{aligned}(\nabla^{\mathbf{h}}_{\overline{X}}\overline{A})(\overline{\alpha_1},\overline{\alpha_2},\ldots,\overline{\alpha_k},\overline{X_1},\overline{X_2},\ldots,\overline{X_l}) &:= \rho_\pounds(\overline{\mathscr{H}}\overline{X})(\overline{\alpha_1},\overline{\alpha_2},\ldots,\overline{\alpha_k},\overline{X_1},\overline{X_2},\ldots,\overline{X_l})\\ &\quad - \sum_{i=1}^k \overline{A}(\overline{\alpha_1},\ldots,\nabla^{\mathbf{h}}_{\overline{X}}\overline{\alpha_i},\ldots,\overline{\alpha_k},\overline{X_1},\overline{X_2},\ldots,\overline{X_l})\\ &\quad - \sum_{i=1}^l \overline{A}(\overline{\alpha_1},\overline{\alpha_2},\ldots,\overline{\alpha_k},\overline{X_1},\ldots,\nabla^{\mathbf{h}}_{\overline{X}}\overline{X_i},\ldots,\overline{X_l}).\end{aligned}$$

Moreover, for $\overline{A} \in \mathscr{T}^k_l(\pi)$, the tensor field $\nabla^{\mathbf{h}}\overline{A} \in \mathscr{T}^k_{l+1}(\pi)$ is defined by the following rule

$$(\nabla^{\mathbf{h}}\overline{A})(\overline{X},\overline{\alpha_1},\overline{\alpha_2},\ldots,\overline{\alpha_k},\overline{X_1},\overline{X_2},\ldots,\overline{X_l}) := (\nabla^{\mathbf{h}}_{\overline{X}}\overline{A})(\overline{\alpha_1},\overline{\alpha_2},\ldots,\overline{\alpha_k},\overline{X_1},\overline{X_2},\ldots,\overline{X_l}).$$

Definition 4.4. A map

$$\begin{cases} D : \Gamma(\pounds^\pi E) \times \Gamma(\pi^*\pi) \longrightarrow \Gamma(\pi^*\pi), \\ \qquad (\widetilde{X},\overline{Y}) \longmapsto D_{\widetilde{X}}\overline{Y}, \end{cases}$$

which satisfies

(i) $D_{f\widetilde{X}+\widetilde{Y}}\overline{Z} = fD_{\widetilde{X}}\overline{Z} + D_{\widetilde{Y}}\overline{Z}$,
(ii) $D_{\widetilde{X}}\widetilde{f}\overline{Z} = \widetilde{f}D_{\widetilde{X}}\overline{Z} + \rho_{\pounds}(\widetilde{X})(\widetilde{f})\overline{Z}$,
(iii) $D_{\widetilde{X}}(\overline{Z}+\overline{W}) = D_{\widetilde{X}}\overline{Z} + D_{\widetilde{X}}\overline{W}$,

is called a $\rho_{\pounds}$-*covariant derivative in* $\Gamma(\pi^*\pi)$.

Theorem 4.1. *Let* $\mathbf{h}$ *be a horizontal endomorphism and* $\overline{\mathscr{H}}$ *be a horizontal map of* π *associated to* $\mathbf{h}$. *Then*

$$\nabla : \Gamma(\pounds^{\pi}E) \times \Gamma(\pi^*\pi) \longrightarrow \Gamma(\pi^*\pi),$$

given by

$$\nabla_{\widetilde{X}}\overline{Y} := \nabla^{\mathbf{v}}_{\overline{\mathscr{V}}\widetilde{X}}\overline{Y} + \nabla^{\mathbf{h}}_{\mathbf{j}\widetilde{X}}\overline{Y}, \tag{4.7}$$

is a $\rho_{\pounds}$-*covariant derivative in* $\Gamma(\pi^*\pi)$, *where* $\widetilde{X} \in \Gamma(\pounds^{\pi}E)$ *and* $\overline{Y} \in \Gamma(\pi^*\pi)$.

Proof. Let $\widetilde{f} \in C^{\infty}(E)$. Then we have

$$\begin{aligned}\nabla_{\widetilde{X}}\widetilde{f}\overline{Y} &= \nabla^{\mathbf{v}}_{\overline{\mathscr{V}}\widetilde{X}}\widetilde{f}\overline{Y} + \nabla^{\mathbf{h}}_{\mathbf{j}\widetilde{X}}\widetilde{f}\overline{Y} = \rho_{\pounds}(\overline{\mathbf{i}\mathscr{V}}\widetilde{X})\widetilde{f} + \rho_{\pounds}(\overline{\mathscr{H}}\mathbf{j}\widetilde{X})\widetilde{f} + \widetilde{f}\nabla^{\mathbf{v}}_{\overline{\mathscr{V}}\widetilde{X}}\overline{Y} + \widetilde{f}\nabla^{\mathbf{h}}_{\mathbf{j}\widetilde{X}}\overline{Y}\\ &= \rho_{\pounds}(\overline{\mathbf{i}\mathscr{V}}\widetilde{X})\widetilde{f} + \rho_{\pounds}(\overline{\mathscr{H}}\mathbf{j}\widetilde{X})\widetilde{f} + \widetilde{f}\nabla_{\widetilde{X}}\overline{Y}.\end{aligned}$$

It is easy to show that $\overline{\mathbf{i}\mathscr{V}}\widetilde{X} = \mathbf{v}\widetilde{X}$ and $\overline{\mathscr{H}}\mathbf{j}\widetilde{X} = \mathbf{h}\widetilde{X}$. Therefore the above equation gives us

$$\nabla_{\widetilde{X}}\widetilde{f}\overline{Y} = \rho_{\pounds}(\mathbf{v}\widetilde{X})\widetilde{f} + \rho_{\pounds}(\mathbf{h}\widetilde{X})\widetilde{f} + \widetilde{f}\nabla_{\widetilde{X}}\overline{Y} = \rho_{\pounds}(\widetilde{X})\widetilde{f} + \widetilde{f}\nabla_{\widetilde{X}}\overline{Y}.$$

Similarly we can show $\nabla_{\widetilde{X}}(\overline{Y}+\overline{Z}) = \nabla_{\widetilde{X}}\overline{Y} + \nabla_{\widetilde{X}}\overline{Z}$ and $\nabla_{\widetilde{f}\widetilde{X}+\widetilde{Y}}\overline{Y} = \widetilde{f}\nabla_{\widetilde{X}}\overline{Z} + \nabla_{\widetilde{Y}}\overline{Z}$. Therefore ∇ is a $\rho_{\pounds}$-covariant derivative in $\Gamma(\pi^*\pi)$. ☐

The $\rho_{\pounds}$-covariant derivative ∇ introduced by the above theorem is called *Berwald derivative generated by* $\mathbf{h}$. Indeed the Berwald derivative is as follows:

$$\nabla_{\widetilde{X}}\overline{Y} = \bar{\mathbf{j}}[\mathbf{v}\widetilde{X}, \overline{\mathscr{H}}\overline{Y}]_{\pounds} + \overline{\mathscr{V}}[\mathbf{h}\widetilde{X}, \bar{\mathbf{i}}\overline{Y}]_{\pounds}, \quad \forall\widetilde{X} \in \Gamma(\pounds^{\pi}E), \quad \forall\overline{Y} \in \Gamma(\pi^*\pi).$$

Using the above equation we can obtain

$$\nabla_{X^V}\widehat{Y} = 0, \qquad \nabla_{X^h}\widehat{Y} = \overline{\mathscr{V}}[X^h, Y^V]_{\pounds}, \tag{4.8}$$

$$\nabla_{\bar{\mathbf{i}}\overline{X}}\overline{Y} = \bar{\mathbf{j}}[\bar{\mathbf{i}}\overline{X}, \overline{\mathscr{H}}\overline{Y}]_{\pounds}, \qquad \nabla_{\overline{\mathscr{H}}\overline{X}}\overline{Y} = \overline{\mathscr{V}}[\overline{\mathscr{H}}\overline{X}, \bar{\mathbf{i}}\overline{Y}]_{\pounds}, \tag{4.9}$$

where X and Y are sections of E and $\overline{X}, \overline{Y} \in \Gamma(\pi^*\pi)$.

Now we consider the local basis $\{e_{\alpha}\}$ of $\Gamma(E)$. Then $\{\widehat{e_{\alpha}}\}$ is a basis of $\Gamma(\pi^*\pi)$, where $\widehat{e_{\alpha}}(u) = (u, e_{\alpha}(\pi(u)))$, for all $u \in E$. Using (2.4), Proposition 4.1, and the definition of $\mathbf{j}$, it is easy to check that

$$\overline{\mathscr{H}}\widehat{e_\alpha}=\delta_\alpha,\quad \bar{\mathbf{i}}\widehat{e_\alpha}=\mathscr{V}_\alpha,\quad \bar{\mathbf{j}}(\delta_\alpha)=\widehat{e_\alpha},\quad \overline{\mathscr{V}}(\mathscr{V}_\alpha)=\widehat{e_\alpha}. \tag{4.10}$$

Also we deduce $\overline{\mathscr{V}}(\delta_\alpha)=0$. Therefore using the above equation, (2.4) and (4.8) we obtain

$$\begin{aligned}\nabla_{\delta_\alpha}\widehat{e_\beta}&=\overline{\mathscr{V}}\,[\,\delta_\alpha,e^V_\beta\,]_{\pounds}=\overline{\mathscr{V}}\,[\,\delta_\alpha,\mathscr{V}_\beta\,]_{\pounds}=-\frac{\partial\mathscr{B}^\gamma_\alpha}{\partial\mathbf{y}^\beta}\widehat{e_\gamma},\\ \nabla_{\mathscr{V}_\alpha}\widehat{e_\beta}&=\bar{\mathbf{j}}\,[\,\mathscr{V}_\alpha,e^h_\beta\,]_{\pounds}=\bar{\mathbf{j}}\,[\,\mathscr{V}_\alpha,\delta_\beta\,]_{\pounds}=0,\end{aligned}$$

and consequently

$$\nabla_{\widetilde{X}}\overline{Y}=\left(\widetilde{X}^\alpha\left((\rho^i_\alpha\circ\pi)\frac{\partial\overline{Y}^\beta}{\partial\mathbf{x}^i}+\mathscr{B}^\gamma_\alpha\frac{\partial\overline{Y}^\beta}{\partial\mathbf{y}^\gamma}\right)-\widetilde{X}^\alpha\overline{Y}^\gamma\frac{\partial\mathscr{B}^\beta_\alpha}{\partial\mathbf{y}^\gamma}+\widetilde{X}^{\overline{\alpha}}\frac{\partial\overline{Y}^\beta}{\partial\mathbf{y}^\alpha}\right)\widehat{e_\beta}, \tag{4.11}$$

where $\widetilde{X}=\widetilde{X}^\alpha\delta_\alpha+\widetilde{X}^{\overline{\alpha}}\mathscr{V}_\alpha\in\Gamma(\pounds^\pi E)$ and $\overline{Y}=\overline{Y}^\beta\widehat{e_\beta}\in\Gamma(\pi^*\pi)$.

Definition 4.5. A $\rho_\pounds$-covariant derivative operator D in $\Gamma(\pi^*\pi)$ is said to be associated to the horizontal map $\overline{\mathscr{H}}$ if $D\delta=\overline{\mathscr{V}}$.

Lemma 4.2. *Let ∇ be the Berwald derivative induced by* $\mathbf{h}$*. Then*

$$\nabla\delta=\overline{\boldsymbol{H}}\circ\bar{\mathbf{j}}+\overline{\mathscr{V}}. \tag{4.12}$$

Proof. Using (ii) of Definition 4.1, (ii) of Definition 4.3 and (4.6) we get

$$(\nabla\delta)(\widetilde{X})=\nabla^{\mathbf{v}}_{\overline{\mathscr{V}}\widetilde{X}}\delta+\nabla^{\mathbf{h}}_{\bar{\mathbf{j}}\widetilde{X}}\delta=\bar{\mathbf{j}}\,[\,\bar{\mathbf{i}}\overline{\mathscr{V}}\widetilde{X},\overline{\mathscr{H}}\delta\,]_{\pounds}+\overline{\mathbf{H}}(\bar{\mathbf{j}}\widetilde{X})=\bar{\mathbf{j}}\,[\,\mathbf{v}\widetilde{X},\overline{\mathscr{H}}\delta\,]_{\pounds}+\overline{\mathbf{H}}(\bar{\mathbf{j}}\widetilde{X}). \tag{4.13}$$

Now let $\widetilde{X}=\widetilde{X}^\alpha\delta_\alpha+\widetilde{X}^{\overline{\alpha}}\mathscr{V}_\alpha$. It is easy to see that $\delta=\mathbf{y}^\alpha\widehat{e_\alpha}$. Then using (4.10) we obtain

$$\bar{\mathbf{j}}\,[\,\mathbf{v}\widetilde{X},\overline{\mathscr{H}}\delta\,]_{\pounds}=\bar{\mathbf{j}}\,[\,\widetilde{X}^{\overline{\alpha}}\mathscr{V}_\alpha,\mathbf{y}^\beta\delta_\beta\,]_{\pounds}=\widetilde{X}^{\overline{\alpha}}\bar{\mathbf{j}}(\delta_\alpha)=\widetilde{X}^{\overline{\alpha}}\widehat{e_\alpha}=\overline{\mathscr{V}}(\widetilde{X}).$$

Setting the above equation in (4.13) implies (4.12). □

Proposition 4.5. *Let S be a spray on $\pounds^\pi E$ and $\mathbf{h}^B_S$ be the Berwald endomorphism generated by it. If $\overline{\mathscr{H}}$ is the horizontal map generated by $\mathbf{h}^B_S$ and ∇ is the Berwald derivative induced by $\mathbf{h}^B_S$, then* $\nabla_S\delta=0$.

Proof. From the above lemma we have

$$\nabla_S\delta=\overline{\mathbf{H}}\bar{\mathbf{j}}(S)+\overline{\mathscr{V}}(S).$$

Using (4.10) it is easy to see that $\bar{\mathbf{j}}S=\mathbf{y}^\alpha\widehat{e_\alpha}=\delta$. Thus we have

$$\nabla_S\delta=\overline{\mathbf{H}}\delta+\overline{\mathscr{V}}(S). \tag{4.14}$$

But (4.6) gives us $\overline{\mathbf{H}}\delta=\overline{\mathscr{V}}\,[\overline{\mathscr{H}}\delta,C]_{\pounds}$. On the other hand, from Lemma 2.1 we have $\mathbf{h}^B_S S=S$. Therefore we get

$$S=\mathbf{h}^B_S S=\overline{\mathscr{H}}\bar{\mathbf{j}}S=\overline{\mathscr{H}}\delta,$$

and consequently $\overline{\mathbf{H}}\delta=\overline{\mathscr{V}}[S,C]_{\pounds}$. Since S is a spray, $[S,C]_{\pounds}=-S$. Therefore $\overline{\mathbf{H}}\delta=-\overline{\mathscr{V}}(S)$. Setting this equation in (4.14) we obtain $\nabla_S\delta=0$. □

4.1. *Torsions and partial curvatures*

Let D be a $\rho_{\pounds}$-covariant derivative in $\Gamma(\pi^*\pi)$. The $(\pi^*\pi)$-valued two-forms

$$T^{\mathbf{h}}(D)(\widetilde{X},\widetilde{Y}) := D_{\widetilde{X}}\bar{\mathbf{j}}\widetilde{Y} - D_{\widetilde{Y}}\bar{\mathbf{j}}\widetilde{X} - \bar{\mathbf{j}}\,[\widetilde{X},\widetilde{Y}]_{\pounds},$$
$$T^{\mathbf{v}}(D)(\widetilde{X},\widetilde{Y}) := D_{\widetilde{X}}\overline{\mathscr{V}}\widetilde{Y} - D_{\widetilde{Y}}\overline{\mathscr{V}}\widetilde{X} - \overline{\mathscr{V}}\,[\widetilde{X},\widetilde{Y}]_{\pounds},$$

are said to be the horizontal and the vertical torsions of D, respectively, where $\widetilde{X}$ and $\widetilde{Y}$ belong to $\Gamma(\pounds^{\pi}E)$. The maps A and B given by

$$A(\overline{X},\overline{Y}) := T^{\mathbf{h}}(D)(\overline{\mathscr{H}}\overline{X},\overline{\mathscr{H}}\overline{Y}), \qquad B(\overline{X},\overline{Y}) := T^{\mathbf{h}}(D)(\overline{\mathscr{H}}\overline{X},\bar{\mathbf{i}}\overline{Y}), \tag{4.15}$$

where $\overline{X}, \overline{Y} \in \Gamma(\pi^*\pi)$, are called the $\mathbf{h}$-horizontal and the $\mathbf{h}$-mixed torsions of D (with respect to $\overline{\mathscr{H}}$), respectively. A will also be mentioned as the torsion of D, while for B we use the term Finsler torsion as well. D is said to be symmetric if $A = 0$ and B is symmetric. The maps R^1, P^1 and Q^1 given by

$$R^1(\overline{X},\overline{Y}) := T^{\mathbf{v}}(D)(\overline{\mathscr{H}}\overline{X},\overline{\mathscr{H}}\overline{Y}), \qquad P^1(\overline{X},\overline{Y}) := T^{\mathbf{v}}(D)(\overline{\mathscr{H}}\overline{X},\bar{\mathbf{i}}\overline{Y}), \tag{4.16}$$
$$Q^1(\overline{X},\overline{Y}) := T^{\mathbf{v}}(D)(\bar{\mathbf{i}}\overline{X},\bar{\mathbf{i}}\overline{Y}), \quad \forall \overline{X},\overline{Y} \in \Gamma(\pi^*\pi), \tag{4.17}$$

are called the $\mathbf{v}$-horizontal, the $\mathbf{v}$-mixed and the $\mathbf{v}$-vertical torsions of D, respectively. Using (4.9), (4.15), (4.16) and (4.17) we can obtain

Lemma 4.3. *Let D be a $\rho_{\pounds}$-covariant derivative in $\Gamma(\pi^*\pi)$. Then all of the partial torsions of the $\rho_{\pounds}$-covariant derivative operator D are tensor fields of type $(1,2)$ on $\Gamma(\pi^*\pi)$. Moreover, for any vector fields $\overline{X}$, $\overline{Y}$ belong $\Gamma(\pi^*\pi)$ we have*

$$A(\overline{X},\overline{Y}) = D_{\overline{\mathscr{H}}\overline{X}}\overline{Y} - D_{\overline{\mathscr{H}}\overline{Y}}\overline{X} - \bar{\mathbf{j}}\,[\overline{\mathscr{H}}\overline{X},\overline{\mathscr{H}}\overline{Y}]_{\pounds},$$
$$B(\overline{X},\overline{Y}) = -D_{\bar{\mathbf{i}}\overline{Y}}\overline{X} - \bar{\mathbf{j}}\,[\overline{\mathscr{H}}\overline{X},\bar{\mathbf{i}}\overline{Y}]_{\pounds} = -D_{\bar{\mathbf{i}}\overline{Y}}\overline{X} + \nabla_{\bar{\mathbf{i}}\overline{Y}}\overline{X},$$
$$R^1(\overline{X},\overline{Y}) = -\overline{\mathscr{V}}\,[\overline{\mathscr{H}}\overline{X},\overline{\mathscr{H}}\overline{Y}]_{\pounds},$$
$$P^1(\overline{X},\overline{Y}) = D_{\overline{\mathscr{H}}\overline{X}}\overline{Y} - \overline{\mathscr{V}}\,[\overline{\mathscr{H}}\overline{X},\bar{\mathbf{i}}\overline{Y}]_{\pounds} = D_{\overline{\mathscr{H}}\overline{X}}\overline{Y} - \nabla_{\overline{\mathscr{H}}\overline{X}}\overline{Y},$$
$$Q^1(\overline{X},\overline{Y}) = D_{\bar{\mathbf{i}}\overline{X}}\overline{Y} - D_{\bar{\mathbf{i}}\overline{Y}}\overline{X} - \overline{\mathscr{V}}\,[\bar{\mathbf{i}}\overline{X},\bar{\mathbf{i}}\overline{Y}]_{\pounds},$$

where ∇ is the Berwald derivative given by (4.7).

Corollary 4.1. *A $\rho_{\pounds}$-covariant derivative in $\Gamma(\pi^*E)$ is the Berwald derivative induced by a given horizontal endomorphism if and only if, its Finsler torsion and $\mathbf{v}$-mixed torsion vanish.*

Using the above lemma we get

$$A(\bar{\mathbf{j}}\widetilde{X},\bar{\mathbf{j}}\widetilde{Y}) = D_{\mathbf{h}\widetilde{X}}\bar{\mathbf{j}}\widetilde{Y} - D_{\mathbf{h}\widetilde{Y}}\bar{\mathbf{j}}\widetilde{X} - \bar{\mathbf{j}}\,[\mathbf{h}\widetilde{X},\mathbf{h}\widetilde{Y}]_{\pounds},$$
$$B(\bar{\mathbf{j}}\widetilde{X},\overline{\mathscr{V}}\widetilde{Y}) = -D_{\mathbf{v}\widetilde{Y}}\bar{\mathbf{j}}\widetilde{X} - \bar{\mathbf{j}}\,[\mathbf{h}\widetilde{X},\mathbf{v}\widetilde{Y}]_{\pounds},$$
$$-B(\bar{\mathbf{j}}\widetilde{Y},\overline{\mathscr{V}}\widetilde{X}) = D_{\mathbf{v}\widetilde{X}}\bar{\mathbf{j}}\widetilde{Y} + \bar{\mathbf{j}}\,[\mathbf{h}\widetilde{Y},\mathbf{v}\widetilde{X}]_{\pounds}.$$

Since $[\mathbf{v}\widetilde{X},\mathbf{v}\widetilde{Y}] \in \Gamma(V\pounds^{\pi}E)$, then $\bar{\mathbf{j}}[\mathbf{v}\widetilde{X},\mathbf{v}\widetilde{Y}] = 0$. Therefore summing the above equations give us

$$A(\bar{\mathbf{j}}\widetilde{X},\bar{\mathbf{j}}\widetilde{Y}) + B(\bar{\mathbf{j}}\widetilde{X},\overline{\mathscr{V}}\widetilde{Y}) - B(\bar{\mathbf{j}}\widetilde{Y},\overline{\mathscr{V}}\widetilde{X}) = D_{\widetilde{X}}\bar{\mathbf{j}}\widetilde{Y} - D_{\widetilde{Y}}\bar{\mathbf{j}}\widetilde{X} - \bar{\mathbf{j}}\,[\widetilde{X},\widetilde{Y}]_{\pounds}$$
$$= T^{\mathbf{h}}(D)(\widetilde{X},\widetilde{Y}).$$

Thus we have

Lemma 4.4. *The horizontal torsion $T^{\mathbf{h}}(D)$ is completely determined by the torsion A and the Finsler torsion B. Indeed, we have*

$$T^{\mathbf{h}}(D)(\widetilde{X},\widetilde{Y}) = A(\bar{\mathbf{j}}\widetilde{X},\bar{\mathbf{j}}\widetilde{Y}) + B(\bar{\mathbf{j}}\widetilde{X},\overline{\mathscr{V}}\widetilde{Y}) - B(\bar{\mathbf{j}}\widetilde{Y},\overline{\mathscr{V}}\widetilde{X}), \quad \forall \widetilde{X},\widetilde{Y} \in \Gamma(\pounds^{\pi}E).$$

Lemma 4.5. *Let D be a $\rho_{\pounds}$-covariant derivative in $\Gamma(\pi^*\pi)$. If D is associated to the horizontal map $\overline{\mathscr{H}}$, then for every section $\overline{X}$ of $\pi^*\pi$ we have*

$$B(\delta,\overline{X}) = 0, \quad P^1(\overline{X},\delta) = -\overline{\mathbf{H}}(\overline{X}).$$

Proof. Since D is associated to the horizontal map $\overline{\mathscr{H}}$, then $D\delta = \overline{\mathscr{V}}$. Therefore using Lemma 4.3 we get

$$\begin{aligned} B(\delta,\overline{X}) &= -D_{\bar{\mathbf{i}}\overline{X}}\delta - \bar{\mathbf{j}}\,[\overline{\mathscr{H}}\delta,\bar{\mathbf{i}}\overline{X}]_{\pounds} = -\overline{\mathscr{V}}(\bar{\mathbf{i}}\overline{X}) - \bar{\mathbf{j}}\,[\overline{\mathscr{H}}\delta,\bar{\mathbf{i}}\overline{X}]_{\pounds} \\ &= -\overline{X} - \bar{\mathbf{j}}\,[\overline{\mathscr{H}}\delta,\bar{\mathbf{i}}\overline{X}]_{\pounds}. \end{aligned} \tag{4.18}$$

Now let $\overline{X} = \overline{X}^{\alpha}\widehat{e_{\alpha}}$. Then we deduce $\bar{\mathbf{i}}\overline{X} = \overline{X}^{\alpha}\mathscr{V}_{\alpha}$ and consequently

$$\bar{\mathbf{j}}\,[\overline{\mathscr{H}}\delta,\bar{\mathbf{i}}\overline{X}]_{\pounds} = \bar{\mathbf{j}}\,[\mathbf{y}^{\alpha}\delta_{\alpha},\overline{X}^{\beta}\mathscr{V}_{\beta}]_{\pounds} = -\bar{\mathbf{j}}(\overline{X}^{\alpha}\delta_{\alpha}) = -\overline{X}^{\alpha}\widehat{e_{\alpha}} = -\overline{X}.$$

Setting the above equation in (4.18) we derive that $B(\delta,\overline{X}) = 0$. Using (4.12) and Lemma 4.3 we get

$$P^1(\overline{X},\delta) = D_{\overline{\mathscr{H}}\overline{X}}\delta - \nabla_{\overline{\mathscr{H}}\overline{X}}\delta = \overline{\mathscr{V}}\,\overline{\mathscr{H}}\overline{X} - (\overline{H}\circ\bar{\mathbf{j}} + \overline{\mathscr{V}})(\overline{\mathscr{H}}\overline{X}) = -\overline{\mathbf{H}}\circ\bar{\mathbf{j}}(\overline{\mathscr{H}}\overline{X}).$$

But we have $\overline{\mathbf{j}\mathscr{H}} = 1_{\Gamma(\pi^*\pi)}$. Therefore the above equation gives us the second part of the assertion. □

Definition 4.6. Let D be a $\rho_{\pounds}$-covariant derivative in $\Gamma(\pi^*\pi)$. Then the maps R, P and Q given by

$$\begin{aligned} R(\overline{X},\overline{Y})\overline{Z} &:= K^D(\overline{\mathscr{H}}\overline{X},\overline{\mathscr{H}}\overline{Y})\overline{Z}, \\ P(\overline{X},\overline{Y})\overline{Z} &:= K^D(\overline{\mathscr{H}}\overline{X},\bar{\mathbf{i}}\overline{Y})\overline{Z}, \\ Q(\overline{X},\overline{Y})\overline{Z} &:= K^D(\bar{\mathbf{i}}\overline{X},\bar{\mathbf{i}}\overline{Y})\overline{Z}, \end{aligned}$$

are said to be the horizontal or Riemann curvature, the mixed or Berwald curvature and the vertical or Berwald-Cartan curvature of D (with respect to $\overline{\mathscr{H}}$), respectively.

Lemma 4.6. *Let D be a $\rho_{\pounds}$-covariant derivative in $\Gamma(\pi^*\pi)$. If D is associated to the horizontal map $\overline{\mathscr{H}}$, then we have*

$$R(\overline{X},\overline{Y})\delta = R^1(\overline{X},\overline{Y}),\ P(\overline{X},\overline{Y})\delta = P^1(\overline{X},\overline{Y}),\ Q(\overline{X},\overline{Y})\delta = Q^1(\overline{X},\overline{Y}),$$

where $\overline{X},\overline{Y} \in \Gamma(\pi^*\pi)$. Moreover, if the Finsler torsion is symmetric, then $Q(\cdot,\cdot)\delta = Q^1 = 0$.

Proof. Since D is associated to the horizontal map $\overline{\mathscr{H}}$, then $D\delta = \overline{\mathscr{V}}$ and therefore

$$D_{\overline{\mathscr{H}X}}\delta = 0, \quad D_{\bar{\mathbf{i}}\overline{X}}\delta = \overline{X}, \ \forall \overline{X} \in \Gamma(\pi^*\pi).$$

Using the above equations, the proof of the first part of the assertion is obvious. Now we prove the second part. From the first part we have

$$Q(\overline{X},\overline{Y})\delta = Q^1(\overline{X},\overline{Y}) = D_{\bar{\mathbf{i}}\overline{X}}\overline{Y} - D_{\bar{\mathbf{i}}\overline{Y}}\overline{X} - \overline{\mathscr{V}}[\bar{\mathbf{i}}\overline{X}, \bar{\mathbf{i}}\overline{Y}]_{\pounds}.$$

Since the Finsler torsion B is symmetric, then

$$0 = B(\overline{X},\overline{Y}) - B(\overline{Y},\overline{X}) = D_{\bar{\mathbf{i}}\overline{X}}\overline{Y} - D_{\bar{\mathbf{i}}\overline{Y}}\overline{X} - \bar{\mathbf{j}}[\overline{\mathscr{H}}\overline{X}, \bar{\mathbf{i}}\overline{Y}]_{\pounds} + \bar{\mathbf{j}}[\overline{\mathscr{H}}\overline{Y}, \bar{\mathbf{i}}\overline{X}]_{\pounds}.$$

Two above equations give us

$$Q(\overline{X},\overline{Y})\delta = \bar{\mathbf{j}}[\overline{\mathscr{H}}\overline{X}, \bar{\mathbf{i}}\overline{Y}]_{\pounds} - \bar{\mathbf{j}}[\overline{\mathscr{H}}\overline{Y}, \bar{\mathbf{i}}\overline{X}]_{\pounds} - \overline{\mathscr{V}}[\bar{\mathbf{i}}\overline{X}, \bar{\mathbf{i}}\overline{Y}]_{\pounds}.$$

Since $\bar{\mathbf{j}}$ is surjective, there exist $\widetilde{X}, \widetilde{Y} \in \Gamma(\pounds^\pi E)$ such that $\overline{X} = \bar{\mathbf{j}}\widetilde{X}$ and $\overline{Y} = \bar{\mathbf{j}}\widetilde{Y}$. Setting these equations in the above equation imply

$$Q(\bar{\mathbf{j}}\widetilde{X}, \bar{\mathbf{j}}\widetilde{Y})\delta = \bar{\mathbf{j}}[\mathbf{h}\widetilde{X}, J\widetilde{Y}]_{\pounds} - \bar{\mathbf{j}}[\mathbf{h}\widetilde{Y}, J\widetilde{X}]_{\pounds} - \overline{\mathscr{V}}[J\widetilde{X}, J\widetilde{Y}]_{\pounds},$$

and consequently

$$\begin{aligned}
\bar{\mathbf{i}}(Q(\bar{\mathbf{j}}\widetilde{X}, \bar{\mathbf{j}}\widetilde{Y})\delta) &= J[\mathbf{h}\widetilde{X}, J\widetilde{Y}]_{\pounds} - J[\mathbf{h}\widetilde{Y}, J\widetilde{X}]_{\pounds} - \mathbf{v}[J\widetilde{X}, J\widetilde{Y}]_{\pounds} \\
&= J[\widetilde{X}, J\widetilde{Y}]_{\pounds} + J[J\widetilde{X}, \widetilde{Y}]_{\pounds} - [J\widetilde{X}, J\widetilde{Y}]_{\pounds} \\
&= -N_J(\widetilde{X}, \widetilde{Y}) = 0.
\end{aligned}$$

Since $\bar{\mathbf{i}}$ is injective, the above equation gives us $Q(\bar{\mathbf{j}}\widetilde{X}, \bar{\mathbf{j}}\widetilde{Y})\delta = 0$ and therefore $Q(\overline{X},\overline{Y})\delta = 0$. □

Now we denote the torsions and the curvatures of the Berwald derivative ∇, by $\overset{\circ}{A}$, $\overset{\circ}{B}$, $\overset{\circ}{R}{}^1$, $\overset{\circ}{P}{}^1$, $\overset{\circ}{Q}{}^1$ and $\overset{\circ}{R}$, $\overset{\circ}{P}$, $\overset{\circ}{Q}$, respectively. Using (4.11) and Lemma 4.3 it is easy to prove the following

Lemma 4.7. *Let ∇ be the Berwald derivative induced by $\mathbf{h}$ and $\{e_\alpha\}$ be a basis of E. Then*

$$\overset{\circ}{A} = \frac{1}{2} t^\gamma_{\alpha\beta} \widehat{e^\alpha} \wedge \widehat{e^\beta} \otimes \widehat{e_\gamma}, \tag{4.19}$$

$$\overset{\circ}{R}{}^1 = -\frac{1}{2} R^\gamma_{\alpha\beta} \widehat{e^\alpha} \wedge \widehat{e^\beta} \otimes \widehat{e_\gamma}, \tag{4.20}$$

$$\overset{\circ}{B} = 0, \ \overset{\circ}{P}{}^1 = 0, \ \overset{\circ}{Q}{}^1 = 0, \tag{4.21}$$

where $\{\widehat{e^\alpha}\}$ is a dual basis of $\{\widehat{e_\alpha}\}$ and $t^\gamma_{\alpha\beta}$ and $R^\gamma_{\alpha\beta}$ are given by (2.13) *and* (2.15).

Using $\overset{\circ}{A}$ and $\overset{\circ}{R^1}$ we introduce the following tensor fields:

$$\begin{cases} \overset{\circ}{A}_\circ : \Gamma(£^\pi E)\times\Gamma(£^\pi E)\to\Gamma(£^\pi E) \\ \overset{\circ}{A}_\circ(\widetilde{X},\widetilde{Y}) = \bar{\mathbf{i}}\,\overset{\circ}{A}(\bar{\mathbf{j}}\widetilde{X},\bar{\mathbf{j}}\widetilde{Y}) \end{cases}, \tag{4.22}$$

$$\begin{cases} \overset{\circ}{R^1}_\circ : \Gamma(£^\pi E)\times\Gamma(£^\pi E)\to\Gamma(£^\pi E) \\ \overset{\circ}{R^1}_\circ(\widetilde{X},\widetilde{Y}) = \bar{\mathbf{i}}\,\overset{\circ}{R^1}(\bar{\mathbf{j}}\widetilde{X},\bar{\mathbf{j}}\widetilde{Y}) \end{cases}. \tag{4.23}$$

Using (4.19) and (4.22) we can obtain

$$\overset{\circ}{A}_\circ(\delta_\alpha,\delta_\beta) = t^\gamma_{\alpha\beta}\mathscr{V}_\gamma,\quad \overset{\circ}{A}_\circ(\mathscr{V}_\alpha,\delta_\beta) = \overset{\circ}{A}_\circ(\mathscr{V}_\alpha,\mathscr{V}_\beta) = 0.$$

Therefore from (2.12) we deduce

$$\overset{\circ}{A}_\circ = \frac{1}{2}t^\gamma_{\alpha\beta}\mathscr{X}^\alpha\wedge\mathscr{X}^\beta\otimes\mathscr{V}_\gamma = t,$$

where t is the weak torsion of $\mathbf{h}$. Similarly using (4.20) and (4.23) we obtain

$$\overset{\circ}{R^1}_\circ = -\frac{1}{2}R^\gamma_{\alpha\beta}\mathscr{X}^\alpha\wedge\mathscr{X}^\beta\otimes\mathscr{V}_\gamma = \Omega,$$

where Ω is the curvature of $\mathbf{h}$ given in (2.14).

Proposition 4.6. *Let ∇ be the Berwald derivative induced by* $\mathbf{h}$ *in $\Gamma(\pi^*\pi)$. Then $\overset{\circ}{A}_\circ = t$ and $\overset{\circ}{R^1}_\circ = \Omega$, where t and Ω are weak torsion and curvature of* $\mathbf{h}$*, respectively.*

Using (4.10), (4.11) and Definition 4.6 we can deduce

Theorem 4.2. *Let ∇ be the Berwald derivative induced by* $\mathbf{h}$ *in $\Gamma(\pi^*\pi)$ and $\{e_\alpha\}$ be a basis of E. Then*

$$\begin{aligned} \overset{\circ}{R} &= \overset{\circ}{R}{}_{\alpha\beta\gamma}{}^{\lambda}\ \widehat{e_\lambda}\otimes\widehat{e^\alpha}\otimes\widehat{e^\beta}\otimes\widehat{e^\gamma}, \\ \overset{\circ}{P} &= \overset{\circ}{P}{}_{\alpha\beta\gamma}{}^{\lambda}\ \widehat{e_\lambda}\otimes\widehat{e^\alpha}\otimes\widehat{e^\beta}\otimes\widehat{e^\gamma}, \\ \overset{\circ}{Q} &= \overset{\circ}{S}{}_{\alpha\beta\gamma}{}^{\lambda}\ \widehat{e_\lambda}\otimes\widehat{e^\alpha}\otimes\widehat{e^\beta}\otimes\widehat{e^\gamma}, \end{aligned}$$

where

$$\begin{aligned} \overset{\circ}{R}{}_{\alpha\beta\gamma}{}^{\lambda} = &-(\rho^i_\alpha\circ\pi)\frac{\partial^2\mathscr{B}^\lambda_\beta}{\partial \boldsymbol{x}^i\partial \boldsymbol{y}^\gamma} - \mathscr{B}^\mu_\alpha\frac{\partial^2\mathscr{B}^\lambda_\beta}{\partial \boldsymbol{y}^\mu\partial \boldsymbol{y}^\gamma} + (\rho^i_\beta\circ\pi)\frac{\partial^2\mathscr{B}^\lambda_\alpha}{\partial \boldsymbol{x}^i\partial \boldsymbol{y}^\gamma} + \mathscr{B}^\mu_\beta\frac{\partial^2\mathscr{B}^\lambda_\alpha}{\partial \boldsymbol{y}^\mu\partial \boldsymbol{y}^\gamma} \\ &+\frac{\partial\mathscr{B}^\mu_\beta}{\partial \boldsymbol{y}^\gamma}\frac{\partial\mathscr{B}^\lambda_\alpha}{\partial \boldsymbol{y}^\mu} - \frac{\partial\mathscr{B}^\mu_\alpha}{\partial \boldsymbol{y}^\gamma}\frac{\partial\mathscr{B}^\lambda_\beta}{\partial \boldsymbol{y}^\mu} + (L^\mu_{\alpha\beta}\circ\pi)\frac{\partial\mathscr{B}^\lambda_\mu}{\partial \boldsymbol{y}^\gamma}, \end{aligned} \tag{4.24}$$

$$\overset{\circ}{P}{}_{\alpha\beta\gamma}{}^{\lambda} = \frac{\partial^2\mathscr{B}^\lambda_\alpha}{\partial \boldsymbol{y}^\beta\partial \boldsymbol{y}^\gamma}, \tag{4.25}$$

$$\overset{\circ}{S}{}_{\alpha\beta\gamma}{}^{\lambda} = 0. \tag{4.26}$$

Using $\overset{\circ}{R}$, $\overset{\circ}{P}$ and $\overset{\circ}{Q}$ we introduce the following tensor fields:

$$\begin{cases} \overset{\circ}{R}_\circ\colon \Gamma(\pounds^\pi E)\times\Gamma(\pounds^\pi E)\to\Gamma(\pounds^\pi E), \\ \quad \overset{\circ}{R}_\circ(\widetilde{X},\widetilde{Y}) = \bar{\mathbf{i}}\,\overset{\circ}{R}(\bar{\mathbf{j}}\widetilde{X},\bar{\mathbf{j}}\widetilde{Y}), \end{cases} \tag{4.27}$$

$$\begin{cases} \overset{\circ}{P}_\circ\colon \Gamma(\pounds^\pi E)\times\Gamma(\pounds^\pi E)\to\Gamma(\pounds^\pi E), \\ \quad \overset{\circ}{P}_\circ(\widetilde{X},\widetilde{Y}) = \bar{\mathbf{i}}\,\overset{\circ}{P}(\bar{\mathbf{j}}\widetilde{X},\bar{\mathbf{j}}\widetilde{Y}), \end{cases} \tag{4.28}$$

$$\begin{cases} \overset{\circ}{Q}_\circ\colon \Gamma(\pounds^\pi E)\times\Gamma(\pounds^\pi E)\to\Gamma(\pounds^\pi E), \\ \quad \overset{\circ}{Q}_\circ(\widetilde{X},\widetilde{Y}) = \bar{\mathbf{i}}\,\overset{\circ}{Q}(\bar{\mathbf{j}}\widetilde{X},\bar{\mathbf{j}}\widetilde{Y}). \end{cases} \tag{4.29}$$

Using the above theorem, (3.1)–(3.3) and (4.27)–(4.29) we derive that

Proposition 4.7. *Let ∇ be the Berwald derivative induced by* $\mathbf{h}$. *Then*

$$\overset{\circ}{R}_\circ = \overset{B}{R}, \quad \overset{\circ}{P}_\circ = \overset{B}{P}, \quad \overset{\circ}{Q}_\circ = \overset{B}{Q},$$

where $\overset{B}{R}$, $\overset{B}{P}$ and $\overset{B}{Q}$ are the horizontal, mixed and vertical curvatures of the Berwald-type connection $(\overset{B}{D},\mathbf{h})$, respectively.

Proposition 4.8. *Let ∇ be the Berwald derivative induced by* $\mathbf{h}$. *Then for sections X, Y and Z of E we have*

$$\overset{\circ}{P}(\widehat{X},\widehat{Y})\widehat{Z} = \overline{\mathscr{V}}[[X^h,Y^V]_\pounds,Z^V]_\pounds.$$

Proof. Let $X = X^\alpha e_\alpha$, $Y = Y^\beta e_\beta$ and $Z = Z^\gamma e_\gamma$ be sections of E. Then we have $\widehat{X} = (X^\alpha\circ\pi)\widehat{e}_\alpha$, $\widehat{Y} = (Y^\beta\circ\pi)\widehat{e}_\beta$ and $\widehat{Z} = (Z^\gamma\circ\pi)\widehat{e}_\gamma$. Therefore, (4.25) implies

$$\overset{\circ}{P}(\widehat{X},\widehat{Y})\widehat{Z} = ((X^\alpha Y^\beta Z^\gamma)\circ\pi)\frac{\partial^2\mathscr{B}^\lambda_\alpha}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma}\widehat{e}_\lambda.$$

Similarly we can obtain

$$\begin{aligned}\overline{\mathscr{V}}[[X^h,Y^V]_\pounds,Z^V]_\pounds &= \overline{\mathscr{V}}[[(X^\alpha\circ\pi)\delta_\alpha,(Y^\beta\circ\pi)\mathscr{V}_\beta]_\pounds,(Z^\gamma\circ\pi)\mathscr{V}_\gamma]_\pounds \\ &= ((X^\alpha Y^\beta Z^\gamma)\circ\pi)\frac{\partial^2\mathscr{B}^\lambda_\alpha}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma}\widehat{e}_\lambda.\end{aligned}$$

Two above equations give us the assertion. □

Definition 4.7. The covariant derivative operator D given by

$$D_{\widetilde{X}}\overline{Y} := \nabla_{\widetilde{X}}\overline{Y} + \frac{1}{n+1}(tr\,\overset{\circ}{P}(\bar{\boldsymbol{j}}\widetilde{X},\overline{Y}))\delta, \tag{4.30}$$

is called the Yano derivative induced by $\overline{\mathscr{H}}$, where $\overset{\circ}{P}$ is the mixed curvature of the Berwald derivative ∇.

Using (4.11) and (4.30) we get

$$D_{\widetilde{X}}\overline{Y} = \left(\widetilde{X}^{\alpha}\left((\rho_{\alpha}^{i}\circ\pi)\frac{\partial\overline{Y}^{\beta}}{\partial\mathbf{x}^{i}}+\mathscr{B}_{\alpha}^{\gamma}\frac{\partial\overline{Y}^{\beta}}{\partial\mathbf{y}^{\gamma}}\right)-\widetilde{X}^{\alpha}\overline{Y}^{\gamma}\frac{\partial\mathscr{B}_{\alpha}^{\beta}}{\partial\mathbf{y}^{\gamma}}+\widetilde{X}^{\overline{\alpha}}\frac{\partial\overline{Y}^{\beta}}{\partial\mathbf{y}^{\alpha}} + \frac{1}{n+1}\widetilde{X}^{\alpha}\overline{Y}^{\gamma}\mathbf{y}^{\beta}\frac{\partial^{2}\mathscr{B}_{\alpha}^{\lambda}}{\partial\mathbf{y}^{\lambda}\partial\mathbf{y}^{\gamma}}\right)\widehat{e}_{\beta},$$

where $\widetilde{X}=\widetilde{X}^{\alpha}\delta_{\alpha}+\widetilde{X}^{\overline{\alpha}}\mathscr{V}_{\alpha}\in\Gamma(\pounds^{\pi}E)$ and $\overline{Y}=\overline{Y}^{\beta}\widehat{e}_{\beta}\in\Gamma(\pi^{*}\pi)$. In particular case we have

$$D_{\delta_{\alpha}}\widehat{e}_{\beta}=(\frac{1}{n+1}\mathbf{y}^{\gamma}\frac{\partial^{2}\mathscr{B}_{\alpha}^{\lambda}}{\partial\mathbf{y}^{\lambda}\partial\mathbf{y}^{\beta}}-\frac{\partial\mathscr{B}_{\alpha}^{\gamma}}{\partial\mathbf{y}^{\beta}})\widehat{e}_{\gamma},\quad D_{\mathscr{V}_{\alpha}}\widehat{e}_{\beta}=0.$$

5. Application to optimal control

Let us consider the following driftless control affine system with quadratic cost in the space $\mathbb{R}^3$:

$$\begin{cases}\dot{x}^1=u^2\\ \dot{x}^2=u^1+u^2x^2\\ \dot{x}^3=u^1+u^2x^3\end{cases}, \tag{5.1}$$

$$\min\int_0^T\mathscr{L}(u(t))dt,\ \mathscr{L}(u)=\frac{1}{2}\left((u^1)^2+(u^2)^2\right),$$

where $\dot{x}^i=\frac{dx^i}{dt}$ and u^1,u^2 are control variables. We are looking for the optimal trajectories starting from the point $(0,1,0)^t$ and parameterized by arclength (minimum time problem) and free endpoint. From the system of differential equations we get $u^1=\dot{x}^2-\dot{x}^1x^2$, $u^2=\dot{x}^1$ and it results the Lagrangian

$$\mathscr{L}(x,\dot{x})=\frac{1}{2}\left(\left(\dot{x}^2-\dot{x}^1x^2\right)^2+\left(\dot{x}^1\right)^2\right),$$

with holonomic constraint

$$\dot{x}^2-\dot{x}^3=\dot{x}^1(x^2-x^3),$$

which leads to the equation $\ln\left|x^2-x^3\right|=x^1+c$. Next, using the Lagrange multiplier $\lambda=\lambda(t)$ we obtain the total Lagrangian (including the constraints) given by

$$\begin{aligned}L(x,\dot{x})&=\mathscr{L}(x,\dot{x})+\lambda(\dot{x}^2-\dot{x}^3-\dot{x}^1x^2+\dot{x}^1x^3)\\&=\frac{1}{2}\left(\left(\dot{x}^2-\dot{x}^1x^2\right)^2+\left(\dot{x}^1\right)^2\right)+\lambda(\dot{x}^2-\dot{x}^3-\dot{x}^1x^2+\dot{x}^1x^3).\end{aligned}$$

We notice that the Hessian matrix of L is singular on tangent bundle $T\mathbb{R}^3$, and L is a degenerate Lagrangian (not regular). The corresponding Euler-Lagrange equations on tangent bundle lead to a complicated system of second-order differential equations. Moreover, because the Lagrangian is not regular, we cannot obtain the explicit coefficients of the semispray S from the symplectic equation $i_S\omega_L=-dE_L$ and it is difficult to find the coefficients of the nonlinear connection induced by Lagrangian function in this case. We will use a different approach, considering the framework of Lie algebroids.

The system can be written in the form

$$\dot{x} = u^1 X_1 + u^2 X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathbb{R}^3,\ X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ X_2 = \begin{pmatrix} 1 \\ x^2 \\ x^3 \end{pmatrix}, \qquad (5.2)$$
$$\min \int_0^T \mathscr{L}(u(t))dt,\ \mathscr{L}(u) = \frac{1}{2}\left((u^1)^2 + (u^2)^2\right).$$

The vector fields are given by

$$X_1 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \quad X_2 = \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3}.$$

The Lie bracket is

$$[X_1, X_2] = \left[\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3}\right] = X_1,$$

and it results that the associated distribution $\Delta = span\{X_1, X_2\}$ is holonomic and has the constant rank 2. Moreover, from the system (5.1) we obtain $\dot{x}^2 - \dot{x}^3 = \dot{x}^1(x^2 - x^3)$ which yields

$$\ln\left|x^2 - x^3\right| = x^1 + c. \qquad (5.3)$$

(c is a constant) and it results that Δ determines a foliation on $\mathbb{R}^3$ given by the surfaces (5.3) and two points can be joined by a optimal trajectory if and only if they are situated on the same leaf. In order to use the framework of Lie algebroids, we consider $E = \Delta$ (holonomic distribution with constant rank), the anchor $\rho : E \to TM$ is the inclusion and $[,]_E$ the induced Lie bracket. In the case of our application, the anchor ρ has the components

$$\rho^i_\alpha = \begin{pmatrix} 0 & 1 \\ 1 & x^1 \\ 1 & x^2 \end{pmatrix},$$

Using the structure equation on Lie algebroid

$$[X_\alpha, X_\beta] = L^\gamma_{\alpha\beta} X_\gamma, \quad \alpha, \beta, \gamma = 1, 2,$$

we obtain the non-zero structure functions

$$L^1_{12} = 1,\ L^1_{21} = -1.$$

The components of the semispray [10]

$$S^\varepsilon = g^{\varepsilon\beta}\left(\rho^i_\beta \frac{\partial \mathscr{L}}{\partial x^i} - \rho^i_\alpha \frac{\partial^2 \mathscr{L}}{\partial x^i \partial u^\beta} y^\alpha - L^\gamma_{\beta\alpha} u^\alpha \frac{\partial \mathscr{L}}{\partial u^\gamma}\right),$$

induced by the Lagrangian $\mathscr{L}(u) = \frac{1}{2}\left((u^1)^2 + (u^2)^2\right)$ on E are given by

$$S^1 = -u^1 u^2,\ S^2 = (u^1)^2.$$

The functions S^α are homogeneous of degree 2 in u and it results that S is a spray. The coefficients of the canonical nonlinear connection $\mathscr{B} = -[S, J]_E$ given by (2.17)

$$\mathscr{B}^\beta_\alpha = \frac{1}{2}\left(\frac{\partial S^\beta}{\partial u^\alpha} - u^\varepsilon L^\beta_{\alpha\varepsilon}\right),$$

have the form

$$\mathscr{B}_1^1 = -u^2,\ \mathscr{B}_2^1 = 0,\ \mathscr{B}_1^2 = u^1,\ \mathscr{B}_2^2 = 0.$$

Also, the non-zero coefficients of the curvature from (2.15) of nonlinear connection $\mathscr{B}$ are given by

$$R_{12}^1 = u^2,\ R_{12}^2 = u^1,\ R_{21}^1 = -u^2,\ R_{21}^2 = -u^1.$$

The Berwald connection is given by

$$\overset{B}{D}_{\delta_1}\ \delta_1 = -\delta_2, \quad \overset{B}{D}_{\delta_1}\ \delta_2 = \delta_1,$$

and the Douglas tensor has the components equal to zero.

The Euler-Lagrange equations on Lie algebroids given by (see [24])

$$\frac{dx^i}{dt} = \rho_\alpha^i u^\alpha, \quad \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial u^\alpha}\right) = \rho_\alpha^i \frac{\partial \mathscr{L}}{\partial x^i} - L_{\alpha\beta}^\varepsilon u^\beta \frac{\partial \mathscr{L}}{\partial u^\varepsilon},$$

lead to the following differential equations

$$\dot{u}^1 = -u^1 u^2, \quad \dot{u}^2 = (u^1)^2,$$

which can be written in the form

$$\frac{dx^i}{dt} = \rho_\alpha^i u^\alpha, \quad \frac{du^\alpha}{dt} = S^\alpha(x,u),$$

and give the integral curves of the spray S.

In order to solve this optimal control problem we can use the Pontryagin Maximum Principle on the cotangent bundle. The Hamiltonian function has the form

$$H(u,x,p) = p_i \dot{x}^i - \mathscr{L} = p_1 u^2 + p_2(u^1 + u^2 x^2) + p_3(u^1 + u^2 x^3) - \frac{1}{2}\left((u^1)^2 + (u^2)^2\right),$$

and with the equations $\frac{\partial H}{\partial u^i} = 0$ we obtain that $p_2 + p_3 = u^1$ and $p_1 + p_2 x^2 + p_3 x^3 = u^2$, which replaced into the expression of Hamiltonian, yields

$$H(x,p) = \frac{1}{2}\left((p_2+p_3)^2 + \left(p_1 + p_2 x^2 + p_3 x^3\right)\right)^2. \tag{5.4}$$

The Hamilton equations on the cotangent bundle

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i},$$

lead to a very complicated system of implicit differential equations. We can use a different approach, involving the framework of Lie algebroids. First, we need by the following result [14]:

Proposition 5.1. *The relation between the Hamiltonian H on the cotangent bundle T^*M and the Hamiltonian $\mathscr{H}$ on dual bundle E^* is given by*

$$H(x,p)=\mathscr{H}(\rho^*(p)),\quad \mu=\rho^*(p),\quad p\in T_x^*M,\quad \mu\in E_x^*.$$

Proof. The Fenchel-Legendre dual of Lagrangian L is the Hamiltonian H given by

$$\begin{aligned} H(x,p) &= \sup_v\{\langle p,v\rangle - L(v)\} = \sup_v\{\langle p,v\rangle - \mathscr{L}(u);\rho(u)=v\} \\ &= \sup_u\{\langle p,\rho(u)\rangle - \mathscr{L}(u)\} = \sup_u\{\langle \rho^*(p),u\rangle - \mathscr{L}(u)\} = \mathscr{H}(\rho^*(p)), \end{aligned}$$

and we get

$$H(x,p)=\mathscr{H}(\mu),\quad \mu=\rho^*(p),$$

or locally

$$\mu_\alpha=\rho_\alpha^{*i}p_i, \tag{5.5}$$

where the Hamiltonian $H(x,p)$ is degenerate on $\operatorname{Ker}\rho^\star\subset T^*M$. □

Using the Legendre transformation associated to the regular Lagrangian $\mathscr{L}=\frac{1}{2}\left((u^1)^2+(u^2)^2\right)$ on Lie algebroid E, we can obtain the nondegenerate (regular) Hamiltonian $\mathscr{H}$ on E^* in the form

$$\mathscr{H}=\frac{1}{2}\left(\mu_1^2+\mu_2^2\right).$$

Using (5.5) we find the Hamiltonian H from (5.4) on the cotangent bundle given by $H(x,p)=\mathscr{H}(\mu)$ with $\mu_\alpha=\rho_\alpha^{*i}p_i$, $\alpha=1,2$

$$\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}=\begin{pmatrix}0 & 1 & 1\\ 1 & x^1 & x^2\end{pmatrix}\begin{pmatrix}p_1\\ p_2\\ p_3\end{pmatrix}.$$

Next, from Hamilton equations on Lie algebroid [4]

$$\frac{dx^i}{dt}=\rho_\alpha^i\frac{\partial\mathscr{H}}{\partial\mu_\alpha},\quad \frac{d\mu_\alpha}{dt}=-\rho_\alpha^i\frac{\partial\mathscr{H}}{\partial x^i}-\mu_\gamma L_{\alpha\beta}^\gamma\frac{\partial\mathscr{H}}{\partial\mu_\beta},$$

we deduce that

$$\begin{aligned} &\dot{x}^1=\mu_2,\ \dot{x}^2=\mu_1+x^2\mu_2,\ \dot{x}^3=\mu_1+x^3\mu_2,\\ &\dot{\mu}_1=-\mu_1\mu_2,\ \dot{\mu}_2=\mu_1^2. \end{aligned}$$

The form of the last relations leads to the following change of variables

$$\mu_1(t)=r(t)\operatorname{sech}\theta(t),\quad \mu_2(t)=r(t)\tanh\theta(t), \tag{5.6}$$

where

$$\sinh\theta=\frac{e^\theta-e^{-\theta}}{2},\ \cosh\theta=\frac{e^\theta+e^{-\theta}}{2},\ \tanh\theta=\frac{\sinh\theta}{\cosh\theta},\ \operatorname{sech}\theta=\frac{1}{\cosh\theta}.$$

The differential equations

$$\dot{\mu}_1 = -\mu_1\mu_2,$$

with the relations (5.6) yields

$$\frac{\dot{r}}{r} - \dot{\theta}\tanh\theta = -r\tanh\theta. \tag{5.7}$$

Also, from the equation

$$\dot{\mu}_2 = \mu_1^2,$$

and (5.6) we get

$$\frac{\dot{r}}{r}\tanh\theta + \dot{\theta}\mathrm{sech}^2\theta = r\mathrm{sech}^2\theta). \tag{5.8}$$

Now, reducing $\dot{\theta}$ from the equations (5.7) and (5.8), we obtain

$$\dot{r} = 0 \Rightarrow r = c, \quad c \in \mathbb{R},$$

and

$$\dot{\theta} = r.$$

Since the optimal trajectories are parameterized by arclength (minimum time problem) the conclusion corresponds exactly to the $1/2$ level of the Hamiltonian and we have

$$\mathscr{H} = \frac{r^2}{2} = \frac{1}{2},$$

which yields

$$r = 1, \quad \theta = t.$$

The equation

$$\dot{\mu}_1 = -\mu_1\dot{x}^1,$$

implies that

$$x^1(t) = \ln\frac{c_1}{\mathrm{sech}t}, \quad c_1 \in \mathbb{R}.$$

Since we are looking for the trajectories starting from the point $(0,1,0)^t$, we have $x^1(0) = 0$ and

$$\ln c_1 = 0 \Rightarrow c_1 = 1,$$

which leads to

$$x^1(t) = \ln\frac{1}{\mathrm{sech}t} = \ln\cosh t.$$

We obtain also that

$$\dot{\mu}_2 = \mu_1\left(\dot{x}^2 - x^2\mu_2\right) = \mu_1\dot{x}^2 + x^2\dot{\mu}_1,$$

and, consequently, $\mu_2 = \mu_1 x^2 + c_2$. Further,

$$x^2(t) = \sinh t + \frac{c_2}{\operatorname{sech} t}.$$

From $x^2(0) = 1$ we obtain that $c_2 = 1$ and this yields

$$x^2(t) = \sinh t + \cosh t.$$

In the same way we get

$$x^3(t) = \sinh t + \frac{c_3}{\operatorname{sech} t}.$$

From $x^3(0) = 0$ we obtain that $c_3 = 0$ and it results

$$x^3(t) = \sinh t.$$

Using (5.1) we have $u^2 = \dot{x}^1$, $u^1 = \dot{x}^3 - u^2 x^3 = \dot{x}^2 - u^2 x^2$ and by direct computation, we obtain the control variables

$$u^2(t) = \frac{\sinh t}{\cosh t}, \quad u^1(t) = \frac{1}{\cosh t}.$$

Finally, we obtain the solution of driftless control affine systems given by

$$x^1(t) = \ln \cosh t, \quad x^2(t) = \sinh t + \cosh t, \quad x^3(t) = \sinh t.$$

The solution is optimal because the Hamiltonian function is convex.

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