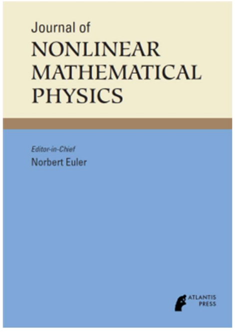

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Gambier lattices and other linearisable systems

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We propose two different appraoches to extending the Gambier mapping to a two-dimensional lattice equation. A first approach relies on a hypothesis of separate evolutions in each of the two directions. We show that known equations like the Startsev-Garifullin-Yamilov equation, the Hietarinta equation, as well as one proposed by the current authors, are in fact Gambier lattices. A second approach, based on the same principle as the Gambier equation, that of two linearisable equations in cascade, constructs a Gambier lattice in the form of a system of two coupled Burgers equations. The (slow) growth properties of the latter are in agreement with its linearisable character.

Keywords: integrable lattices, Gambier mapping, singularity confinement, growth properties.

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1. Introduction

Linearisable systems are the simplest instances of integrability. However 'simple' is far from meaning 'trivial' and the study of linearisable systems is both interesting and demanding. Calogero [1] devoted particular attention to linearisable systems, coining the expression C-integrable in order to designate this large class of integrable systems. A system is said to be C-integrable if its solution can be obtained from a finite system of nondifferential equations (possibly nonlinear) and a finite system of linear differential ones. The definition is tailored to differential systems but it carries over in a straightforward way to a discrete setting.

In a series of papers we have explored the linearisable domain for difference equations focusing predominantly on second-order mappings. We have, in particular, shown that linearisability for these systems appears under three different guises. The first class of linearisable mappings [2] are the ones called projective. Its canonical form is

$$x_{n+1}x_nx_{n-1} + a_nx_nx_{n-1} + b_nx_{n-1} + c_n = 0, \tag{1.1}$$

where a_n, b_n, c_n are functions of the independent variable. In order to linearise (1.1) we introduce a Cole-Hopf transformation $x_n = w_{n+1}/w_n$ obtaining the linear equation

$$w_{n+2} + a_nw_{n+1} + b_nw_n + c_nw_{n-1} = 0. \tag{1.2}$$

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The second class of linearisable systems is that of the Gambier mappings [3]. In analogy to the continuous Gambier equation, which is a system of two Riccati equations coupled in cascade, these systems are two coupled homographic mappings in cascade. Their generic form is

$$\frac{a_n x_n x_{n+1} + b_n x_n + c_n x_{n+1} + d_n}{e_n x_n x_{n+1} + f_n x_n + g_n x_{n+1} + h_n} = y_n. \tag{1.3}$$

Without loss of generality we can assume that y_n is constant, i.e. obeys the equation $y_n = y_{n-1}$. The third class comprises the mappings for which we coined the term 'third kind'. They were first discovered in [4] by applying limiting procedures to discrete Painlevé equations. In this case we have a nonlinear mapping

$$F(x_{n+1}, x_n, x_{n-1}) = m, \tag{1.4a}$$

and an associated linear equation

$$\frac{a_n x_{n+1} + b_n x_n + c_n x_{n-1} + d_n}{e_n x_{n+1} + f_n x_n + g_n x_{n-1} + h_n} = k, \tag{1.4b}$$

where $m,\ k$ are constants. Taking the discrete derivatives of (1.4a) and (1.4b) we require that the third-order mappings they lead to be the same up to nonessential factors. Thanks to this relation one can construct the solution of the nonlinear mapping (1.4a), from that of the linear one. An interesting result concerning the third kind mappings was presented in [5] where we have shown that all the known ones can be explicitly integrated.

This paper is devoted to the study of linearisable equations akin to those of the second class described above, namely the Gambier mappings. We shall address the question whether one can obtain two-dimensional, i.e. lattice, equations of Gambier type. As we shall show the answer to this question is positive and we shall present two different realisations of this program.

2. A first Gambier lattice equation

Before proceeding to the construction of a two-dimensional Gambier equation it is interesting to have a second look at (1.3). One possible choice of a simplified form leads to the following Gambier equation

$$x_{n+1} \frac{a_n x_n + b_n}{c_n x_n + d_n} = y_n \quad \text{with} \quad y_n = y_{n-1}, \tag{2.1}$$

where the $a_n, b_n, \ldots$ are not the same as in (1.3) (but we decided to use the same letters in order to avoid the proliferation of different symbols). A first possible construction of a 2-dimensional equation is to keep the same form, assume that the variables and the parameters are fuctions of two independent variables, say (n,m), and posit that the evolution y is constant in one of the evolution directions. We obtain thus the system

$$x_{n+1,m} \frac{a_{n,m} x_{n,m} + b_{n,m}}{c_{n,m} x_{n,m} + d_{n,m}} = y_{n,m} \quad \text{with} \quad y_{n,m} = y_{n,m-1}. \tag{2.2}$$

The integration of (2.2) is elementary. One finds that $y_{n,m}$ is a function of n only, whereupon one has to integrate a homographic equation for $x_{n,m}$, something that can be done by reducing the latter to a linear equation through a discrete Cole-Hopf transformation.

Eliminating y we obtain for the Gambier lattice equation the form:

$$x_{n+1,m+1}(a_{n,m+1}x_{n,m+1}+b_{n,m+1})(c_{n,m}x_{n,m}+d_{n,m})$$
$$=x_{n+1,m}(a_{n,m}x_{n,m}+b_{n,m})(c_{n,m+1}x_{n,m+1}+d_{n,m+1}). \quad (2.3)$$

Having obtained this form for the Gambier lattice equation it is sensible to ask whether among the known integrable (linearisable) lattice equations there exist some which can be cast into the Gambier form (2.3). It turns out that this is indeed the case.

In [6] we had studied an equation proposed by Hietarinta [7] and showed that it was a linearisable one. The autonomous form of that equation can be written, after some elementary manipulations, as

$$x_{n+1,m+1}(x_{n,m}+A)(x_{n,m+1}+B)=x_{n,m+1}(x_{n,m}+B)(x_{n+1,m}+A). \quad (2.4)$$

Given the forms of (2.4) and (2.3) it does not appear possible to cast directly the former into a special form of the latter. However if we consider the special case $A=0$ of (2.4) then we find that the resulting equation

$$x_{n+1,m+1}x_{n,m}(x_{n,m+1}+B)=x_{n,m+1}x_{n+1,m}(x_{n,m}+B), \quad (2.5)$$

is indeed of the form (2.3) provided one takes $d=0$, $a=c=1$ and $b=B$. The same procedure allows to bring the non-autonomous extension of (2.4) to a Gambier form. Our starting point is

$$x_{n+1,m+1}(x_{n,m}+A_{n,m})(x_{n,m+1}+B_{n,m+1})=x_{n,m+1}(x_{n,m}+B_{n,m})(x_{n+1,m}+A_{n+1,m}). \quad (2.6)$$

Note that the form (2.6) is different from the one given in [7] but the two are related by a homographic transformation. Taking again the limit $A_{n,m}=0$ we find the equation

$$x_{n+1,m+1}x_{n,m}(x_{n,m+1}+B_{n,m+1})=x_{n,m+1}x_{n+1,m}(x_{n,m}+B_{n,m}), \quad (2.7)$$

which is indeed of the form (2.3) with $d=0$, $a=c=1$ and $b_{n,m}=B_{n,m}$.

Next we turn to an equation proposed by Startsev [8] and extensively studied by Garifullin and Yamilov [9]. It has the form

$$x_{n+1,m+1}(x_{n,m+1}-1)=x_{n+1,m}(x_{n,m}-1). \quad (2.8)$$

Without any further transformation we recognise a Gambier form obtained from (2.3) with $a=1$, $b=-1$, $c=0$, $d=1$. Clearly (2.3) constitutes the non-autonomous extension of the Startsev-Garifullin-Yamilov equation. At this point a remark is in order. While one can integrate (2.8) following the Gambier procedure another, it is possible to obtain the exact solution of (2.8) in closed form. In order to do this we shall use the relation of the Startsev-Garifullin-Yamilov equation to the discrete Liouville equation derived by Hirota [10]. The latter has the form

$$u_{n+1,m+1}u_{n,m}=u_{n,m+1}u_{n+1,m}-1. \quad (2.9)$$

Another discrete analogue of the Liouville equation does exist. It has the form

$$w_{n+1,m+1}w_{n,m}=(w_{n,m+1}-1)(w_{n+1,m}-1), \quad (2.10)$$

and was presented by Adler and Startsev [11] who attribute it to Hirota. Equation (2.10) can be obtained from the nonlinear form of the Hirota-Miwa equation

$$v_{n+1,m,k}v_{n,m+1,k}(v_{n+1,m+1,k+1}-1)(v_{n,m,k-1}-1)$$

$$= v_{n+1,m+1,k}v_{n,m,k}(v_{n+1,m,k}-1)(v_{n,m+1,k}-1), \tag{2.11}$$

by putting to zero the terms with third index $k\pm 1$ and taking $w_{n,m}=v_{n,m,k}^{-1}$. Equation (2.10) was later obtained by Hydon and Viallet [12] who did not recognize its relation to the Liouville equation. In fact (2.10) and (2.9) are connected by a Miura transformation of the form

$$w_{n,m} = u_{n,m+1}u_{n+1,m}, \tag{2.12}$$

which means that (2.9) is the 'modified' form of (2.10). The existence of the Miura transformation allowed us [13] to give the explicit integration of the Adler-Startsev-Hirota equation (2.10) starting from the explicit solution of the Hirota-Liouville one. Here we shall use the solution of the latter and construct the solution of the Startsev-Garifullin-Yamilov equation (2.8). Starting from $u_{n,m}$ satisfying the Liouville equation we introduce the auxiliary variable

$$y_{n,m} = \frac{u_{n,m}u_{n,+1,m+1}}{u_{n+1,m}u_{n,m+1}} = 1-\frac{1}{u_{n+1,m}u_{n,m+1}}, \tag{2.13}$$

and obtain $x_{n,m}$ in terms of the latter as $x_{n,m}=y_{n-1,m}(y_{n,m}-1)/(y_{n,m}y_{n-1,m}-1)$. This allows us to obtain the relation between the Startsev-Garifullin-Yamilov and the discrete Liouville equation

$$x_{n,m} = \frac{u_{n-1,m}}{u_{n+1,m}-u_{n-1,m}}. \tag{2.14}$$

The solution of the discrete Liouville equation (already given by Hirota) involves two free functions $f(n)$ and $g(m)$ with which one constructs by quadrature the functions $h(n)$ and $k(m)$, given by

$$h(n+1)-h(n) = f(n+1)f(n), \tag{2.15a}$$

$$k(m+1)-k(m) = g(m+1)g(m). \tag{2.15b}$$

Using these quantities one expresses the solution of the discrete Liouville equation as

$$u_{n,m} = \frac{h(n)+k(m)}{f(n)g(m)}. \tag{2.16}$$

From this expression we can write simply the solution of (2.8) as

$$x_{n,m} = \frac{f(n+1)(h(n)+g(m)-f(n-1)f(n))}{(h(n)+g(m))(f(n-1)+f(n+1))}. \tag{2.17}$$

which, as expected, involves two free functions.

3. Another Gambier lattice equation

The choice (2.2) for the Gambier equation is clearly not the only one. Any choice which can bring the left-hand side of (1.3) to a more convenient form can be equally acceptable. In [14] we had obtained a linearisable equation as a special limit of the Adler-Bobenko-Suris family of integrable lattices. In the, initial, autonomous form this equation is

$$x_{n+1,m+1}x_{n+1,m}x_{n,m+1}+x_{n+1,m+1}x_{n,m+1}x_{n,m}-x_{n+1,m+1}x_{n+1,m}x_{n,m}-x_{n+1,m}x_{n,m+1}x_{n,m}$$
$$-x_{n+1,m+1}-x_{n+1,m}+x_{n,m+1}+x_{n,m}=0. \tag{3.1}$$

It is indeed a Gambier lattice equation. We put

$$y_{n,m} = \frac{x_{n+1,m}x_{n,m}+1}{x_{n+1,m}+x_{n,m}} \tag{3.2}$$

and find that (3.1) corresponds to the conservation $y_{n,m} = y_{n,m-1}$.

This is clearly different from the form (2.2) so the question that naturally arises is how would (3.1) look under a canonical form (2.2). The change of variables

$$x_{n,m} = \frac{u_{n,m}+1}{u_{n,m}-1}, \tag{3.3}$$

in the quantity $(y_{n,m}+1)/(y_{n,m}-1)$ transforms (3.2) into a form akin to that of (2.2). The corresponding equation is

$$u_{n+1,m+1}u_{n,m+1} = u_{n+1,m}u_{n,m}. \tag{3.4}$$

Given its very simple form it is now elementary to give its general solution. We introduce two free functions $f(n)$ and $g(m)$, integrate (3.4) to $u_{n+1,m}u_{n,m} = f(n+1)f(n)$, whereupon the general solution is given by $u_{n,m} = f(n)g(m)^{(-1)^n}$.

The deautonomisation of (3.1) is straightforward. We start from (3.4) for u and introduce a variable x through the homography

$$x_{n,m} = \frac{a_{n,m}u_{n,m}+b_{n,m}}{c_{n,m}u_{n,m}+d_{n,m}}. \tag{3.5}$$

We obtain finally the equation

$$\begin{aligned}
&x_{n,m}x_{n,m+1}x_{n+1,m}x_{n+1,m+1}(c_{n,m}c_{n+1,m}d_{n,m+1}d_{n+1,m+1}-c_{n,m+1}c_{n+1,m+1}d_{n,m}d_{n+1,m})\\
&+x_{n,m}x_{n,m+1}x_{n+1,m}(a_{n+1,m+1}c_{n,m+1}d_{n,m}d_{n+1,m}-b_{n+1,m+1}c_{n,m}c_{n+1,m}d_{n,m+1})\\
&+x_{n,m}x_{n,m+1}x_{n+1,m+1}(-a_{n+1,m}c_{n,m}d_{n,m+1}d_{n+1,m+1}+b_{n+1,m}c_{n,m+1}c_{n+1,m+1}d_{n,m})\\
&+x_{n,m}x_{n+1,m}x_{n+1,m+1}(a_{n,m+1}c_{n+1,m+1}d_{n,m}d_{n+1,m}-b_{n,m+1}c_{n,m}c_{n+1,m}d_{n+1,m+1})\\
&+x_{n,m+1}x_{n+1,m}x_{n+1,m+1}(-a_{n,m}c_{n+1,m}d_{n,m+1}d_{n+1,m+1}+b_{n,m}c_{n,m+1}c_{n+1,m+1}d_{n+1,m})\\
&+x_{n,m}x_{n,m+1}(a_{n+1,m}b_{n+1,m+1}c_{n,m}d_{n,m+1}-a_{n+1,m+1}b_{n+1,m}c_{n,m+1}d_{n,m})\\
&+x_{n,m}x_{n+1,m}(-a_{n,m+1}a_{n+1,m+1}d_{n,m}d_{n+1,m}+b_{n,m+1}b_{n+1,m+1}c_{n,m}c_{n+1,m})\\
&+x_{n,m}x_{n+1,m+1}(-a_{n,m+1}b_{n+1,m}c_{n+1,m+1}d_{n,m}+a_{n+1,m}b_{n,m+1}c_{n,m}d_{n+1,m+1})\\
&+x_{n,m+1}x_{n+1,m+1}(a_{n,m}a_{n+1,m}d_{n,m+1}d_{n+1,m+1}-b_{n,m}b_{n+1,m}c_{n,m+1}c_{n+1,m+1})\\
&+x_{n,m+1}x_{n+1,m}(a_{n,m}b_{n+1,m+1}c_{n+1,m}d_{n,m+1}-a_{n+1,m+1}b_{n,m}c_{n,m+1}d_{n+1,m})\\
&+x_{n+1,m}x_{n+1,m+1}(a_{n,m}b_{n,m+1}c_{n+1,m}d_{n+1,m+1}-a_{n,m+1}b_{n,m}c_{n+1,m+1}d_{n+1,m})\\
&+x_{n,m}(a_{n,m+1}a_{n+1,m+1}b_{n+1,m}d_{n,m}-a_{n+1,m}b_{n,m+1}b_{n+1,m+1}c_{n,m})\\
&+x_{n,m+1}(-a_{n,m}a_{n+1,m}b_{n+1,m+1}d_{n,m+1}+a_{n+1,m+1}b_{n,m}b_{n+1,m}c_{n,m+1})\\
&+x_{n+1,m}(-a_{n,m}b_{n,m+1}b_{n+1,m+1}c_{n+1,m}+a_{n,m+1}a_{n+1,m+1}b_{n,m}d_{n+1,m})\\
&+x_{n+1,m+1}(-a_{n,m}a_{n+1,m}b_{n,m+1}d_{n+1,m+1}+a_{n,m+1}b_{n,m}b_{n+1,m}c_{n+1,m+1})\\
&+a_{n,m}a_{n+1,m}b_{n,m+1}b_{n+1,m+1}-a_{n,m+1}a_{n+1,m+1}b_{n,m}b_{n+1,m}=0.
\end{aligned} \tag{3.6}$$

Equation (3.6) has the same aspect as what Adler-Bobenko-Suris [15] call the Q4 equation, but, of course, the precise structure of its coefficients is such that it remains linearisable.

Having two different choices for the Gambier lattice, namely (2.2) and (3.2), one may wonder whether there exists a more general form, constructed always on the hypothesis of the separation of evolution in the n and m directions. For this we go back to the form (1.3) of the Gambier mapping. Clearly this form has been arbitrarily simplified, since we chose the equation for $y_n = y_{n-1}$, which skews the evolution in one direction. In order to put x and y on the same footing, we consider the original definition of a Gambier mapping, that of two homographic equations in cascade, i.e. we assume thus that y obeys a homographic equation and an equation like (1.3) gives another homography for x. Given that we allow for a full freedom for y it is clear that the equation for x can be simplified without loss of generality and cast in the form encountered in (2.1). Putting all these considerations together we can now propose a more general Gambier lattice obtained under the hypothesis of separation of evolutions:

$$x_{n+1,m}\frac{a_{n,m}x_{n,m}+b_{n,m}}{c_{n,m}x_{n,m}+d_{n,m}} = y_{n,m} \quad \text{with} \quad y_{n,m+1} = \frac{f_{n,m}y_{n,m}+g_{n,m}}{h_{n,m}y_{n,m}+k_{n,m}}. \tag{3.7}$$

Eliminating y we can obtain an equation for x alone. It has the form

$$\begin{gathered}
x_{n,m}x_{n,m+1}x_{n+1,m}x_{n+1,m+1}a_{n,m}a_{n,m+1}h_{n,m} \\
-x_{n,m}x_{n,m+1}x_{n+1,m}a_{n,m}c_{n,m+1}d_{n,m}f_{n,m} + x_{n,m}x_{n,m+1}x_{n+1,m+1}c_{n,m}a_{n,m+1}k_{n,m} \\
+x_{n,m}x_{n+1,m}x_{n+1,m+1}a_{n,m}b_{n,m+1}h_{n,m} + x_{n,m+1}x_{n+1,m}x_{n+1,m+1}a_{n,m+1}b_{n,m}h_{n,m} \\
-x_{n,m}x_{n,m+1}c_{n,m}c_{n,m+1}g_{n,m} + x_{n,m}x_{n+1,m}a_{n,m}d_{n,m+1}f_{n,m} + x_{n,m}x_{n+1,m+1}b_{n,m+1}c_{n,m}k_{n,m} \\
+x_{n,m+1}x_{n+1,m+1}a_{n,m+1}d_{n,m}k_{n,m} - x_{n,m+1}x_{n+1,m}b_{n,m}c_{n,m+1}f_{n,m} + x_{n+1,m}x_{n+1,m+1}b_{n,m}b_{n,m+1}h_{n,m} \\
-x_{n,m}c_{n,m}d_{n,m+1}g_{n,m} - x_{n,m+1}c_{n,m+1}d_{n,m}g_{n,m} - x_{n+1,m}b_{n,m}d_{n,m+1}f_{n,m} + x_{n+1,m+1}b_{n,m+1}d_{n,m}k_{n,m} \\
-d_{n,m}d_{n,m+1}g_{n,m} = 0.
\end{gathered} \tag{3.8}$$

This is the generic form of a two-dimensional equation of Gambier type one can obtain under the hypothesis of the separation of evolutions. We remark that, just as (3.6), equation (3.8) has the same aspect as the Q4 equation.

4. A Gambier-Burgers lattice equation

In the previous sections we introduced a 2-dimensional Gambier system by assuming a separation of the evolutions along the n and m directions. As we saw, several known linearisable systems belong to this class. In this section we shall adopt a different point of view, going back to the original definition of the Gambier equation [16], which is a system of two Riccati equations coupled in cascade. This means that we have a first Riccati for a single variable, say y, and the solution of this first equation appears in the coefficients of the second Riccati for x. The approach of Gambier is to have y appear linearly in the coefficients of the equation for x. Since the Riccati equation can be linearised by a Cole-Hopf transformation, we have at the end a first linear equation for one variable and a second linear equation where the solution of the first appears explicitly in the coefficients. The previous considerations carry over to the discrete domain in exactly the same way, the homographic mapping being the discrete analogue of the Riccati equation.

How about the 2-dimensional extension of the argument of the previous paragraph? Two-dimensional equations linearisable by a Cole-Hopf transformation do exist, the classical example being the Burgers equation. In a discrete setting the latter has the form

$$x_{n,m+1} = x_{n,m} \frac{1 + x_{n+1,m}}{1 + x_{n,m}}. \tag{4.1}$$

In [17] we studied the non-autonomous extension of the discrete Burgers equation and showed that its linearisable character is preserved if instead of (4.1) we consider

$$x_{n,m+1} = x_{n,m} \frac{1 + z_{n+1,m} x_{n+1,m}}{1 + z_{n,m} x_{n,m}}, \tag{4.2}$$

where $z_{n,m}$ is an otherwise unspecified function of its arguments. Now, if we assume that $z_{n,m}$ satisfies an equation of the form (4.1), i.e.

$$z_{n,m+1} = z_{n,m} \frac{1 + z_{n+1,m}}{1 + z_{n,m}}. \tag{4.3}$$

we have a perfect realisation the Gambier approach. Equation (4.3) is linearised by putting $z_{n,m} = Q_{n+1,m}/Q_{n,m}$, leading to

$$Q_{n,m+1} = Q_{n,m} + Q_{n+1,m} \tag{4.4}$$

with the appropriate choice of gauge. Similarly, putting $x_{n,m} = P_{n+1,m}/P_{n,m}$, we find from (4.2) the linear equation

$$P_{n,m+1} = P_{n,m} + z_{n,m} P_{n+1,m}. \tag{4.5}$$

Solving (4.5) for $z_{n,m}$ and substituting back into (4.3) we can obtain an equation for P only. We find thus

$$\frac{P_{n+1,m}(P_{n,m+2} - P_{n,m+1})}{P_{n+1,m+1}(P_{n,m+1} - P_{n,m})} = \frac{P_{n+1,m}(P_{n+2,m} + P_{n+1,m+1} - P_{n+1,m})}{P_{n+2,m}(P_{n+1,m} + P_{n,m+1} - P_{n,m})}. \tag{4.6}$$

The study of singularities of the discrete Burgers equation be it autonomous as (4.1) or nonautonomous like (4.2) is straightforward. Assuming that for some point we have $x_{n+1,m} = -1$, in the autonomous case, or $x_{n+1,m} = -1/z_{n+1,m}$, in the nonautonomous one, we find $x_{n,m+1} = 0$, $x_{n+1,m+1} = \infty$, whereupon the singularity is confined.

It is interesting at this point to study the growth properties of the solution of the Gambier-Burgers system (4.2)-(4.3). In order to do this we compute the growth of some initial condition of equation (4.6) and use the relation between P and x in order to obtain the degree growth of the latter. The initial conditions are given on two adjacent lines on the (n,m) plane. We choose for $x_{n,-1} = f(n)$, $x_{-1,m} = g(m)$ and $x_{0,m} = h(m)$ where f, g, h are arbitrary functions of n or m respectively and take $x_{n,0} = pk(n)/q$, where $k(n)$ is another arbitrarily chosen function of n. We iterate (4.6) and compute the homogeneous degree of $x_{n,m}$ in p, q given by the ratio $P_{n+1,m}/P_{n,m}$. Here are the results:

$$
\begin{array}{cccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
0 & 30 & 49 & 71 & 94 & 115 & 134 & 148 & 156 & 163 & 164 & \cdots \\
0 & 26 & 42 & 60 & 78 & 93 & 105 & 112 & 118 & 119 & 119 & \cdots \\
0 & 22 & 35 & 49 & 62 & 71 & 77 & 82 & 83 & 83 & 83 & \cdots \\
0 & 18 & 28 & 38 & 46 & 50 & 54 & 55 & 55 & 55 & 55 & \cdots \\
0 & 10 & 14 & 17 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & \cdots \\
0 & 6 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & \cdots \\
0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \cdots \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots
\end{array}
\qquad (4.7)
$$

We remark that in the m (vertical) direction the degree grows linearly, albeit after some steps. In the n (horizontal) direction on the other hand the degree saturates after a sufficient number of iteration steps. This (very) slow-growth behaviour is what one would intuitively expect given the linearisable character of the Gambier-Burgers equation. In the one-dimensional case the growth of the Gambier mappings is covered by the Diller-Favre [18] theorem. According to the latter a linearisable mapping with confined singularities has zero degree growth. In practice what this means is that a Gambier mapping will exhibit linear growth up to the point of confinement at which point the growth is arrested. Apparently the situation is more complicated in the two-dimensional case (and, anyhow, no theorem analogous to the Diller-Favre one does exist).

5. Conclusions

In this paper we have addressed the question of generalising the Gambier mapping to two dimensions. Since a general setting for this is lacking we opted for a heuristic approach. Two different strategies were adopted. In the first one we started from the one-dimensional case and went to two dimensions assuming that the evolution was separated, i.e. the equation for y was giving an evolution along the m direction while that of x handled the evolution along n. Even with this simplifying assumption there exists a plethora of choices for the form of the Gambier lattice. We have examined two of them showing that they correponded to already known equations. The second strategy consisted in considering a coupled systems of lattice equations which could be linearised by a Cole-Hopf-like transformation in analogy to what happens in the continuous and the discrete one-dimensional cases. We proposed thus a coupled Burgers system that has exactly the linearisability property we were looking for.

Questions do remain open at this point. Although we did present realisations of a Gambier-type lattice, we still do not know what would be the most general form of the latter. The equations presented in sections 2 and 3 are quite different from those of section 4 and thus none of them can be a candidate for the general Gambier lattice. A serious setback in the quest for the latter is also the relative paucity of results on two-dimensional linearisable systems. On a different point now, we must admit that we were both intrigued and reconforted by the results of the study of the degree growth of the Gambier-Burgers system. Why is the degree growing linearly in one direction and saturating in the other one? No clear explanation does exist for this and it is clear that more studies are necessary before one can formulate hypotheses that have a chance to be correct. We intend to return to this point in some future work of ours.

References

[1] F. Calogero, in *What is Integrability?*, ed., V. Zakharov, (Springer, New York, 1990) 1).

[2] A. Ramani, B. Grammaticos, G. Karra, Linearizable mappings, *Physica A*, **180** (1992) 115.

[3] B. Grammaticos, A. Ramani and S. Lafortune, The Gambier Mapping revisited, *Physica A*, **253** (1998) 260.

[4] A. Ramani, B. Grammaticos and Y. Ohta, Discrete integrable systems from continuous Painlevé equations through limiting procedures, *Nonlinearity*, **13** (2000) 1073.

[5] A. Ramani, B. Grammaticos and J. Satsuma, On the explicit integration of a class of linearisable mappings, *J. Phys. A*, **45** (2012) 365202.

[6] A. Ramani, N. Joshi, B. Grammaticos and T. Tamizhmani, Deconstructing an integrable lattice equation, *J. Phys. A* **39** (2006) L145.

[7] J. Hietarinta, A new two-dimensional lattice model that is 'consistent around a cube', *J. Phys. A*, **37** (2004) L67.

[8] S. Ya. Startsev, On non-point invertible transformations of difference and differential-difference equations, *SIGMA*, **6** (2010) 092.

[9] R.N. Garifullin and R.I. Yamilov, On a series of Darboux integrable discrete equations on the square lattice, *Ufimsk. Mat. Zh.*, **4** (2012) 177.

[10] R. Hirota, Nonlinear partial difference equations V: nonlinear equations reducible to linear equations, *J. Phys. Soc. Japan* **46** (1979) 312.

[11] V.E. Adler and S.Ya. Startsev, Discrete analogue of the Liouville equation, *Theor. Math. Phys.*, **121** (1999) 1484.

[12] P.E. Hydon and C.-M. Viallet, Asymmetric integrable quad-graph equations, *Applicable Analysis*, **89** (2010) 493.

[13] C. Scimiterna, B. Grammaticos and A. Ramani, On two integrable lattice equations and their interpretation, *J. Phys. A*, **44** (2011) 032002.

[14] B. Grammaticos and A. Ramani, Singularity confinement property for the (non-autonomous) Adler-Bobenko-Suris integrable lattice equations, *Lett. Math. Phys.*, **92** (2010) 33.

[15] V. Adler, A. Bobenko and Yu. Suris, Classification of integrable equations on quad-graphs. The consistency approach, *Comm. Math. Phys.*, **233** (2003) 513.

[16] B. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes, *Acta Math.*, **33** (1910) 1.

[17] S. Tremblay, B. Grammaticos and A. Ramani, Integrable lattice equations and their growth properties, *Phys. Lett. A*, **278** (2001) 319.

[18] J. Diller and C. Favre, Dynamics of bimeromorphic maps of surfaces, *Amer. J. Math.*, **123** (2001) 1135.