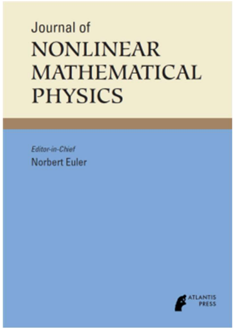

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Trigonal Toda Lattice Equation

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In this article, we give the trigonal Toda lattice equation,

$$-\frac{1}{2}\frac{d^3}{dt^3}q_\ell(t) = \mathrm{e}^{q_{\ell+1}(t)} + \mathrm{e}^{q_{\ell+\zeta_3}(t)} + \mathrm{e}^{q_{\ell+\zeta_3^2}(t)} - 3\mathrm{e}^{q_\ell(t)},$$

for a lattice point $\ell \in \mathbb{Z}[\zeta_3]$ as a directed 6-regular graph where $\zeta_3 = \mathrm{e}^{2\pi\sqrt{-1}/3}$, and its elliptic solution for the curve $y(y-s) = x^3$, $(s \neq 0)$.

Keywords: Toda lattice equation; directed 6-regular graph; Eisenstein integers.

2000 Mathematics Subject Classification: 33E05, 37K10, 11B99, 05C20.

1. Introduction

The elliptic functions have high symmetries and generate many interesting relations. In the celebrated paper [11], Toda derived the Toda lattice equation based on the addition formula of the elliptic functions. Using the addition formulae of hyperelliptic curves [3], the hyperelliptic quasi-periodic solutions of the Toda lattice equation are also obtained as in [7, 9]. The derivation in [7, 9] can be regarded as a natural generalization of Toda's original one. The addition formulae for the Toda lattice equation are essential.

Recently Eilbeck, Matsutani and Ônishi introduced a new addition formula for the Weierstrass $\wp$ functions on an elliptic curve E, $y(y-s) = x^3$, which is called the equiharmonic elliptic curve [10]. The curve E has the automorphism, the cyclic group action of order three as a Galois action [4], i.e., $\widehat{\zeta_3}(x,y) = (\zeta_3 x, y)$, where $\zeta_3 = \mathrm{e}^{2\pi\sqrt{-1}/3}$.

In this article, we use the new addition formula on E in [4] and derive a non-linear differential and difference equation following the derivation in [7, 9, 11] as shown in Proposition 3.1. Thus we call it *the trigonal Toda lattice equation*. The trigonal Toda lattice equation consists of the third order differential and the trigonal difference operators, which reflects the cyclic symmetry of the curve. The difference operator agrees with the graph Laplacian of a directed 6-regular graph associated with Eisenstein integers $\mathbb{Z}[\zeta_3]$. The trigonal Toda lattice equation is defined over the infinite directed 6-regular graph c.f. Proposition 3.2. It means that we provide the trigonal Toda lattice equation and its elliptic function solution as a special solution. Since the lattice given by the infinite directed 6-regular graph appear in models in statistical mechanics, e.g., [1], the new Toda lattice equation might show a nonlinear excitation of such models.

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The contents are as follows. In Section 2, we show the properties of the elliptic curve E. We derive a new differential-difference equation, or the trigonal Toda lattice equation, and its elliptic function solution as an identity in the meromorphic functions on E in Section 3. We give some discussions in Section 4.

2. Properties of the equiharmonic elliptic curve

Let us consider an elliptic curve E given by the affine equation

$$y(y-s) = x^3, \tag{2.1}$$

which is called the equiharmonic elliptic curve [4, 10]. E has the automorphism associated with the cyclic group of order three as the Galois action on E; $\widehat{\zeta}_3 : E \to E$, $\widehat{\zeta}_3((x,y)) = (\zeta_3 x, y)$ where $\zeta_3 = \mathrm{e}^{2\pi\sqrt{-1}/3}$; the action $\widehat{\zeta}_3$ on E is invariant. We call it the trigonal cyclic symmetry. The affine equation is expressed by

$$\left(y - \frac{s}{2}\right)^2 = \left(x + \sqrt[3]{\frac{s^2}{4}}\right)\left(x + \zeta_3\sqrt[3]{\frac{s^2}{4}}\right)\left(x + \zeta_3^2\sqrt[3]{\frac{s^2}{4}}\right).$$

By letting $e_j = -\zeta_3^{1-j}\sqrt[3]{\dfrac{s^2}{4}}$, it corresponds to the Weierstrass standard form

$$(\wp')^2 = 4\wp^3 - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3),$$

where $g_3 = -4s^2$ using the Weierstrass $\wp$-function,

$$\wp(u) = x(u), \quad \wp'(u) = 2y(u) - s, \quad y(u) = \frac{1}{2}(\wp'(u) + s),$$

for the elliptic integral

$$u = \int_\infty^{(x,y)} du, \quad du = \frac{dx}{2y - s}.$$

It is known that since the image of the incomplete elliptic integral agrees with the complex plane $\mathbb{C}$, the $\wp$-function (and thus x and y) is expressed by the Weierstrass sigma function,

$$\wp(u) = -\frac{d^2}{du^2}\log\sigma(u), \quad \left(y(u) = -\frac{1}{2}\left(\frac{d^3}{du^3}\log\sigma(u) + s\right)\right).$$

It means that $x(u)$ and $y(u)$ are considered as meromorphic functions on $\mathbb{C}$. The trigonal cyclic symmetry induces the action on u and sigma function, i.e., for $u \in \mathbb{C}$,

$$\sigma(\zeta_3 u) = \zeta_3\sigma(u), \quad \wp(\zeta_3 u) = \zeta_3\wp(u), \quad \wp'(\zeta_3 u) = \wp'(u).$$

Eilbeck, Matsutani and Ônishi showed an addition formula of the elliptic sigma function of the curve E [4],

$$\frac{\sigma(u-v)\sigma(u-\zeta_3 v)\sigma(u-\zeta_3^2 v)}{\sigma(u)^3\sigma(v)^3} = (y(u) - y(v)). \tag{2.2}$$

In this article, we consider the curve E and this formula (2.2).

Let the elliptic integral from the infinity point ∞ to $(x,y) = (0,s)$ denoted by ω_s, and similarly that to $(0,0)$ by ω_0,

$$\omega_s = \int_\infty^{(0,s)} du, \quad \omega_0 = \int_\infty^{(0,0)} du, \quad du = \frac{dx}{2y-s}.$$

The complete elliptic integrals of the first and the second kinds are given by

$$\omega_i := \int_\infty^{e_i} du, \quad \eta_i = \int_\infty^{e_i} x du, \quad (i = 1, 2, 3).$$

Following Weierstrass' convention

$$\omega' = \omega_1, \quad \omega'' = \omega_3, \quad \eta' = \eta_1, \quad \eta'' = \eta_3,$$

they satisfy the relations [5, 10]

$$\omega'' = \zeta_3 \omega', \quad \eta'' = \zeta_3^2 \eta',$$
$$\omega_1 + \omega_2 + \omega_3 = 0, \quad \eta_1 + \eta_2 + \eta_3 = 0, \quad \eta'\omega'' - \eta''\omega' = \frac{\pi\sqrt{-1}}{2}.$$

Further for the branch points $(0,0)$ and $(s,0)$, the following relations hold:

Lemma 2.1.

$$x(\omega_0) = x(\omega_s) = 0, \quad y(\omega_0) = 0, \quad y(\omega_s) = s,$$
$$\omega_s = \frac{1-\zeta_3^2}{3} 2\omega', \quad \omega_0 = -\omega_s, \quad \omega' = \frac{3}{2}\frac{1}{\zeta_3 - \zeta_3^2}\frac{\Gamma\left(\frac{1}{3}\right)^2}{s^{1/3}\Gamma\left(\frac{2}{3}\right)}, \quad \eta' = \frac{\pi\sqrt{-1}}{3\sqrt{3}}\frac{s^{1/3}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)^2}.$$

Proof. See [5, Appendix C]. □

The image of the incomplete elliptic integrals is acted by $\mathrm{SL}(2,\mathbb{Z})$ and the cyclic group ζ_3. For $u, v \in \mathbb{C}$ $(v \neq 0)$, we define a lattice

$$\mathscr{Z}_{v,u} := \mathbb{Z}[\zeta_3]v + u := \{\ell_1 v + \ell_2 \zeta_3 v + u \mid \ell_1, \ell_2 \in \mathbb{Z}\}. \tag{2.3}$$

Noting $\zeta_6 = \zeta_3 + 1$ for $\zeta_6 := \mathrm{e}^{2\pi\sqrt{-1}/6}$, $\mathbb{Z}[\zeta_3] = \mathbb{Z}[\zeta_6]$. Since $\mathscr{Z}_{2\omega',0}$ agrees with the lattice of the periodicity, i.e., $x(u+L) = x(u)$, $y(u+L) = y(u)$ for $L \in \mathscr{Z}_{2\omega',0}$, the Jacobian $\mathscr{J}_E$ of the curve E is given by

$$\mathscr{J}_E = \mathbb{C}/\mathscr{Z}_{2\omega',0}, \quad \kappa : \mathbb{C} \to \mathscr{J}_E.$$

These points of the integrals for the branch points of the curve E are illustrated in Figure 1. We also regard x and y as meromorphic functions on $\mathscr{J}_E$ due to their periodicity.

Further Lemma 2.1 shows that ω_s and ω_0 belong to $\frac{1}{3}\mathscr{Z}_{2\omega',0}$.

3. The trigonal Toda lattice equation

The addition formula (2.2) gives the following lemma:

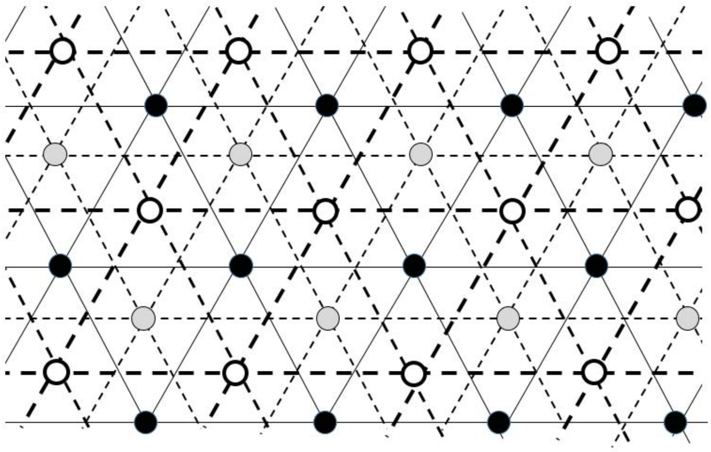


Fig. 1. The lattice points of $\mathscr{Z}_{2\omega',0}$: The lattice points of $\mathscr{Z}_{2\omega',0}$ are denoted by the black dots, ω_0 and ω_s with $\mathscr{Z}_{2\omega',0}$ translations are denoted by gray dots and white dots respectively.

Lemma 3.1. *The quantity*

$$q(u,v) := \log(y(u)-y(v)), \quad \left(y(u)-y(v) = \mathrm{e}^{q(u,v)}\right)$$

satisfies the relation

$$-\frac{1}{2}\frac{d^3}{du^3}q(u,v) = \mathrm{e}^{q(u-v,v)} + \mathrm{e}^{q(u-\zeta_3 v,v)} + \mathrm{e}^{q(u-\zeta_3^2 v,v)} - 3\mathrm{e}^{q(u,v)}.$$

Proof. We consider the logarithm of both sides of (2.2) and differentiate both side three times with respect to u. Then we obtain

$$-\frac{d^3}{du^3}\log(y(u)-y(v)) = 2(y(u-v)+y(u-\zeta_3 v)+y(u-\zeta_3^2 v)) - 6(y(u)).$$

□

The right hand side in Lemma 3.1 should be expressed by a difference operator. In order to express it, we prepare the geometry associated with Lemma 3.1.

We fix the complex numbers u_0 and v_0. We regard $\mathscr{Z}_{v_0,u_0}$ as the set of nodes $\mathscr{N}_{v_0,u_0}$ of an infinite directed (oriented) 6-regular graph $\mathscr{G}_{v_0,u_0}$ whose incoming degree and outgoing degree at each node are three, $\mathscr{N}_{v_0,u_0} = \mathscr{Z}_{v_0,u_0}$ [6]; $\mathscr{G}_{v_0,u_0}$ is illustrated in Figure 2. Every node n_ℓ in $\mathscr{N}_{v_0,u_0}$ is labeled by an Eisenstein integer $\ell \in \mathbb{Z}[\zeta_3]$.

It is noted that every $v_0 \in \mathbb{C}$ is decomposed to $v_0 = v_0'\omega' + v_0''\omega''$ using $v_0', v_0'' \in \mathbb{R}$. Further the quotient set of the lattice points modulo $\mathscr{Z}_{2\omega',0}$ is denoted by $\mathscr{N}_{v_0,u_0}/\mathscr{Z}_{2\omega',0} = \kappa(\mathscr{N}_{v_0,u_0})$. The following are obvious:

Lemma 3.2.

(1) *For $v_0 = v_0'\omega' + v_0''\omega''$ of $v_0', v_0'' \in \mathbb{Q} \cap [0,2]$, the cardinality $|\mathscr{N}_{v_0,u_0}/\mathscr{Z}_{2\omega',0}|$ is finite for every $u_0 \in \mathbb{C}$, and*

(2) *for $v_0 = v_0'\omega' + v_0''\omega''$ of $v_0', v_0'' \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,2]$, $\mathscr{N}_{v_0,u_0}/\mathscr{Z}_{2\omega',0}$ is dense in $\mathscr{J}_E$ for every $u_0 \in \mathbb{C}$.*

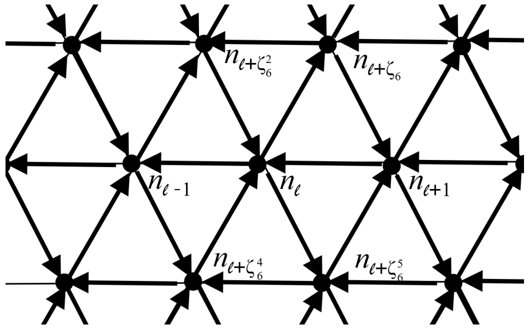


Fig. 2. The directed graph $\mathscr{G}_{v_0,u_0}$.

Let us introduce the function spaces, Ω and $\log\Omega$,

$$\Omega := \{Q : \mathbb{C}\times\mathscr{N}_{v_0,u_0} \to \mathbb{P} \mid \text{ meromorphic}\}, \quad \log\Omega := \{q : \mathbb{C}\times\mathscr{N}_{v_0,u_0} \to \mathbb{P} \mid \mathrm{e}^q \in \Omega\}.$$

For an Eisenstein integer $\ell \in \mathbb{Z}[\zeta_3]$ or $n_\ell \in \mathscr{N}_{v_0,u_0}$, $t \in \mathbb{C}$ and fixed u_0, $v_0 \in \mathbb{C}$, let us consider an element in $\log\Omega$,

$$q_\ell(t; u_0, v_0) := q(t + u_0 - \ell v_0, v_0) = \log(y(t + u_0 - \ell v_0) - y(v_0)), \tag{3.1}$$

which is denoted by $q_\ell(t)$ for brevity.

Lemma 3.1 gives the nonlinear relation on $\log\Omega$:

Proposition 3.1. *For $n_\ell \in \mathscr{N}_{v_0,u_0}$, $t \in \mathbb{C}$ and fixed u_0, $v_0 \in \mathbb{C}$, $q_\ell(t) = q_\ell(t; u_0, v_0)$ satisfies the relation,*

$$-\frac{1}{2}\frac{d^3}{dt^3} q_\ell(t) = \mathrm{e}^{q_{\ell+1}(t)} + \mathrm{e}^{q_{\ell+\zeta_3}(t)} + \mathrm{e}^{q_{\ell+\zeta_3^2}(t)} - 3\mathrm{e}^{q_\ell(t)}. \tag{3.2}$$

It is emphasized that (3.2) can be regarded as a differential-difference non-linear equation and its special solution is given by (3.1) for the elliptic curve (2.1). Its derivation is basically the same as Toda's original derivation of Toda lattice equation [11] and that in [7, 9]. Further it is related to the infinite graph $\mathscr{G}_{v_0,u_0}$. Thus we will call this relation *the trigonal Toda lattice equation*.

We recall $\zeta_6 = \zeta_3 + 1$. For a given $n_\ell \in \mathscr{N}_{v_0,u_0}$, let us consider subgraph $\mathscr{G}_\ell \subset \mathscr{G}_{v_0,u_0}$ given by its nodes $\mathscr{N}_\ell := \{n_\ell, n_{\ell+1}, n_{\ell+\zeta_6}, n_{\ell+\zeta_6^2}, \ldots, n_{\ell+\zeta_6^5}\}$ $(\subset \mathscr{N}_{v_0,u_0})$; $\mathscr{N}_\ell$ consists of the center point n_ℓ with a hexagon $n_{\ell+\zeta_6^i}$ $(i = 0, 1, \ldots, 5)$. The submatrix of the incoming adjacency matrix $\mathscr{A}_{\text{in}}$ for $\mathscr{G}_\ell$ is given by

$$\mathscr{A}_{\text{in}}|_{\mathscr{G}_\ell} = \begin{pmatrix} 0&1&0&1&0&1&0\\ 0&0&1&0&0&0&1\\ 1&0&0&0&0&0&0\\ 0&0&1&0&1&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&1\\ 1&0&0&0&0&0&0 \end{pmatrix}.$$

The incoming degree matrix is given by the diagonal matrix $\mathscr{D}_{\text{in}}$ whose diagonal element is three. Thus we define the incoming Laplacian [6],

$$\Delta_{\text{in}} := \mathscr{D}_{\text{in}} - \mathscr{A}_{\text{in}}.$$

Let us consider the functions $q \in \log \Omega$ and $e^q \in \Omega$ whose components at n_ℓ are given by $q_\ell(t)$ and $\mathrm{e}^{q_\ell(t)}$. We regard them as column vectors for each $\ell \in \mathbb{Z}[\zeta_3]$. Then the Laplacian acts on the vector spaces.

Using the incoming Laplacian, Proposition 3.1 is reduced to the following formula.

Proposition 3.2. *Using the above notations,* (3.2) *is written by*

$$\frac{d^3}{dt^3}q(t) = -\Delta_{\mathrm{in}} e^q(t).$$

It turns out that the trigonal Toda lattice equation consists of the third order differential operators and trigonal graph Laplacian, which is a natural generalization of the original Toda lattice equation [11], though it has not ever obtained as far as we know.

As we obtain the equation, we will consider its solutions (3.1), especially their initial condition u_0 and the configurations $\mathscr{G}_{v_0,u_0}$:

Remark 3.1.

(1) The domain of the solution e^{q_ℓ} of (3.1) is the Jacobian $\mathscr{J}_E$; for $L \in \mathscr{Z}_{2\omega',0}$, $e^{q_\ell}(t+L) = e^{q_\ell}(t)$. From Lemma 2.1, the periods $2\omega'$ and $2\omega''$ are scaled by $s^{-1/3}$. Further in the projection $\pi : E \to \mathbb{P}$, $(\pi((x,y)) = y)$, which determines the three special points $(0,s,\infty)$ in $\mathbb{P}$, the range of the solution e^{q_ℓ} as a meromorphic function on E is also parameterized by s via y and the governing equation (2.1). It is easy to find the s-dependence of e^{q_ℓ} and thus we may fix s as a finite real number.

(2) For $v_0(\neq 0)$ such that $v_0 \neq \omega_0$, $q(u,v_0)$ as a function with respect to u diverges only at the points in $\mathscr{Z}_{2\omega',0}$ and $\bigcup_{i=0}^{2}(\zeta_3^i v_0 + \mathscr{Z}_{2\omega',0})$. Their union is denoted by $\mathscr{S}_{v_0}$. It means that for an $\ell \in \mathbb{Z}[\zeta_3]$, if the orbit of $q_\ell(t;u_0,v_0)$ in t avoids $\mathscr{S}_{v_0}$, the value of $q_\ell(t;u_0,v_0)$ is finite.

Let us consider its orbit whose value is finite value. We restrict its domain $\mathbb{C} \times \mathscr{Z}_{v_0,u_0}$ to its real subspace $\mathbb{R}\mathrm{e}^{\alpha\sqrt{-1}} \times \mathscr{Z}_{v_0,u_0}$ for a certain unit direction $\mathrm{e}^{\alpha\sqrt{-1}}$ (i.e., $|\mathrm{e}^{\alpha\sqrt{-1}}| = 1$),

For the case (2) in Lemma 3.2, there are infinite many points at which $|q_\ell(t;u_0,v_0)|$ is greater than for every given positive number $1/\varepsilon$. Thus we should employ the case (1) in Lemma 3.2.

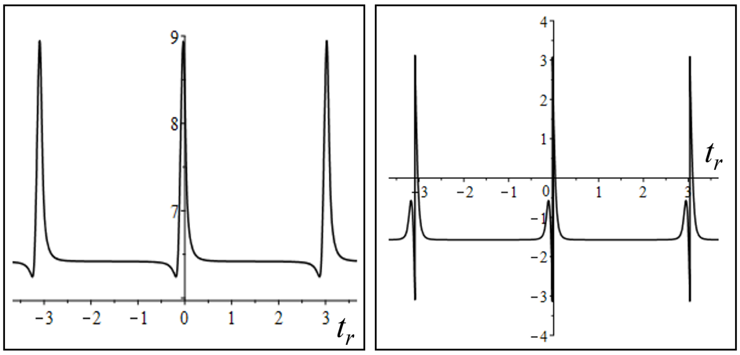
(3) Let us assume that $K := |\mathscr{N}_{v_0,u_0}/\mathscr{Z}_{2\omega',0}|$ is finite. For a certain direction $\mathrm{e}^{\alpha\sqrt{-1}}$ in the complex plane and $u_0 \neq 0$, we find the subspace $\mathrm{e}^{\alpha\sqrt{-1}}\mathbb{R}$ in $\mathbb{C}$ such that every $q_\ell(t_r\mathrm{e}^{\alpha\sqrt{-1}};u_0,v_0)$ does not diverge for each $\ell \in \mathbb{Z}[\zeta_3]$ and $t \in \mathbb{R}$, and satisfies the trigonal Toda lattice equation,

$$\frac{d^3}{dt_r^3}q(t_r\mathrm{e}^{\alpha\sqrt{-1}}) = -\mathrm{e}^{3\alpha\sqrt{-1}}\Delta_{\mathrm{in}} e^q(t_r\mathrm{e}^{\alpha\sqrt{-1}}). \tag{3.3}$$

The conditions on $\mathrm{e}^{\alpha\sqrt{-1}}$ and u_0 correspond to the conditions on the embedding ι of $\mathbb{R}^K$ into $\mathscr{J}_E$ such that the image of ι is compact and disjoint from $\mathscr{S}_{v_0}/\mathscr{Z}_{2\omega',0}$. Under these conditions, we have the complex valued finite solutions of the trigonal Toda lattice equation (3.3).

An elliptic function solution of this equation is illustrated in Figure 3 for $v_0 = (1+\zeta_6)\omega'/13$, $u_0 = v_0/2$, $\mathrm{e}^{\alpha\sqrt{-1}} = \omega'/|\omega'|$ and $s = 1.0$.

Let us consider the continuum limit of the the trigonal Toda lattice equation as follows:



(a) (b)

Fig. 3. An elliptic solution of the trigonal Toda lattice equation $q_0(t_r \mathrm{e}^{\alpha\sqrt{-1}})$ at $\ell = 0$ for $v_0 = (1+\zeta_6)\omega'/13$, $u_0 = v_0/2$, $\mathrm{e}^{\alpha\sqrt{-1}} = \omega'/|\omega'|$ and $s = 1.0$, and $t_r \in \mathbb{R}$: (a) shows its real part whereas (b) corresponds to its imaginary part.

Remark 3.2.

(1) It is noted that $q(u, v_0)$ diverges for the limit $v_0 \to 0$ and thus for this elliptic function solution $q_\ell(t)$, we cannot obtain the continuum limit of the graph Laplacian Δ_{in} and of the trigonal Toda lattice equation.

(2) The elliptic curve E becomes the three rational curves for the limit $s \to 0$ as in [5, Appendix C], and y behaves like $y = -\dfrac{1}{u^3} + \dfrac{1}{2}s^2u^3 + o(s^3)$ [4]. In the limit, the trigonal Toda lattice equation does not satisfy.

4. Discussion

We derived the trigonal Toda lattice equation in Propositions 3.1 and 3.2 based on the addition formula (2.2) for the curve E associated with the automorphism of the curve. It is associated with the lattice, or the directed 6-regular graph, given by the Eisenstein integers $\mathbb{Z}[\zeta_3]$. It means that we have an nonlinear equation on the lattice and its elliptic function solution. Since there are physical models based on the triangle lattice given by the infinite 6-regular graph [1], this trigonal Toda lattice equation might describe a nonlinear excitation in the models.

The third order differential equation reminds us of the Chazy equation, which is a third order ordinary differential equation and posses Painlevé property [2]. However the trigonal Toda lattice equation cannot have a non-trivial continuum limit because E becomes the three rational curves for the limit $s \to 0$ [5] and $q_\ell(t; u_0, v_0)$ diverges for the limit $v_0 \to 0$ as in Remark 3.2. In other words, we could not directly argue the integrablity of the trigonal Toda lattice equation using the Chazy equation, even though both elliptic function solutions are closely related. It means, in this stage, that it is not obvious whether the trigonal Toda lattice equation is an integrable equation as a time-development equation, and thus it is an open problem to determine the behavior of its solution for every initial state as an initial value problem.

However the addition theorem in [3, (A.3)] for the genus three curve can be regarded as a generalization of the addition formula (2.2) for the cyclic action $\widehat{\zeta_3}$ on curves. Thus it is expected that the trigonal Toda lattice equation might have algebro-geometric solutions of algebraic curves of higher genus. Further this approach could be generalized to more general curves, e.g., the genus three curve [3] and more general curves with a trigonal cyclic group [8].

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