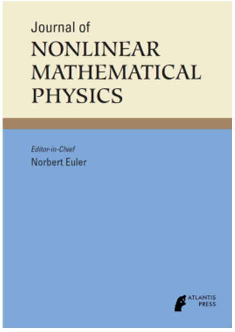


ATLANTIS PRESS

**Journal of Nonlinear Mathematical Physics**

**ISSN (Online): 1776-0852** **ISSN (Print): 1402-9251**
**Journal Home Page: https://www.atlantis-press.com/journals/jnmp**

# Asymptotics behavior for the integrable nonlinear Schrödinger equation with quartic terms: Cauchy problem

Lin Huang

**To cite this article**: Lin Huang (2020) Asymptotics behavior for the integrable nonlinear Schrödinger equation with quartic terms: Cauchy problem, Journal of Nonlinear Mathematical Physics 27:4, 592–615, DOI: https://doi.org/10.1080/14029251.2020.1819605

**To link to this article**: https://doi.org/10.1080/14029251.2020.1819605

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 27, No. 4 (2020) 592–615

# Asymptotics behavior for the integrable nonlinear Schrödinger equation with quartic terms: Cauchy problem

Lin Huang
*School of Science, Hangzhou Dianzi University,*
*Hangzhou 310018, P. R. China*
*lin.huang@hdu.edu.cn & linhuang@kth.se*

Received 10 August 2019

Accepted 3 January 2020

We consider the Cauchy problem of integrable nonlinear Schrödinger equation with quartic terms on the line. The first part of the paper considers the Riemann-Hilbert formula via the unified method(also known as the Fokas method). The second part of the paper establishes asymptotic formulas for the solution of initial value problem using the nonlinear steepest descent method(also known as the Deift-Zhou method).

*Keywords*: Integrable nonlinear Schrödinger equation with quartic terms; Long-time asymptotics; Nonlinear steepest descent method.

2000 Mathematics Subject Classification: 37K15, 41A60, 35P25.

## 1. Introduction

The long-time asymptotics of initial value problem of integrable nonlinear evolution equations can be analyzed by means of the nonlinear steepest descent method introduced by Deift and Zhou [19]. In the context of initial value problems, the Riemann-Hilbert (RH) problem is formulated on the basis of certain spectral functions whose definitions involve the initial data of the solution [3, 40]. In this way, the long-time asymptotics for the solutions of decay initial value problem of the mKdV equation and the Schrödinger equation were analyzed respectively by Deift, Its and Zhou [17, 19]. This method developed by several authors [18, 27] and has already been used for:

(a) For the integrable equation with $2\times 2$ spectral problem (i.e., $2\times 2$ Lax pair);

- (i) Decay initial value problem: modified KdV equation [19]; defocusing NLS(nonlinear Schrödinger equation) [17]; sine-Gordon equation [15]; Camassa-Holm equation [11]; modified NLS equation [29]; Fokas-Lenells equation [38]; Hirota equation [26]; short pulse equation [37];
- (ii) Nondecaying initial value problem: focusing NLS equation [8]; derivative NLS equation [39]; modified KdV equation [30]; Camassa-Holm equation [32];
- (iii) Decay initial boundary value problem: modified KdV equation [27]; derivative NLS equation [2], Hirota equation [23]; sine-Gordon equation [24];
- (iv) Time-periodic boundary value problem: focusing NLS equation [5–8], stimulated Raman scattering [33];
- (v) Nonzero boundary conditions at infinity: focusing NLS equation [4] focusing Kundu–Eckhaus equation [36].

(b) For the integrable equation (system) with $3\times 3$ spectral problem (i.e., $3\times 3$ Lax pair);

(i) Decay initial value problem: Degasperis-Procesi equation [10], coupled nonlinear Schrödinger equations [21]; Boussinesq equation [16]; Sasa-satsuma equation [22, 25, 28].

(ii) Decay initial boundary value problem: Degasperis-Procesi equation [9].

The aim of this paper is to analyze long-time asymptotics for the initial value problem of the integrable nonlinear Schrödinger equation with quartic terms(NLSQ),

$$iq_t+\frac{1}{2}q_{xx}-|q|^2q+\gamma(q_{xxxx}-6q_x^2q^*-4q|q_x|^2-8q_{xx}|q|^2-2q_{xx}^*q^2+6q|q|^4)=0, \tag{1.1a}$$

$$\text{the initial data } q(x,0)=q_0(x)\in\mathscr{S}(\mathbb{R}) \tag{1.1b}$$

where $\gamma$ is positive constant and the $\mathscr{S}(\mathbb{R})$ is denote the Schwartz class on the line. This equation was first analysed by Lakshmanan, Porsezian, Danielin in papers [31, 34, 35], and it is a modified nonlinear Schrödinger equation that takes into account fourth order dispersion. The NLSQ equation was considered by many scientists, they obtained some especial solutions for the equation, for example, soliton solution [1, 12], Breather solutions and rogue wave [13]. More recently, the quintic equation of the hierarchy has been considered in [14]. The authors show that a breather solution and get the locus of the eigenvalues in the complex plane which converts breathers into solitons. Considering the following Lax pair representation [1]:

$$\psi_x=M\psi,\quad \psi_t=N\psi. \tag{1.2}$$

where

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{1.3}$$

$$M=i\lambda\sigma_3+\begin{pmatrix}0 & ir\\ iq & 0\end{pmatrix}\equiv i\lambda\sigma_3+\mathbf{U}, \tag{1.4}$$

$$N=N_4\lambda^4+N_3\lambda^3+N_2\lambda^2+N_1\lambda+N_0$$

where

$$N_4=\begin{pmatrix}-8i\gamma & 0\\ 0 & 8i\gamma\end{pmatrix},\quad N_3=\begin{pmatrix}0 & -8i\gamma r\\ -8i\gamma q & \end{pmatrix},\quad N_2=i\begin{pmatrix}1+4\gamma qr & 4i\gamma r_x\\ -4i\gamma q_x & -1-4\gamma qr\end{pmatrix}$$

$$N_1=i\begin{pmatrix}-2i\gamma(r_xq-q_xr) & r+4\gamma qr^2+2\gamma r_{xx}\\ q+4\gamma q^2r+2\gamma q_{xx} & 2i\gamma(r_xq-q_xr)\end{pmatrix},\quad N_0=\begin{pmatrix}A_1 & B_1\\ C_1 & -A_1\end{pmatrix}$$

$$A_1=i(-\frac{1}{2}qr-3\gamma q^2r^2-\gamma(r_{xx}q-q_xr_x+q_{xx}r))$$

$$B_1=-\frac{i}{2}r_x-6i\gamma qrr_x-i\gamma r_{xxx},\quad C_1=\frac{i}{2}q_x+6i\gamma qrq_x+i\gamma q_{xxx}.$$

The zero-curvature equation reproduces the following equation,

$$iq_t+\frac{1}{2}q_{xx}+q^2r+\gamma(q_{xxxx}+6r^2q^3+4qr_xq_x+2q^2r_{xx}+6rq_x^2+8rqq_{xx})=0 \tag{1.5}$$

If we take $r=-q^*$, then the equation (1.5) can be write as (1.1a).

Our main results are shown in Section 2. They state in the form of two theorems (Theorem 1-2). Theorem 2.1 is concerned with the construction of solutions of the initial value problem, and the proof relies on the RH techniques(also call Fokas method). Section 3 is devotes to Theorem 2.2. The proof is based on the nonlinear steepest descent approach of the Deift and Zhou [19].

## 2. Main results

The first theorem presents how to get the solutions of (1.1) can be constructed starting from the reflect coefficient function $r(\lambda)$. Let $\mathscr{S}(\mathbb{R})$ denote the Schwartz class of smooth rapidly decaying functions.

**Theorem 2.1 (*Construction of solutions*).** *Suppose $r(\lambda) \in \mathscr{S}(\mathbb{R})$. Define the $2\times 2$-matrix valued jump matrix $J(x,t,\lambda)$ by*

$$J(\lambda) = \begin{pmatrix} 1-|r(\lambda)|^2 & -\overline{r(\lambda)}e^{2(i\lambda x-(8i\gamma\lambda^4-i\lambda^2)t)} \\ r(\lambda)e^{-2(i\lambda x-(8i\gamma\lambda^4-i\lambda^2)t)} & 1 \end{pmatrix} \tag{2.1}$$

*Then the $2\times 2$-matrix RH problem*

- *$M(x,t,\lambda)$ is analytic in $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*
- *The boundary value $M_\pm(x,t,\lambda)$ at $\mathbb{R}$ satisfy the jump condition*

$$M_+(x,t,\lambda) = M_-(x,t,\lambda)J(x,t,\lambda), \quad \lambda \in \mathbb{R},$$

- *Behavior at $\infty$*

$$M(x,t,\lambda) = \mathbb{I} + O\left(\frac{1}{\lambda}\right), \qquad \text{as } \lambda \to \infty.$$

*has a unique solution for each $(x,t)\in\mathbb{R}^2$ and the limit $\lim_{\lambda\to\infty}(\lambda M(x,t,\lambda))_{12}$ exists for each $(x,t)\in\mathbb{R}^2$. Moreover, the function $q(x,t)$ defined by*

$$q(x,t) = 2i\lim_{\lambda\to\infty}(\lambda M(x,t,\lambda))_{12} \tag{2.2}$$

*is a smooth function of $(x,t)\in\mathbb{R}^2$ with rapid decay as $|x|\to\infty$ which satisfies the integrable nonlinear Schrödinger equation with quartic terms (1.1) for $(x,t)\in\mathbb{R}^2$.*

*Proof.* See Section 3. □

The second theorem gives the long-time asymptotics of the solutions constructed in Theorem 2.1 in the sector $\mathscr{P} = \left\{(x,t) \mid \left(\frac{x}{t}\right)^2 < \frac{1}{27\gamma}\right\}$.

**Theorem 2.2.** *Assuming* $\mathscr{P}=\left\{(x,t)\mid\left(\frac{x}{t}\right)^2<\frac{1}{27\gamma}\right\}$, *let* $q(x,t)$ *be the solution of the Cauchy problem* (1.1), *then as* $t\to\infty$ *and* $(x,t)\in\mathscr{P}$, *we have*

$$
\begin{aligned}
q(x,t)=&\frac{1}{\sqrt{8t(1+8\gamma\lambda_0^2)}}(((32\lambda_0^2t(1+8\gamma\lambda_0^2))^{-i\nu(\lambda_0)/2}e^{it(2\lambda_0^2(4\gamma\lambda_0^2+1))}e^{\chi_0(\lambda_0)})^2(M_1^A)_{12})\\
&+\frac{1}{\sqrt{8t(1+8\gamma\lambda_2^2)}}(((32\lambda_2^2t(1+8\gamma\lambda_2^2))^{-i\nu(\lambda_2)/2}e^{it(2\lambda_2^2(4\gamma\lambda_2^2+1))}e^{\chi_2(\lambda_2)})^2(M_1^B)_{12})\\
&+\frac{1}{2\sqrt{t(1+8\gamma\lambda_1^2)}}(((16\lambda_1^2t(1+8\gamma\lambda_1^2))^{-i\nu(\lambda_1)/2}e^{it(2\lambda_1^2(4\gamma\lambda_1^2+1))}e^{\chi_1(\lambda_1)})^2(M_1^C)_{12})\\
&+O\left(\frac{\log t}{t}\right).
\end{aligned}\tag{2.3}
$$

*In the region* $0<\max\{|\lambda_0|,|\lambda_1|,|\lambda_2|\}<M$, *M is a positive constant. Where* $\lambda_0$, $\lambda_1$, $\lambda_2$ *are three stationary points of phase function which defined in* (4.1). $(M_1^A)_{12}$, $(M_1^B)_{12}$, $(M_1^C)_{12}$, $\nu(\lambda_0)$, $\nu(\lambda_1)$, $\nu(\lambda_2)$ *are defined by* (4.32) ,(4.31), (4.33) *and* (4.6), *respectively* , $\chi_0(\lambda_0)$, $\chi_1(\lambda_1)$, $\chi_2(\lambda_2)$, *are defined by following*

$$
\begin{aligned}
\chi_0(\lambda_0)=&\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_0}\ln\left(\frac{1-|r(\mu)|^2}{1-|r(|\lambda_0|)|^2}\right)\frac{d\mu}{\mu-\lambda_0}\\
\chi_2(\lambda_2)=&\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2}\ln\left(\frac{1-|r(\mu)|^2}{1-|r(|\lambda_2|)|^2}\right)\frac{d\mu}{\mu-\lambda_2}\\
\chi_1(\lambda_1)=&\frac{1}{2\pi i}\int_{i\lambda_1\pm i\infty}^{i\lambda_1}\ln\left(\frac{1-|r(\mu)|^2}{1-|r(|\lambda_1|)|^2}\right)\frac{d\mu}{\mu-\lambda_1}.
\end{aligned}
$$

*Proof.* See Section 4 □

## 3. The proof of Theorem 2.1

In this part, we only give a sketch how to prove Theorem 2.1 from the inverse scattering transformation of NLSQ equation.

We extend the column vector $\psi$ to a $2\times2$ matrix and letting

$$\psi=\Psi e^{(i\lambda x-(8i\gamma\lambda^4-i\lambda^2)t)\sigma_3},$$

then the Lax pair (1.2) can be rewritten as follows,

$$\Psi_x-i\lambda[\sigma_3,\Psi]=\mathbf{U}\Psi,\tag{3.1a}$$

$$\Psi_t+(8i\gamma\lambda^4-i\lambda^2)[\sigma_3,\Psi]=V\Psi\tag{3.1b}$$

where $\mathbf{U}$ is defined by (1.4) and $V$ is defined follows

$$V=N_3\lambda^3+(N_2-\sigma_3)\lambda^2+N_1\lambda+N_0\tag{3.2}$$

which can be written in total differential form,

$$d(e^{[-i\lambda x+(8i\gamma\lambda^4-i\lambda^2)t)]\widehat{\sigma}_3}\Psi(x,t,\lambda))=e^{[-i\lambda x+(8i\gamma\lambda^4-i\lambda^2)t]\widehat{\sigma}_3}W(x,t,\lambda)\Psi,$$

where

$$\begin{aligned} W(x,t,\lambda) &= \mathbf{U}dx + Vdt \\ &= \begin{pmatrix} 0 & -i\overline{q} \\ iq & 0 \end{pmatrix} dx + (-\lambda \mathbf{U} + \frac{1}{2}\mathbf{V} + \gamma \mathbf{V_p})dt. \end{aligned}$$

In order to formulate a RH problem for the solution of the inverse spectral problem,we seek solutions of the spectral problem which approach the $2 \times 2$ identity matrix as $\lambda \to \infty$.

Throughout this section, assuming that $q(x,t)$ is sufficiently smooth. Defining two solutions of (3.1a) and (3.1b) by

$$\Psi_j(x,t,\lambda) = \mathbb{I} + \int_{(x_j,t_j)}^{(x,t)} e^{\,[i\lambda(x-x')-(8i\gamma\lambda^4-i\lambda^2)(t-t')\,]\widehat{\sigma}_3}(\mathbf{U}dx' + Vdt'), \qquad j = 1,2,$$

where $(x_1,t_1) = (-\infty,t), (x_2,t_2) = (+\infty,t)$.

This choice implies following,

$$\begin{aligned} &\Gamma_1 : x > x', \quad \text{i.e.,} \quad x - x' > 0, \\ &\Gamma_2 : x < x', \quad \text{i.e.,} \quad x - x' < 0. \end{aligned}$$

It means that we can write $\Psi_j$ as:

$$\begin{aligned} \Psi_1 &= \mathbb{I} + \int_{-\infty}^{x} e^{-i\lambda(x-x')\widehat{\sigma}_3}\mathbf{U}(x',t,\lambda)\Psi_1(x',t,\lambda)dx', \\ \Psi_2 &= \mathbb{I} - \int_{x}^{\infty} e^{-i\lambda(x-x')\widehat{\sigma}_3}\mathbf{U}(x',t,\lambda)\Psi_2(x',t,\lambda)dx'. \end{aligned}$$

It can be shown that the second column of $\Psi_j$ is bounded and analytic in the following domain:

$$\begin{aligned} &[\Psi_1]_2 \text{ in low-half plane,} \quad [\Psi_1]_1 \text{ in up-half plane,} \\ &[\Psi_2]_1 \text{ in low-half plane,} \quad [\Psi_2]_2 \text{ in up-half plane.} \end{aligned}$$

According to the ordinary differential equation theory, it follows that the two solutions of (3.1) have linear relations,

$$\Psi_1(x,t,\lambda) = \Psi_2(x,t,\lambda)e^{[i\lambda x-(8i\gamma\lambda^4-i\lambda^2)t]\widehat{\sigma}_3}S(\lambda), \tag{3.3}$$

where

$$S(\lambda) = \begin{pmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{pmatrix}.$$

It is easy to show that the following symmetry for $\Psi(x,t,\lambda)$:

$$\Psi(x,t,\lambda) = \sigma_1\overline{\Psi(x,t,\overline{\lambda})}\sigma_1, \quad \sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The symmetry implies as follows,

$$S(\lambda) = \begin{pmatrix} a(\lambda) & \overline{b(\overline{\lambda})} \\ b(\lambda) & \overline{a(\overline{\lambda})} \end{pmatrix}.$$

Taking the sectionally meromorphic function $M(x,t,\lambda)$ defined by

$$M(x,t,\lambda)=\begin{cases}\left(\dfrac{[\Psi_1]_1}{a(\lambda)},[\Psi_2]_2\right), & \operatorname{Im}\lambda>0,\\ \left([\Psi_2]_1,\dfrac{[\Psi_1]_2}{\overline{a(\bar{\lambda})}}\right), & \operatorname{Im}\lambda<0,\end{cases}$$

where $[\cdot]_1$ and $[\cdot]_2$ denote the first and second column, respectively.

By the analytic conditions of $\Psi_1,\Psi_2,a(\lambda)$. It means that the function $M(x,t,\lambda)$ is well-defined, and (3.3) can be rewritten as

$$M_+(x,t,\lambda)=M_-(x,t,\lambda)J(\lambda),$$

where the jump matrix $J(\lambda)$ is given by (2.1) with the function $r(\lambda)=b(\lambda)/a(\lambda)$.

**Remark 3.1.** Noticing that the jump matrix $J(\lambda)$ in (2.1) is Hermite and positive. By the Zhou' law [41], we can get the vanishing lemma for the Riemann-Hilbert problem $M(x,t,\lambda)$. It means that the associated homogeneous RH problem has only the zero solution. In other wards, the solution of Riemann-Hilbert problem exists.

Substituting the asymptotic expansion

$$\Psi=D+\frac{\Psi_1}{\lambda}+\frac{\Psi_2}{\lambda^2}+\frac{\Psi_3}{\lambda^3}+O\left(\frac{1}{\lambda^4}\right),\quad \lambda\to\infty$$

into (3.1) and comparing the coefficients of $\lambda$, we find

$$O(\lambda):\quad i[\sigma_3,D]=0, \tag{3.4}$$
$$O(1):\quad D_x+i[\sigma_3,\Psi_1]=UD. \tag{3.5}$$

Hence one can get $D=\mathbb{I}$.

Inserting into the off-diagonal elements of (3.5), we can get

$$q(x,t)=2i(\Psi_1)_{12},$$

that is,

$$q(x,t)=2i\lim_{\lambda\to\infty}(\lambda M(x,t,\lambda))_{12}.$$

where $M$ is the solution of the following RH problem: Given $r(\lambda),\lambda\in\mathbb{R}$, find a $2\times2$ matrix-value function $M(x,t,\lambda)$ such that

- $M(x,t,\lambda)$ is analytic in $\lambda\in\mathbb{C}\setminus\mathbb{R}$.
- The boundary value $M_\pm(x,t,\lambda)$ at $\Sigma$ satisfy the jump condition

$$M_+(x,t,\lambda)=M_-(x,t,\lambda)e^{(i\lambda x-(8i\gamma\lambda^4-i\lambda^2)t)\hat{\sigma}_3}J(x,t,\lambda),\quad \lambda\in\mathbb{R}, \tag{3.6}$$

  where the jump matrix $J(x,t,\lambda)$ is defined in terms of $r(\lambda)$ by (2.1).
- Behavior at $\infty$

$$M(x,t,\lambda)=\mathbb{I}+O\left(\frac{1}{\lambda}\right),\qquad \text{as } \lambda\to\infty.$$

**Lemma 3.1.** *Define* $q(x,t)$ *by* (2.2)*, then*

$$\begin{cases} M_x(x,t,\lambda) - i\lambda[\sigma_3, M(x,t,k)] = \mathbf{U}M(x,t,\lambda), \\ M_t(x,t,\lambda) + (8i\gamma\lambda^4 - i\lambda^2)[\sigma_3, M(x,t,k)] = VM(x,t,\lambda). \end{cases} \tag{3.7}$$

*where* $\mathbf{U}$ *and* $V$ *are defined in terms of* $q(x,t)$ *and the derivative of* $q(x,t)$ *by* (1.4) *and* (3.2)*, respectively.*

*Proof.* We can use the Foaks method and the dressing method to prove the lemma. The argument is analogous to that Lemma 4.1 in [25], so the proof will be omitted. □

The compatibility condition of (3.7) shows that $q(x,t)$ satisfies (1.1a). The proof of Theorem 2.1 is complete.

## 4. The proof of Theorem 2.2

In order to analyze the long-time behavior of the solution of the NLSQ equation, first we should split the jump matrix into an appropriate upper/lower triangular form which can help us to localize the problem to the neighborhood of the stationary point. An appropriate rescaling then reduces an oscillatory RH problem to a RH problem with constant coefficients, which can be solved explicitly in terms of classical functions.

### 4.1. *The augmented RH problem*

We begin with the decomposition of the complex $\lambda$-plane according to the signature of the real part of the phase of the conjugating exponential of the oscillatory RH problem , $\mathrm{Re}(2it\theta(\lambda))$, where

$$\theta(\lambda) = -\frac{x}{t}\lambda + 8\gamma\lambda^4 - \lambda^2, \tag{4.1}$$

Assuming $x/t := \xi$ satisfies the inequality,

$$\xi^2 < \frac{1}{27\gamma},$$

then the phase function $\theta(\lambda)$ have three real stationary points $\lambda_0 > \lambda_1 > \lambda_2$. The signature table of $\mathrm{Re}(2it\theta(\lambda))$ is given in Figure 1.

The main purpose of this section is to reformulate the original RH problem (Lemma 2.1) as an equivalent RH problem (see Lemma 4.1) on the augmented contour $\Sigma$ (see Figure 2),

$$\Sigma = L \cup \bar{L} \cup \mathbb{R},$$

where,

$$\begin{aligned} L = &\{\lambda; \lambda = \lambda_0 + \frac{\lambda_0 u}{2} e^{\frac{3\pi i}{4}}, u \in (-\infty, \sqrt{2}]\} \cup \{\lambda; \lambda = \lambda_2 + \frac{|\lambda_2| u}{2} e^{-\frac{i\pi}{4}}, u \in (-\infty, \sqrt{2}]\} \\ &\cup \{\lambda; \lambda = \lambda_1 + \frac{|\lambda_1| u}{2} e^{\frac{i\pi}{4}}, u \in (-\infty, \sqrt{2}]\}. \end{aligned} \tag{4.2}$$

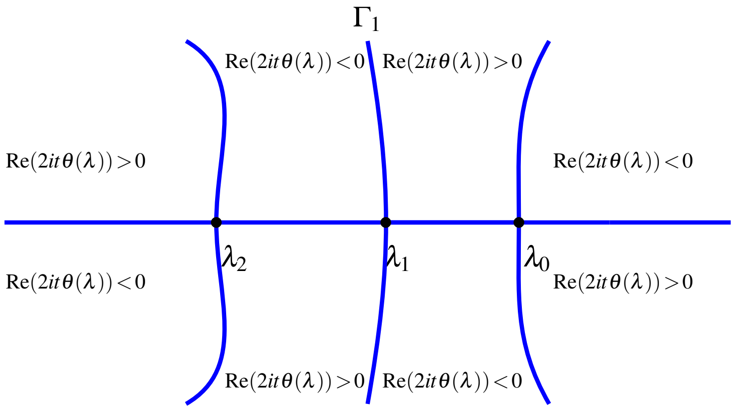


Fig. 1. The signature of $\mathrm{Re}(2it\theta(\lambda))$

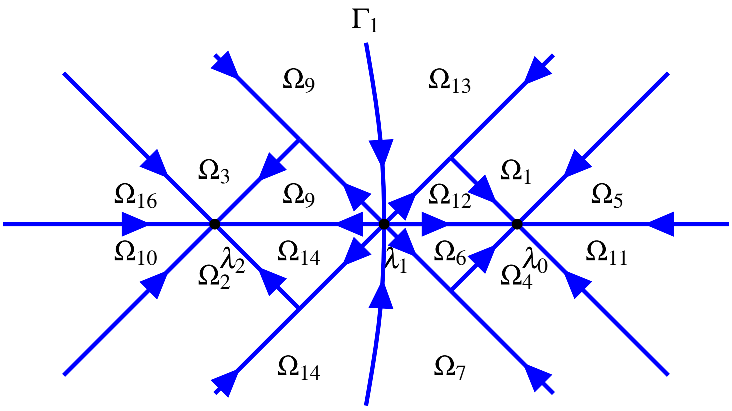


Fig. 2. Augmented contour $\Sigma$.

In order to define the conjugation matrices on $\Sigma$ and exploit the analyses in [3, 19], we need to formulate two technical propositions: the first concerns the triangular factorisation of the conjugation matrices of the original RH problem (3.6), and the second pertains to a special decomposition for the reflection coefficient $r(\lambda)$. The jump matrices (3.6) have following form,

$$e^{-it\theta(\lambda)\widehat{\sigma_3}}G(\lambda) = e^{-it\theta(\lambda)\widehat{\sigma_3}}\begin{pmatrix}1 & r(\lambda)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\overline{r(\overline{\lambda})} & 1\end{pmatrix},$$

for $\lambda \in (-\infty, \lambda_2) \cup (\lambda_0, +\infty)$, and, the form

$$\begin{aligned} e^{-it\theta(\lambda)\widehat{\sigma_3}}G(\lambda) &= e^{-it\theta(\lambda)\widehat{\sigma_3}}\begin{pmatrix}1 & 0\\ -\overline{r(\overline{\lambda})}(1-\overline{r(\overline{\lambda})}r(\lambda))^{-1} & 1\end{pmatrix} \\ &\times \begin{pmatrix}(1-\overline{r(\overline{\lambda})}r(\lambda)) & 0\\ 0 & (1-\overline{r(\overline{\lambda})}r(\lambda))^{-1}\end{pmatrix}\begin{pmatrix}1 & r(\lambda)(1-\overline{r(\overline{\lambda})}r(\lambda))^{-1}\\ 0 & 1\end{pmatrix}. \end{aligned} \tag{4.3}$$

for $\lambda \in (\lambda_2, \lambda_0) \cup \{(\lambda_1 - i\infty, \lambda + i\infty) \in \Gamma_1\}$.

In order to eliminate the diagonal matrix between the lower/upper triangular factors in (4.3), by the scheme in [19], introduce the function $\delta(\lambda)$ which solves the scalar RH problem:

$$\delta_+(\lambda) = \begin{cases} \delta_-(\lambda) = \delta(\lambda), & \lambda \in (-\infty, \lambda_2) \cup (\lambda_0, +\infty), \\ \delta_-(\lambda)(1 - \overline{r(\overline{\lambda})} r(\lambda)), & \lambda \in (\lambda_2, \lambda_0) \cup \{(\lambda_1 - i\infty, \lambda + i\infty) \in \Gamma_1\}, \end{cases} \tag{4.4}$$
$$\delta(\lambda) \to 1 \text{ as } \lambda \to \infty.$$

According to the Plemelj formulae, it is straightforward to show that the solution of RH problem (4.4) is given by

$$\delta(\lambda) = \left(\tfrac{\lambda-\lambda_0}{\lambda-\lambda_1}\right)^{i\nu(\lambda_0)} \left(\tfrac{\lambda-\lambda_2}{\lambda-\lambda_1}\right)^{i\nu(\lambda_2)} e^{\chi_+(\lambda)} e^{\chi_-(\lambda)} e^{\widehat{\chi}_+(\lambda)} e^{\widehat{\chi}_-(\lambda)}, \tag{4.5}$$

where

$$\nu(\lambda_0) = -\frac{1}{2\pi} \log(1 - |r(\lambda_0)|^2), \tag{4.6a}$$
$$\nu(\lambda_1) = -\frac{1}{2\pi} \log(1 - |r(\lambda_1)|^2), \tag{4.6b}$$
$$\nu(\lambda_2) = -\frac{1}{2\pi} \log(1 - |r(\lambda_2)|^2), \tag{4.6c}$$

and

$$\chi_+(\lambda) = \frac{1}{2\pi i} \int_{\lambda_1}^{\lambda_0} \ln\left(\frac{1 - |r(\mu)|^2}{1 - |r(\lambda_0)|^2}\right) \frac{d\mu}{(\mu - \lambda)}, \quad \chi_-(\lambda) = \frac{1}{2\pi i} \int_{\lambda_1}^{\lambda_2} \ln\left(\frac{1 - |r(\mu)|^2}{1 - |r(\lambda_0)|^2}\right) \frac{d\mu}{(\mu - \lambda)},$$
$$\widehat{\chi}_+(\lambda) = \int_{\lambda_1 + i\infty}^{\lambda_1} \frac{\ln(1 - \overline{r(\overline{\mu})} r(\mu))}{(\mu - \lambda)} \frac{d\mu}{2\pi i}, \quad \widehat{\chi}_-(\lambda) = \int_{\lambda_1 - i\infty}^{\lambda_1} \frac{\ln(1 - \overline{r(\overline{\mu})} r(\mu))}{(\mu - \lambda)} \frac{d\mu}{2\pi i}$$

**Proposition 4.1.** *Let*

$$f(z) = \begin{cases} \frac{-\overline{r(\lambda)}}{1 - |r(\lambda)|^2}, & \lambda \in (\lambda_2, \lambda_0) \cup \{(\lambda_1 - i\infty, \lambda + i\infty) \in \Gamma_1\}, \\ r(\lambda), & \lambda \in (-\infty, \lambda_2) \cup (\lambda_0, \infty). \end{cases}$$

*Then $f$ has a decomposition*

$$f(\lambda) = h_I(\lambda) + h_{II}(\lambda) + R(\lambda), \quad \lambda \in \mathbb{R},$$

*where $R(\lambda)$ is piecewise rational and $h_{II}(\lambda)$ has an analytic continuation to L. Moreover, in the domain* $\max\{|\lambda_2|, |\lambda_1|, |\lambda_0|\} < M$*, the following estimates are valid as $t \to \infty$,*

$$|e^{-2it\theta(\lambda)} h_I(\lambda)| \leqslant \frac{c}{(1 + |\lambda|^2) t^\ell}, \qquad \lambda \in \mathbb{R},$$
$$|e^{-2it\theta(\lambda)} h_{II}(\lambda)| \leqslant \frac{c}{(1 + |\lambda|^2) t^\ell}, \qquad \lambda \in L,$$
$$|e^{-2it\theta(\lambda)} R(\lambda)| \leqslant C e^{-\overline{c}\varepsilon^2}, \qquad \lambda \in L_\varepsilon,$$

*where $L$ is given by* (4.2) *and $L_\varepsilon$ is defined by follows*

$$L_\varepsilon = \{\lambda; \lambda = \lambda_0 + \frac{\lambda_0 u}{2} e^{\frac{3\pi i}{4}}, u \in (\varepsilon, \sqrt{2}]\} \cup \{\lambda; \lambda = \lambda_2 + \frac{|\lambda_2| u}{2} e^{-\frac{i\pi}{4}}, u \in (\varepsilon, \sqrt{2}]\},$$
$$\cup \{\lambda; \lambda = \lambda_1 + \frac{|\lambda_1| u}{2} e^{\frac{i\pi}{4}}, u \in (\varepsilon, \sqrt{2}]\},$$

*Then the conjugates*

$$\overline{f(\lambda)} = \overline{h_I(\lambda)} + \overline{h_{II}(\overline{\lambda})} + \overline{R(\overline{\lambda})}$$

*gives the same estimates for* $e^{2it\theta(\lambda)}\overline{h_I(\lambda)}$, $e^{2it\theta(\lambda)}\overline{h_{II}(\lambda)}$ *and* $e^{2it\theta(\lambda)}\overline{R(\overline{\lambda})}$ *on* $\mathbb{R}\cup\overline{L}$.

*Proof.* The argument is analogously as in the proof of Proposition 1.92 in [19] by expanding $\rho(\lambda)$ in terms of a rational polynomial approximation in the neighbourhood of the real, first-order stationary phase points, $\lambda_0$, $\lambda_1$, $\lambda_2$. □

**Lemma 4.1.** *Let* $m(x,t;\lambda)$ *be the solution of the RH problem formulated in* (3.6). *Set* $m^{\Delta}(x,t;\lambda) \equiv m(x,t;\lambda)(\Delta(\lambda))^{-1}$, *where*

$$\Delta(\lambda) \equiv (\delta(\lambda))^{\sigma_3},$$

*and* $\delta(\lambda)$ *is given by* (4.5). *Define*

$$m^{\sharp}(x,t;\lambda) \equiv \begin{cases} m^{\Delta}(x,t;\lambda), & \lambda \in \Omega_1\cup\Omega_2\cup\Omega_3\cup\Omega_4, \\ m^{\Delta}(x,t;\lambda)(\mathrm{I}-(w_-^a)_{x,t,\delta})^{-1}, & \lambda \in \Omega_5\cup\Omega_6\cup\Omega_7\cup\Omega_8\cup\Omega_9\cup\Omega_{10}, \\ m^{\Delta}(x,t;\lambda)(\mathrm{I}+(w_+^a)_{x,t,\delta})^{-1}, & \lambda \in \Omega_{11}\cup\Omega_{12}\cup\Omega_{13}\cup\Omega_{14}\cup\Omega_{15}\cup\Omega_{16}. \end{cases}$$

*Then* $m^{\sharp}(x,t;\lambda)$ *solves the following (augmented) RH problem on* $\Sigma$,

$$\begin{gathered} m_+^{\sharp}(x,t;\lambda) = m_-^{\sharp}(x,t;\lambda)v_{x,t,\delta}^{\sharp}(\lambda), \quad \lambda \in \Sigma, \\ m^{\sharp}(x,t;\lambda) \to \mathrm{I} \text{ as } \lambda \to \infty, \end{gathered} \tag{4.7}$$

*where*

$$v_{x,t,\delta}^{\sharp}(\lambda) \equiv (\mathrm{I}-(w_-^{\sharp})_{x,t,\delta})^{-1}(\mathrm{I}+(w_+^{\sharp})_{x,t,\delta}) = \begin{cases} (\mathrm{I}-(w_-^0)_{x,t,\delta})^{-1}(\mathrm{I}+(w_+^0)_{x,t,\delta}), & \lambda\in\mathbb{R}, \\ (\mathrm{I}+(w_+^a)_{x,t,\delta}), & \lambda\in L, \\ (\mathrm{I}-(w_-^a)_{x,t,\delta})^{-1}, & \lambda\in\overline{L}, \end{cases}$$

*and*

$$\begin{aligned} (w_{\pm}^{(0,a)})_{x,t,\delta} &= (\delta_{\pm}(\lambda))^{\widehat{\sigma_3}} \exp\left\{-it\theta(\lambda)\widehat{\sigma_3}\right\} w_{\pm}^{(0,a)}, \\ w_+^0 &= \begin{pmatrix} 0 & h_I(\lambda) \\ 0 & 0 \end{pmatrix}, \quad w_+^a = \begin{pmatrix} 0 & (h_{II}(\lambda)+R(\lambda)) \\ 0 & 0 \end{pmatrix}, \\ w_-^0 &= \begin{pmatrix} 0 & 0 \\ -\overline{h_I(\lambda)} & 0 \end{pmatrix}, \quad w_-^a = \begin{pmatrix} 0 & 0 \\ -(\overline{h_{II}(\overline{\lambda})}+\overline{R(\overline{\lambda})}) & 0 \end{pmatrix} \end{aligned}$$

*Proof.* In terms of the function $m^{\Delta}(x,t;\lambda)$ defined in the Lemma, the original oscillatory RH problem (3.6) can be rewritten in the following form,

$$\begin{gathered} m_+^{\Delta}(x,t;\lambda) = m_-^{\Delta}(x,t;\lambda)(\mathrm{I}-(w_-)_{x,t,\delta})^{-1}(\mathrm{I}+(w_+)_{x,t,\delta}), \quad \lambda \in \mathbb{R}, \\ m^{\Delta}(x,t;\lambda) \to \mathrm{I} \text{ as } \lambda \to \infty, \end{gathered}$$

where

$$(w_{\pm})_{x,t,\delta} = (\delta_{\pm}(\lambda))^{\widehat{\sigma_3}} \exp\left\{-it\theta(\lambda)\widehat{\sigma_3}\right\} w_{\pm},$$

and

$$w_+ = \begin{pmatrix} 0 & \rho(\lambda) \\ 0 & 0 \end{pmatrix}, \quad w_- = \begin{pmatrix} 0 & 0 \\ -\overline{\rho(\bar{\lambda})} & 0 \end{pmatrix}.$$

Then the new RH problem is equivalent to the RH problem (3.6). □

## 4.2. *RH Problem on the Truncated Contour*

In this section we show how to convert the RH problem (4.7) on $\Sigma$ to a RH problem on a truncated contour with controlled error terms.

From the above section we have

$$\begin{aligned}\widetilde{q}(x,t) &= 2i \lim_{\lambda\to\infty} (zm^{\Delta}(x,t,\lambda))_{12} \\ &= i \lim_{\lambda\to\infty} \lambda([\sigma_3, m^{\Delta}(x,t,z)])_{12} \\ &= i \lim_{\lambda\to\infty} \lambda([\sigma_3, m^{\sharp}(x,t,z)])_{12}.\end{aligned}$$

We can take the limit $z\to\infty$ in $\Omega_1$, where $m^{\Delta}(x,t,z) = m^{\sharp}(x,t,z)$, so

$$\widetilde{q}(x,t) = i \lim_{\lambda\to\infty} \lambda([\sigma_3, m^{\sharp}(x,t,\lambda)])_{12}. \tag{4.8}$$

The RH problem (4.7) can be solved as follows (see, for example, [3]). Let

$$(C_{\pm}f)(\lambda) = \int_{\Sigma} \frac{f(\zeta)}{\zeta-\lambda_{\pm}} \frac{d\zeta}{2\pi i}, \quad \lambda\in\Sigma, f\in L^2(\Sigma), \tag{4.9}$$

denote the Cauchy operator on $\Sigma$ oriented as in Figure 2. Thus, for example, for $\lambda > \lambda_0$ we have $(C_+f)(\lambda) = \lim_{\varepsilon\downarrow 0}\int_{\Sigma} \frac{f(\zeta)}{\zeta-(\lambda-i\varepsilon)}\frac{d\zeta}{2\pi i}$, etc. As is well known, the operators $C_{\pm}$ are bounded from $L^2(\Sigma)$ to $L^2(\Sigma)$, and $C_+ - C_- = 1$. Also, by scaling, we know that the bounds on $C_{\pm}: L^2(\Sigma)\to L^2(\Sigma)$ are independent of the stationary points.

Define

$$C_{w^{1'}_{x,t,\delta}} f = C_+(f(w^{1'}_-)_{x,t,\delta}) + C_-(f(w^{1'}_+)_{x,t,\delta}), \tag{4.10}$$

for $2\times 2$ matrix-valued functions $f$. By Proposition 4.1, $C_{w^{1'}_{x,t,\delta}}$ is a bounded map from $L^2(\Sigma)+L^{\infty}(\Sigma)$ into $L^2(\Sigma)$. Let $\mu^{1'} = \mu^{1'}(z;x,t) \in L^2(\Sigma)+L^{\infty}(\Sigma)$ be the solution of the basic inverse equation

$$\mu^{1'} = \mathbb{I} + C_{w^{1'}_{x,t,\delta}}\mu^{1'}. \tag{4.11}$$

By the method of Beals and Coifman,

$$m^{\sharp}(x,t,\lambda) = \mathbb{I} + \int_{\Sigma} \frac{\mu^{1'}(\zeta;x,t)w^{1'}_{x,t,\delta}(\zeta)}{\zeta-\lambda}\frac{d\zeta}{2\pi i}, \quad \lambda\in\mathbb{C}\backslash\Sigma \tag{4.12}$$

is the unique solution of the RH problem (4.7). Indeed,

$$\mu^{1'}_{\pm} = \mathbb{I} + C_{\pm}(\mu^{1'}w^{1'}_{x,t,\delta}) = \mu^{1'}(b^{1'}_{\pm})_{x,t,\delta}$$

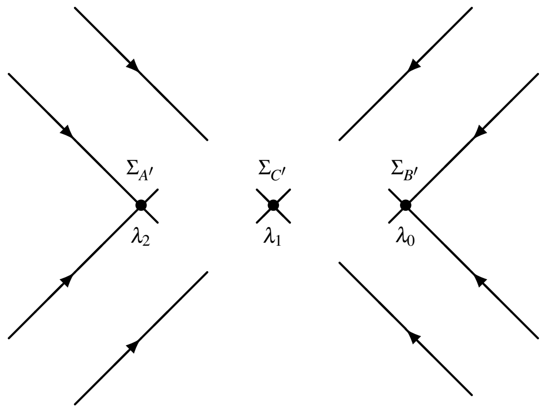


Fig. 3. Truncated contour $\Sigma'$.

Substituting formula (4.12) into (4.8), we learn that

$$\begin{aligned} q(x,t) &= -\Big(\int_{\Sigma_1}[\sigma_3,\mu^{1'}(\zeta;x,t)w^{1'}_{x,t,\delta}(\zeta)]\frac{d\zeta}{2\pi i}\Big)_{12} \\ &= -\Big(\int_{\Sigma_1}[\sigma_3,((\mathbb{I}-C_{w^{1'}_{x,t,\delta}})^{-1})(\zeta)w^{1'}_{x,t,\delta}(\zeta)]\frac{d\zeta}{2\pi i}\Big)_{12}. \end{aligned}$$

Let $w^e:\Sigma\to M(2,\mathbb{C})$ be a sum of three terms

$$w^e = w^a + w^b + w^c. \tag{4.13}$$

we then have the following:

$$\begin{cases} w^a = w^{1'}_{x,t,\delta}|\mathbb{R} \text{ is supported on } \mathbb{R} \text{ and is composed of terms of type} \\ \quad h_I \text{ and } \overline{h}_I. \\ w^b \text{ is supported on } L\cup\overline{L} \text{ and is composed of the contribution to } w^{1'}_{x,t,\delta} \\ \quad \text{from terms of type } h_{II} \text{ and } \overline{h}_I. \\ w^c \text{ is supported on } L_\varepsilon\cup\overline{L}_\varepsilon \text{ and is composed of the contribution to } w^{1'}_{x,t,\delta} \\ \quad \text{from terms of type } R \text{ and } \overline{R}. \end{cases}$$

Set

$$\Sigma' = \Sigma\backslash(\mathbb{R}\cup L_\varepsilon\cup\overline{L}_\varepsilon)$$

with the orientation as in Figure.3. Define $w'$ through

$$w^{1'}_{x,t,\delta} = w' + w^e.$$

Observe that $w'=0$ on $\Sigma\backslash\Sigma'$.

**Lemma 4.2.** *For arbitrary $l\in\mathbb{Z}_{\geqslant 1}$ and sufficiently small $\varepsilon\in\mathbb{R}_{>0}$, as $t\to+\infty$ such that $\lambda = \min\{|\lambda_0|,|\lambda_1|,|\lambda_2|\} > M$,*

$$||w^a||_{\mathscr{L}_k^{(2\times2)}(\mathbb{R})} \leqslant \frac{\underline{c}}{(\lambda_l^2 t)^l},\ ||w^b||_{\mathscr{L}_k^{(2\times2)}(L\cup\overline{L})} \leqslant \frac{\underline{c}}{(\lambda_l^2 t)^l},\ ||w^c||_{\mathscr{L}_k^{(2\times2)}(L_{\widehat{\delta}}\cup\overline{L_\varepsilon})} \leqslant \underline{c}e^{-2\lambda_l^4\varepsilon^2 t},$$
$$||w'||_{\mathscr{L}_2^{(2\times2)}(\Sigma)} \leqslant \frac{\underline{c}}{(\lambda_l^2 t)^{1/4}},\ \ ||w'||_{\mathscr{L}_1^{(2\times2)}(\Sigma)} \leqslant \frac{\underline{c}}{\sqrt{\lambda_l^2 t}},$$

*where $k\in\{1,2,\infty\}$.*

*Proof.* By the conclusion of Proposition 4.1, the following estimates,

$$(w_{\pm}^{\sharp})_{x,t,\delta},\ w_{x,t,\delta}^{\sharp} \in \mathscr{L}_k^{(2\times2)}(\Sigma)\cap\mathscr{L}_\infty^{(2\times2)}(\Sigma),\ \ k\in\{1,2\},$$

and analogous calculations as in Lemma 2.13 of [19] □

**Definition 4.1.** Denote by $\mathscr{N}(\cdot)$ the space of bounded linear operators acting in $\mathscr{L}_2^{(2\times2)}(\cdot)$.

**Lemma 4.3.** *As $t\to+\infty$ such that $\lambda_0>M$, $(\underline{\mathbf{Id}}-C_{w^{\sharp}_{x,t,\delta}})^{-1}\in\mathscr{N}(\Sigma)\Leftrightarrow(\underline{\mathbf{Id}}-C_{w'})^{-1}\in\mathscr{N}(\Sigma)$.*

*Proof.* Using the consequence of the following inequality, $||C_{w^{\sharp}_{x,t,\delta}}-C_{w'}||_{\mathscr{N}(\Sigma)}\leqslant\underline{c}||w^e||_{\mathscr{L}_\infty^{(2\times2)}(\Sigma)}$, the fact that $||w^e||_{\mathscr{L}_\infty^{(2\times2)}(\Sigma)}(\leqslant\underline{c}|x|^{-l})\leqslant\underline{c}(\lambda_r^2 t)^{-l}\leqslant\underline{c}$ ((4.13) and Lemma 4.2 ), and the second resolvent identity. □

**Proposition 4.2.** *If $(\underline{\mathbf{Id}}-C_{w'})^{-1}\in\mathscr{N}(\Sigma)$, then for arbitrary $l\in\mathbb{Z}_{\geqslant1}$, as $t\to+\infty$ such that $\lambda_0>M$,*

$$q(x,t)=-i\left(\int_\Sigma[\sigma_3,((\underline{\mathbf{Id}}-C_{w'})^{-1}\mathrm{I})(\xi)w'(\xi)]\frac{d\xi}{2\pi i}\right)_{21}+\mathscr{O}\left(\frac{\underline{c}}{t^l}\right).$$

*Proof.* Using second resolvent identity and analogous calculations as (2.27) and Proposition 2.63 in [19] (or, Proposition 5.1 in [29], Proposition 3.19 in [26]) □

In the sense of appropriately defined operator norms, let us show that it may always choose to minus (or plus) a part of contour on which the jump matrix is $\mathrm{I}+\mathscr{O}(\frac{\underline{c}}{(\lambda_0^2 t)^l})$, $l\in\mathbb{Z}_{\geqslant1}$ and arbitrarily large, and without altering the RH problem,

Let:

- $R_{\Sigma'}:\mathscr{L}_2^{(2\times2)}(\Sigma)\to\mathscr{L}_2^{(2\times2)}(\Sigma')$ denote the restriction map;
- $\underline{\mathbf{I}}_{\Sigma'\to\Sigma}:\mathscr{L}_2^{(2\times2)}(\Sigma')\to\mathscr{L}_2^{(2\times2)}(\Sigma)$ denote the embedding;
- $C_{w'}^{\Sigma}:\mathscr{L}_2^{(2\times2)}(\Sigma)\to\mathscr{L}_2^{(2\times2)}(\Sigma)$ denote the operator in (4.10) with $w\leftrightarrow w'$;
- $C_{w'}^{\Sigma'}:\mathscr{L}_2^{(2\times2)}(\Sigma')\to\mathscr{L}_2^{(2\times2)}(\Sigma')$ denote the operator in (4.10) with $w\leftrightarrow w'|_{\Sigma'}$;
- $C_{w'}^{E}:\mathscr{L}_2^{(2\times2)}(\Sigma')\to\mathscr{L}_2^{(2\times2)}(\Sigma)$ denote the restriction of $C_{w'}^{\Sigma}$ to $\mathscr{L}_2^{(2\times2)}(\Sigma')$;
- $\underline{\mathbf{Id}}_{\Sigma'}$ and $\underline{\mathbf{Id}}_{\Sigma}$ denote, respectively, the identity operators on $\mathscr{L}_2^{(2\times2)}(\Sigma')$ and $\mathscr{L}_2^{(2\times2)}(\Sigma)$.

**Lemma 4.4.**

$$\begin{gathered}C_{w'}^{\Sigma}C_{w'}^{E}=C_{w'}^{E}C_{w'}^{\Sigma'},\\(\underline{\mathbf{Id}}_{\Sigma'}-C_{w'}^{\Sigma'})^{-1}=R_{\Sigma'}(\underline{\mathbf{Id}}_{\Sigma}-C_{w'}^{\Sigma})^{-1}\underline{\mathbf{I}}_{\Sigma'\to\Sigma},\\(\underline{\mathbf{Id}}_{\Sigma}-C_{w'}^{\Sigma})^{-1}=\underline{\mathbf{Id}}_{\Sigma}+C_{w'}^{E}(\underline{\mathbf{Id}}_{\Sigma'}-C_{w'}^{\Sigma'})^{-1}R_{\Sigma'}.\end{gathered}\tag{4.14}$$

*Proof.* See Lemma 2.56 in [19]. □

**Proposition 4.3.** *If $(\underline{\mathbf{Id}}_{\Sigma}-C_{w'}^{\Sigma})\in\mathscr{N}(\Sigma)$, then for arbitrary $l\in\mathbb{Z}_{\geqslant1}$, as $t\to+\infty$ such that $\lambda_0>M$,*

$$q(x,t)=-i\left(\int_{\Sigma'}[\sigma_3,((\underline{\mathbf{Id}}_{\Sigma'}-C_{w'}^{\Sigma'})^{-1}I)(\xi)w'(\xi)]\frac{d\xi}{2\pi i}\right)_{21}+\mathscr{O}\left(\frac{\underline{c}}{t}\right).\tag{4.15}$$

*Proof.* The boundedness of $||(\underline{\mathbf{Id}}_{\Sigma'}-C_{w'}^{\Sigma'})^{-1}||_{\mathscr{N}(\Sigma')}$ follows from the assertion of the Lemma and (4.14) : the remainder is a consequence of Proposition 4.2. □

From Proposition 4.3 we obtain that the asymptotic behavior can be constructed by the following RH problem on the contour $\Sigma'$,

$$\begin{aligned} m_+^{\Sigma'}(x,t;\lambda) &= m_-^{\Sigma'}(x,t;\lambda)v_{x,t,\delta}^{\Sigma'}(\lambda), \ \ \lambda\in\Sigma', \\ m^{\Sigma'}(x,t;\lambda) &\to \mathrm{I} \ \text{ as } \ \lambda\to\infty, \ \ \lambda\in\mathbb{C}\setminus\Sigma', \end{aligned} \tag{4.16}$$

where $v_{x,t,\delta}^{\Sigma'}(\lambda) = (\mathrm{I}-(w_-^{\Sigma'})_{x,t,\delta})^{-1}(\mathrm{I}+(w_+^{\Sigma'})_{x,t,\delta})$, with

$$(w_+^{\Sigma'})_{x,t,\delta} = (\delta(\lambda))^{\widehat{\sigma_3}}e^{-it\theta(\lambda)\widehat{\sigma_3}}R(\lambda)\begin{pmatrix}0&1\\0&0\end{pmatrix}, \ \ (w_-^{\Sigma'})_{x,t,\delta}=0, \ \ \lambda\in L\backslash L_\varepsilon,$$

$$(w_+^{\Sigma'})_{x,t,\delta}=0, \ \ (w_-^{\Sigma'})_{x,t,\delta} = -(\delta(\lambda))^{\widehat{\sigma_3}}e^{-it\theta(\lambda)\widehat{\sigma_3}}\overline{R(\lambda)}\begin{pmatrix}0&0\\1&0\end{pmatrix}, \ \ \lambda\in\overline{L}\backslash\overline{L_\varepsilon}.$$

Denote $(w^{\Sigma'})_{x,t,\delta} = (w_+^{\Sigma'})_{x,t,\delta}+(w_-^{\Sigma'})_{x,t,\delta}$, so that $(w^{\Sigma'})_{x,t,\delta} = w'|_{\Sigma'}$; then, using the Beals-Coifman theory, the solution of RH problem (4.16) is shown as follows,

$$m^{\Sigma'}(x,t;\lambda) = \mathrm{I}+\int_{\Sigma'}\frac{\mu^{\Sigma'}(x,t;\xi)(w^{\Sigma'}(\xi))_{x,t,\delta}}{(\xi-\lambda)}\frac{d\xi}{2\pi i}, \ \ \lambda\in\mathbb{C}\setminus\Sigma',$$

where $\mu^{\Sigma'}(x,t;\lambda)\equiv(\underline{\mathbf{Id}}_{\Sigma'}-C_{w'}^{\Sigma'})^{-1}\mathrm{I}$.

### 4.3. *RH Problem on the Disjoint Crosses*

In this section, we will show that how to separate out the contributions of the three crosses in $\Sigma'$ to the solution $q(x,t)$ in the formula (4.15). The main result in this part is Proposition 4.5.

Let us introduce some notations which for exact formulations. Taking $\Sigma'$ as the disjoint union of the three crosses, $\Sigma_{A'}$, $\Sigma_{B'}$, and $\Sigma_{C'}$, extend the contours $\Sigma_{A'}$, $\Sigma_{B'}$, and $\Sigma_{C'}$ (with orientations unchanged) to the follows,

$$\widehat{\Sigma}_{A'} = \{\lambda;\lambda=\lambda_2+\tfrac{|\lambda_2|u}{2}e^{\pm\frac{i\pi}{4}}, u\in\mathbb{R}\}, \ \ \widehat{\Sigma}_{B'} = \{\lambda;\lambda=\lambda_0+\tfrac{\lambda_0 u}{2}e^{\pm\frac{3\pi i}{4}}, u\in\mathbb{R}\},$$
$$\widehat{\Sigma}_{C'} = \{\lambda;\lambda=\lambda_1+\tfrac{|\lambda_1|u}{2}e^{\pm\frac{i\pi}{4}}, u\in\mathbb{R}\},$$

and define by $\Sigma_A$, $\Sigma_B$, and $\Sigma_C$, respectively, the contours $\{\lambda;\lambda=\lambda_1+\frac{|\lambda_{1p}|u}{2}e^{\pm\frac{i\pi}{4}}, u\in\mathbb{R}\}$ oriented inward as in $\Sigma_{A'}$ and $\widehat{\Sigma}_{A'}$, inward as in $\Sigma_{B'}$ and $\widehat{\Sigma}_{B'}$, and inward/outward as in $\Sigma_{C'}$ and $\widehat{\Sigma}_{C'}$.

Let us prepare the following operators, for $k\in\{A,B,C\}$,

$$N_k\colon \mathscr{L}^2(\widehat{\Sigma}_{k'})\to\mathscr{L}^2(\Sigma_k), \ \ f(\lambda)\mapsto(N_kf)(\lambda)=f(\lambda_k+\varepsilon_k),$$

where

$$\lambda_A=\lambda_2, \ \ \lambda_B=\lambda_0, \ \ \lambda_C=\lambda_1,$$
$$\varepsilon_\ell = \lambda(8t(1+8\gamma\lambda_\ell^2))^{-1/2}, \ \ell=A,B,C.$$

Taking action of the operators $N_k$ on function $\delta(\lambda)e^{-it\theta(\lambda)}$, it find that, for $k\in\{A,B,C\}$, $I_A\equiv(-\infty,\lambda_A)$, $I_B\equiv(\lambda_B,+\infty)$, and $I_C\equiv(\lambda_A,\lambda_B)$,

$$N_k\{\delta(\lambda)e^{-it\theta(\lambda)}\} = \delta_k^0\delta_k^1(\lambda), \ \ \mathrm{Re}(\lambda)\in I_k,$$

where,

$$\delta_A^0 = (8t\lambda_2^2(1+8\gamma\lambda_2^2))^{-\frac{i\nu}{2}} \exp\left\{2i\lambda_2^2(1+8\gamma\lambda_2^2)t + \sum_{m\in\mathscr{M}} \chi_m(\lambda_k)\right\}, \tag{4.17}$$

$$\delta_B^0 = (8t\lambda_0^2(1+8\gamma\lambda_0^2))^{-\frac{i\nu}{2}} \exp\left\{2i\lambda_0^2(1+8\gamma\lambda_0^2)t + \sum_{m\in\mathscr{M}} \chi_m(\lambda_k)\right\}, \tag{4.18}$$

$$\delta_C^0 = \exp\left\{\sum_{m\in\mathscr{M}} \chi_m(\lambda_1)\right\}, \tag{4.19}$$

$$\delta_B^1(\lambda) = \frac{(\lambda\lambda_0)^{i\nu}(\varepsilon_B+2\lambda_B)^{i\nu}}{(\varepsilon_B+\lambda_B)^{2i\nu}} \exp\left\{-\frac{i\lambda^2}{2}(1+\frac{\varepsilon_B}{2\lambda_0})^2 + \sum_{m\in\mathscr{M}} (\chi_m(\lambda_B+\varepsilon_B)-\chi_m(\lambda_B))\right\},$$

$$\delta_A^1(\lambda) = \frac{(\lambda\lambda_2)^{i\nu}(\varepsilon_A+2\lambda_A)^{i\nu}}{(\varepsilon_A+\lambda_A)^{2i\nu}} \exp\left\{-\frac{i\lambda^2}{2}(1+\frac{\varepsilon_A}{2\lambda_2})^2 + \sum_{m\in\mathscr{M}} (\chi_m(\lambda_A+\varepsilon_A)-\chi_m(\lambda_A))\right\},$$

$$\delta_C^1(\lambda) = ((\varepsilon_C-\lambda_0)(\varepsilon_C-\lambda_2))^{i\nu} \exp\left\{\frac{i\lambda^2}{2}(1-\frac{2\varepsilon_C}{\lambda_2})(1-\frac{2\varepsilon_C}{\lambda_0}) + \sum_{m\in\mathscr{M}} (\chi_m(\varepsilon_C)-\chi_m(\lambda_1))\right\},$$

with $\mathscr{M} \equiv \{A,B,+,-\}$, and

$$\chi_k(\lambda_\ell) = \frac{1}{2\pi i}\int_{\lambda_C}^{\lambda_k} \ln\left(\frac{1-|r(\mu)|^2}{1-|r(|\lambda_\ell|)|^2}\right)\frac{d\mu}{(\mu-\lambda_\ell)}$$

$$\chi_\pm(\lambda_\ell) = \frac{i}{2\pi}\int_{\pm\infty}^{\lambda_C} \ln|i\mu-\lambda_l|\,d\ln(1+|r(i\mu)|^2),$$

$$\chi_k(\lambda_C) = \frac{i}{2\pi}\int_{\lambda_C}^{\lambda_k} \ln|\mu|\,d\ln(1-|r(\mu)|^2),$$

$$\chi_\pm(\lambda_C) = \int_{i\lambda_C\pm i\infty}^{i\lambda_C} \frac{\ln(1+|r(\mu)|^2)}{\mu}\frac{d\mu}{2\pi i}.$$

Set

$$\Delta_k^0 = (\delta_k^0)^{\sigma_3},$$

and let $\widetilde{\Delta_k^0}$ denote right multiplication by $\Delta_k^0$:

$$\widetilde{\Delta_k^0}\phi \equiv \phi\Delta_k^0.$$

Denote

$$w^{k'}(\lambda) = \begin{cases} (w^{\Sigma'})_{x,t,\delta}, & \lambda\in\Sigma_{k'}, \\ 0, & \lambda\in\Sigma'\setminus\Sigma_{k'}, \end{cases} \quad \text{and} \quad \widehat{w}^{k'}(\lambda) = \begin{cases} w^{k'}, & \lambda\in\Sigma_{k'}, \\ 0, & \lambda\in\widehat{\Sigma}_{k'}\setminus\Sigma_{k'}. \end{cases}$$

Then we have,

$$(w^{\Sigma'})_{x,t,\delta} = \sum_{k\in\{A,B,C\}} w^{k'}, \quad C^{\Sigma'}_{(w^{\Sigma'})_{x,t,\delta}} \equiv C^{\Sigma'}_{w'} = \sum_{k\in\{A,B,C\}} C^{\Sigma'}_{w^{k'}} \equiv \sum_{k\in\{A,B,C\}} C^{\Sigma_{k'}}_{w^{k'}}.$$

In the remainder of this section , we remove the special notation for $w^{k'}|_{\Sigma'_{k'}}$. We first show that some technical results concerning the operators $C^{\Sigma_{k'}}_{w^{k'}}$ and $C^{\widehat{\Sigma}_{k'}}_{\widehat{w}^{k'}}$.

**Proposition 4.4.** *For $k \in \{A,B,C\}$,*

$$C^{\widehat{\Sigma}_{k'}}_{\widehat{w}^{k'}} = (N_k)^{-1}(\widetilde{\Delta^0_k})^{-1}C^{\Sigma_k}_{w^k}(\widetilde{\Delta^0_k})N_k, \quad w^k \equiv (\Delta^0_k)^{-1}(N_k\widehat{w}^{k'})(\Delta^0_k),$$

*where*

$$C^{\Sigma_k}_{w^k}|_{\mathscr{L}_2^{(2\times 2)}(\overline{L_k})} = -C_+(\cdot(\overline{R(\varepsilon_k+\lambda_k)}(\delta^1_k(\lambda))^{-2}\sigma_-)),$$
$$C^{\Sigma_k}_{w^k}|_{\mathscr{L}_2^{(2\times 2)}(L_k)} = C_-(\cdot(R(\varepsilon_k+\lambda_k)(\delta^1_k(\lambda))^{2}\sigma_+)).$$

*The contours $L_k$ are defined following,*

$$L_l \equiv \{\lambda;\lambda = \tfrac{\lambda_\ell u}{2}(8t(1+8\gamma\lambda_\ell^2))^{1/2}e^{-\frac{i\pi}{4}}, u\in(-\varepsilon,+\infty)\}, \quad \ell\in\{A,B\},$$
$$L_C \equiv \{\lambda;\lambda = \tfrac{|\lambda_1|u}{2}(8t(1+8\gamma\lambda_1^2))^{1/2}e^{\frac{i\pi}{4}}, u\in\mathbb{R}\},$$

*it means that $\Sigma_{k'} = L_k \cup \overline{L_k}$.*

*Proof.* Analogous to the prove of Lemma 3.5 in [19], Proposition 6.1 in [29] or Lemma 3.20 in [26]. □

We introduce following expressions

$$R(\lambda_j+) \equiv \lim_{\mathrm{Re}\lambda>\lambda_j} R(z) = \bar{r}(\lambda_j);$$
$$R(\lambda_j-) \equiv \lim_{\mathrm{Re}\lambda<\lambda_j} R(z) = -\bar{r}(\lambda_j)(1-|r(\lambda_j)|^2).$$

**Lemma 4.5.** *Let $\kappa\in(0,1)$. Then $\forall\,\lambda\in\overline{L_k}\subset\Sigma_k$, as $t\to+\infty$ such that $\lambda_0>M$,*

$$|\overline{R(\varepsilon_k+\lambda_k)}(\delta^1_k(\lambda))^{-2} - \overline{R(\lambda_k\pm)}(\lambda)^{-2i\nu}e^{i\mathrm{sgn}(k)\lambda^2}|$$
$$\leqslant C(\lambda_k)|e^{i\gamma/2\lambda^2}|\left(\frac{1}{(t\lambda_k^2(1+8\gamma\lambda_k^2))^{1/2}} + \frac{\log(t|c(\lambda_0,\lambda_1,\lambda_2)|)}{(t|\lambda_k|)^{1/2}}\right)$$

*and $\forall\,\lambda\in L_k\subset\Sigma_k$,*

$$|R(\varepsilon_k+\lambda_k)(\delta^1_k(\lambda))^{2} - R(\lambda_k\pm)(\lambda)^{2i\nu}e^{-i\mathrm{sgn}(k)\lambda^2}|$$
$$\leqslant C(\lambda_k)|e^{-i\gamma/2\lambda^2}|\left(\frac{1}{(t\lambda_k^2(1+8\gamma\lambda_k^2))^{1/2}} + \frac{\log(t|c(\lambda_0,\lambda_1,\lambda_2)|)}{(t|\lambda_k|)^{1/2}}\right)$$

*where $L_k$ (resp. $\overline{L_k}$), $k\in\{A,B,C\}$, are defined in Proposition 6.1, $u\in(-\varepsilon,+\infty)$, with $0<\varepsilon<\sqrt{2}$.*

*Proof.* Analogous of Lemma 3.35 in [19] (or Lemma 6.1 in [29]). □

**Lemma 4.6 ([29]).** *For general operators $C^{\Sigma'}_{w^{k'}}$, $k\in\{1,2,\dots,N\}$, if $(\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{k'}})^{-1}$ exist, then*

$$(\underline{\mathbf{Id}}_{\Sigma'}+\sum_{1\leqslant\alpha\leqslant N}C^{\Sigma'}_{w^{\alpha'}}(\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{\alpha'}})^{-1})(\underline{\mathbf{Id}}_{\Sigma'}-\sum_{1\leqslant\beta\leqslant N}C^{\Sigma'}_{w^{\beta'}})=$$
$$\underline{\mathbf{Id}}_{\Sigma'}-\sum_{1\leqslant\alpha\leqslant N}\sum_{1\leqslant\beta\leqslant N}(1-\delta_{\alpha\beta})(\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{\alpha'}})^{-1}C^{\Sigma'}_{w^{\alpha'}}C^{\Sigma'}_{w^{\beta'}},$$

*and*

$$(\underline{\mathbf{Id}}_{\Sigma'}-\sum_{1\leqslant\beta\leqslant N}C^{\Sigma'}_{w^{\beta'}})(\underline{\mathbf{Id}}_{\Sigma'}+\sum_{1\leqslant\alpha\leqslant N}C^{\Sigma'}_{w^{\alpha'}}(\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{\alpha'}})^{-1})=$$
$$\underline{\mathbf{Id}}_{\Sigma'}-\sum_{1\leqslant\alpha\leqslant N}\sum_{1\leqslant\beta\leqslant N}(1-\delta_{\alpha\beta})C^{\Sigma'}_{w^{\alpha'}}C^{\Sigma'}_{w^{\beta'}}(\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{\beta'}})^{-1},$$

*where $\delta_{\alpha\beta}$ is the Kronecker delta.*

**Lemma 4.7.** *For $\alpha\neq\beta\in\{A',B',C'\}$, as $t\to+\infty$ such that $\lambda_0>M$,*

$$||C^{\Sigma'}_{w^\alpha}C^{\Sigma'}_{w^\beta}||_{\mathscr{N}(\Sigma')}\leqslant\frac{\underline{c}}{\lambda_0\sqrt{t}},\quad ||C^{\Sigma'}_{w^\alpha}C^{\Sigma'}_{w^\beta}||_{\mathscr{L}^{(2\times2)}_\infty(\Sigma')\to\mathscr{L}^{(2\times2)}_2(\Sigma')}\leqslant\frac{\underline{c}}{(\lambda_0^2t)^{1/4}\sqrt{\lambda_0^4t}}.$$

*Proof.* Analogous to Lemma 3.5 in [19]. □

**Proposition 4.5.** *If, for $k\in\{A,B,C\}$, $(\underline{\mathbf{Id}}_{\Sigma_{k'}}-C^{\Sigma_{k'}}_{w^{k'}})^{-1}\in\mathscr{N}(\Sigma_{k'})$, then as $t\to+\infty$ such that $\lambda_0>M$,*

$$P(x,t)$$
$$=-i\sum_{k\in\{A,B,C\}}\left(\int_{\Sigma_{k'}}[\sigma_3,((\underline{\mathbf{Id}}_{\Sigma_{k'}}-C^{\Sigma_{k'}}_{w^{k'}})^{-1}\mathrm{I})(\xi)w^{k'}(\xi)]\frac{d\xi}{2\pi i}\right)_{21}+\mathscr{O}\left(\frac{C(\lambda_0,\lambda_1,\lambda_2)\underline{c}}{t}\right).\quad(4.20)$$

*Proof.* Analogous of (2.27) in [19] and the second resolvent identity, one writes

$$(\underline{\mathbf{Id}}_{\Sigma'}-\sum_{k\in\{A,B,C\}}C^{\Sigma'}_{w^{k'}})^{-1}=\underline{\mathbf{D}}_{\Sigma'}+\underline{\mathbf{D}}_{\Sigma'}(\underline{\mathbf{Id}}_{\Sigma'}-\underline{\mathbf{E}}_{\Sigma'})^{-1}\underline{\mathbf{E}}_{\Sigma'},\quad(4.21)$$

where

$$\underline{\mathbf{D}}_{\Sigma'}\equiv\underline{\mathbf{Id}}_{\Sigma'}+\sum_{k\in\{A,B,C\}}C^{\Sigma'}_{w^{k'}}(\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{k'}})^{-1},$$
$$\underline{\mathbf{E}}_{\Sigma'}\equiv\sum_{\alpha,\beta\in\{A,B,C\}}(1-\delta_{\alpha\beta})C^{\Sigma'}_{w^{\alpha'}}C^{\Sigma'}_{w^{\beta'}}(\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{\beta'}})^{-1}.$$

From the Lemma 4.2,

$$\|C^{\Sigma'}_{w^{k'}}\|_{\mathscr{N}(\Sigma')}\leqslant\|w^{k'}\|_{\mathscr{L}^{(2\times2)}_2(\Sigma')}\leqslant\frac{\underline{c}}{(\lambda_0^2t)^{1/4}}\leqslant\underline{c},$$

and, take assumption $||(\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{k'}})^{-1}||_{\mathscr{N}(\Sigma')}<\infty$, by means of Lemma 4.7, it is show that, as $t\to+\infty$ then,

$$||\underline{\mathbf{D}}_{\Sigma'}||_{\mathscr{N}(\Sigma')}\leqslant\underline{c},\quad\text{and}\quad||(\underline{\mathbf{Id}}_{\Sigma'}-\underline{\mathbf{E}}_{\Sigma'})^{-1}||_{\mathscr{N}(\Sigma')}\leqslant\underline{c},$$

as $\lambda_0 > M$. Taking account of the second resolvent identity, it is simple to show that,

$$||\underline{\mathbf{E}}_{\Sigma'}\mathrm{I}||_{\mathscr{L}_2^{(2\times2)}(\Sigma')} \leqslant \sum_{\alpha,\beta\in\{A,B,C\}}(1-\delta_{\alpha\beta})||C^{\Sigma'}_{w^{\alpha'}}C^{\Sigma'}_{w^{\beta'}}\mathrm{I}||_{\mathscr{L}_2^{(2\times2)}(\Sigma')}$$
$$+\sum_{\alpha,\beta\in\{A,B,C\}}(1-\delta_{\alpha\beta})||C^{\Sigma'}_{w^{\alpha'}}C^{\Sigma'}_{w^{\beta'}}||_{\mathscr{N}(\Sigma')}||(\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{\beta'}})^{-1}||_{\mathscr{N}(\Sigma')}||C^{\Sigma'}_{w^{\beta'}}\mathrm{I}||_{\mathscr{L}_2^{(2\times2)}(\Sigma')}.$$

According to Lemma 4.2,

$$||C^{\Sigma'}_{w^{k'}}\mathrm{I}||_{\mathscr{L}_2^{(2\times2)}(\Sigma')} \leqslant ||w^{k'}||_{\mathscr{L}_2^{(2\times2)}(\Sigma')} \leqslant \frac{\underline{c}}{(1+8\gamma\lambda_0^2 t)^{1/4}},$$

by the Lemma 4.7 and the second resolvent identity,

$$||\underline{\mathbf{E}}_{\Sigma'}\mathrm{I}||_{\mathscr{L}_2^{(2\times2)}(\Sigma')} \leqslant \tfrac{\underline{c}}{(1+8\gamma\lambda_0^2)t^{3/4}}.$$

According to the Cauchy-Schwarz inequality and Lemma 4.2,

$$||\underline{\mathbf{E}}_{\Sigma'}w^{\Sigma'}||_{\mathscr{L}_1^{(2\times2)}(\Sigma')} \leqslant ||\underline{\mathbf{E}}_{\Sigma'}\mathrm{I}||_{\mathscr{L}_2^{(2\times2)}(\Sigma')}||w^{\Sigma'}||_{\mathscr{L}_2^{(2\times2)}(\Sigma')} \leqslant \tfrac{\underline{c}}{(1+8\gamma\lambda_0^2)t};$$

hence, recalling (4.21), $(\underline{\mathbf{Id}}_{\Sigma'}-\sum_{k\in\{A,B,C\}}C^{\Sigma'}_{w^{k'}})^{-1}\in\mathscr{N}(\Sigma')$. From Proposition 4.2 and the above estimates such that,

$$\begin{aligned}&\int_{\Sigma'}((\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w'})^{-1}\mathrm{I})(\xi)w^{\Sigma'}(\xi)d\xi \\ &\quad=\int_{\Sigma'}(\underline{\mathbf{D}}_{\Sigma'}\mathrm{I})w^{\Sigma'}(\xi)d\xi+\mathscr{O}(\frac{\underline{c}}{(1+8\gamma\lambda_0^2)t})+\mathscr{O}\left(\frac{\underline{c}}{((1+8\gamma\lambda_0^2)t)^l}\right)\end{aligned}\tag{4.22}$$

as $t\to+\infty$, for $\lambda_0 > M$ and arbitrary $l\in\mathbb{Z}_{\geqslant1}$. Recall that $w^{\Sigma'}=\sum_{k\in\{A,B,C\}}w^{k'}$, the integral on the right-hand side of (4.22) can be written following:

$$\int_{\Sigma'}(\underline{\mathbf{D}}_{\Sigma'}\mathrm{I})w^{\Sigma'}(\xi)d\xi=\int_{\Sigma'}(\underline{\mathbf{Id}}_{\Sigma'}\sum_{k\in\{A,B,C\}}w^{k'}+\sum_{\alpha,\beta\in\{A,B,C\}}C^{\Sigma'}_{w^{\alpha'}}((\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{\alpha'}})^{-1}\mathrm{I})w^{\beta'})(\xi)d\xi.\tag{4.23}$$

By Lemma 4.7 and the assumption that $||(\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{k'}})^{-1}||_{\mathscr{N}(\Sigma')}<\infty$. Applying the second resolvent identity to the right-hand side of (4.23), it is show that,

$$\begin{aligned}\int_{\Sigma'}((\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w'})^{-1}\mathrm{I})(\xi)w^{\Sigma'}(\xi)d\xi &= \sum_{k\in\{A,B,C\}}\int_{\Sigma'}((\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{k'}})^{-1}\mathrm{I})(\xi)w^{k'}(\xi)d\xi \\ &\quad+\mathscr{O}\left(\frac{c(\lambda_0,\lambda_1,\lambda_2)\underline{c}}{t}\right),\end{aligned}\tag{4.24}$$

for arbitrary $l\in\mathbb{Z}_{\geqslant1}$. Now, from Lemma 4.4,

$$(\underline{\mathbf{Id}}_{\Sigma_{k'}}-C^{\Sigma_{k'}}_{w^{k'}})^{-1}=R_{\Sigma_{k'}}(\underline{\mathbf{Id}}_{\Sigma'}-C^{\Sigma'}_{w^{k'}})^{-1}\underline{\mathbf{I}}_{\Sigma'_{k'}\to\Sigma};\tag{4.25}$$

Then, substituting identity (4.25) into (4.24), and recalling (4.21) and (4.22), the proof is complete. □

**Lemma 4.8.** *For $k\in\{A,B,C\}$, $(\underline{\mathbf{Id}}_{\Sigma_{k'}}-C^{\Sigma_{k'}}_{w^{k'}})^{-1}\in\mathscr{N}(\Sigma_{k'})$.*

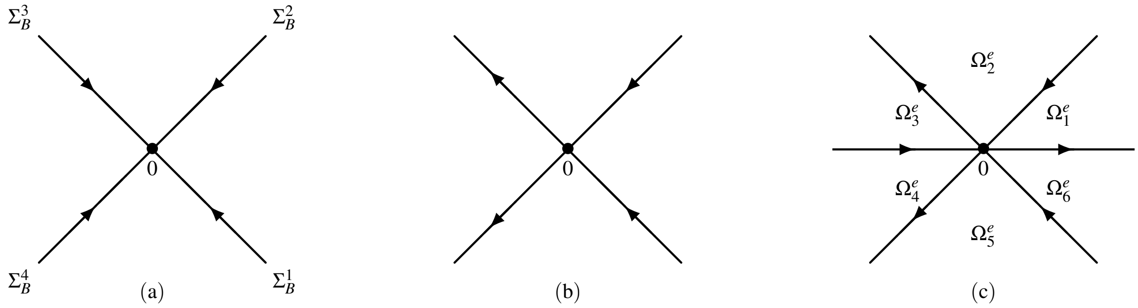

Fig. 4. (a) $\Sigma_B$; (b) $\Sigma_{B,r}$; and (c) $\Sigma_{ex} \equiv \Sigma_{B,r} \cup \mathbb{R}$.

**Remark 4.1.** The Lemma 4.8 was proved in [19, 29]: In order to obtain the explicit asymptotic formulae presented in Theorem 2.2, we need a model RH problem which arises in crosses.

*Proof.* We only consider the case $k=B$, the cases $k=A$ and $C$ follow in an analogous manner. From Lemma 4.3, by the fact that the boundedness of $(1_{\widehat{\Sigma}_{B'}} - C_{\omega^{B'}}^{\widehat{\Sigma}_{B'}})^{-1}$ , it follows that the boundedness of $(1_{\Sigma_{B'}} - C_{\omega^{B'}}^{\Sigma_{B'}})^{-1}$. We note that

$$(1_{\widehat{\Sigma}_{B'}} - C_{\omega^{B'}}^{\widehat{\Sigma}_{B'}})^{-1} = (N_B)^{-1}\widetilde{(\Delta_B^0)}^{-1}(1_{\Sigma_{B'}} - C_{\omega^{B'}}^{\Sigma_{B'}})^{-1}\widetilde{(\Delta_B^0)}N_B, \tag{4.26}$$

and then the boundedness of $(1_{\widehat{\Sigma}_{B'}} - C_{\omega^{B'}}^{\widehat{\Sigma}_{B'}})^{-1}$ follows from the boundedness of $(1_{\Sigma_B} - C_{\omega^B}^{\Sigma_B})^{-1}$.

Set

$$\omega^B = (\Delta_B^0)^{-1}(N_B\widehat{\omega}^{B'})\Delta_B^0,$$

so that

$$C_{\omega^B}^{\Sigma_B} = C_+(\cdot\omega_-^B) + C_-(\cdot\omega_+^B).$$

On $\Sigma_B$, we have the diagram in Figure.4(a)

Set $J^{B^0} = (b_-^{B^0})^{-1}b_+^{B^0} = (\mathbb{I} - \omega_-^{B^0})^{-1}(\mathbb{I} + \omega_+^{B^0})$. Defining as usual $\omega^{B^0} = \omega_+^{B^0} + \omega_-^{B^0}$, and using Lemma 4.5, it is show that

$$||\omega^B - \omega^{B^0}||_{L^\infty(\Sigma_B \cap L^1(\Sigma_B) \cap L^2(\Sigma_B)} \leqslant C(\lambda_0)t^{-\frac{1}{2}}. \tag{4.27}$$

Hence, as $t \to \infty$,

$$||C_{\omega^B}^{\Sigma_B} - C_{\omega^{B^0}}^{\Sigma_B}||_{L^2(\Sigma_B)} \leqslant C(\lambda_1)t^{-\frac{1}{2}}, \tag{4.28}$$

and consequently, one sees that the boundedness of $(1_{\Sigma_B} - C_{\omega^B}^{\Sigma_B})^{-1}$ follows from the boundedness of $(1_{\Sigma_B} - C_{\omega^{B^0}}^{\Sigma_B})^{-1}$ as $t \to \infty$.

Then reorient $\Sigma_B$ to $\Sigma_{B,r}$ as Figure. 4(b). A simple computation shows that the jump matrix $J^{B,r} = (b_-^{B,r})^{-1}(b_+^{B,r}) = (\mathbb{I} - \omega_-^{B,r})^{-1}(\mathbb{I} + \omega_+^{B,r})$ on $\Sigma_{B,r}$ is determined by

$$\omega_{\pm}^{B,r}(\lambda) = -\omega_{\mp}^{B^0}(\lambda), \quad for \quad \mathrm{Re}\lambda > 0,$$

and

$$\omega_{\pm}^{B,r}(\lambda) = \omega_{\pm}^{B^0}(\lambda), \quad for \quad \mathrm{Re}\lambda < 0.$$

The third step is that extending $\Sigma_{B,r} \to \Sigma_e = \Sigma_{B,r} \cup \mathbb{R}$ with the orientation on $\Sigma_{B,r}$ as Figure. 4(c) and the orientation on $\mathbb{R}$ from $-\infty$ to $\infty$. And the jump $J^e = (b_-^e)^{-1} b_+^e = (\mathbb{I} - \omega_-^e)^{-1}(\mathbb{I} + \omega_+^e)$ with

$$\omega^e(\lambda) = \omega^{B,r}(\lambda), \quad \lambda \in \Sigma_{B,r},$$

$$\omega^e(\lambda) = 0, \quad \lambda \in \mathbb{R}.$$

Set $C_{\omega^e}$ on $\Sigma_e$. Once again, by Lemma 4.4, it is sufficient to bound $(1_{\Sigma_e} - C_{\omega^e})^{-1}$ on $L^2(\Sigma_e)$.

Then define a piecewise-analytic matrix function $\phi$ as follows:

$$\widetilde{M}^{(\lambda_0)} = M^{(\lambda_0)} \phi,$$

where

$$\phi = \begin{cases} \lambda^{i\nu\sigma_3}, & \lambda \in \Omega_2^e, \Omega_5^e, \\ \lambda^{i\nu\sigma_3} \begin{pmatrix} 1 & 0 \\ -r(\lambda_0) e^{-i\frac{\lambda^2}{2}} & 1 \end{pmatrix}, & \lambda \in \Omega_1^e, \\ \lambda^{i\nu\sigma_3} \begin{pmatrix} 1 & -\overline{r(\lambda_0)} e^{i\frac{\lambda^2}{2}} \\ 0 & 1 \end{pmatrix}, & \lambda \in \Omega_6^e, \\ \lambda^{i\nu\sigma_3} \begin{pmatrix} 1 & \frac{\overline{r(\lambda_0)}}{1-|r(\lambda_0)|^2} e^{i\frac{\lambda^2}{2}} \\ 0 & 1 \end{pmatrix}, & \lambda \in \Omega_3^e, \\ \lambda^{i\nu\sigma_3} \begin{pmatrix} 1 & 0 \\ -\frac{\overline{r(\lambda_0)}}{1-|r(\lambda_0)|^2} e^{-i\frac{\lambda^2}{2}} & 1 \end{pmatrix}, & \lambda \in \Omega_4^e. \end{cases}$$

Thus, we can get the RH problem of $\widetilde{M}^{(\lambda_0)}$

$$\widetilde{M}_+^{(\lambda_0)}(x,t,k) = \widetilde{M}_-^{(\lambda_0)}(x,t,k) J^{e,\phi},$$

$$\widetilde{M}^{(\lambda_0)}(x,t,k) = \left( \mathbb{I} + \frac{M_1^{B^0}}{\lambda} + O(\frac{1}{\lambda^2}) \right) \lambda^{i\nu\sigma_3}, \; \lambda \to \infty.$$

where

$$J^{e,\phi} = \begin{cases} \begin{pmatrix} 1 - |r(\lambda_0)|^2 & \overline{r(\lambda_0)} e^{-i\frac{\lambda^2}{2}} \\ -r(\lambda_0) e^{i\frac{\lambda^2}{2}} & 1 \end{pmatrix}, & z \in \mathbb{R}, \\ \mathbb{I}, & \lambda \in \Sigma_{B,r}. \end{cases}$$

On $\mathbb{R}$ we have

$$J^{e,\phi} = (b_-^{e,\phi})^{-1} b_+^{e,\phi} = (\mathbb{I} - \omega_-^{e,\phi})^{-1} (\mathbb{I} + \omega_+^{e,\phi}) = \begin{pmatrix} 1 & e^{-\frac{i\lambda^2}{2}} \overline{r}(\lambda_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{\frac{i\lambda^2}{2}} r(\lambda_0) & 1 \end{pmatrix}.$$

this completes the proof. □

## 4.4. *Model RH Problem*

In this section, we will convert the evaluation of the integral in the Lemma 4.3 into three solvable RH problems on $\mathbb{R}$.

For $j \in \{A, B, C\}$, define

$$M^j(z) = \mathbb{I} + \int_{\Sigma_j} \frac{((1_{\Sigma_j} - C^{\Sigma_j}_{\omega^{j^0}})^{-1}\mathbb{I})(\xi)\omega^{j^0}(\xi)}{\xi - z}\frac{d\xi}{2\pi i}, \quad z \in \mathbb{C}\backslash\Sigma_j.$$

Then, $M^j(z)$ solves the RH problem

$$\begin{cases} M^j_+(z) = M^l_-(z)J^j(z) = M^j_-(z)(\mathbb{I} - \omega^{j^0}_-)^{-1}(\mathbb{I} + \omega^{j^0}_+), & z \in \Sigma_j, \\ M^j(z) \to \mathbb{I}, & z \to \infty. \end{cases}$$

If we take the asymptotic expansion,

$$M^j(z) = \mathbb{I} + \frac{M^j_1}{z} + O(z^{-2}), \quad z \to \infty,$$

then

$$M^j_1 = -\int_{\Sigma_j} ((1_{\Sigma_j} - C^{\Sigma_j}_{\omega^j})^{-1}\mathbb{I})(\xi)\omega^j(\xi)\frac{d\xi}{2\pi i}. \tag{4.29}$$

Substituting into (4.20) of Proposition 4.5 and observe that inequalities (4.27) and (4.28) (and their analogues for $\Sigma_A$ and $\Sigma_C$), we obtain

$$\begin{aligned} q(x,t) =& \frac{1}{\sqrt{8t(1+8\gamma\lambda_0^2)}}(\delta_B^0)^{-2}(m_1^B)_{12} + \frac{1}{\sqrt{8t(1+8\gamma\lambda_2^2)}}(\delta_A^0)^{-2}(m_1^A)_{12} \\ &+ \frac{1}{\sqrt{8t(1+8\gamma\lambda_1^2)}}(\delta_C^0)^{-2}(m_1^C)_{12} + O\left(C(\lambda_0,\lambda_1,\lambda_2)\frac{\log t}{t}\right). \end{aligned}$$

where, the function $C(\lambda_0, \lambda_1, \lambda_2)$ is bounded in the sector $\mathscr{P}$ defined in Theorem 2.2, i.e.,

$$\begin{aligned} q(x,t) =& \frac{1}{\sqrt{8t(1+8\gamma\lambda_0^2)}}(\delta_B^0)^{-2}(m_1^B)_{12} + \frac{1}{\sqrt{8t(1+8\gamma\lambda_2^2)}}(\delta_A^0)^{-2}(m_1^A)_{12} \\ &+ \frac{1}{\sqrt{8t(1+8\gamma\lambda_1^2)}}(\delta_C^0)^{-2}(m_1^C)_{12} + O\left(\frac{\log t}{t}\right). \end{aligned} \tag{4.30}$$

Analogous of the references [19, 26, 29], when we consider the case $B$, write

$$\Psi = \widetilde{M}^{(\lambda_0)} e^{-i\frac{\lambda^2}{4}\sigma_3} = \widehat{\Psi}\lambda^{i\nu\sigma_3} e^{-i\frac{\lambda^2}{4}\sigma_3}.$$

We have,

$$\Psi_+(\lambda) = \Psi_-(\lambda)\widetilde{J}^{(\lambda_0)}, \quad \lambda \in \mathbb{R},$$

where

$$J^{(\lambda_0)} = \begin{pmatrix} 1 - |r(\lambda_0)|^2 & \overline{r(\lambda_0)} \\ -r(\lambda_0) & 1 \end{pmatrix}.$$

By taking the derivative of $\lambda$ and Liouville theorem, it is easy to show that,

$$\frac{d\Psi}{d\lambda}+\frac{1}{2}i\lambda\sigma_3\Psi=\beta\Psi,$$

where

$$\beta^B=\frac{i}{2}[\sigma_3,M_1^B]=\begin{pmatrix} o & \beta_{12}^B \\ \beta_{21}^B & 0 \end{pmatrix}.$$

Following [19] (P.350-352), we have

$$\beta_{12}^B=-\frac{e^{-\frac{\pi}{2}\nu(\lambda_0)}}{r(\lambda_0)}\frac{\sqrt{2\pi}e^{i\frac{\pi}{4}}}{\Gamma(-i\nu(\lambda_0))}.$$

Hence,

$$(M_1^B)_{12}=-i\beta_{12}^B=i\frac{e^{-\frac{\pi}{2}\nu(\lambda_0)}}{r(\lambda_0)}\frac{\sqrt{2\pi}e^{i\frac{\pi}{4}}}{\Gamma(-i\nu(\lambda_0))}. \tag{4.31}$$

The proof of the case $A$ and case $C$ follows in a similar manner, we can get

$$(M_1^A)_{12}=-i\beta_{12}^A=i\frac{e^{-\frac{\pi}{2}\nu(\lambda_2)}}{r(\lambda_2)}\frac{\sqrt{2\pi}e^{i\frac{\pi}{4}}}{\Gamma(-i\nu(\lambda_2))}, \tag{4.32}$$

$$(M_1^C)_{12}=-i\beta_{12}^C=i\frac{e^{-\frac{\pi}{2}\nu(\lambda_1)}}{r(\lambda_1)}\frac{\sqrt{2\pi}e^{i\frac{\pi}{4}}}{\Gamma(-i\nu(\lambda_1))}. \tag{4.33}$$

where $\Gamma(\cdot)$ denotes the standard Gamma function.

*Proof of Theorem* 2.2. Substituting formulas (4.32), (4.31), (4.33), (4.17), (4.18), (4.19) into (4.30), then we can get the equation (2.3), i.e., the theorem 2.2 is proven.

## Acknowledgement

The author thank the two referees for valuable comments. Support is acknowledged from the National Science Foundation of China, Grant No. {11901141, 11671095}.

## References

[1] A. Ankiewicz, N. Akhmediev, Higher-order integrable evolution equation and its soliton solutions, *Phys. Lett. A*, **378** (2014) 358–361.
[2] L. Arruda, J. Lenells, Long-time asymptotics for the derivative nonlinear Schrödinger equation on the half-line, *Nonlinearity*, **30** (2017) 4141–4172.
[3] R. Beals, R. Coifman, Scattering and inverse scattering for first order systems, *Commun. Pure Appl. Math.*, **37** (1984) 39–90.
[4] G. Biondini, D. Mantzavinos, Long–time asymptotics for the focusing nonlinear schrödinger equation with nonzero boundary conditions at infinity and asymptotic stage of modulational instability, *Commun. Pure Appl. Math.*, **70** (2017) 2300–2365.
[5] A. Boutet de Monvel, A.R. Its, and V.P. Kotlyarov, Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition on the half-line, *Commun. Math. Phys.*, **290** (2009) 479–522.

[6] A. Boutet de Monvel, V.P. Kotlyarov, The focusing nonlinear Schrödinger equation on the quarter plane with time-periodic boundary condition:a Riemann-Hilbert approach, *J. Inst. Math. Jussieu*, **6** (2007) 579–611.

[7] A. Boutet de Monvel, V.P. Kotlyarov, and D. Shepelsky, Decaying long-time asymptotics for the focusing NLS equation with periodic boundary condition, *Int. Math. Res. Notices*, **2009** (2009) 547–577.

[8] A. Boutet de Monvel, V.P. Kotlyarov, and D. Shepelsky, Focusing NLS equation:Long-time dynamics of step-like initial data, *Int. Math. Res. Notices*, **2011** (2011) 1613–1653.

[9] A. Boutet de Monvel, J. Lenells, and D. Shepelsky, Long-time asymptotics for the Degasperis-Procesi equation on the half-line, *Ann. Inst. Fourier (Grenoble)*, **69** (2019) 171–230.

[10] A. Boutet de Monvel, D. Shepelsky, A Riemann-Hilbert approach for the Degasperis-Procesi equation, *Nonlinearity*, **26** (2013) 2081–2107.

[11] A. Boutet de Monvel, D. Shepelsky, *Long-time asymptotics of the Camassa-Holm equation on the line*, Integrable systems and random matrices, 99–116, *Contemp. Math.*, **458**, Amer. Math. Soc., Providence, RI, 2008.

[12] A. Chowdury, D.J. Kedziora, A. Ankiewicz and N. Akhmediev, Soliton solutions of an integrable nonlinear Schrödinger equation with quintic terms, *Phys. Rev. E*, **90** (2014) 032922.

[13] A. Chowdury, D.J. Kedziora, A. Ankiewicz and N. Akhmediev, Breather solutions of the integrable quintic nonlinear Schrödinger equation and their interactions, *Phys. Rev. E*, **91** (2015) 022919.

[14] A. Chowdury, D.J. Kedziora, A. Ankiewicz and N. Akhmediev, Breather-to-soliton conversions described by the quintic equation of the nonlinear Schrödinger hierarchy, *Phys. Rev. E*, **91** (2015) 032928.

[15] P. Cheng, S. Venakides, and X. Zhou, Long-time asymptotics for the pure radiation solution of the sine-Gordon equation, *Comm. Partial Differential Equations*, **24** (1999), 1195–1262.

[16] C. Charlier, J. Lenells and D. Wang, The “good” Boussinesq equation: long-time asymptotics, arXiv:2003.04789, 2020.

[17] P. Deift, A.R. Its, and X. Zhou, Long-time asymptotics for integrable nonlinear wave equations, *Important developments in soliton theory*, 181–204, Springer Ser. Nonlinear Dynam., Springer, Berlin, 1993.

[18] P. Deift, S. Venakides, and X. Zhou, An extension of the steepest descent method for Riemann-Hilbert problems: the small dispersion limit of the Korteweg-de Vries (KdV) equation, *PNAS*, **95** (1998), 450–454.

[19] P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problem problems. Asymptotics for the MKdV equation, *Ann. Math.*, **137** (1993) 295–368.

[20] A.S. Fokas, A unified transform method for solving linear and certain nonlinear PDEs, *Proc. Roy. Soc. Lond. A*, **453** (1997) 1411–1443.

[21] X. Geng, H. Liu, The nonlinear steepest descent method to long-time asymptotics of the coupled nonlinear Schrödinger equation, *J. Nonlinear Sci.*, **28** (2018) 739–763.

[22] B. Guo, N. Liu, Long-time asymptotics for the Sasa-Satsuma equation via nonlinear steepest descent method, *J. Math. Phys.*, **60** (2019) 011504.

[23] B. Guo , N. Liu and Y. Wang, Long-time asymptotics for the Hirota equation on the half-line, *Nonlinear Analysis*, **174** (2018) 118–140.

[24] L. Huang, J. Lenells, Construction of solutions and asymptotics for the sine-Gordon equation in the quarter plane, *J. Integrable Syst.*, **3** (2018) , xyy014, 92 pp.

[25] L. Huang, J. Lenells, Asymptotics for the Sasa–Satsuma equation in terms of a modified Painlevé II transcendent, *J. Differ. Equations*, **268** (2020) 7480–7504.

[26] L. Huang, J. Xu and E. Fan, Long-time asymptotic for the hirota equation via nonlinear steepest descent method, *Nonlinear Analysis: Real World Applications*, **26** (2015) 229–262.

[27] J. Lenells, The nonlinear steepest descent nethod: asymptotics for initial-boundary value problem, *SIAM J. Math Anal.*, **48** (2016) 2076–2118.

[28] H. Liu, X. Geng and B. Xue, The Deift-Zhou steepest descent method to long-time asymptotics for the Sasa-Satsuma equation, *J. Differ. Equations*, **265** (2018) 5984–6008.

[29] A.V. Kitaev, A.H. Vartanian, Leading-order temporal asymptotics of the modified nonlinear Schrödinger equation: solitonless sector, *Inverse problems*, **13** (1997) 1311–1339.

[30] V. Kotlyarov, A. Minakov, Riemann-Hilbert problem to the modified Korteveg-de Vries equation: Long-time dynamics of the steplike initial data, *J. Math. Phys.*, **51** (2010) 093506.

[31] M. Lakshmanan, K. Porsezian and M. Daniel, Effect of discreteness on the continuum limit of the Heisenberg spin chain, *Phys. Lett. A*, **133** (1988) 483–488.

[32] A. Minakov, Riemann-Hilbert problem for Camassa-Holm equation with step-like initial data, *J. Math. Anal. Appl.*, **429** (2015) 81–104.

[33] E. Moskovchenko, V. Kotlyarov, Periodic boundary data for an integrable model of stimulated Raman scattering: long-time asymptotic behavior, *J. Phys. A*, **45** (2010) 055205.

[34] K. Porsezian, M. Daniel and M. Lakshmanan, On the integrability aspects of the one-dimensional classical continuum isotropic Heisenberg spin chain, *J. Math. Phys.*, **33** (1992) 1807–1816.

[35] K. Porsezian, Completely integrable nonlinear Schrödinger type equations on moving space curves, *Phys. Rev. E*, **55** (1997) 3785–3789.

[36] D. Wang, B. Guo and X. Wang, Long-time asymptotics of the focusing Kundu-Eckhaus equation with nonzero boundary conditions, *J. Differ. Equations*, **266** (2018) 5209–5253.

[37] J. Xu, Long-time asymptotics for the short pulse equation, *J. Differ. Equations*, **265** (2018) 3494–3532.

[38] J. Xu, E.G. Fan, Leading-order temporal asymptotics of the Fokas-Lenells Equation without solitons, *J. Differ. Equations*, **259** (2015) 1098–1148.

[39] J. Xu, E.G. Fan and Y. Chen, Long-time asymptotic for the derivative nonlinear Schrödinger equation with step-like initial value, *Math. Phys. Anal. Geom.*, **16** (2013) 253–288.

[40] V.E. Zakharov, A. Shabat, A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem, I and II, *Funct. Anal. Appl.*, **8** (1974) 226–235 and **13** (1979) 166–174.

[41] X. Zhou, The Riemann–Hilbert problem and inverse scattering, *SIAM J. Math Anal.*, **20** (1989), 966–986.

Co-published by Atlantis Press and Taylor & Francis