



Research Article

# A Note on the D-trigonometry and the Relevant D-Fourier Expansions

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## ABSTRACT

Considering the diamond, i.e. the square inclined at an angle of 45°, it is possible to define the analogues of circular functions and to construct formulas that translate the trigonometric ones. The relative D-trigonometric functions have geometric shapes closely related to the corresponding classical ones, so that the orthogonality property can also be proven and D-Fourier expansions follow easily. Possible applications can be found in the representation piece-wise linear functions in a simpler way form compared to ordinary Fourier analysis.

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## 1. INTRODUCTION

According to the work of Johan Gielis book [1], Mladinić [2] shows the possibility to introduce a trigonometry based on different geometrical shapes, generalizing the circle. Actually it should be possible, starting, for example, from a regular polygon centred at the origin, to introduce functions similar to the trigonometric *sine* and *cosine* and to find the equations that generalize the circular formulas. However, in order to recover the analogues of the simplest classical equations, it is necessary to choose the exact position of the polygon adjusted appropriately. This is not done in the above mentioned article [2], and this is probably why no equation is found in that article.

Among the considered figures, it seems that the most simple one is the *diamond*, defined by the parametric equations

$$\begin{cases} x(t) = \arcsin(\cos t) \\ y(t) = \arcsin(\sin t) \end{cases} \quad (1)$$

where  $t \in [0, 2\pi]$ , (see Figures 1 and 2).

Actually the *diamond* is a very particular case (for  $n = 1$ ) of the Lamé curves, as considered in Gielis [1].

Note that the basic equation  $\sin(\pi/2 - t) = \cos t$  turns into

$$\arcsin\left[\sin\left(\frac{\pi}{2} - t\right)\right] = \arcsin(\cos t), \quad (2)$$

i.e. the two graphs coincide up to a translation of  $\pi/2$ .

This is a fundamental property that is not valid for the graphs introduced in Mladinić [2], and which allows to build most of the results proved in the following.

## 2. BASIC GRAPHS

In this section we show the basic shapes of the D-trigonometric functions, noting the strict analogy with that related to the circle. A number of figures are reported in what follows, in order to justify this statement.

For example, we find the same periodicity for the functions in which the angle is a multiple or a submultiple.

- The periodicity of the functions  $\arcsin(\sin mt)$  and  $\arcsin(\cos mt)$  is  $2\pi/m$ .
- The periodicity of the functions  $\arcsin(\sin(t/m))$  and  $\arcsin(\cos(t/m))$  is  $2m\pi$ .

Furthermore, the areas delimited by the graphs behave in a very similar way. This fact has an immediate effect on the calculation of the corresponding integrals and allows to determine the relevant orthogonality properties.

All these facts are shown in what follows, mainly by the graphical point of view, also noting that the considered functions are related with the possible motion of a bicycle with square wheels on a suitable ground, as it is recalled in [3].

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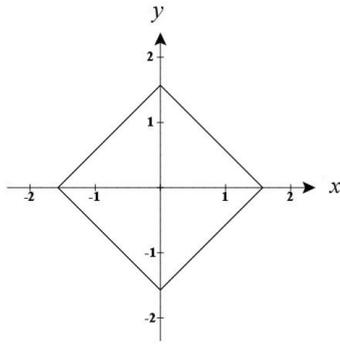


Figure 1 | The diamond.

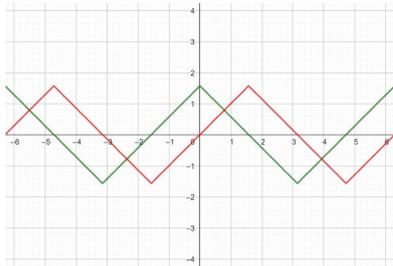


Figure 2 | Green:  $x(t)$ , red:  $y(t)$ .

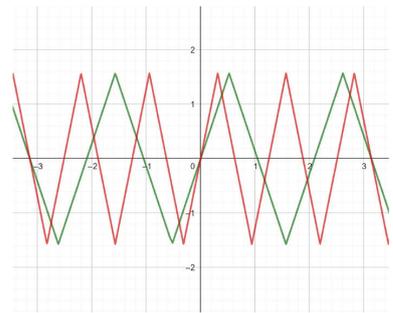


Figure 3 | Green:  $\arcsin(\sin(3t))$ , red:  $\arcsin(\sin(5t))$ .

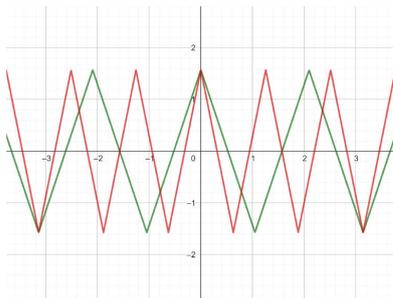


Figure 4 | Green:  $\arcsin(\cos(3t))$ , red:  $\arcsin(\cos(5t))$ .

Changing the angle into a its multiple, the graphs of functions exhibit the same character of the circular functions.

In what follows some examples are shown.

In Figure 3 the graph of  $\arcsin(\sin(3t))$  is represented in green, and that of  $\arcsin(\sin(5t))$  is represented in red.

In Figure 4 the graph of  $\arcsin(\cos(3t))$  is represented in green, and that of  $\arcsin(\cos(5t))$  is represented in red.

In Figure 5 the graph of  $\arcsin(\sin(3t))$  is represented in green, and that of  $\arcsin(\cos(5t))$  is represented in red.

In Figure 6 the graph of  $\arcsin(\sin(t))$  is represented in green, that of  $\arcsin(\sin(2t))$  in red and that of  $\arcsin(\sin(t/2))$  in light blue.

In Figure 7 the graph of  $\arcsin(\cos(t))$  is represented in green, that of  $\arcsin(\cos(2t))$  in red and that of  $\arcsin(\cos(t/2))$  in light blue.

In Figure 8 it is shown the analogy with the circular tan and cotan functions.

In Figure 9 it is shown the fundamental identity  $\arcsin(\sin^2(t)) + \arcsin(\cos^2(t)) = D(t)$ , where  $D(t)$  is a periodic function similar to the type of profile under which the diamond can roll without crawling Derby SJ, et al. [3]. For the circle this profile is obviously a straight line (in the graph  $y(t) \equiv 1$ ).

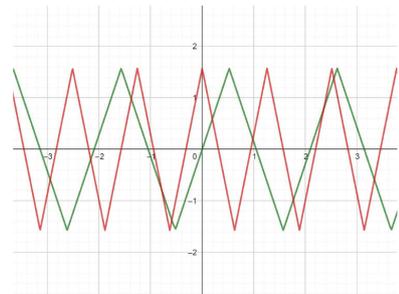


Figure 5 | Green:  $\arcsin(\sin(3t))$ , red:  $\arcsin(\cos(5t))$ .

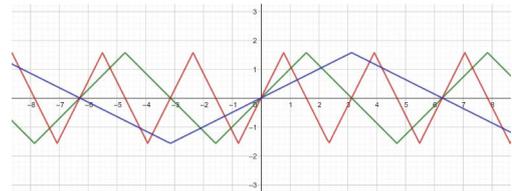


Figure 6 | Green:  $\arcsin(\sin t)$ , red:  $\arcsin(\sin(2t))$ , blue:  $\arcsin(\sin(t/2))$ .

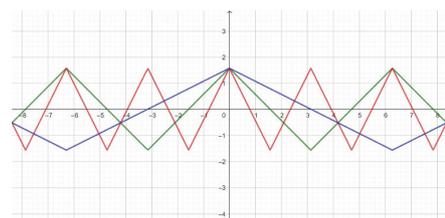


Figure 7 | Green:  $\arcsin(\cos t)$ , red:  $\arcsin(\cos(2t))$ , blue:  $\arcsin(\cos(t/2))$ .

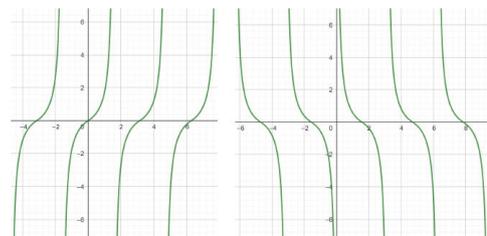


Figure 8 |  $\arcsin(\sin t)/\arcsin(\cos t)$ .

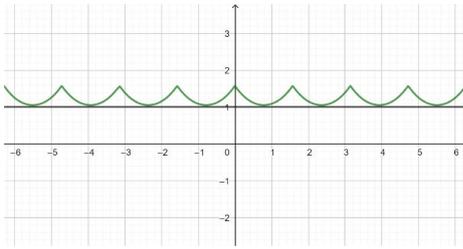


Figure 9 | Green:  $\arcsin(\sin^2 t) + \arcsin(\cos^2 t)$ .

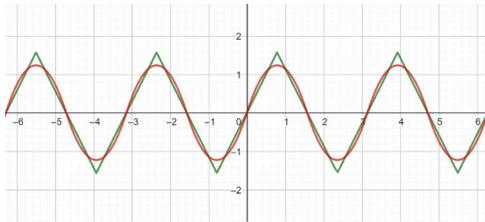


Figure 10 | Green:  $\arcsin(\sin(2t))$ , red:  $2 \arcsin(\sin(t)) \arcsin(\cos(t))$ .

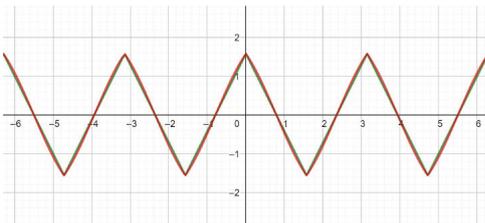


Figure 11 | Green:  $\arcsin(\cos(2t))$ , red:  $2 \arcsin(\cos^2(t)) - \arcsin(\sin^2(t))$ .

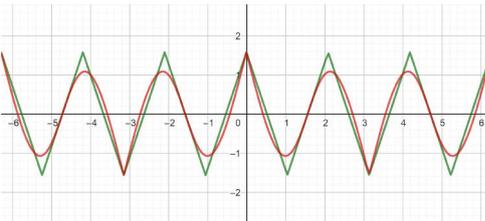


Figure 12 | Green:  $\arcsin(\cos(3t))$ , red:  $4 \arcsin(\cos^3(t)) - 3 \arcsin(\cos(t))$ .

### 3. ANALOGY WITH THE CIRCULAR MULTIPLICATION FORMULAS

In Figure 10 we compare the two functions:

- $\arcsin(\sin(2t))$ , in green,
- $2 \arcsin(\sin(t)) \arcsin(\cos(t))$ , in red.

In Figure 11 we compare the two functions:

- $\arcsin(\cos(2t))$ , in green,
- $2 \arcsin(\cos^2(t)) - \arcsin(\sin^2(t))$ , in red.

In Figure 12 we compare the two functions:

- $\arcsin(\cos(3t))$ , in green,
- $4 \arcsin(\cos^3(t)) - 3 \arcsin(\cos(t))$ , in red.

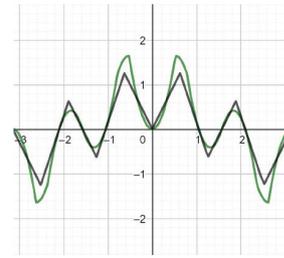


Figure 13 | Green:  $\arcsin(\sin 3t) \arcsin(\sin 2t)$ , black:  $\frac{1}{2} [\arcsin(\cos t) - \arcsin(\cos 5t)]$ .

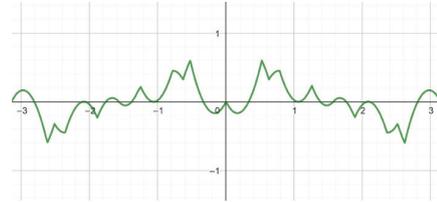


Figure 14 |  $\arcsin(\sin 3t) \arcsin(\sin 2t) - \frac{1}{2} [\arcsin(\cos t) - \arcsin(\cos 5t)]$ .

### 4. ANALOGY WITH THE CIRCULAR FORMULAS OF PROSTAFERESIS

In particular cases it is possible to show the analogy of the circular prostaferesis formulas [4] with those of the diamond case.

In Figure 13 the graphs of the functions

- $\arcsin(\sin 3t) \arcsin(\sin 2t)$ , in green

and

- $\frac{1}{2} [\arcsin(\cos t) - \arcsin(\cos 5t)]$ , in black

are compared, showing the analogy with the circular equation

$$\sin 3t \sin 2t = \frac{1}{2} (\cos t - \cos 5t)$$

In Figure 14 it is shown the even function, representing the difference

$$\arcsin(\sin 3t) \arcsin(\sin 2t) - \frac{1}{2} [\arcsin(\cos t) - \arcsin(\cos 5t)]$$

In Figure 15 the graphs of the functions

- $\arcsin(\cos 5t) \arcsin(\cos 2t)$ , in green

and

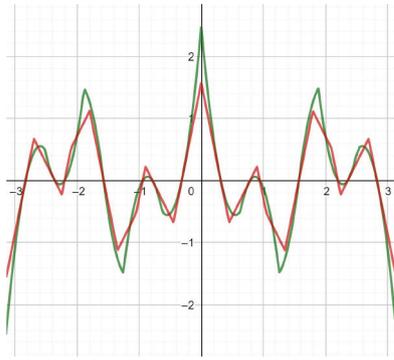
- $\frac{1}{2} [\arcsin(\cos 7t) + \arcsin(\cos 3t)]$ , in red

are compared, showing the analogy with the circular equation:

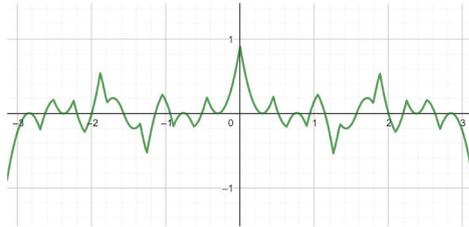
$$\cos 5t \cos 2t = \frac{1}{2} (\cos 7t + \cos 3t)$$

In Figure 16 it is shown the even function, representing the difference

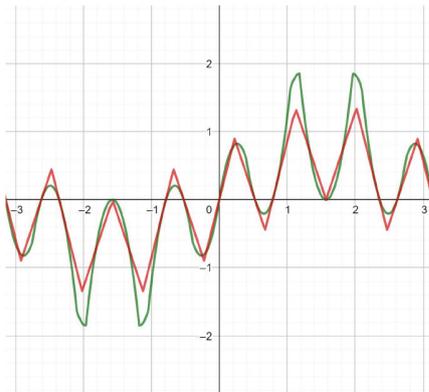
$$\arcsin(\cos 5t) \arcsin(\cos 2t) - \frac{1}{2} [\arcsin(\cos 7t) + \arcsin(\cos 3t)]$$



**Figure 15** Green:  $\arcsin(\cos 5t) \arcsin(\cos 2t)$ , red:  $1/2[\arcsin(\cos 7t) + \arcsin(\cos 3t)]$ .



**Figure 16**  $\arcsin(\cos 5t) \arcsin(\cos 2t) - 1/2[\arcsin(\cos 7t) + \arcsin(\cos 3t)]$ .



**Figure 17** Green:  $\arcsin(\sin 4t) \arcsin(\cos 3t)$ , red:  $1/2 [\arcsin(\sin 7t) + \arcsin(\sin t)]$ .

In **Figure 17** the graphs of the functions

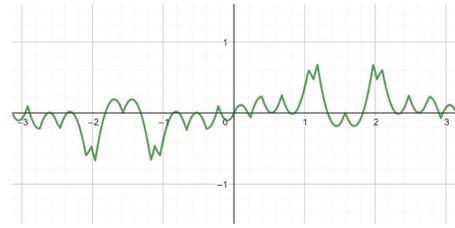
- $\arcsin(\sin 4t)\arcsin(\cos 3t)$ , in green
- and
- $\frac{1}{2}[\arcsin(\sin 7t) + \arcsin(\sin t)]$ , in red

are compared, showing the analogy with the circular equation

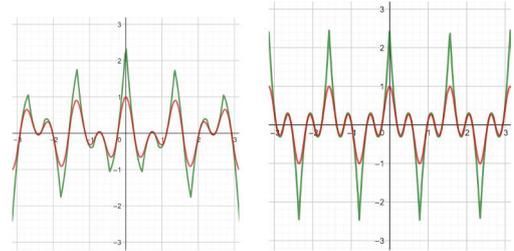
$$\sin(4t) \cos(3t) = \frac{1}{2}(\sin 7t + \sin t).$$

In **Figure 18** it is shown the odd function, representing the difference

$$\arcsin(\sin(4t))\arcsin(\cos(3t)) - \frac{1}{2} [\arcsin(\sin(7t)) + \arcsin(\sin(t))]$$



**Figure 18**  $\arcsin(\sin(4t)) \arcsin(\cos(3t)) - 1/2[\arcsin(\sin 7t) + \arcsin(\sin t)]$ .



**Figure 19** Left:  $\cos(2t) \cos(7t)$ , in green VS  $\arcsin(\cos(2t)) \arcsin(\cos(7t))$ , in red; right:  $\cos(4t) \cos(8t)$ , in green VS  $\arcsin(\cos(4t)) \arcsin(\cos(8t))$ , in red.

## 5. ORTHOGONALITY PROPERTIES

The orthogonality of the functions  $\arcsin(\sin nt)$  and  $\arcsin(\cos mt)$ ,  $\forall m, n$  is a trivial consequence of the symmetry of the interval and the odd symmetry of the product, so that:

$$\int_{-\pi}^{\pi} \arcsin(\sin nt) \arcsin(\cos mt) dt = 0, \quad (\forall n, m). \quad (3)$$

It is also possible to prove the orthogonality properties,  $\forall n \neq m$ :

$$\int_{-\pi}^{\pi} \arcsin(\cos nt) \arcsin(\cos mt) dt = 0, \quad (4)$$

$$\int_{-\pi}^{\pi} \arcsin(\sin nt) \arcsin(\sin mt) dt = 0.$$

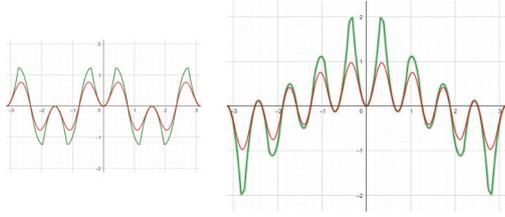
To this aim, it is sufficient to observe that the orthogonality property of the circular functions, that is  $\forall m \neq n$ , the definite integral in  $(-\pi, \pi)$  of the products  $\cos mt \cos nt$  or  $\sin mt \sin nt$  follows from the graph of these functions, for which the positive values of the areas above the  $t$  axis are compensated, by symmetry, by the negative values below.

Since the products  $\arcsin(\cos mt)\arcsin(\cos nt)$  and  $\arcsin(\sin mt) \arcsin(\sin nt)$  are piece-wise linear functions which exhibit a high variation in the considered interval, the numerical computation, by considering all sub-intervals in which the function is linear, is not possible in general. However, it is possible to compare the product  $\arcsin(\cos mt)\arcsin(\cos nt)$  with  $\cos mt \cos nt$  and the product  $\arcsin(\sin mt) \arcsin(\sin nt)$  with  $\sin mt \sin nt$ . A graphical comparison, for every fixed  $m$  and  $n$ , will prove the same behaviour of the two functions.

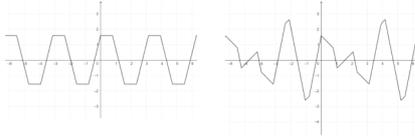
A few particular examples are shown below, but the same result can be checked for every different values of  $m$  and  $n$ , in order to prove the orthogonality of functions for these values.

Let us consider here some special cases.

In **Figure 19** (left) the graph of the function  $\cos(2t)\cos(7t)$ , in green, is compared to that of  $\arcsin(\cos(2t))\arcsin(\cos(7t))$ , in red.



**Figure 20** Left:  $\sin(2t)\sin(4t)$ , in green VS  $\arcsin(\sin(2t))\arcsin(\sin(4t))$ , in red; right:  $\sin(4t)\sin(5t)$ , in green VS  $\arcsin(\sin(4t))\arcsin(\sin(5t))$ , in red.



**Figure 21** Left:  $\arcsin(\sin(2t)) + \arcsin(\cos(2t))$ , right:  $\arcsin(\sin(2t)) + \arcsin(\cos(3t))$ .

In [Figure 19](#) (right) the graph of the function  $\cos(4t)\cos(8t)$ , in green, is compared to that of  $\arcsin(\cos(4t))\arcsin(\cos(8t))$ , in red.

In [Figure 20](#) (left) the graph of the function  $\sin(2t)\sin(4t)$ , in green, is compared to that of  $\arcsin(\sin(2t))\arcsin(\sin(4t))$ , in red.

In [Figure 20](#) (right) the graph of the function  $\sin(4t)\sin(5t)$ , in green, is compared to that of  $\arcsin(\sin(4t))\arcsin(\sin(5t))$ , in red.

## 6. EXAMPLES OF D-TRIGONOMETRIC POLYNOMIALS

Noting that by using combinations of the above considered functions, i.e. considering D-trigonometric polynomials, it is possible to represent with a closed form many piece-linear functions that are usually expressed by trigonometric series. A few examples are shown in the figures below.

In [Figure 21](#) (left) it is shown the graph of function

$$y = \arcsin(\sin(2t)) + \arcsin(\cos(2t))$$

and in [Figure 21](#) (right) the graph of the function

$$y = \arcsin(\sin(2t)) + \arcsin(\cos(3t))$$

In [Figure 22](#) (left) it is shown the graph of the function

$$y = \arcsin(\sin(11t)) + \arcsin(\cos(13t))$$

and in [Figure 22](#) (right) the graph of the function

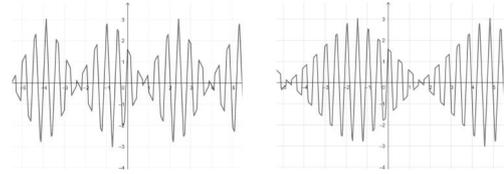
$$y = \arcsin(\sin(12t)) + \arcsin(\cos(13t))$$

The shape of the combinations shown in the above figures suggests the possibility to represent graphs with high variation linear functions, typical of sounds or seismic phenomena, with D-Fourier series expansions, that is expansions in  $\arcsin(\cos t)$  and  $\arcsin(\sin t)$  functions.

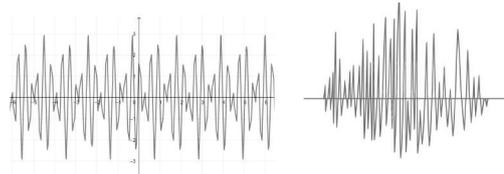
In [Figure 23](#) (right) a sound wave, and in [Figure 23](#) (left) the periodic function, combination of D-trigonometric functions:

$$y = \arcsin(\cos(21t)) + \arcsin(\sin(15t))$$

are represented.



**Figure 22** Left:  $\arcsin(\sin(11t)) + \arcsin(\cos(13t))$ , right:  $\arcsin(\sin(12t)) + \arcsin(\cos(13t))$ .



**Figure 23** Left:  $\arcsin(\cos(21t)) + \arcsin(\sin(15t))$ , right: a sound wave.

## 7. D-FOURIER POLYNOMIAL EXPANSIONS

As it seems that only a small number of D-trigonometric functions are sufficient to approximate graphs of complicate shapes, in what follows we consider only D-Fourier polynomial expansions, avoiding the complicate problems of series' convergence.

Given a function  $f(t) \in L^2(-\pi, \pi)$ , put:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^N a_k \arcsin(\cos kt) + b_k \arcsin(\sin kt), \quad (5)$$

where  $N$  is the highest frequency of the signal  $f(t)$ .

Then the coefficients are computed by the using Fourier's method [5].

For  $k = 0$ , by [Eqs. \(3\)](#) and [\(4\)](#), it results:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt, \quad (6)$$

and, for  $k \geq 1$ , the coefficients are derived by taking the scalar products:

$$(f(t), \arcsin(\cos kt)) = a_k \int_{-\pi}^{\pi} \arcsin^2(\cos kt) dt, \quad (7)$$

$$(f(t), \arcsin(\sin kt)) = b_k \int_{-\pi}^{\pi} \arcsin^2(\sin kt) dt, \quad (8)$$

so that:

$$a_k = \frac{(f(t), \arcsin(\cos kt))}{\|\arcsin(\cos nt)\|^2}, \quad (9)$$

$$b_k = \frac{(f(t), \arcsin(\sin kt))}{\|\arcsin(\sin nt)\|^2}. \quad (10)$$

The integrals in the denominators of [Eqs. \(9\)](#) and [\(10\)](#) are given by:

$$\begin{aligned} \int_{-\pi}^{\pi} \arcsin^2(\sin kt) dt &= \int_{-\pi}^{\pi} \arcsin^2(\cos kt) dt \\ &= 2 \int_{-1}^1 \frac{(\arcsin x)^2}{\sqrt{1-x^2}} dx \simeq 5.16771... \end{aligned} \quad (11)$$

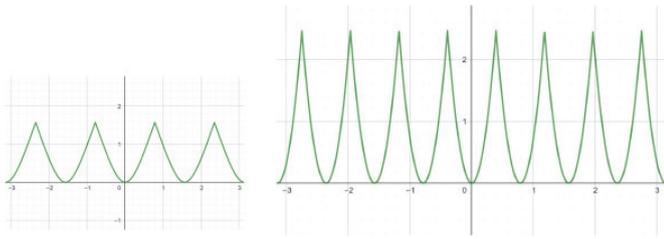


Figure 24 Left:  $\arcsin(\sin(2t))^2$ , right:  $\arcsin(\sin(4t))^2$ .

**Remark 7.1.**

Note that for computing the integral in Eq. (11) it is necessary to divide the interval into several parts and to use the additivity property of the integral with respect to the integration set.

Two examples can be seen in the next figure, where the interval is divided into four and eight parts.

In Figure 24 the graphs of the functions  $\arcsin(\sin(2t))^2$ , (on the left) and  $\arcsin(\sin(4t))^2$ , (on the right), are shown.

Then, introducing the constant  $D := 5.16771\dots$ , we find:

$$a_k = \frac{1}{D} \int_{-\pi}^{\pi} f(t) \arcsin(\cos kt) dt, \tag{12}$$

$$b_k = \frac{1}{D} \int_{-\pi}^{\pi} f(t) \arcsin(\sin kt) dt. \tag{13}$$

Therefore, Eq. (5) writes:

$$f(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{k=1}^N \left[ \frac{1}{D} \left( \int_{-\pi}^{\pi} f(t) \arcsin(\cos kt) dt + \int_{-\pi}^{\pi} f(t) \arcsin(\sin kt) dt \right) \right], \tag{14}$$

where  $1/D \approx 0,1935\dots$ , is a constant which corresponds to  $1/\pi \approx 0,3183$ , appearing in the Fourier trigonometric coefficients.

## 8. CONCLUSION

The D-trigonometric functions, that is the analogues of the circular function, related to a diamond instead of a circle have been introduced. These functions exhibit a character strictly related to the classical trigonometric functions, and it is possible to write for them equations corresponding to the circular ones. Deriving the orthogonal properties, by exploiting the symmetry properties of the relevant graphs, the D-Fourier polynomial expansions can be easily proven. It is worth to note that finite combinations of D-trigonometric functions allow to write the piece-wise linear functions in a more simple way with respect to their Fourier expansions.

## CONFLICTS OF INTEREST

The author declare that he has not received funds from any institution and that he has no conflicts of interest.

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