Journal of Nonlinear Mathematical Physics
In Press, Corrected Proof
DOI: https://doi.org/10.2991/jnmp.k.201104.001; ISSN 1402-9251; eISSN 1776-0852
https://www.atlantis-press.com/journals/jnmp

Research Article

Compatible Poisson Structures and bi-Hamiltonian Systems Related to Low-dimensional Lie Algebras

Gh. Haghighatdoost[1,*], S. Abdolhadi-Zangakani[2], J. Abedi-Fardad[1]

[1]*Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz 53714-161, Iran*
[2]*Department of Mathematics, University of Bonab, Tabriz, Iran*

ARTICLE INFO

Article History

Received 29 June 2020
Accepted 31 October 2020

Keywords

Compatible Poisson structures
bi-Hamiltonian system
Lie groups

ABSTRACT

In this work, we give a method to construct compatible Poisson structures on Lie groups by means of structure constants of their Lie algebras and some vector field. In this way we calculate some compatible Poisson structures on low-dimensional Lie groups. Then, using a theorem by Magri and Morosi, we obtain new integrable bi-Hamiltonian systems with two-, four- and six-dimensional symplectic real Lie groups as phase spaces.

1. INTRODUCTION

The pioneering work in the integrable bi-Hamiltonian systems is done by Magri [9] and then followed by the fundamental papers by Gelfand and Dorfman [6], Kosmann-Schwarzbach and Magri [8] and Magri and Morosi [11]. These works show that integrability of many systems in mathematical physics, mechanics, and geometry is closely related to their bi-Hamiltonian structures. It is shown that many classical systems have the bi-Hamiltonian structure, at the same time by using the bi-Hamiltonian methods many new nontrivial and interesting examples of integrable systems have been found (for more details see [5]). As we know, the study of bi-Hamiltonian structure is based on the very simple notion of compatible Poisson structures. It is proved that bi-Hamiltonian structure is very powerful in the theory of integrable Hamiltonian systems not only for finding new examples, but also for the integration of systems, constructing separation of variables and description of properties of solutions (see [6–10] for a review). In Abedi-Fardad et al. [2], by the adjoint representation of Lie algebra authors have calculated some compatible Poisson structure and bi-Hamiltonian systems on Lie groups as phase space. In this work, we give a method to construct compatible Poisson structures on Lie groups by means of structure constants of its Lie algebras and some vector field **X**. Then we obtain new integrable bi-Hamiltonian systems by using Magri–Morosi theorem [11], for which the Lie group is the phase space.

The structure of this paper is as follows. In Section 2, we briefly review the definitions and notations of compatible Poisson structures and integrable bi-Hamiltonian systems. In Section 3, we give a method to obtain the compatible Poisson structures on low-dimensional Lie groups by means of structure constants of Lie algebras and some vector field **X**. In Section 4, we obtain these structures on two, four and nilpotent six-dimensional symplectic real Lie algebras. Finally, in Section 5, we obtain some compatible Poisson structures and integrable bi-Hamiltonian systems with two, four and nilpotent six-dimensional symplectic real Lie groups as phase spaces.

2. PRELIMINARIES

In this section, we recall some basic definitions and notations on compatible Poisson structures and integrable bi-Hamiltonian systems.

Definition: [4] A bi-Hamiltonian manifold **M** is a smooth manifold endowed with two compatible bi-vectors **P** and **P′** such that

$$[\mathbf{P},\mathbf{P}]=0,\quad [\mathbf{P},\mathbf{P}']=0,\quad [\mathbf{P}',\mathbf{P}']=0\,, \tag{1}$$

where $[.,.]$ is the Schouten bracket.

The Poisson bracket corresponding to the Poisson bi-vector has the form

$$\{f(x),\,g(x)\}=<df\,,\mathbf{P}\,dg>=\mathbf{P}^{ij}(x)\frac{\partial f(x)}{\partial x_i}\frac{\partial g(x)}{\partial x_j}\,, \tag{2}$$

*Corresponding author. Email: gorbanali@azaruniv.ac.ir

for all $f, g \in F(M)$ and similar brackets $\{.,.\}'$ to $\mathbf{P}'$. The brackets $\{.,.\}'$ and $\{.,.\}'$ satisfies the Jacobi identity: i.e. $\forall\, f, g, h \in C^\infty(\mathbf{M})$,

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=[\mathbf{P},\mathbf{P}]^{ijk}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial x_k}=0, \tag{3}$$

if $[\mathbf{P}, \mathbf{P}] = 0$ and vice versa. A family of functions $\{H_i\}$ on the manifold $\mathbf{M}$ that are in bi-involution with respect to these compatible Poisson brackets,

$$\{H_i, H_j\} = \{H_i, H_j\}' = 0, \tag{4}$$

is called bi-integrable system or generalized bi-Hamiltonian system [12]. So to introduce the bi-Hamiltonian structure on the manifold $\mathbf{M}$, we must determine a pair of compatible and independent Poisson bi-vectors $\mathbf{P}$ and $\mathbf{P}'$.

Definition: [14] In the coordinate basis, $T_p\mathbf{M}$ spanned by $\{e_\mu\} = \{\partial_\mu\}$ and $T^*_p\mathbf{M}$ by $\{dx^\mu\}$, let us consider their linear combinations,

$$e_i = e_i^{\mu}\,\partial_\mu, \quad \Theta^i = e^i_{\mu}\,dx^\mu, \quad e_i^{\mu} \in GL(m, R), \tag{5}$$

where $m = dim(\mathbf{M})$, e_i^{μ} and e^i_{μ} are non-singular $(m \times m)$-matrices. In other words, $\{e_i\}$ is the frame of basis vectors which is obtained by a $GL(m, R)$-rotation of the basis $\{e_\mu\}$. In the above e^i_{μ} is inverse of e_i^{μ} and we have

$$e^i_{\mu}\, e_i^{\nu} = \delta^{\nu}_{\mu}, \quad e^i_{\mu} e^{\mu}_j = \delta^i_j, \tag{6}$$

the bases $\{e_i\}$ and $\{\Theta^i\}$ are called the non-coordinate bases and coefficients e_i^{μ} are called the vielbeins. We have

$$[e_i, e_j] = f_{ij}^{\ k} e_k, \tag{7}$$

where $f_{ij}^{\ k}$ is a function of coordinates of the manifold $\mathbf{M}$. When $\mathbf{M}$ is a Lie group $\mathbf{G}$, we suppose that coefficients $f_{ij}^{\ k}$ be the structure constants of the Lie algebra $\mathbf{g}$ of the Lie group $\mathbf{G}$ and also e_i are (left or right) invariant vector fields on the group $\mathbf{G}$ (for more details see [14]). We have

$$f_{ij}^{\ k} = e^k_{\nu}\,(e_i^{\mu}\,\partial_\mu\, e_j^{\nu} - e_j^{\mu}\,\partial_\mu\, e_i^{\nu}). \tag{8}$$

Consider a symplectic manifold $\mathbf{M}$ endowed with a second compatible Poisson bracket. An important class of bi-Hamiltonian manifold occurs when one of the compatible Poisson structures is invertible, then one can define a linear map $\mathbf{N}$: $TM \to TM$ acting on the tangent bundle by Magri et al. [12]

$$\mathbf{N} = \mathbf{P}'\, \mathbf{P}^{-1}. \tag{9}$$

Also by using Magri–Morosi's theorem as follows, one can find the Hamiltonian and integrals of motions of bi-Hamiltonian systems.

Theorem (Magri–Morosi): [3,4] *A remarkable consequence of the compatibility of* $\mathbf{P}$ *and* $\mathbf{P}'$ *is that the torsion of Nijenhuis tensor* $\mathbf{N}$*, i.e.*

$$\mathbf{T}_{\mathrm{N}}(X,Y)=[\mathbf{N}X,\mathbf{N}Y]-\mathbf{N}[\mathbf{N}X,Y]-\mathbf{N}[X,\mathbf{N}Y]+\mathbf{N}^2[X,Y], \tag{10}$$

identically vanishes, where X and Y are arbitrary vector fields and the bracket $[X, Y]$ *denotes the Lie bracket (commutator) of vector fields. One of the main properties of* $\mathbf{N}$ *is that the normalized traces of the powers of* $\mathbf{N}$

$$H_k = \frac{1}{2k} Tr\mathbf{N}^k, \tag{11}$$

are in involution and satisfy Lenard–Magri recurrence relations [12]

$$\mathbf{P}'\, dH_i = \mathbf{P}\, dH_{i+1}. \tag{12}$$

3. COMPATIBLE POISSON STRUCTURES ON LOW-DIMENSIONAL LIE GROUPS

In this section, we give a method to obtain compatible Poisson structures on low-dimensional Lie groups by means of structure constants of related Lie algebras and some vector field $\mathbf{X}$. For this purpose, we write the Poisson structure $\mathbf{P}$ (which is presented in Abedi-Fardad et al. [2]) in terms of the non-coordinate basis as

$$\mathbf{P}^{\mu\nu} = e_i^{\ \mu} e_j^{\ \nu} P^{ij}, \tag{13}$$

where P^{ij}'s are constant antisymmetric matrix[1]. One can rewrite the Schouten bracket [**P**, **P**] = 0 in the following matrix forms [11]:

$$P\ X_i\ P^{ij} + P\ Y^j\ P + P^{ij}\ X_i^{\ t}\ P = 0, \tag{14}$$

where $(X_i)_j^{\ k} = -f_{ij}^{\ \ k}$ and $(Y^k)_{ij} = -f_{ij}^{\ \ k}$. In this way, having the structure constants of the Lie algebra **g**, one can solve (14) and obtain constant antisymmetric matrix P. Then, by substituting P in Eq. (13) we can obtain Poisson structure **P** of the Lie group **G**. The list of some Poisson structure **P** obtained by Eq. (14) are brought in Abedi-Fardad et al. [2].

Now, according to Tsiganov [18,19] and Vershilov and Tsiganov [21], let us suppose that the desired second Poisson bi-vector $\mathbf{P}'$ is the Lie derivative of **P** along some unknown vector field **X**

$$\mathbf{P}' = L_X(\mathbf{P})\,, \tag{15}$$

which must satisfy the equation

$$[\mathbf{P}', \mathbf{P}'] = [L_X(\mathbf{P}), L_X(\mathbf{P})] = 0 \Leftrightarrow [L_X^2(\mathbf{P}), \mathbf{P}] = 0\,, \tag{16}$$

with respect to the Schouten bracket [.,.]. By (15) bi-vector $\mathbf{P}'$ is compatible with a given bi-vector **P**, i.e. $[\mathbf{P}, \mathbf{P}'] = 0$. Obviously, Eq. (16) is too difficult to be solved, because it has infinitely many solutions labeled by different separated coordinates (see [9] and [10]). To solve Eq. (16), we will use the vector field [9]

$$\mathbf{X} = \mathbf{X}^{\mu}\partial_{\mu}. \tag{17}$$

In this way the relation (15) has the following form:

$$\mathbf{P}'^{\mu\nu} = \mathbf{X}^{\lambda}(\partial_{\lambda}\mathbf{P}^{\mu\nu}) + (\partial_{\lambda}\mathbf{X}^{\mu})\mathbf{P}^{\nu\lambda} + \partial_{\lambda}\mathbf{X}^{\nu}\mathbf{P}^{\lambda\mu}. \tag{18}$$

We suppose that

$$\mathbf{X}^{\mu} = X^i e_i^{\ \mu}, \tag{19}$$

where X^i is a linear function of group coordinates x_i of the Lie group **G**. Note that one can obtain another set of vector field and second Poisson bi-vector by setting X_i as a quadratic function of the Lie group coordinates and in this way obtain new bi-Hamiltonian systems. Now using (8), (13) and (19) one can rewrite the relation (18) as follows[2]:

$$P'^{mn} = X^i f_{ij}^{\ \ m} P^{jn} + X^i f_{ik}^{\ \ n} P^{mk} - (\partial_{\lambda} X^m) e_k^{\ \lambda} P^{kn} - (\partial_{\lambda} X^n) e_j^{\ \lambda} P^{mj}. \tag{20}$$

Also, we can rewrite Eq. (16) in the following matrix forms [11]:

$$P'\ X_i\ P'^{i\gamma} + P'\ Y^{\gamma}\ P' + P'^{i\gamma}\ X_i^{\ t}\ P' + (e^t P')^{k\gamma}\,\partial_k P' + A + B = 0, \tag{21}$$

where e^t is a transpose of the vielbein e_{α}^{μ} and A and B have the following forms:

$$A = \begin{pmatrix} (e^t P')^{k1}\partial_k P'^{1\gamma} & (e^t P')^{k1}\partial_k P'^{2\gamma} & \cdots & (e^t P')^{k1}\partial_k P'^{m\gamma} \\ (e^t P')^{k2}\partial_k P'^{1\gamma} & & & \\ . & . & & . \\ . & . & & . \\ & . & & . \\ (e^t P')^{km}\partial_k P'^{1\gamma} & (e^t P')^{km}\partial_k P'^{2\gamma} & \cdots & (e^t P')^{km}\partial_k P'^{m\gamma} \end{pmatrix}, \tag{22}$$

$$B = \begin{pmatrix} (e^t P')^{k1}\partial_k P'^{\gamma 1} & (e^t P')^{k2}\partial_k P'^{\gamma 1} & \cdots & (e^t P')^{km}\partial_k P'^{\gamma 1} \\ (e^t P')^{k1}\partial_k P'^{\gamma 2} & & & \\ . & . & & . \\ . & . & & . \\ & . & & . \\ (e^t P')^{k1}\partial_k P'^{\gamma m} & (e^t P')^{k2}\partial_k P'^{\gamma m} & \cdots & (e^t P')^{km}\partial_k P'^{\gamma m} \end{pmatrix}. \tag{23}$$

[1]The study of non-constant P^{ij} can be a new problem.
[2]Note that here P^{ij}'s are constant.

In this way, having the structure constants of the Lie algebra **g** and using the relation (20), we can solve the matrix equation (21) in order to obtain $\mathbf{P}'$ and then by inserting $\mathbf{P}'$ in Eq. (13) and using the related vielbeins, the second Poisson bi-vector $\mathbf{P}'$ on Lie groups is obtained.

Note that in Abedi-Fardad et al. [2], by the adjoint representation of the Lie algebra, the authors have calculated some compatible Poisson structure and bi-Hamiltonian systems on Lie groups as phase space. Indeed, the Schouten bracket (1) has been rewritten in the matrix forms (2.17), (2.21) and (2.22) in Abedi-Fardad et al. [2] and they have obtained the set of compatible Poisson bracket by setting $\mathbf{P}'$ as linear functions of the Lie group coordinates and new bi-Hamiltonian systems. In this work according to Tsiganov [18,19] and Vershilov and Tsiganov [21] we consider

$$\mathbf{P}' = L_{\mathbf{X}}(\mathbf{P}),\ \mathbf{X} = \mathbf{X}^{\mu}\partial_{\mu},$$

and by using adjoint representation, we rewrite (15) in the matrix form (18) and obtain completely different new compatible Poisson bracket and bi-Hamiltonian systems on Lie groups as phase space.

The method which is used in Abedi-Fardad et al. [2] is completely different from the method of current work. By the new method we can find new integrable bi-Hamiltonian systems, for example, for two-dimensional symplectic real Lie groups, but by using the method which applied in Abedi-Fardad et al. [2] one cannot obtain them for the two-dimensional cases.

Our new results are not isomorphic to the systems that have been found in Abedi-Fardad et al. [2]. In this work we suppose vector field **X** to be linear. Study of non-linear vector field can be a new complicated question, but maybe gives us some newer systems.

4. SOME COMPATIBLE POISSON STRUCTURE ON LOW-DIMENSIONAL LIE ALGEBRAS

In this section, we will consider all of the two, four and nilpotent six-dimensional symplectic real Lie algebras and solve matrix equation (21) in order to obtain the vector field **X** and $\mathbf{P}'$. For this purpose, we use the classification of two-, four- and six-dimensional real Lie algebras (A_2, A_4 and A_6) which have been presented in Patera et al. [15]. Let us consider an example; for Lie algebra $A_2 \oplus A_2$ we have the following non-zero commutators:

$$[e_1, e_2] = e_2,\ [e_3, e_4] = e_4.$$

Also, according to Mojaveri and Rezaei-Aghdam [12] for Lie algebra $A_2 \oplus A_2$ the matrix $(e_i^{\ j})$ and P are as follows:

$$(e_i^{\ j}) = \begin{pmatrix} 1 & x_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x_4 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & p_{12} & 0 & 0 \\ * & 0 & 0 & p_{24} \\ * & * & 0 & p_{34} \\ * & * & * & 0 \end{pmatrix}.$$

Substituting $f_{ij}^{\ k}$, P and $e_i^{\ j}$ in Eq. (20) and solving (21) one can obtain the vector field **X** and Poisson structure P' for this Lie algebra. One of the solutions has the following form:

$$\begin{aligned} X^1 &= a_1 x_1 - \frac{d_2 p_{12} x_2}{p_{24}}, \\ X^2 &= b_1 x_1 + b_2 x_2 + b_3 x_3 + \frac{c_4 + p_{24} x_4}{p_{34}}, \\ X^3 &= c_3 x_3 + c_4 x_4, \\ X^4 &= d_1 x_1 + d_2 x_2 + d_3 x_3 + d_4 x_4, \end{aligned}$$

$$P' = \begin{pmatrix} 0 & -a_1 p_{12} - b_2 p_{12} + a_1 p_{12} x_1 & 0 & 0 \\ * & 0 & 0 & d_1 p_{12} - b_2 p_{24} - d_4 p_{24} - b_3 p_{34} + a_1 p_{24} x_1 + c_3 p_{24} x_3 \\ 0 & * & 0 & -c_3 p_{34} - d_4 p_{34} + c_3 p_{34} x_3 \\ * & * & * & 0 \end{pmatrix}.$$

In this way, we have obtained vector field **X** and compatible Poisson structure on two, four and nilpotent six-dimensional symplectic real Lie algebras. The results are summarized in Table 1[3]. Note that in the Table 1 we present some Lie algebras in which we can construct integrable bi-Hamiltonian systems over their related Lie groups. Also, all parameters a_i, b_i, c_i, d_i, e_i, f_i and p_{ij} are arbitrary real constants.

[3]Note that in Abedi-Fardad et al. [11] the Poisson structure P on Lie algebras have been given.

Table 1 | Some vector fields and compatible Poisson structures on two, four and nilpotent six-dimensional symplectic real Lie algebras

g	Vector field X	Non-zero Poisson structure relations P′
A_2	$X^1 = a_1x_1 + a_2x_2$ $X^2 = b_1x_1 + b_2x_2$	$\{x_1, x_2\} = -a_1p_{12} - b_2p_{12} + a_1p_{12}x_1$ $\{x_1, x_2\} = -a_2p_{12} - d_3x_3p_{12} + \frac{a_2x_4p_{12}}{2}$
$A_{4,1}$	$X^1 = a_2x_2 + a_3x_3 + a_4x_4$ $X^2 = b_1x_1 + b_2x_2 - c_3x_3 + b_4x_4$ $X^3 = -b_1x_1 - b_2x_2 + c_3x_3 + c_4x_4$ $X^4 = d_3x_3 - (a_2x_4)/2$	$\{x_1, x_3\} = \frac{a_2p_{14}}{2} - a_4p_{14} - b_4p_{14} - c_4p_{14} + b_4x_4p_{14} + c_4x_4p_{14}$ $\{x_2, x_3\} = -a_2p_{23} - a_3p_{23} - d_3p_{23} - 2d_3x_3p_{23}$
$A_{4,3}$	$X^1 = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$ $X^2 = b_1x_1 + b_2x_2 - b_3x_3 + b_4x_4$ $X^3 = -a_1x_1 - a_2x_2 + (d_3 - a_3) + c_4x_4$ $X^4 = d_3x_3 - b_2x_4$	$\{x_1, x_2\} = -b_2p_{12} - b_1e_4^xp_{12} - d_3p_{12}x_3 + b_2p_{12}x_4$ $\{x_1, x_4\} = (-a_4 + b_2 - b_4 - c_4)p_{14} - b_1e_4^xp_{14} + a_3p_{14}x_3 - d_3p_{14}x_3 + (-a_3 + d_3)p_{14}x_3$ $+ (a_4p_{14} + b_2p_{14} + c_4p_{14})x_4$ $\{x_2, x_3\} = (-a_3 - b_2 - b_3 - d_3 + a_3 - d_3)p_{23} - d_3p_{23}x_3$
$II \oplus R$	$X^1 = d_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$ $X^2 = b_4x_4$ $X^3 = c_2x_2$ $X^4 = d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4$	$\{x_1, x_2\} = -a_2p_{12} - c_2p_{12} - d_2p_{12} - c_2p_{12}x_2$ $\{x_1, x_4\} = -a_3p_{13} - d_3p_{13} + b_4p_{13}x_4$ $\{x_3, x_4\} = -a_3p_{34} - a_4p_{34} - b_4p_{34} - d_3p_{34} - d_4p_{34} + b_4p_{34}x_4$
$III \oplus R$	$X^1 = -a_3x_2 + a_3x_3 + a_4x_4$ $X^2 = b_1x_1 + b_2x_2 + b_3x_3 + a_4x_4$ $X^3 = c_1x_1 - b_2x_2 - (2a_3 + b_3 + d_3)x_3$ $X^4 = d_1x_1 + d_3x_3 + d_4x_4$	$\{x_1, x_2\} = -p_{13}(b_1 + c_1 + d_1 - 2b_1x_1 - 2c_1x_1 + 2d_3x_3 + a_3(-1 + 8x_3))$ $\{x_1, x_3\} = -p_{13}(b_1 + c_1 + d_1 - 2b_1x_1 - 2c_1x_1 + 2d_3x_3 + a_3(-1 + 8x_3))$ $\{x_2, x_4\} = p_{24}(-d_4 + a_3(1 + 2x_2 - 2x_3) - 2a_4(1 + x_4))$
$VI_0 \oplus R$	$X^1 = -b_1x_1 - b_2x_2 + a_3x_3 + a_4x_4$ $X^2 = b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4$ $X^3 = c_1x_1 - c_2x_2$ $X^4 = (-c_1 - c_2 - d_2)x_1 + d_2x_2 + d_3x_3 + d_4x_4$	$\{x_1, x_2\} = -2p_{12}(c_1x_1 + c_2x_2)$ $\{x_3, x_4\} = p_{34}(-b_3 - b_4 - d_3 - d_4 + a_3(-1 + x_3) + b_3x_3 + a_4(-1 + x_4) + b_4x_4)$
$A_2 \oplus A_2$	$X^1 = a_1x_1 - (d_2p_{12}x_2)/p_{24}$ $X^2 = b_1x_1 + b_2x_2 + b_3x_3 + (c_4p_{24}x_4)/p_{34}$ $X^3 = c_3x_3 + c_4x_4$ $X^4 = d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4$	$\{x_1, x_2\} = -a_1p_{12} - b_2p_{12} + a_1p_{12}x_1$ $\{x_2, x_4\} = d_1p_{12} - b_2p_{24} - d_4p_{24} - b_3p_{34} + a_1p_{24}x_1 + c_3p_{24}x_3$ $\{x_3, x_4\} = c_3p_{34} - d_4p_{34} + c_3p_{34}x_3$
$A_{6,7}$	$X^1 = a_1x_1 + a_3x_3,$ $X^2 = -2a_1x_1 + b_3x_3,$ $X^3 = c_2x_2 + c_3x_3 + c_6x_6$ $X^4 = d_1x_1 + c_2x_2 + (-c_3 - h_4)x_3 - c_6x_6$ $X^5 = f_1x_1 + f_2x_2 + f_3x_3 + f_5x_5 + f_6x_6,$ $X^6 = h_1x_1 + h_2x_2 + h_3x_3 + h_4x_4 - f_5x_5 + (-c_3 - f_6)x_6$	$\{x_1, x_5\} = a_1p_{15} - d_1p_{15} - f_1p_{15} - h_1p_{15} - d_1p_{15}x_1 - c_3p_{15}x_3 - (-c_3 - h_4)p_{15}x_3 - h_4p_{15}x_3$ $\{x_2, x_6\} = (-f_2 - (-c_3 - f_6) - f_6 - h_2 - c_2x_2 - c_3x_3 - (-c_3 - f_6)x_3 - f_6x_3 - c_6x_6)p_{26}$ $\{x_3, x_4\} = (-a_3 - b_3 - c_3 - f_3 - h_3 - (-c_3 - h_4) - h_4 + 2a_3x_3 + b_3x_3)p_{34}$ $\{x_4, x_5\} = (-h_4 + a_1x_1 + a_3x_3)p_{45}$
$A_{6,25}$	$X^1 = a_3x_3 + \frac{1}{2}(-c_6 - f_5)x_6,$ $X^2 = b_1x_1,$ $X^3 = c_1x_1 - 2a_3x_3 + c_4x_4 + c_5x_5 + c_6x_6,$ $X^4 = d_5x_5,$ $X^5 = f_2x_2 + a_3x_3 + f_4x_4 + f_5x_5 + \frac{1}{2}(-c_6 - f_5)x_6,$ $X^6 = h_1x_1 + h_2x_2 + h_4x_4 + h_5x_5$	$\{x_1, x_3\} = (-b_1 - c_1 - h_1 - b_1x_1)p_{13}$ $\{x_2, x_4\} = (-c_4 - f_2 - f_4 - h_2 - h_4 - f_2x_2 - f_4x_4 - c_6x_5 - (-c_6 - f_5)x_5 - f_5x_5)p_{24}$ $\{x_3, x_6\} = -c_6p_{36} - (-c_6 - f_5)p_{36}$ $\{x_4, x_5\} = (-h_4 + a_1x_1 + a_3x_3)p_{45}$ $\{x_5, x_6\} = (-c_5 - c_6 - d_5 - (-c_6 - f_5) - f_5 - h_5 + d_5x_5)p_{56}$

5. COMPATIBLE POISSON STRUCTURES AND INTEGRABLE BI-HAMILTONIAN SYSTEMS ON TWO, FOUR AND NILPOTENT SIX-DIMENSIONAL SYMPLECTIC REAL LIE GROUPS

In this section, we construct the compatible Poisson structures and integrable bi-Hamiltonian systems with real Lie groups separately as follows. Substituting P' in Eq. (13) and using the related vielbeins, the compatible Poisson structure $\mathbf{P}$ is obtained. Using relations (15) and (16), the compatible Poisson structure $\mathbf{P}'$ and vector field $\mathbf{X}$ on Lie groups are obtained. In this way we find new bi-Hamiltonian systems over two, four and nilpotent six-dimensional symplectic real Lie groups as phase spaces.

Lie group A_2:

Substituting X^i in Eq. (19) and P' in Eq. (13) one can obtain the vector field $\mathbf{X}$ and compatible Poisson structures $\mathbf{P}$ and $\mathbf{P}'$ as follows:

$$\mathbf{X}^1 = a_1x_1 + a_2x_2 + a_1x_1x_2 + a_2x_2^2,$$
$$\mathbf{X}^2 = b_1x_1 + b_2x_2,$$

$$\mathbf{P}=\begin{pmatrix}0 & p_{12}\\ * & 0\end{pmatrix},$$

$$\mathbf{P}'=\begin{pmatrix}0 & -a_1p_{12}-b_2p_{12}+a_1p_{12}x_1\\ * & 0\end{pmatrix}.$$

By means of Eqs. (9) and (11), the integral of motion can be found for this Lie group as follows:

$$H=-a_1-b_2+a_1x_1.$$

Lie group $A_{4,1}$:

Similar to previous case, from (13) and (19) we calculate the vector field **X** and compatible Poisson structures **P** and **P′** as follows:

$$\mathbf{X}^1=a_2x_2+a_3x_3+a_4x_4,$$
$$\mathbf{X}^2=b_1x_1+b_2x_2-c_3x_3+b_4x_4+b_1x_1x_4+b_2x_2x_4-c_3x_3x_4+b_4x_4^2,$$
$$\mathbf{X}^3=-b_1x_1-b_2x_2+c_3x_3+c_4x_4-b_1x_1x_4-b_2x_2x_4+c_3x_3x_4+c_4x_4^2-\frac{1}{2}b_1x_1x_4^2-\frac{1}{2}b_2x_2x_4^2+\frac{1}{2}c_3x_3x_4^2+(c_4x_4^3)/2,$$
$$\mathbf{X}^4=d_3x_3-(a_2x_4)/2,$$

$$\mathbf{P}=\begin{pmatrix}0 & \frac{p_{23}x_4^2}{2}+p_{12} & p_{23}x_4 & p_{14}\\ * & 0 & p_{23} & 0\\ * & * & 0 & 0\\ * & * & * & 0\end{pmatrix},$$

$$\mathbf{P}'=\begin{pmatrix}0 & p_{12} & p_{13} & p_{14}\\ * & 0 & p_{23} & 0\\ * & * & 0 & 0\\ * & * & * & 0\end{pmatrix},$$

where

$$\mathbf{p_{12}}=-\frac{1}{2}a_2p_{23}x_4^2-\frac{1}{2}a_3p_{23}x_4^2-\frac{1}{2}d_3p_{23}x_4^2-d_3p_{23}x_3x_4^2+\frac{a_2p_{12}x_4}{2}-a_2p_{12}-d_3p_{12}x_3,$$
$$\mathbf{p_{13}}=-p_{23}(a_2+a_3+d_3+2d_3x_3)x_4,$$
$$\mathbf{p_{14}}=\frac{1}{2}p_{14}(a_2-2(a_4-(b_4+c_4)(-1+x_4))),$$
$$\mathbf{p_{23}}=-a_2p_{23}-a_3p_{23}-d_3p_{23}-2d_3x_3p_{23}.$$

For this Lie group, by means of (9) and (11), the integrals of motion can be found as follows:

$$H_1=-\frac{a_2}{2}-a_3-a_4-b_4-c_4-d_3-2d_3x_3+b_4x_4+c_4x_4,$$
$$H_2=\frac{1}{2}((a_2+a_3+d_3+2d_3x_3)^2+\left(\frac{a_2}{2}-a_4+(b_4+c_4)(-1+x_4))^2\right).$$

Lie group $A_{4,3}$:

Again, substituting X^i in (19) and P' in (13) one can obtain the vector field **X** and compatible Poisson structures **P** and **P′** for this Lie group as follows:

$$\mathbf{X}^1=a_1e^{x_4}x_1+a_2e^{x_4}x_2+a_3e^{x_4}x_3+a_4e^{x_4}x_4,$$
$$\mathbf{X}^2=b_1x_1+b_2x_2+b_3x_3+b_4x_4,$$
$$\mathbf{X}^3=-a_1x_1-a_2x_2(d_3-a_3)x_3+c_4x_4-a_1x_1x_4-a_2x_2x_4-a_3x_3x_4+c_4x_4^2,$$
$$\mathbf{X}^4=d_3x_3-b_2x_4,$$

$$\mathbf{P}=\begin{pmatrix}0 & p_{12}e^{x_4} & 0 & p_{14}e^{x_4}\\ * & 0 & p_{23} & 0\\ * & * & 0 & 0\\ * & * & * & 0\end{pmatrix},$$

$$\mathbf{P}' = \begin{pmatrix} 0 & p_{12} & 0 & p_{14} \\ * & 0 & p_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{p_{12}} &= -b_2 e^{x_4} p_{12} - b_1 e^{2x_4} p_{12} - d_3 e^{x_4} x_3 p_{12} + b_2 e^{x_4} x_4 p_{12},\\
\mathbf{p_{14}} &= -a_4 e^{x_4} p_{14} + b_2 e^{x_4} p_{14} - b_4 e^{x_4} p_{14} - c_4 e^{x_4} p_{14} - b_1 e^{2x_4} p_{14} + a_4 e^{x_4} x_4 p_{14} + b_2 e^{x_4} x_4 p_{14} + c_4 e^{x_4} x_4 p_{14},\\
\mathbf{p_{23}} &= -b_2 p_{23} - b_3 p_{23} - 2d_3 p_{23} - d_3 x_3 p_{23}.
\end{aligned}$$

Also the integrals of motion can be found as follows:

$$\begin{aligned}
H_1 &= -a_4 - b_3 - b_4 - c_4 - 2d_3 - b_1 e^{x_4} - d_3 x_3 + a_4 x_4 + b_2 x_4 + c_4 x_4,\\
H_2 &= \frac{1}{2}((b_2 + b_3 + d_3(2 + x_3))^2 + (a_4 + b_4 + c_4 + b_1 e^{x_4} - a_4 x_4 - c_4 x_4 - b_2(1 + x_4))^2).
\end{aligned}$$

Lie group $A_2 \oplus A_2$:

Substituting X^i in Eq. (19) and P' in Eq. (13) one can obtain the vector field $\mathbf{X}$, compatible Poisson structures $\mathbf{P}$, $\mathbf{P}'$ for this Lie group are obtained as follows:

$$\begin{aligned}
\mathbf{X}^1 &= a_1 x_1 - \frac{d_2 p_{12} x_2}{p_{24}} + a_1 x_1 x_2 - \frac{d_2 p_{12} x_2^2}{p_{24}},\\
\mathbf{X}^2 &= b_1 x_1 + b_2 x_2 + b_3 x_3 + \frac{c_4 p_{24} x_4}{p_{34}},\\
\mathbf{X}^3 &= c_3 x_3 + c_4 x_4 + c_3 x_3 x_4 + c_4 x_4^2,\\
\mathbf{X}^4 &= d_1 x_1 + d_2 x_2 + d_3 x_3 + d_4 x_4,
\end{aligned}$$

$$\mathbf{P} = \begin{pmatrix} 0 & p_{12} & 0 & 0 \\ * & 0 & 0 & p_{24} \\ * & * & 0 & p_{34} \\ * & * & * & 0 \end{pmatrix},$$

$$\mathbf{P}' = \begin{pmatrix} 0 & (-a_1 - b_1 - 2c_1 + a_1 x_1 - b_1 x_1) p_{12} & 0 & 0 \\ * & 0 & 0 & a_1 p_{24} x_1 + c_1 p_{24} x_1 + b_3 p_{24} x_3 + c_3 p_{24} x_3 \\ * & * & 0 & -2b_3 p_{34} - c_3 p_{34} - d_3 p_{34} + c_3 x_3 p_{34} - d_3 x_3 p_{34} \\ * & * & * & 0 \end{pmatrix}.$$

By means of (9) and (11), the integral of motion can be found for this Lie group as follows:

$$\begin{aligned}
H_1 &= -a_1 - b_1 - 2b_3 - 2c_1 - c_3 - d_3 + a_1 x_1 - b_1 x_1 + c_3 x_3 - d_3 x_3,\\
H_2 &= \frac{1}{2}((a_1 + b_1 + 2c_1 - a_1 x_1 + b_1 x_1)^2 + (2b_3 + c_3 + d_3 - c_3 x_3 + d_3 x_3)^2).
\end{aligned}$$

Lie group $\mathrm{III} \oplus \mathbf{R}$:

Also for this Lie group, from (13) and (19) one can obtain the vector field $\mathbf{X}$, compatible Poisson structures $\mathbf{P}$, $\mathbf{P}'$ as follows:

$$\begin{aligned}
\mathbf{X}^1 &= -a_3 x_2 + 2a_3 x_2^2 + a_3 x_3 - 2a_3 x_3^2 + a_4 x_4 - 2a_4 x_2 x_4 - 2a_4 x_3 x_4,\\
\mathbf{X}^2 &= b_1 x_1 + b_2 x_2 + b_3 x_3 + a_4 x_4,\\
\mathbf{X}^3 &= c_1 x_1 - b_2 x_2 - (2a_3 + b_3 + d_3) x_3,\\
\mathbf{X}^4 &= d_1 x_1 + d_3 x_3 + d_4 x_4,
\end{aligned}$$

$$\mathbf{P} = \begin{pmatrix} 0 & p_{13} & p_{13} & 0 \\ * & 0 & 0 & p_{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix},$$

$$\mathbf{P}' = \begin{pmatrix} 0 & \mathbf{p}_{12} & \mathbf{p}_{13} & 0 \\ * & 0 & 0 & \mathbf{p}_{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix},$$

where

$$\begin{aligned} \mathbf{p_{12}} &= a_3 p_{13} - b_1 p_{13} - c_1 p_{13} - d_1 p_{13} + 2b_1 x_1 p_{13} + 2c_1 x_1 p_{13} - 8a_3 x_3 p_{13} - 2d_3 x_3 p_{13}, \\ \mathbf{p_{13}} &= a_3 p_{13} - b_1 p_{13} - c_1 p_{13} - d_1 p_{13} + 2b_1 x_1 p_{13} + 2c_1 x_1 p_{13} - 8a_3 x_3 p_{13} - 2d_3 x_3 p_{13}, \\ \mathbf{p_{24}} &= a_3 p_{24} - 2a_4 p_{24} - d_4 p_{24} + 2a_3 x_2 p_{24} - 2a_3 x_3 p_{24} - 2a_4 x_4 p_{24}. \end{aligned}$$

By means of (9) and (11), the integral of motion can be found for this Lie group as follows:

$$\begin{aligned} H_1 &= 2a_3 - 2a_4 - b_1 - c_1 - d_1 - d_4 + 2b_1 x_1 + 2c_1 x_1 + 2a_3 x_2 - 10a_3 x_3 - 2d_3 x_3 - 2a_4 x_4, \\ H_2 &= \frac{1}{2}((b_1 + c_1 + d_1 - 2b_1 x_1 - 2c_1 x_1 + 2d_3 x_3 + a_3(-1 + 8x_3))^2 + (d_4 + a_3(-1 - 2x_2 + 2x_3) + 2a_4(1 + x_4))^2). \end{aligned}$$

Lie group II ⊕ R:

Through substituting X^i in Eq. (19) and P' in Eq. (13) one can obtain the vector field **X** and compatible Poisson structures **P** and **P′** for this Lie group as follows:

$$\begin{aligned} \mathbf{X}^1 &= -d_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4, \\ \mathbf{X}^2 &= b_4 x_4 + b_4 x_3 x_4, \\ \mathbf{X}^3 &= c_2 x_2, \\ \mathbf{X}^4 &= d_1 x_1 + d_2 x_2 + d_3 x_3 + d_4 x_4, \end{aligned}$$

$$\mathbf{P} = \begin{pmatrix} 0 & p_{12} & p_{13} & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & p_{34} \\ * & * & * & 0 \end{pmatrix},$$

$$\mathbf{P}' = \begin{pmatrix} 0 & \mathbf{p_{12}} & \mathbf{p_{13}} & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & \mathbf{p_{34}} \\ * & * & * & 0 \end{pmatrix},$$

where

$$\begin{aligned} \mathbf{p_{12}} &= -a_2 p_{12} - c_2 p_{12} - d_2 p_{12} - c_2 x_2 p_{12}, \\ \mathbf{p_{13}} &= -a_3 p_{13} - d_3 p_{13} + b_4 x_4 p_{13}, \\ \mathbf{p_{34}} &= -a_3 p_{34} - a_4 p_{34} - b_4 p_{34} - d_3 p_{34} - d_4 p_{34} + b_4 x_4 p_{34}. \end{aligned}$$

By means of (9) and (11), the integrals of motion can be found for this Lie group as follows:

$$\begin{aligned} H_1 &= -a_2 - a_3 - a_4 - b_4 - c_2 - d_2 - d_3 - d_4 - c_2 x_2 + b_4 x_4, \\ H_2 &= \frac{1}{2}((a_2 + c_2 + d_2 + c_2 x_2)^2 + (a_3 + a_4 + b_4 + d_3 + d_4 - b_4 x_4)^2). \end{aligned}$$

Lie group $\mathbf{VI_0} \oplus$ R:

Again, substituting X^i in Eq. (19) and P' in Eq. (13) one can obtain the vector field **X** and compatible Poisson structures **P** and **P′** as follows:

$$\begin{aligned} \mathbf{X}^1 &= (-b_1 x_1 - b_2 x_2 + a_3 x_3 + a_4 x_4)(\mathrm{Cos}h[x_3] + \mathrm{Sin}h[x_3]), \\ \mathbf{X}^2 &= (b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4)(\mathrm{Cos}h[x_3] + \mathrm{Sin}h[x_3]), \\ \mathbf{X}^3 &= c_1 x_1 + c_2 x_2, \\ \mathbf{X}^4 &= -(c_1 + c_2 + d_2) x_1 + d_2 x_2 + d_3 x_3 + d_4 x_4, \end{aligned}$$

$$\mathbf{P}=\begin{pmatrix} 0 & p_{12}\cos h^2(x_3)-p_{12}\sin h^2(x_3) & 0 & 0\\ * & 0 & 0 & 0\\ * & * & 0 & p_{34}\\ * & * & * & 0\end{pmatrix},$$

$$\mathbf{P}'=\begin{pmatrix} 0 & \mathbf{p_{12}} & 0 & 0\\ * & 0 & 0 & 0\\ * & * & 0 & \mathbf{p_{34}}\\ * & * & * & 0\end{pmatrix},$$

where

$$\mathbf{p_{12}}=-2c_1p_{12}x_1\cos h^2(x_3)-2c_2p_{12}x_2\cos h^2(x_3)+2c_1p_{12}x_1\sin h^2(x_3)+2c_2p_{12}x_2\sin h^2(x_3),$$
$$\mathbf{p_{34}}=-a_3p_{34}-a_4p_{34}-b_3p_{34}-b_4p_{34}-d_3p_{34}-d_4p_{34}+a_3x_3p_{34}+b_3x_3p_{34}+a_4x_4p_{34}+b_4x_4p_{34}.$$

By means of (9) and (11), the integrals of motion can be found for this Lie group as follows:

$$H_1=-a_3-a_4-b_3-b_4-d_3-d_4-2c_1x_1-2c_2x_2+a_3x_3+b_3x_3+a_4x_4+b_4x_4,$$
$$H_2=\frac{1}{4}(8(c_1x_1+c_2x_2)^2+2(a_3+a_4+b_3+b_4+d_3+d_4-a_3x_3-b_3x_3-a_4x_4-b_4x_4)^2).$$

Lie group $A_{6,7}$:

Substituting X^i in Eq. (19) and P' in Eq. (13) one can obtain the vector field **X** and compatible Poisson structures **P** and **P′** as follows:

$$\mathbf{X}^1=a_1x_1+a_3x_3+a_1x_1x_3+a_3x_3^2+a_1x_1x_4+a_3x_3x_4,$$
$$\mathbf{X}^2=-2a_1x_1+b_3x_3-2a_1x_1x_3+b_3x_3^2,$$
$$\mathbf{X}^3=c_2x_2+c_3x_3+c_6x_6,$$
$$\mathbf{X}^4=d_1x_1-c_2x_2-(c_3+h_4)x_3+d_6x_6,$$
$$\mathbf{X}^5=f_1x_1+f_2x_2+f_3x_3+f_5x_5+f_6x_6,$$
$$\mathbf{X}^6=h_1x_1+h_2x_2+h_3x_3+h_4x_4-f_5x_5-(c_3+f_6)x_6,$$

$$\mathbf{P}=\begin{pmatrix} 0 & 0 & 0 & 0 & p_{15} & 0\\ * & 0 & 0 & 0 & 0 & p_{16}\\ * & * & 0 & p_{34} & 0 & 0\\ * & * & * & 0 & p_{45}+p_{15}x_3 & 0\\ * & * & * & * & 0 & 0\\ * & * & * & * & * & 0\end{pmatrix},$$

$$\mathbf{P}'=\begin{pmatrix} 0 & 0 & 0 & 0 & p_{12} & 0\\ * & 0 & 0 & 0 & 0 & p_{26}\\ * & * & 0 & p_{34} & 0 & 0\\ * & * & * & 0 & p_{45} & 0\\ * & * & * & * & 0 & 0\\ * & * & * & * & * & 0\end{pmatrix},$$

where

$$\mathbf{p_{15}}=(a_1-d_1-f_1-h_1-d_1x_1)p_{15},$$
$$\mathbf{p_{26}}=(c_3-f_2-h_2-c_2x_2-c_6x_6)p_{26},$$
$$\mathbf{p_{34}}=(-a_3-b_3-f_3-h_3+2a_3x_3+b_3x_3)p_{34},$$
$$\mathbf{p_{45}}=-h_4p_{45}+a_1p_{45}x_1+a_1p_{15}x_3-d_1p_{15}x_3-f_1p_{15}x_3-h_1p_{15}x_3+a_3p_{45}x_3-d_1p_{15}x_1x_3.$$

By means of (9) and (11), the integrals of motion can be found for this Lie group as follows:

$$H_1 = a_1 - a_3 - b_3 + c_3 - d_1 - f_1 - f_2 - f_3 - h_1 - h_2 - h_3 - d_1x_1 - c_2x_2 + 2a_3x_3 + b_3x_3 - c_6x_6,$$
$$H_2 = \frac{1}{2}((-a_1 + d_1 + f_1 + h_1 + d_1x_1)^2 + (a_3 + b_3 + f_3 + h_3 - 2a_3x_3 - b_3x_3)^2 + (-c_3 + f_2 + h_2 + c_2x_2 + c_6x_6)^2),$$
$$H_3 = \frac{1}{3}(-(-a_1 + d_1 + f_1 + h_1 + d_1x_1)^3 - (a_3 + b_3 + f_3 + h_3 - 2a_3x_3 - b_3x_3)^3 - (-c_3 + f_2 + h_2 + c_2x_2 + c_6x_6)^3).$$

Lie group $A_{6,25}$:

Finally, the vector field **X**, compatible Poisson structures **P**, **P′** and the integrals of motion of this Lie group are obtained as follows:

$$\mathbf{X}^1 = a_3x_3 + a_3x_2x_3 - 1/2(c_6 + f_5)x_6 - 1/2(c_6 + f_5)x_2x_6,$$
$$\mathbf{X}^2 = b_1x_1,$$
$$\mathbf{X}^3 = c_1x_1 - 2a_3x_3 + c_4x_4 + c_5x_5 + c_6x_6,$$
$$\mathbf{X}^4 = d_5x_5 + d_5x_5^2,$$
$$\mathbf{X}^5 = f_2x_2 + a_3x_3 + f_4x_4 + f_5x_5 - 1/2(c_6 + f_5)x_6,$$
$$\mathbf{X}^6 = h_1x_1 + h_2x_2 + h_4x_4 + h_5x_5,$$
$$\mathbf{P} = \begin{pmatrix} 0 & 0 & p_{13} & 0 & 0 & 0 \\ * & 0 & 0 & p_{24} & 0 & p_{24}x_5 \\ * & * & 0 & 0 & 0 & p_{36} \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & p_{56} \\ * & * & * & * & * & 0 \end{pmatrix},$$

where

$$\mathbf{P}' = \begin{pmatrix} 0 & 0 & p_{13} & 0 & 0 & 0 \\ * & 0 & 0 & p_{24} & 0 & p_{26} \\ * & * & 0 & 0 & 0 & p_{36} \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & p_{56} \\ * & * & * & * & * & 0 \end{pmatrix},$$
$$\mathbf{p_{13}} = (-b_1 - c_1 - h_1 - b_1x_1)p_{13},$$
$$\mathbf{p_{24}} = (-c_4 - f_2 - f_4 - h_2 - h_4 - f_2x_2 - f_4x_4)p_{24},$$
$$\mathbf{p_{26}} = (-c_4x_5 - f_2x_5 - f_4x_5 - h_2x_5 - h_4x_5 - f_2x_2x_5 - f_4x_4x_5)p_{24},$$
$$\mathbf{p_{36}} = f_5p_{36},$$
$$\mathbf{p_{56}} = -c_5p_{56} - d_5p_{56} - h_5p_{56} + d_5p_{56}x_5.$$

By means of (9) and (11), the integrals of motion can be found for this Lie group as follows:

$$H_1 = -b_1 - c_1 - c_4 - c_5 - d_5 - f_2 - f_4 - h_1 - h_2 - h_4 - h_5 - b_1x_1 - f_2x_2 - f_4x_4 + d_5x_5,$$
$$H_2 = \frac{1}{2}((b_1 + c_1 + h_1 + b_1x_1)^2 + (c_4 + f_2 + f_4 + h_2 + h_4 + f_2x_2 + f_4x_4)^2 + c_5 + d_5 + h_5 - d_5x_5)^2),$$
$$H_3 = \frac{1}{3}(-(b_1 + c_1 + h_1 + b_1x_1)^3 - (c_4 + f_2 + f_4 + h_2 + h_4 + f_2x_2 + f_4x_4)^3 - (c_5 + d_5 + h_5 - d_5x_5)^3).$$

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

ACKNOWLEDGMENT

The corresponding author would like to thank Professor Norbert Euler for kindly and worthy comments. The authors would like to thank the referees for carefully reading the paper and their valuable comments.

REFERENCES

[1] J. Abedi-Fardad, A. Rezaei-Aghdam, Gh. Haghighatdoost, Classification of four-dimensional real Lie bialgebras of symplectic type and their Poisson-Lie groups, Theor. Math. Phys. 190 (2017), 1–17.
[2] J. Abedi-Fardad, A. Rezaei-Aghdam, Gh. Haghighatdoost, Some compatible Poisson structures and integrable bi-Hamiltonian systems on four dimensional and nilpotent six dimensional symplectic real Lie groups, J. Nonlinear Math. Phys. 24 (2017), 149–170.
[3] A. Bolsinov, Compatible Poisson brackets on Lie algebras and completeness of families of functions in involution, Izv. Akad. Nauk SSSR Ser. Mat. 55 (1991), 68–92.
[4] A.V. Bolsinov, A.A. Oshemkov, Bi-Hamiltonian structures and singularities of integrable systems. Regular Chaotic Dyn. 14 (2009), 431–454.
[5] V.G. Drinfel'd, V.V. Sokolov, Lie algebras and equations of Korteweg-de Vries type, J. Sov. Math. 30 (1985), 1975–2036.
[6] I.M. Gelfand, I.Y. Dorfman, Hamiltonian operators and algebraic structures related to them, Funct. Anal. Appl. 13 (1979), 248–262.
[7] A. Ibort, F. Magri, G. Marmo. Bihamiltonian structures and stäckel separability, J. Geom. Phys. 33 (2000), 210–228.
[8] Y. Kosmann-Schwarzbach, F. Magri, Poisson-Nijenhuis structures, Ann. Inst. H. Poincare Phys Theor. 53 (1990), 35–81.
[9] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19 (1978), 1156–1162.
[10] F. Magri, P. Casati, G. Falqui, M. Pedroni, Eight lectures on integrable systems, in: Y. Kosmann-Schwarzbach, B. Grammaticos, K.M. Tamizhmani (Eds.), Integrability of Nonlinear Systems, 2nd edition, Lecture Notes in Physics, vol. 495, Springer, Berlin, Heidelberg, 1997, pp. 256–296.
[11] F. Magri, C. Morosi, A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds, Dipartimento di Matematica F. Enriques, 1984.
[12] B. Mojaveri, A. Rezaei-Aghdam, (4+1)-Dimensional homogeneous anisotropic string cosmological models, Int. J. Mod. Phys. A 27 (2012), 1250032.
[13] M. Nakahara, Geometry, Topology and Physics, 2nd edition, Institute of Physics Publishing, 2003.
[14] G.P. Ovando, Four dimensional symplectic Lie algebras, Beitr. Algebra Geom. 47 (2006), 419–434.
[15] J. Patera, R.T. Sharp, P. Winternitz, H. Zassenhaus, Invariants of real low dimension Lie algebras, J. Math. Phys. 17 (1976), 986–994.
[16] A.M. Perelomov, Integrable systems of classical mechanics and Lie algebras, Birkhäuser, Basel, 1990.
[17] A.V. Tsiganov, On the two different bi-Hamiltonian structures for the Toda lattice, J. Phys. A: Math. Theor. 40 (2007), 6395–6406.
[18] A.V. Tsiganov, On bi-Hamiltonian geometry of the Lagrange top, J. Phys. A: Math. Theor. 41 (2008), 315212.
[19] A.V. Tsiganov, On bi-Hamiltonian structure of some integrable systems on $so^*(4)$, J. Nonlinear Math. Phys. 15 (2008), 171–185.
[20] A.V. Tsiganov, On bi-integrable natural Hamiltonian systems on the Riemannian manifolds, J. Nonlinear Math. Phys. 18 (2011), 245–268.
[21] A.V. Vershilov, A.V. Tsiganov, On bi-Hamiltonian geometry of some integrable systems on the sphere with cubic integral of motion, J. Phys. A: Math. Theor. 42 (2009), 12.