Journal of Nonlinear Mathematical Physics
Vol. **28**(**1**); March (2021), *pp.* 27–37
DOI: https://doi.org/10.2991/jnmp.k.200922.004; ISSN 1402-9251; eISSN 1776-0852
https://www.atlantis-press.com/journals/jnmp

Research Article

The (N + 1)-Dimensional Burgers Equation: A Bilinear Extension, Vortex, Fusion and Rational Solutions

Hongli An[1,*], Engui Fan[2], Manwai Yuen[3]

[1]*College of Science, Nanjing Agricultural University, Nanjing 210095, People's Republic of China*
[2]*School of Mathematical Sciences, Fudan University, Shanghai 200433, People's Republic of China*
[3]*Department of Mathematics and Information Technology, The Education University of Hong Kong, Tai Po, New Territories, Hong Kong*

ARTICLE INFO

Article History

Received 14 September 2019
Accepted 05 March 2020

Keywords

The (N + 1)-dimensional Burgers system
bilinear formulation
generalized Cole-Hopf transformation
vortex solutions
multiple fusion solutions
rational solutions

Mathematics Subject Classification 2010

35D99
35C05
35C99

ABSTRACT

In this paper, by introducing a fractional transformation, we obtain a bilinear formulation for the (N + 1)-dimensional Burgers equation. Such a formulation constitutes a bilinear extension to the (N + 1)-dimensional Cole-Hopf transformation between the (N + 1)-dimensional Burgers system and generalized heat equation. As applications of the bilinear extension to the Cole-Hopf transformation, four types of physically interesting exact solutions are constructed, which contain vortex solutions, multiple fusions, rational solutions and triangular rational solutions. The behaviors of these solutions are analyzed and simulated. Interestingly, the type of fusion solutions has recently found applications in organic membrane, macromolecule material, even-clump DNA, nuclear physics and plasmas physics et al.

1. INTRODUCTION

Exact solutions to Partial Differential Equations (PDEs) are important. Not only because they can serve as useful tools to test the effectiveness of numerical algorithms of the PDEs, but also they can help us to better understand various phenomena in nature described by the PDEs and then lead to further applications. Therefore, to study solutions of PDEs has always been an interesting and important work. Up to now, lots of effective methods have been developed, including inverse scattering approach [1], Darboux and Bäcklund Transformation [26,28], Hirota bilinear method [14,18], Lie symmetry method [5], Wronskian and Casoratian technique [31,33], variable separation method [27,38], various tanh methods [11,44] and so on [9,23–25]. Among them, the Hirato bilinear method has been considered as the most simple and direct method to construct solutions.

It is noticed that among the PDEs, the Burgers equation is among the simplest models:

$$u_t + uu_x - \mu u_{xx} = 0, \tag{1.1}$$

where u denotes the fluid velocity and μ is the viscosity coefficient. This equation was originally introduced by Bateman [4] and subsequently investigated by Burgers [6,7]. Investigation shows that the Burgers equation has occurred in many branches of physics, such as fluid, gas dynamics, acoustics waves, traffic flow, population growth, density and electromagnetic waves et al (see [2,3,10,12,13,16,20,21,37]). It is known that under the Cole-Hopf transformation

$$u = -2\mu(\log f)_x, \tag{1.2}$$

the Burgers equation is mapped into the heat equation [17]

$$f_t - \mu f_{xx} = 0, \tag{1.3}$$

*Corresponding author. Email: kaixinguoan@163.com

which is an important physical model used to solve Black–Scholes equation, to study random walks and Brownian motion.

Recently, in turbulence, Frisch and Burgulence introduced a general $(N+1)$-Dimensional vector Burgers (NDB) system [22]:

$$u_t + (u\cdot\nabla)u - \mu\Delta u = 0. \tag{1.4}$$

In the above, $u = (u_1, u_2, \cdots, u_N)^T$ is the velocity vector of fluid and μ denotes the viscosity coefficient. While $\nabla = (\partial_{x_1}, \cdots, \partial_{x_N})^T$ represents a Hamilton gradient operator and $\Delta = \sum_{j=1}^{N}\partial_{x_j}^2$ is the Laplacian operator. It is noted that when $N = 1$, the system (1.4) is reducible to the classical Burgers equation (1.1), which admits the soliton fusion solutions [43]. When $N = 2$, it becomes the coupled Burgers equation

$$\begin{cases} u_{1t} + u_1u_{1x} + u_2u_{1y} = u_{1xx} + u_{1yy}, \\ u_{2t} + u_1u_{2x} + u_2u_{2y} = u_{2xx} + u_{2yy}, \end{cases} \tag{1.5}$$

which admits travelling wave solutions [19,29,39,42]. It is also noticed that the coupled integrable Burgers equations have been studied earlier in Ma and Zhou [34] wherein some rational solutions were given. Interestingly, a kind of similar rational solutions called lumps has been analyzed recently pretty systematically in Ma and Zhou [35].

Based on the fact that the Burgers equation (1.1) admits the Cole-Hopf transformation, the authors in Chen et al. [8] extended the transformation to an $(N+1)$-dimensional Cole-Hopf transformation

$$u_j = -2\mu(\log f)_{x_j},\ j = 1, \cdots, N \tag{1.6}$$

which lead to the $(N+1)$-dimensional heat equation

$$f_t - \mu\Delta f = 0. \tag{1.7}$$

However, they did not discuss the bilinear formulation for the NDB equation. It is known that for most nonlinear equations, the bilinear forms can be obtained under a logarithmic type (log-type) transformation. For example, by using the transformation

$$u = -2(\log f)_{xx},$$

the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0$$

can be written into a bilinear formulation. Since the log-type transformation exists in the NDB equation (1.4), a natural question comes whether a bilinear form or a generalized bilinear operator introduced by Ma [32] exists in this NDB system, which constitutes a primary motivation to undertake the present work. Another strong motivation of our work arises from the significance of the fusion solution, which has found their applications in many physical models, such as in organic membrane, macromolecule material [40], in even-clump DNA [15] and in nuclear physics et al. [41]. Wang et al. [43] showed that the (1 + 1)-dimensional Burgers equation admits the fusion solution via the Painléve expansion method. Recently, they also have found that the Sharma–Tasso–Olver–Burgers equation admits the fusion solutions via introducing a velocity resonance mechanics [45]. Therefore, it motives us to consider whether the NDB system, like the (1 + 1)-dimensional Burgers equation, admits fusion solutions. If so, it would be interesting to use these solutions to explain/predict some physical phenomenon. The third motivation arises from the work of Ma [30], in which the author showed that the higher-dimensional integrable systems admit the general Darboux transformations. This inspires us to consider whether the general Darboux transformations exist in the NDB equation.

With the above questions bearing in mind, we expand the investigation on the NDB equation. By introducing a fractal transformation, we establish a bilinear formulation for the NDB equation, which can be regarded as a "quasi-linearization" of NDB system or a bilinear extension to the Cole-Hopf transformation. As its applications, we construct several types of physically interesting exact solutions, including the vortex solutions, multiple fusions, rational solutions and triangular rational solutions.

This paper is arranged as follows. In Section 2, the bilinear extension is given for the NDB equation by using a rational transformation. In Section 3, special reductions of the bilinear extension are discussed and several types of exact physically interesting solutions are obtained. Numerical analysis is made on the behaviors of the solution given. Finally a short conclusion is attached.

2. A BILINEAR FORMULATION TO THE NDB EQUATION

It is shown in section that the NDB equation admits a bilinear formulation by using the $(N+1)$-dimensional fractional transformation. To show its effectiveness, two examples are given.

In order to construct the bilinear formulation of the NDB equation (1.4), we first write into a scalar form

$$u_{i,t} + \sum_{j=1}^{N} u_j u_{i,x_j} = \mu\Delta u_i,\ i = 1, 2, \cdots, N. \tag{2.1}$$

Now, to bilinearize the Eq. (2.1), a fractional transformation is introduced via

$$u_i = \frac{g_i}{f}, i = 1, \cdots, N, \tag{2.2}$$

where g_i and f are functions to be determined.

Direct calculation shows that

$$\frac{\partial u_i}{\partial t} = \frac{g_{i,t} f - f_t g_i}{f^2}, \tag{2.3}$$

$$\sum_{j=1}^{N} u_j \frac{\partial u_i}{\partial x_j} = \sum_{j=1}^{N} \frac{f g_j g_{i,x_j} - g_i g_j f_{x_j}}{f^3}, \tag{2.4}$$

$$\mu \Delta u_i = \mu \sum_{j=1}^{N} \frac{g_{i,x_j x_j} f - 2 f_{x_j} g_{i,x_j} + g_i f_{x_j x_j}}{f^2} + \mu \sum_{j=1}^{N} \frac{2 f_{x_j}^2 g_i - 2 f g_i f_{x_j x_j}}{f^3}. \tag{2.5}$$

Substituting (2.3)–(2.5) into (2.1) yields

$$\begin{aligned} u_{i,t} + \sum_{j=1}^{N} u_j u_{i,x_j} - \mu \Delta u_i = {} & \frac{1}{f^2} \left\{ (g_{i,t} f - f_t g_i) - \mu \sum_{j=1}^{N} (g_{i,x_j x_j} f - 2 f_{x_j} g_{i,x_j} + g_i f_{x_j x_j}) + \sum_{j=1}^{N} (g_{i,x_j} g_j - g_i g_{j,x_j}) \right\} \\ & + \frac{g_i}{f^3} \sum_{j=1}^{N} [f (g_j + 2\mu f_{x_j})_{x_j} - f_{x_j} (g_j + 2\mu f_{x_j})]. \end{aligned} \tag{2.6}$$

Let

$$(g_{i,t} f - f_t g_i) - \mu \sum_{j=1}^{N} (g_{i,x_j x_j} f - 2 f_{x_j} g_{i,x_j} + g_i f_{x_j x_j}) + \sum_{j=1}^{N} (g_{i,x_j} g_j - g_i g_{j,x_j}) = 0, \tag{2.7}$$

$$\sum_{j=1}^{N} [f (g_j + 2\mu f_{x_j})_{x_j} - f_{x_j} (g_j + 2\mu f_{x_j})] = 0, \tag{2.8}$$

which is equivalent to the bilinear form

$$\left(D_t - \mu \sum_{j=1}^{N} D_{x_j}^2 \right) g_i \cdot f + \sum_{j=1}^{N} D_{x_j} g_i \cdot g_j = 0, \tag{2.9}$$

$$\sum_{j=1}^{N} D_{x_j} f \cdot (g_j + 2\mu f_{x_j}) = 0, i = 1, \cdots, N, \tag{2.10}$$

where D_{x_j} and D_t are the bilinear derivative operators defined via

$$D_x^\alpha D_t^\beta (f \cdot g) = (\partial_x - \partial_{x'})^\alpha (\partial_t - \partial_{t'})^\beta f(x,t) g(x',t') |_{x'=x,\, y'=y,\, t'=t}.$$

At this stage, the bilinear formulation of the NDB equation has been constructed by using the N-dimensional fractal transformation. This result is quite novel. In the following, we shall take $N = 1$ and $N = 2$, namely the classical Burgers (1.1) and coupled Burgers equations (1.5), as examples to show the effectiveness and feasibility.

Example 2.1. For the (1 + 1)-dimensional Burgers equation (1.1), by using the formula

$$u = \frac{g_1}{f},$$

and noticing that $D_x g_1 \cdot g_1 = 0$, we can obtain the bilinear formula of the Burgers system (1.1), which is

$$\begin{aligned} &\left(D_t - \mu D_x^2 \right) f \cdot g_1 = 0, \\ &D_x f \cdot (g_1 + 2\mu f_x) = 0. \end{aligned}$$

Example 2.2. For the coupled Burgers equation (1.5), via the formula (2.2), (2.9) and (2.10), we get that under the transformation

$$u_1 = \frac{g_1}{f}, \quad u_2 = \frac{g_2}{f},$$

the coupled Burgers system (1.5) admits the bilinear formula:

$$\begin{aligned}
&\left(D_t - \mu D_{x_1}^2 - \mu D_{x_2}^2\right) g_1 \cdot f + D_{x_1} g_1 \cdot g_2 = 0,\\
&\left(D_t - \mu D_{x_1}^2 - \mu D_{x_2}^2\right) g_2 \cdot f + D_{x_2} g_2 \cdot g_1 = 0,\\
&D_{x_1} f \cdot (g_1 + 2\mu f_{x_1}) + D_{x_2} f \cdot (g_2 + 2\mu f_{x_2}) = 0.
\end{aligned}$$

3. REDUCTIONS OF THE BILINEAR FORMULATION

In this section, special reductions of bilinear equations (2.9) and (2.10) are considered and thereby a generalized $(N+1)$-dimensional heat equation is derived.

It is noticed that the bilinear equations (2.9) and (2.10) are equivalent to (2.7) and (2.8). For convenience, we shall go with the latter two equations. For Eq. (2.8), if we take

$$f(g_j + 2\mu f_{x_j})_{x_j} - f_{x_j}(g_j + 2\mu f_{x_j}) = 0, \quad j = 1, \cdots, N, \tag{3.1}$$

which can be written in a form of

$$\frac{(g_j + 2\mu f_{x_j})_{x_j}}{g_j + 2\mu f_{x_j}} = \frac{f_{x_j}}{f}. \tag{3.2}$$

Integrating with respect to x_j produces

$$g_j = -2\mu f_{x_j} - 2\mu c_j f, \quad j = 1, \cdots, N, \tag{3.3}$$

where c_j is an arbitrary constant. On substituting it into (2.2), we obtain that

$$u_j = -2\mu(\log f)_{x_j} - 2\mu c_j, \quad j = 1, \cdots, N. \tag{3.4}$$

Interestingly, the expression of u_j readily satisfies the irrotational condition

$$\nabla \times u = \sum_{i,j=1}^{N} \left(u_{j,x_i} - u_{i,x_j}\right) e_i \wedge e_j = 0, \tag{3.5}$$

where e_i $(i = 1, 2, \cdots, N)$ are the basis of the Euclidean space $\mathbf{R}^N$.

Now we turn back to consider Eq. (2.9). Inserting (3.3) into (2.9) leads to a vector equation

$$f(f_t - \mu\Delta f - 2\mu c \cdot \nabla f)_{x_i} - f_{x_i}(f_t - \mu\Delta f - 2\mu c \cdot \nabla f) = 0, \tag{3.6}$$

where $c = (c_1, \cdots, c_N)$ is an N-dimensional vector. It is found that (3.6) gives a generalized heat equation

$$f_t - \mu\Delta f - 2\mu c \cdot \nabla f - c_0 f = 0, \tag{3.7}$$

where c_0 is an constant of integration.

Remark 3.1.

What needs to point out is that Eq. (3.7) is the model describing the 1-dimensional unsteady convective mass transfer with a first-order volume chemical reaction in a continuous medium that moves with a constant velocity. A similar equation is used to analyze the corresponding 1-dimensional thermal processes in a moving medium with volume heat release proportional to temperature. See p. 283, in Polyanin [36].

Without loss of generality, we may take $c_0 = 0$. Since under the transformation $f \to e^{c_0 t} f$, we have

$$f_t - \mu\Delta f - 2\mu c \cdot \nabla f = 0. \tag{3.8}$$

Remark 3.2.

This Eq. (3.8) is called a convective heat equation, which is encountered in 1-dimensional nonstationary problems of convective mass transfer in a continuous medium that moves with a constant velocity with no absorption or release of substance. See, p. 280 in Polyanin [36].

In the case of $c = 0$, Eq. (3.6) becomes

$$f_t - \mu\Delta f - c_0 f = 0.$$

This equation comes from the heat equation with an additional term to account for radiative loss of heat, which depends upon the excess temperature at a given point compared with the surroundings.

It is seen that the formula (3.4), compared with (1.2) and (1.6), is indeed a more general $(N + 1)$-dimensional Cole-Hopf transformation. And Eq. (3.8) is a more general linear equation than the heat equation (1.7).

Example 3.1. For the $(1 + 1)$-dimensional Burgers equation (1.1), by using formula (3.4) and (3.8), we obtain that under the Cole-Hopf transformation

$$u = -2\mu(\log f)_x + c_1,$$

the Burgers equation (1.1) is linearized into linear heat equation

$$f_t - \mu f_{xx} - 2\mu c_1 f_x = 0,$$

which is related to the classical heat equation

$$w_t - \mu w_{xx} = 0$$

under transformation $f = e^{-c_1 x - \mu c_1^2 t} w(x, t)$.

4. APPLICATIONS FOR EXACT SOLUTIONS TO THE NDB EQUATION

This section is devoted to using the more general $(N + 1)$-dimensional Cole-Hopf transformation (3.4) to construct exact solutions of the NDB equation.

4.1. The Vortex Solutions

To solve Eq. (3.8), the Fourier transformation is introduced via

$$\hat{f}(\xi) = \int e^{i\xi \cdot x} f(x) dx, \tag{4.1}$$

where $\xi \cdot x = \xi_1 x_1 + \cdots + \xi_N x_N$. Its Fourier inverse transformation takes a form of

$$f(x) = \frac{1}{(2\pi)^N} \int e^{-i\xi \cdot x} \hat{f}(\xi) d\xi, \tag{4.2}$$

When applying the Fourier transform (4.1) to (3.8), it produces

$$\hat{f}_t + \mu |\xi|^2 \hat{f} + 2i\mu c \cdot \xi \hat{f} = 0, \tag{4.3}$$

where $c = (c_1, \cdots, c_N)$ is a constant vector. It is shown that (4.3) admits a solution

$$\hat{f}(\xi) = \exp\left[-\mu t(|\xi|^2 + 2ic \cdot \xi)\right]. \tag{4.4}$$

Taking the Fourier inverse transformation to the above equation and then we obtain a special solution of the heat equation (3.8), which is

$$f(x, t) = \frac{1}{(4\pi\mu t)^{N/2}} \exp\left[-\sum_{j=1}^{N} \frac{(x_j + 2\mu c_j t)^2}{4\mu t}\right], \tag{4.5}$$

It is noticed that the heat equation (3.8) is linear and 1 is also its solution. According to the superposition principle, the combination of 1 and (4.5), namely

$$f(x,t) = 1 + \frac{1}{(4\pi\mu t)^{N/2}} \exp\left[-\sum_{j=1}^{N} \frac{(x_j + 2\mu c_j t)^2}{4\mu t}\right] \equiv 1 + \exp(\eta), \tag{4.6}$$

is also a solution of (3.8). In the above

$$\eta = -\sum_{j=1}^{N} \frac{(x_j + 2\mu c_j t)^2}{4\mu t} - \frac{N}{2}\ln(4\pi\mu t). \tag{4.7}$$

Therefore, substituting (4.6) into the Cole-Hopf transformation (3.4), leads to the solution for the NDB system

$$u_j = \frac{x_j + 2\mu c_j t}{t}[1 + \tanh(\eta/2)] - 2\mu c_j,\ j = 1, 2, \cdots, N. \tag{4.8}$$

Surprisingly, such a kind of solutions (4.8) exhibits an interesting physical properties. Here we take $N = 2$ and the solution for the coupled Burgers system is

$$\begin{aligned} u_1 &= \frac{x_1 + 2\mu c_1 t}{t}[1 + \tanh(\eta/2)] - 2\mu c_1, \\ u_2 &= \frac{x_2 + 2\mu c_2 t}{t}[1 + \tanh(\eta/2)] - 2\mu c_2. \end{aligned} \tag{4.9}$$

Since the structure of u_1 and u_2 takes the same form, here we only take u_1 as illustrative example to draw the pictures. The behaviors of the solution u_1 are exhibited in Figure 1. From figures, we can see that such a solution exhibits a localized wave propagation, being a soliton form in x_1-axis but a vortex form in x_2-axis.

4.2. The Multiple Fusion Solutions

For constructing the multiple fusion solutions, we introduce the variable transformation via

$$\eta_i = \sum_{j=1}^{N} \alpha_{ij} x_j + \mu t \sum_{j=1}^{N} (\alpha_{ij}^2 + 2c_j\alpha_{ij}), \quad i = 1, 2, \cdots, N, \tag{4.10}$$

where α_{ij} $(i, j = 1, 2, \cdots, N)$ are arbitrary constants.

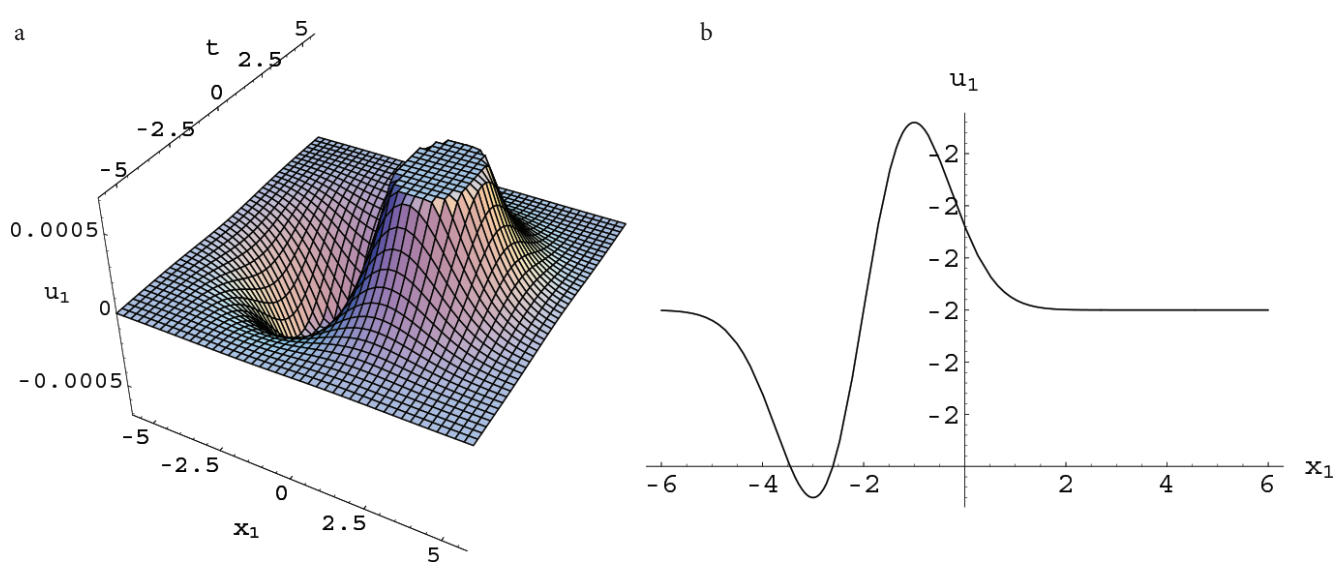

Figure 1 | Plots for u_1 in the solution (4.9). (a) Perspective view of the wave. (b) The propagation along x_1 direction.

One may readily verify that

$$1, \exp(\eta_i), i = 1, 2, \cdots, N \tag{4.11}$$

are all solutions of the linear heat equation (3.8). Therefore, according to the superposition formula, their combination is also the solution of the heat equation, which is

$$f = 1 + \sum_{i=1}^{N} \exp(\eta_i). \tag{4.12}$$

Insertion of (4.12) into (3.4) produces a travelling wave solution to the NDB equation (1.4), which is in the form of

$$u_j = \frac{-2\mu \sum_{i=1}^{N} \alpha_{ij} \exp(\eta_i)}{1 + \sum_{i=1}^{N} \exp(\eta_i)} - 2\mu c_j, \; j = 1, 2, \cdots, N. \tag{4.13}$$

Interestingly, unlike the first type solution (4.8), such a solution (4.13) shows N-fusion phenomenon. For convenience, we take $N = 2$ as example to show the results, wherein the two-travelling wave solution to the (2 + 1)-dimensional Burgers system is given by

$$u_j = \frac{\alpha_{1j} e^{\eta_1} + \alpha_{2j} e^{\eta_2}}{1 + e^{\eta_1} + e^{\eta_2}} - 2\mu c_j, \; j = 1, 2. \tag{4.14}$$

The 2-soliton fusion phenomenon of u_1 can be clearly seen in Figure 2. From the pictures, we can see that there are two single kink-soliton collide and fusion to one at certain time. Such fusion phenomenon are very important, which have been observed in many physical models, like in organic membrane, macromolecule material, even-clump DNA, nuclear physics and Sr-Ba-Ni oxidation crystal (see [15,40,41]). Deep analysis shows that for all the ranges of the parameters a_{1j} and a_{2j}, only fusion can occur in the 2-solitons expressed by (4.14). Both elastic scattering and fission can not happen.

4.3. The Rational Solutions

Here, we aim to seek to the polynomial solution of heat equation (3.8) in the form of

$$f(x, t) = \sum_{j=1}^{N} (x_j + A_j t)^2 + Bt, \tag{4.15}$$

where A_j and B are constants to be determined.

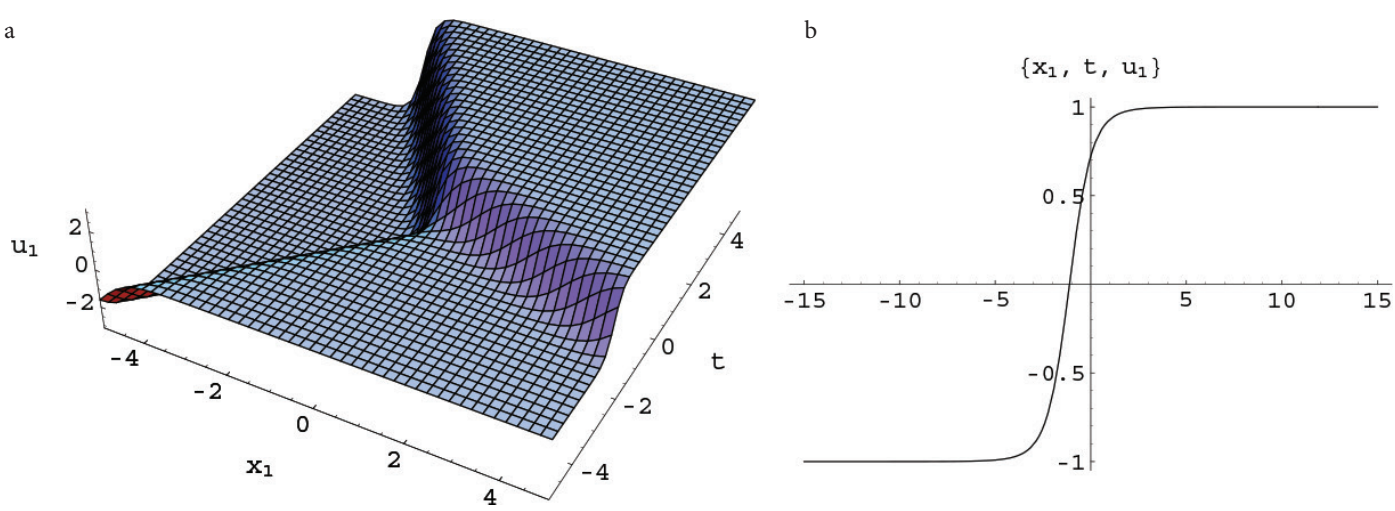

Figure 2 | Plots of u_1 in the solution (4.14). (a) Perspective view of the wave. (b) The propagation along x direction.

Substituting (4.15) into (3.8) produces

$$2\sum_{j=1}^{N} A_j(x_j + A_j t) + B - 2n\mu + 4\mu\sum_{j=1}^{N} c_j(x_j + A_j t) = 0. \tag{4.16}$$

which leads to

$$A_j = 2\mu c_j, \qquad B = 2N\mu.$$

So the solution of the heat equation (3.8) is

$$f(x,t) = \sum_{j=1}^{N}(x_j + 2\mu c_j t)^2 + 2N\mu t. \tag{4.17}$$

Inserting it into the Cole-Hopf transformation (3.4), the rational solution for the NDB system is obtained as follows

$$u_j = \frac{-4\mu\sum_{j=1}^{N}(x_j + 2\mu c_j t)}{\sum_{j=1}^{N}(x_j + 2\mu c_j t)^2 + 2N\mu t} - 2\mu c_j, \quad j = 1,\cdots,N. \tag{4.18}$$

For $t > 0$ and $x \to \pm\infty$, it shows that the solution u_j tends to $-2\mu c_j$.

For example, setting $N = 1$, the Burgers equation (1.1) admits a solution of

$$u = \frac{-4\mu(x + 2\mu ct)}{(x + 2\mu ct)^2 + 2\mu t} - 2\mu c, \tag{4.19}$$

whose structure can be seen in Figure 3.

4.4. The Triangular Rational Solutions

Now, we search for a kind of triangular function solutions for the heat equation (3.8) in the form of

$$f(x,t) = 1 + \sum_{j=1}^{N} e^{A_j t}\cos(x_j + B_j t), \tag{4.20}$$

where A_j and B_j are constants to be determined.

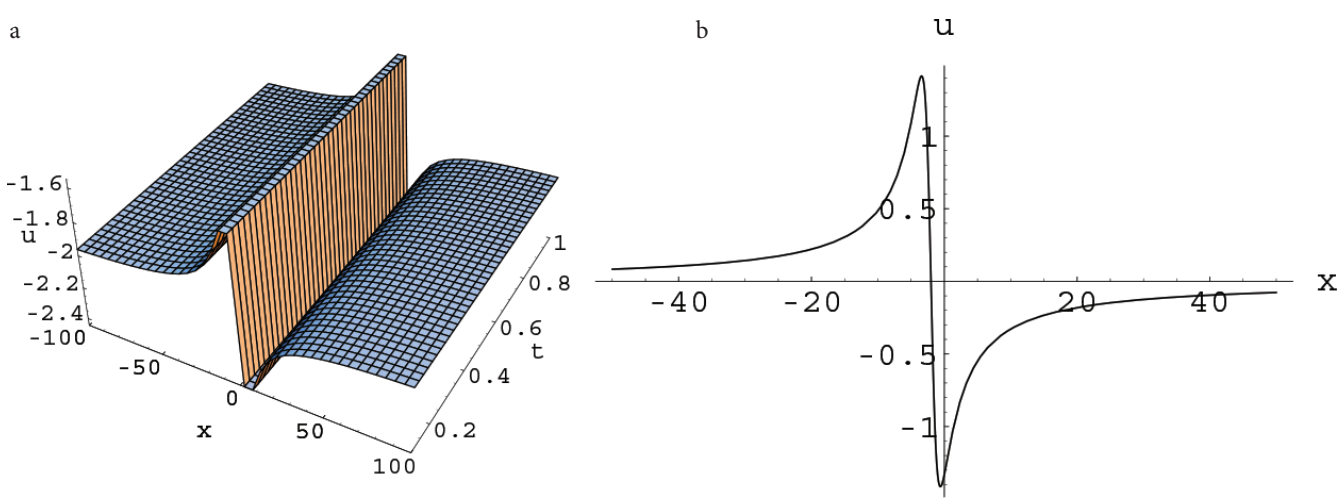

Figure 3 | Plots of u in the solution (4.19). (a) Perspective view of the wave. (b) The propagation along x direction.

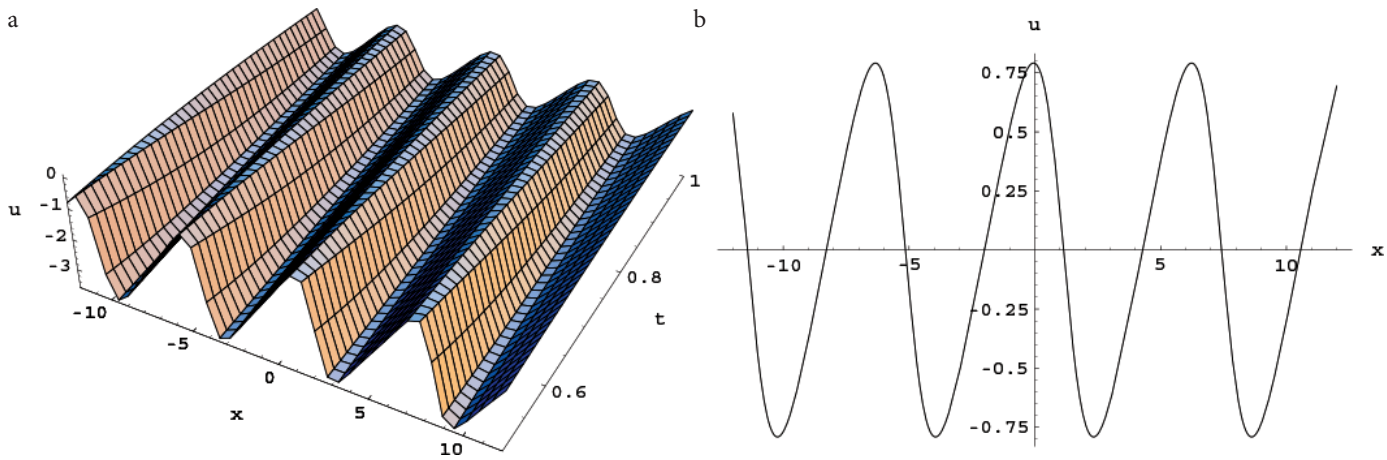

Figure 4 | Plots of u in the solution (4.24). (a) Perspective view of the wave. (b) The propagation along x direction.

By substituting (4.20) into (3.8), we have

$$\sum_{j=1}^{N}(A_j+\mu)\cos(x_j+B_jt)+\sum_{j=1}^{N}(2\mu c_j-B_j)\sin(x_j+B_jt)=0, \tag{4.21}$$

which shows

$$A_j=-\mu,\quad B_j=2\mu c_j.$$

So the generalized heat equation (3.8) admits a solution

$$f(x,t)=1+e^{-\mu t}\sum_{j=1}^{N}\cos(x_j+2\mu c_jt). \tag{4.22}$$

By using the Cole-Hopf transformation (3.4), we then find a kind of triangular rational solutions for NDB system

$$u_j=\frac{2\mu e^{-\mu t}\sum_{j=1}^{N}\sin(x_j+2\mu c_jt)}{1+e^{-\mu t}\sum_{j=1}^{N}\cos(x_j+2\mu c_jt)}-2\mu c_j,\quad j=1,\cdots,N. \tag{4.23}$$

When $N=1$, we get a triangular rational solution for the Burger equation (1.1)

$$u=\frac{2\mu e^{-\mu t}\sin(x+2\mu ct)}{1+e^{-\mu t}\cos(x+2\mu ct)}-2\mu c, \tag{4.24}$$

which is bounded for $t>0$ and $\mu>0$, which can be clearly seen in Figure 4.

5. CONCLUSION

In this paper, the bilinear formulation for the $(N+1)$-dimensional Burgers equation is established by using a fractal transformation. A special reduction of the bilinear formulation is introduced and thereby the generalized Cole-Hopf transformation is obtained, which leads to the general heat equation. When $N=1$, the reduced general heat equation coincides with the some physical models (see Remarks 3.1 and 3.2). As applications of the N-dimensional Cole-Hopf transformation, some physically interesting solutions, such as vortex solutions, multiple fusion and two different types of rational solutions are given. Numerical analysis shows that these solutions exhibit physically interesting behaviors. However, there are some interesting problems that deserve further investigation. For example, since the momentum equation of the NDB system shares some similarities to that of the N-dimensional incompressible/compressible Euler equation and Navier–Stokes (NS)

equation. Therefore, a natural question is that whether there exists appropriate bilinear formulations for the Euler and NS equations. If the answer is positive, then it means that we can quasi-linearize the Euler and NS equations. Based on the importance and wide applications of the two equations, these questions are worth deep investigations in the future.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

ACKNOWLEDGMENTS

The authors would like to express their sincere thanks to the referees for their kind comments and valuable suggestions. This work is supported by the National Science Foundation of China under grant No. 11775116, No. 11671095, No. 51879045, Jiangsu Qinglan high-level talent Project and Dean's Research Fund from the Education University of Hong Kong under grant No. 2018-19, FLASS/DRF/IRS-5.

REFERENCES

[1] M.J. Ablowitz, P.A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scatting, Cambridge University Press, New York, 1991.
[2] M.J. Ablowitz, S. De Lillo, The Burgers equation under deterministic and stochastic forcing, Phys. D 92 (1996), 245–259.
[3] A. Balogh, D.S. Gilliam, V.I. Shubov, Stationary solutions for a boundary controlled Burgers' equation, Math. Comput. Model. 33 (2001), 21–37.
[4] H. Bateman, Some recent researches on the motion of fluids, Monthly Weather Rev. 43 (1915), 163–170.
[5] G.W. Bluman, S. Kumei, Symmetries and Differential Equations, Spinger-Verlag, New York, 1989.
[6] J.M. Burgers, Mathematical examples illustrating relations occuring in the theory of turbulent fluid motion, verhand. Trans. Roy. Neth. Acad. Sci. Amsterdam 17 (1939), 1–53.
[7] J.M. Burgers, A mathematical model illustrating the theory of turbulence, Adv. Appl. Mech. 1 (1948), 171–199.
[8] Y. Chen, E. Fan, W. Yuen, The Hopf–Cole transformation, topological solitons and multiple fusion solutions for the n-dimensional Burgers system, Phys. Lett. A 380 (2016), 9–14.
[9] Y. Chen, Q. Wang, A series of new soliton-like solutions and double-like periodic solutions of a (2+1)-dimensional dispersive long wave equation, Chaos Solitons Fractals 23 (2005), 801–807.
[10] J. Doyle, M.J. Englefield, Similarity solutions of a generalized Burgers equation, IMA J. Appl. Math. 44 (1990), 145–153.
[11] E. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000), 212–218.
[12] Y.T. Gao, X.G. Xu, B. Tian, Variable-coefficient forced Burgers system in nonlinear fluid mechanics and its possibly observable effects, Int. J. Mod. Phys. C 14 (2003), 1207–1222.
[13] V. Gurarie, A. Migdal, Instantons in the Burgers equation, Phys. Rev. E Stat. Phys. Plasmas Fluids Relat. Interdiscip. Topics 54 (1996), 4908–4914.
[14] R. Hirota, Exact solution of the Korteweg-de Vries equation for multiple collisions of solitions, Phys. Rev. Lett. 27 (1971), 1192–1194.
[15] M. Hisakado, Breather trapping mechanism in piecewise homogeneous DNA, Phys. Lett. A 227 (1997), 87–93.
[16] W.P. Hong, On Bäcklund transformation for a generalized Burgers equation and solitonic solutions, Phys. Lett. A 268 (2000), 81–84.
[17] E. Hopf, The partial differential equation $u_t + uu_x = \mu_{xx}$, Commun. Pure Appl. Math. 3 (1950), 201–230.
[18] X.B. Hu, Generalized Hirota's bilinear equations and their soliton solutions, J. Phys. A: Math. Gen. 26 (1993), L465–L471 .
[19] D. Kaya, A. Yokus, A decomposition method for finding solitary and periodic solutions for a coupled higher-dimensional Burgers equations, Appl. Math. Comput. 164 (2005), 857–864.
[20] J.G. Kingston, C. Sophocleous, On point transformations of a generalised Burgers equation, Phys. Lett. A 155 (1991), 15–19.
[21] R.A. Kraenkel, J.G. Pereira, M.A. Manna, Nonlinear surface-wave excitations in the Bénard-Marangoni system, Phys. Rev. A 46 (1992), 4786–4790.
[22] M. Lesieur, A. Yaglom, F. David, New Trends in Turbulence, Springer EDP-Sciences, Berlin, 2001, pp. 341–383.
[23] X.Y. Li, Q.L. Zhao, A new integrable symplectic map by the binary nonlinearization to the super AKNS system, J. Geom. Phys. 121 (2017), 123–137.
[24] Y. Liu, H. Dong, Y. Zhang, Solutions of a discrete integrable hierarchy by straightening out of its continuous and discrete constrained flows, Anal. Math. Phys. 9 (2019), 465–481.
[25] S. Liu, Z. Fu, S. Liu, Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A 289 (2001), 69–74.
[26] Q.P. Liu, M. Mañas, Darboux transformations for super-symmetric KP hierarchies, Phys. Lett. B 485 (2000), 293–300.
[27] S.Y. Lou, L.L Chen, Formal variable separation approach for nonintegrable models, J. Math. Phys. 40 (1999), 6491–6500.
[28] S.Y. Lou, J.Z. Lu, Special solutions from variable separation approach: Davey-Stewartson equation, J. Phys. A: Math. Gen. 29 (1996), 4209–4215.
[29] Z. Lü, H. Zhang, New applications of a further extended tanh method, Phys. Lett. A 2004 (324), 293–298.
[30] W.M. Ma, Darboux transformations for a Lax integrable system in $2n$ dimensions, Lett. Math. Phys. 39 (1997), 33–49.

[31] W.X. Ma, Complexiton solutions to integrable equations, Nonlinear Anal. 63 (2005), e2461–e2471.
[32] W.X. Ma, Generalized bilinear differential equations, Stud. Nonlinear Sci. 2 (2011), 140–144.
[33] W.X. Ma, Y. You, Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions, Trans. Amer. Math. Soc. 357 (2005), 1753–1778.
[34] W.X. Ma, Z.X. Zhou, Coupled integrable systems associated with a polynomial spectral problem and their Virasoro symmetry algebras, Prog. Theoret. Phys. 96 (1996), 449–457.
[35] W.X. Ma, Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, J. Diff. Eqn. 264 (2018), 2633–2659.
[36] A.D. Polyanin, V.E. Nazaikinskii, Handbook of Linear Partial Differential Equations for Engineers and Scientists, Chapman and Hall/CRC Press,Virginia Beach, 2002.
[37] C. Qu, Allowed transformations and symmetry classes of variable coefficient Burgers equations, IMA J. Appl. Math. 54 (1995), 203–225.
[38] C. Rogers, W.K. Schief, Bäcklund and Darboux Transformations Geometry and Modern Applications in Soliton Theory, Cambridge University Press, New York, 2002.
[39] M. Salerno, On the phase manifold geometry of the two-dimensional Burgers equations, Phys. Lett. A 121 (1987), 15–18.
[40] V.N. Serkin, V.M. Chapela, J. Percino, T.L. Belyaeva, Nonlinear tunneling of temporal and spatial optical solitons through organic thin films and polymeric waveguides, Optics Commun. 192 (2001), 237–244.
[41] G. Stoitcheva, L. Ludu, J.P. Draayer, Antisolution model for fission model, Math. Comput. Simul. 55 (2001), 621–625.
[42] K.M. Tamizhmani, P. Punithavathi, Similarity reductions and painleve property of the coupled higher dimensional Burgers equation, Int. J. Non-Linear Mech. 26 (1991), 427–438.
[43] S. Wang, X.y. Tang, S.Y. Lou, Soliton fission and fusion: Burgers equation and Sharma–Tasso–Olver equation, Chaos Solitons Fractals 21 (2004), 231–239.
[44] Z. Yan, New explicit travelling wave solutions for two new integrable coupled nonlinear evolution equations, Phys. Lett. A 292 (2001), 100–106.
[45] Z. Yan, S. Lou, Soliton molecules in Sharma–Tasso–Olver–Burgers equation, Appl. Math. Lett. 104 (2020), 106271–106277.