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Research Article

Nurowski's Conformal Class of a Maximally Symmetric (2,3,5)-Distribution and its Ricci-flat Representatives

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ABSTRACT

We show that the solutions to the second-order differential equation associated to the generalised Chazy equation with parameters $k = 2$ and $k = 3$ naturally show up in the conformal rescaling that takes a representative metric in Nurowski's conformal class associated to a maximally symmetric (2,3,5)-distribution $\left(\text{described locally by a certain function } \varphi(x, q) = \dfrac{q^2}{H''(x)}\right)$ to a Ricci-flat one.

The article concerns the occurrence of the $k = 2$ and $k = 3$ generalised Chazy equation in a geometric setting, closely connected to the occurrence of the solutions of the generalised Chazy equation with parameters $k = \frac{2}{3}$ and $k = \frac{3}{2}$ respectively. We first discuss the set-up in which the differential equations will appear. This concerns the theory of maximally non-integrable rank 2 distribution $\mathcal{D}$ on a 5-manifold M. The maximally non-integrable condition of $\mathcal{D}$ determines a filtration of the tangent bundle TM given by

$$\mathcal{D} \subset [\mathcal{D}, \mathcal{D}] \subset [\mathcal{D}, [\mathcal{D}, \mathcal{D}]] \cong TM.$$

The distribution $[\mathcal{D},\mathcal{D}]$ has rank 3 while the full tangent space TM has rank 5, hence such a geometry is also known as a (2,3,5)-distribution. Let M_{xyzpq} denote the 5-dimensional mixed order jet space $J^{2,0}(\mathbb{R},\mathbb{R}^2) \cong J^2(\mathbb{R},\mathbb{R})\times\mathbb{R}$ with local coordinates given by $(x, y, z, p, q) = (x, y, z, y',y'')$ (see also [15], [16]). Let $\mathcal{D}_{\varphi(x,y,z,y',y'')}$ denote the maximally non-integrable rank 2 distribution on M_{xyzpq} associated to the underdetermined differential equation $z' = \varphi(x, y, z, y',y'')$. This means that the distribution is annihilated by the following three 1-forms

$$\omega_1 = dy - pdx, \qquad \omega_2 = dp - qdx, \qquad \omega_3 = dz - \varphi(x,y,z,p,q)dx.$$

Such a distribution $\mathcal{D}_{\varphi(x,y,z,y',y'')}$ is said to be in Monge normal form (see page 90 of [15]). In Section 5 of [11], it is shown how to associate canonically to such a (2,3,5)-distribution a conformal class of metrics of split signature (2,3) (henceforth known as Nurowski's conformal structure or Nurowski's conformal metrics) such that the rank 2 distribution is isotropic with respect to any metric in the conformal class. The method of equivalence [5] (also see the introduction to [3], Section 5 of [11], [14] and [10]) produces the 1-forms $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ that gives a coframing for Nurowski's metric. These 1-forms satisfy the structure equations

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$$\begin{aligned} d\theta_1 &= \theta_1 \wedge (2\Omega_1 + \Omega_4) + \theta_2 \wedge \Omega_2 + \theta_3 \wedge \theta_4, \\ d\theta_2 &= \theta_1 \wedge \Omega_3 + \theta_2 \wedge (\Omega_1 + 2\Omega_4) + \theta_3 \wedge \theta_5, \\ d\theta_3 &= \theta_1 \wedge \Omega_5 + \theta_2 \wedge \Omega_6 + \theta_3 \wedge (\Omega_1 + \Omega_4) + \theta_4 \wedge \theta_5, \\ d\theta_4 &= \theta_1 \wedge \Omega_7 + \frac{4}{3}\theta_3 \wedge \Omega_6 + \theta_4 \wedge \Omega_1 + \theta_5 \wedge \Omega_2, \\ d\theta_5 &= \theta_1 \wedge \Omega_7 - \frac{4}{3}\theta_3 \wedge \Omega_5 + \theta_4 \wedge \Omega_3 + \theta_5 \wedge \Omega_4, \end{aligned} \tag{0.1}$$

where $(\Omega_1, \ldots, \Omega_7)$ and two additional 1-forms (Ω_8, Ω_9) together define a rank 14 principal bundle over the 5-manifold M (see [5] and Section 5 of [11]). A representative metric in Nurowski's conformal class [11] is given by

$$g = 2\theta_1\theta_5 - 2\theta_2\theta_4 + \frac{4}{3}\theta_3\theta_3. \tag{0.2}$$

When g has vanishing Weyl tensor, the distribution is called maximally symmetric and has split G_2 as its group of local symmetries. For further details, see the introduction to [3] and Section 5 of [11]. For further discussion on the relationship between maximally symmetric (2,3,5)-distributions and the automorphism group of the split octonions, see Section 2 of [15].

The historically important example is the Hilbert-Cartan distribution obtained when $\varphi(x, y, z, p, q) = q^2$ [5]. This distribution gives the flat model of a (2,3,5)-distribution and is associated to the Hilbert-Cartan equation $z' = (y'')^2$ (see Section 5 of [11] for a discussion of this equation). When $\varphi(x, y, z, p, q) = q^m$, we obtain the distribution associated to the equation $z' = (y'')^m$. For such distributions, Nurowski's metric [11] given by (0.2) has vanishing Weyl tensor precisely when $m \in \left\{-1, \frac{1}{3}, \frac{2}{3}, 2\right\}$. For the values of $m = -1, \frac{1}{3}$ and $\frac{2}{3}$ the maximally symmetric distributions are all locally diffeomorphic to the $m = 2$ Hilbert-Cartan case.

In this article, we consider distributions of the form $\varphi(x, y, z, p, q) = \dfrac{q^2}{H''(x)}$. The Weyl tensor vanishes in the case where $H(x)$ satisfies the 6th-order ordinary differential equation (ODE) known as Noth's equation [3]. For such maximally symmetric distributions we find the corresponding Ricci-flat representatives in Nurowski's conformal class. This involves solving a second-order differential equation (see Proposition 35 of [15]) to find the conformal scale in which the Ricci tensor of the conformally rescaled metric vanishes, which turns out to be related to the solutions of Noth's equation. The 6th-order ODE can be solved by the generalised Chazy equation with parameter $k = \frac{3}{2}$ and its Legendre dual is another 6th-order ODE that can be solved by the generalised Chazy equation with parameter $k = \frac{2}{3}$ [12].

We find the second-order differential equation that determines the conformal scale for Ricci-flatness involves solutions of the generalised Chazy equation with parameter $k = 3$ and in the dual case $k = 2$. This is the content of Theorems 3.1 and 3.2 in Section 3. We also give few remarks concerning the case for other parameters of k in Section 4.

The aim of finding Ricci-flat representatives is motivated by the consideration that in the Ricci-flat, conformally flat case, we might be able to integrate the structure equations and redefine local coordinates to obtain the Hilbert-Cartan distribution. This is possible for the distributions of the form $\varphi(x, y, z, p, q) = q^m$, with $m \in \left\{-1, \frac{1}{3}, \frac{2}{3}\right\}$ (see [9]), but would require further investigations in the general setting.

The computations here are done using the indispensable DifferentialGeometry package in Maple 2018.

1. DERIVING THE EQUATION FOR RICCI-FLATNESS

We consider the rank 2 distribution $\mathcal{D}_{\varphi(x,q)}$ on M_{xyzpq} associated to the underdetermined differential equation $z' = \varphi(x, y'')$ where $\varphi(x, y'') = \dfrac{(y'')^2}{H''(x)}$ and $H''(x)$ is a non-zero function of x. This is to say that the distribution $\mathcal{D}_{\varphi(x,q)}$ is annihilated by the three 1-forms

$$\begin{aligned} \omega_1 &= dy - p dx, \\ \omega_2 &= dp - q dx, \\ \omega_3 &= dz - \varphi(x, q) dx, \end{aligned}$$

where $\varphi(x,q) = \dfrac{q^2}{H''(x)}$. These three 1-forms are completed to a coframing on M_{xyzpq} by the additional 1-forms

$$\omega_4 = dq - \frac{H^{(3)}}{H''} q dx, \quad \omega_5 = -\frac{H''}{2} dx.$$

Taking appropriate linear combinations, we let

$$\theta_1 = \omega_3 - \frac{2}{H''}q\omega_2, \qquad \theta_2 = \omega_1, \qquad \theta_3 = \left(\frac{2}{H''}\right)^{\frac{1}{3}}\omega_2,$$

with

$$\theta_4 = \left(\frac{2}{H''}\right)^{\frac{2}{3}}\omega_4 + a_{41}\theta_1 + a_{42}\theta_2 + a_{43}\theta_3$$

and

$$\theta_5 = \left(\frac{2}{H''}\right)^{\frac{2}{3}}\omega_5 + a_{51}\theta_1 + a_{52}\theta_2 + a_{53}\theta_3.$$

Imposing Cartan's structure equations (0.1) on $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ then gives the constraints $a_{51} = a_{53} = 0$ and $a_{41} = a_{52}$, which we can set both to be zero, and we also find $a_{42} = \frac{1}{30}\frac{2^{\frac{2}{3}}(3H''H^{(4)} - 5(H^{(3)})^2)}{(H'')^{\frac{8}{3}}}$ and $a_{43} = -\frac{2^{\frac{1}{3}}H^{(3)}}{3(H'')^{\frac{4}{3}}}$. We obtain the 1-forms $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ that give a coframing for a metric in Nurowski's conformal class [11], related to the 1-forms $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$ as follows:

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{pmatrix} = \begin{pmatrix} 0 & -\frac{2q}{H''} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{2}{H''}\right)^{\frac{1}{3}} & 0 & 0 & 0 \\ \frac{2^{\frac{2}{3}}(3H''H^{(4)} - 5(H^{(3)})^2)}{30(H'')^{\frac{8}{3}}} & -\frac{2^{\frac{2}{3}}H^{(3)}}{3(H'')^{\frac{5}{3}}} & 0 & \left(\frac{2}{H''}\right)^{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{2}{H''}\right)^{\frac{2}{3}} \end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\\ \omega_5\end{pmatrix}.$$

The metric $g = 2\theta_1\theta_5 - 2\theta_2\theta_4 + \frac{4}{3}\theta_3\theta_3$ is conformally flat, i.e. the metric g has vanishing Weyl tensor if and only if $H(x)$ is a solution to the 6th-order nonlinear differential equation

$$10(H'')^3H^{(6)} - 70(H'')^2H^{(3)}H^{(5)} - 49(H'')^2(H^{(4)})^2 + 280H''(H^{(3)})^2H^{(4)} - 175(H^{(3)})^4 = 0. \tag{1.1}$$

This equation is called Noth's equation [3]. In this case the distribution of the form $\mathcal{D}_{\varphi(x,q)}$ is maximally symmetric and in the paper we will concern ourselves with the problem of finding Ricci-flat representatives in the conformal class of metrics associated to this distribution.

The explicit form of the metric given by the distribution $\mathcal{D}_{\varphi(x,q)}$ is as follows. If we replace $H''(x) = e^{\int \frac{2}{3}P(x)dx}$, then we find that equation (1.1) reduces to the $k = \frac{3}{2}$ generalised Chazy equation

$$P''' - 2PP'' + 3P'^2 - \frac{4}{36 - \left(\frac{3}{2}\right)^2}(6P' - P^2)^2 = 0,$$

and we find that the conformally rescaled metric $\tilde{g} = 2^{-\frac{2}{3}}(H'')^{\frac{2}{3}}g$ has the form

$$\tilde{g} = -\frac{2}{15}\left(P' - \frac{4}{9}P^2\right)\omega_1\omega_1 + \frac{4}{9}P\omega_1\omega_2 + \frac{4}{3}\omega_2\omega_2 + 2\omega_3\omega_5 - 2\omega_1\omega_4 - 4qe^{\int -\frac{2}{3}Pdx}\omega_2\omega_5.$$

We can reexpress this metric as

$$\tilde{g} = -\frac{2}{15}\left(P' - \frac{1}{6}P^2\right)\omega_1\omega_1 + \frac{4}{3}\left(\frac{P}{6}\omega_1 + \omega_2\right)\left(\frac{P}{6}\omega_1 + \omega_2\right) + 2\omega_3\omega_5 - 2\omega_1\omega_4 - 4qe^{\int -\frac{2}{3}Pdx}\omega_2\omega_5.$$

By defining the new coframes

$$\tilde{\omega}_3 = e^{\int \frac{2P}{3}dx}\omega_3,$$
$$\tilde{\omega}_5 = e^{-\int \frac{2P}{3}dx}\omega_5,$$

and making the further substitution $Q = P^2 - 6P'$, we get the following cosmetic improvement for $\tilde{g}$:

$$\tilde{g} = \frac{1}{45}Q\omega_1\omega_1 + \frac{4}{3}\left(\frac{P}{6}\omega_1 + \omega_2\right)\left(\frac{P}{6}\omega_1 + \omega_2\right) + 2\tilde{\omega}_3\tilde{\omega}_5 - 2\omega_1\omega_4 - 4q\omega_2\tilde{\omega}_5.$$

From this we can rescale the metric $\tilde{g}$ further by a conformal factor Ω to obtain a Ricci-flat representative. When $\mathrm{Ric}(\Omega^2\tilde{g}) = 0$, we say that $\Omega^2\tilde{g}$ is a Ricci-flat representative of Nurowski's conformal class. We find that $\Omega^2\tilde{g}$ is Ricci-flat when Ω satisfies the second-order differential equation

$$\Omega''\Omega - 2(\Omega')^2 - \frac{2}{3}P\Omega\Omega' - \frac{1}{18}P^2\Omega^2 - \frac{1}{30}Q\Omega^2 = 0.$$

We make the substitution $\Omega = \frac{1}{\rho}e^{-\frac{1}{3}\int Pdx}$ to obtain

$$\rho'' - \frac{1}{45}Q\rho = 0, \tag{1.2}$$

where $\rho(x)$ is to be determined.

The function $H(x)$ is related to another function $F(\tilde{x})$ by a Legendre transformation [3], [12]. We say that $F(\tilde{x})$ is the Legendre dual of $H(x)$ determined by the relation $H(x) + F(\tilde{x}) = x\tilde{x}$. This implies $\tilde{x} = H'(x)$ with $d\tilde{x} = H''dx$ and $H'' = \frac{1}{F_{\tilde{x}\tilde{x}}}$. We can make use of this transformation to write $dx = F_{\tilde{x}\tilde{x}}d\tilde{x}$. The Legendre dual of the distribution $\mathcal{D}_{\varphi(x,q)}$ is therefore given by the annihilator of the three 1-forms

$$\omega_1 = dy - pF_{\tilde{x}\tilde{x}}d\tilde{x},$$
$$\omega_2 = dp - qF_{\tilde{x}\tilde{x}}d\tilde{x},$$
$$\omega_3 = dz - q^2(F_{\tilde{x}\tilde{x}})^2d\tilde{x}$$

on the mixed jet space with local coordinates $(\tilde{x}, y, z, p, q)$. Relabelling $\tilde{x}$ with x, we have

$$\omega_1 = dy - pF''dx,$$
$$\omega_2 = dp - qF''dx,$$
$$\omega_3 = dz - q^2(F'')^2dx.$$

Here F now becomes a function of x. These three 1-forms are completed to a coframing on M with local coordinates (x, y, z, p, q) by the additional 1-forms

$$\omega_4 = dq + \frac{F'''}{F''}qdx, \quad \omega_5 = -\frac{1}{2}dx.$$

(These are the Legendre transformed 1-forms ω_4 and ω_5). Similar as before, we consider the linear combinations

$$\theta_1 = \omega_3 - 2F''q\omega_2, \qquad \theta_2 = \omega_1, \qquad \theta_3 = (2F'')^{\frac{1}{3}}\omega_2,$$

with

$$\theta_4 = (2F'')^{\frac{2}{3}}\omega_4 + b_{41}\theta_1 + b_{42}\theta_2 + b_{43}\theta_3$$

and

$$\theta_5 = (2F'')^{\frac{2}{3}}\omega_5 + b_{51}\theta_1 + b_{52}\theta_2 + b_{53}\theta_3.$$

Imposing Cartan's structure equations (0.1) on $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ once again gives $b_{51} = b_{53} = 0$ and $b_{41} = b_{52}$, which we set to be zero. We also have

$$b_{42} = -\frac{1}{30}\frac{2^{\frac{2}{3}}(3F''F^{(4)} - 4(F^{(3)})^2)}{(F'')^{\frac{10}{3}}} \quad \text{and} \quad b_{43} = \frac{2^{\frac{1}{3}}F^{(3)}}{3(F'')^{\frac{5}{3}}}.$$

We obtain the 1-forms $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ that give a coframing for a metric in Nurowski's conformal class [11], related to the 1-forms $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$ as follows:

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{pmatrix} = \begin{pmatrix} 0 & -2F''q & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & (2F'')^{\frac{1}{3}} & 0 & 0 & 0\\ -\frac{2^{\frac{2}{3}}(3F''F^{(4)} - 4(F^{(3)})^2)}{30(F'')^{\frac{10}{3}}} & \frac{2^{\frac{2}{3}}F^{(3)}}{3(F'')^{\frac{4}{3}}} & 0 & (2F'')^{\frac{2}{3}} & 0\\ 0 & 0 & 0 & 0 & (2F'')^{\frac{2}{3}} \end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\\ \omega_5\end{pmatrix}.$$

A representative metric of Nurowski's conformal class is again given by (0.2). The condition that the metric g is conformally flat, i.e. the metric g has vanishing Weyl tensor, occurs when $F(x)$ is a solution to the nonlinear differential equation

$$10(F'')^3F^{(6)} - 80(F'')^2F^{(3)}F^{(5)} - 51(F'')^2(F^{(4)})^2 + 336F''(F^{(3)})^2F^{(4)} - 224(F^{(3)})^4 = 0. \tag{1.3}$$

If we replace $F''(x) = e^{\int \frac{1}{2}P(x)dx}$, then we find that the conformally rescaled metric $\tilde{g} = 2^{\frac{1}{3}}(F'')^{-\frac{2}{3}}g$ has the form

$$\tilde{g} = \frac{1}{30}(6P' - P^2)e^{\int -Pdx}\omega_1\omega_1 - \frac{2}{3}Pe^{-\int \frac{1}{2}Pdx}\omega_1\omega_2 + \frac{8}{3}\omega_2\omega_2 + 4\omega_3\omega_5 - 4\omega_1\omega_4 - 8qe^{\int \frac{1}{2}Pdx}\omega_2\omega_5. \tag{1.4}$$

Here equations (1.3) is reduced to the generalised Chazy equation

$$P''' - 2PP'' + 3P'^2 - \frac{4}{36 - \left(\frac{2}{3}\right)^2}(6P' - P^2)^2 = 0$$

for $P(x)$ with parameter $k = \frac{2}{3}$. From the form of the metric $\tilde{g}$ we can locally rescale the metric again by a conformal factor to obtain Ricci-flat representatives.

We find that the Ricci tensor of $\Omega^2\tilde{g}$ is zero when Ω satisfies

$$40\Omega''\Omega - 80(\Omega')^2 - 6\Omega^2P' + \Omega^2P^2 = 0.$$

If we make the substitution $\Omega = \frac{1}{\eta}$, then we obtain the differential equation

$$\eta'' - \frac{1}{40}Q\eta = 0 \tag{1.5}$$

where $Q = P^2 - 6P'$ and η is to be determined. From the form of the metric $\tilde{g}$ in (1.4), we can also define new coframes by

$$\tilde{\omega}_1 = e^{-\int \frac{P}{2}dx}\omega_1 = \frac{dy}{F''} - p\,dx,$$
$$\tilde{\omega}_2 = \omega_2 = dp - qF''dx,$$
$$\tilde{\omega}_3 = e^{-\int \frac{P}{2}dx}\omega_3 = \frac{dz}{F''} - q^2F''dx,$$
$$\tilde{\omega}_4 = e^{\int \frac{P}{2}dx}\omega_4 = F''dq + qF'''dx,$$
$$\tilde{\omega}_5 = e^{\int \frac{P}{2}dx}\omega_5 = -\frac{F''}{2}dx.$$

We have used that $e^{-\int \frac{P}{2}dx} = \frac{1}{F''}$. Also replacing $6P' - P^2 = -Q$, this gives the cosmetic improvement for $\tilde{g}$:

$$\tilde{g} = -\frac{Q}{30}\tilde{\omega}_1\tilde{\omega}_1 - \frac{2P}{3}\tilde{\omega}_1\tilde{\omega}_2 + \frac{8}{3}\tilde{\omega}_2\tilde{\omega}_2 + 4\tilde{\omega}_3\tilde{\omega}_5 - 4\tilde{\omega}_1\tilde{\omega}_4 - 8q\tilde{\omega}_2\tilde{\omega}_5.$$

We now investigate the solutions to (1.2) and (1.5). They are given by Theorems 3.1 and 3.2. We first review some results about the solutions to the generalised Chazy equation.

2. GENERALISED CHAZY EQUATION

The generalised Chazy equation with parameter k is given by

$$y''' - 2yy'' + 3y'^2 - \frac{4}{36-k^2}(6y' - y^2)^2 = 0$$

and Chazy's equation

$$y''' - 2yy'' + 3y'^2 = 0$$

is obtained in the limit as k tends to infinity. The generalised Chazy equation was introduced in [6], [7] and studied more recently in [8], [1], [2] and [4]. The generalised Chazy equation with parameters $k = \frac{2}{3}, \frac{3}{2}, 2$ and 3 was also further investigated in [13]. The solution to the generalised Chazy equation is given as follows (see also Table 2 in Section 3.3 of [4] and Proposition 2.2 of [13]). Let

$$w_1 = -\frac{1}{2}\frac{d}{dx}\log\frac{s'}{s(s-1)},$$
$$w_2 = -\frac{1}{2}\frac{d}{dx}\log\frac{s'}{s-1},$$
$$w_3 = -\frac{1}{2}\frac{d}{dx}\log\frac{s'}{s},$$

where $s = s(\alpha, \beta, \gamma, x)$ is a solution to the Schwarzian differential equation

$$\{s, x\} + \frac{1}{2}(s')^2 V = 0 \tag{2.1}$$

and

$$\{s, x\} = \frac{d}{dx}\left(\frac{s''}{s'}\right) - \frac{1}{2}\left(\frac{s''}{s'}\right)^2$$

is the Schwarzian derivative with the potential V given by

$$V = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2 + \gamma^2 - \alpha^2 - 1}{s(s-1)}. \tag{2.2}$$

The combination $y = -2w_1 - 2w_2 - 2w_3$ solves the generalised Chazy equation when

$$(\alpha,\beta,\gamma) = \left(\frac{1}{3},\frac{1}{3},\frac{2}{k}\right) \text{ or } \left(\frac{2}{k},\frac{2}{k},\frac{2}{k}\right). \tag{2.3}$$

This combination corresponds to cases 1(b) and 3(b) of Table 2 in [4]. The combination $y = -w_1 - 2w_2 - 3w_3$ solves the generalised Chazy equation when

$$(\alpha,\beta,\gamma) = \left(\frac{1}{k},\frac{1}{3},\frac{1}{2}\right) \text{ or } \left(\frac{1}{k},\frac{2}{k},\frac{1}{2}\right) \text{ or } \left(\frac{1}{k},\frac{1}{3},\frac{3}{k}\right), \tag{2.4}$$

with permutations of w_1, w_2 and w_3 in y corresponding to permutations of the values α, β and γ in (α, β, γ). This combination corresponds to cases 1(a), 2(a) and 2(b) of Table 2 in [4]. The combination $y = -w_1 - w_2 - 4w_3$ solves the generalised Chazy equation whenever

$$(\alpha,\beta,\gamma) = \left(\frac{1}{k},\frac{1}{k},\frac{4}{k}\right) \text{ or } \left(\frac{1}{k},\frac{1}{k},\frac{2}{3}\right), \tag{2.5}$$

again permuting w_1, w_2 and w_3 in y corresponds to permuting the values α, β, γ in (α, β, γ). This combination corresponds to cases 2(c) and 3(a) of Table 2 in [4]. Following [1], the functions w_1, w_2 and w_3 satisfy the following system of differential equations:

$$\begin{aligned} w_1' &= w_2w_3 - w_1(w_2+w_3) + \tau^2, \\ w_2' &= w_3w_1 - w_2(w_3+w_1) + \tau^2, \\ w_3' &= w_1w_2 - w_3(w_1+w_2) + \tau^2, \end{aligned} \tag{2.6}$$

where

$$\tau^2 = \alpha^2(w_1-w_2)(w_3-w_1) + \beta^2(w_2-w_3)(w_1-w_2) + \gamma^2(w_3-w_1)(w_2-w_3).$$

The second-order differential equation associated to the generalised Chazy equation with parameter k is given by

$$u_{ss} + \frac{1}{4}Vu = 0 \tag{2.7}$$

with the same potential V as given in (2.2) and (α, β, γ) is one of the triples in (2.3), (2.4) or (2.5). The equation (2.7) corresponds to the general solution of the Schwarzian differential equation (2.1) after interchanging dependent and independent variables [8]. In this case $x = \frac{u_2}{u_1}$ where u_1 and u_2 are linearly independent solutions to (2.7). Using the further substitution $u(s) = (s-1)^{\frac{1-\gamma}{2}} s^{\frac{1-\beta}{2}} z(s)$, the equation (2.7) can be brought to the hypergeometric differential equation

$$s(1-s)z_{ss} + (c-(a+b+1)s)z_s - abz = 0$$

with

$$a = \frac{1}{2}(1-\alpha-\beta-\gamma), \qquad b = \frac{1}{2}(1+\alpha-\beta-\gamma), \qquad c = 1-\beta.$$

From the differential equations (2.6), we can recover s by $s = \frac{w_1 - w_3}{w_2 - w_3}$. From this we deduce $s' = 2(w_1 - w_2)s$ and we also obtain the relation $ds = 2(w_1 - w_2)sdx$.

3. MAIN RESULTS: SOLVING THE EQUATIONS FOR RICCI-FLATNESS

In this section we give the general solution to the differential equation (1.2) where $Q = P^2 - 6P'$ and P is a solution of the $k = \frac{3}{2}$ generalised Chazy equation in Theorem 3.1 and the general solution to the differential equation (1.5) where again $Q = P^2 - 6P'$ and P is a solution of the $k = \frac{2}{3}$ generalised Chazy equation in Theorem 3.2. We first prove the following theorem.

Theorem 3.1. *The solution to the differential equation*

$$\rho'' - \frac{1}{45}Q\rho = 0,$$

where $Q = P^2 - 6P'$ *and* P *is a solution to the* $k = \frac{3}{2}$ *generalised Chazy equation, is given by* $\rho = \frac{u}{v}$ *where* v *is the solution to the second-order differential equation associated to the* $k = \frac{3}{2}$ *generalised Chazy equation and* u *is a solution to the second-order differential equation associated to the* $k = 3$ *generalised Chazy equation.*

Proof. To prove the claim, we consider the second-order differential equation of the form

$$v_{ss} + \frac{1}{4}Vv = 0 \tag{3.1}$$

associated to the generalised Chazy equation with parameter $k = \frac{3}{2}$, where V is the function given by

$$V = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)}$$

and (α, β, γ) is one of the triples in (2.3), (2.4) or (2.5) with $k = \frac{3}{2}$. We find that $v = v(s(x))$ as a function of x satisfies

$$\begin{aligned} &v_{xx} - 2(w_1 - w_2 - w_3)v_x - ((\alpha^2-1)w_1^2 + (\beta^2-1)w_2^2 + (\gamma^2-1)w_3^2)v \\ &+((\alpha^2+\beta^2-\gamma^2-1)w_1w_2 + (\alpha^2-\beta^2+\gamma^2-1)w_1w_3 - (\alpha^2-\beta^2-\gamma^2+1)w_2w_3)v = 0. \end{aligned} \tag{3.2}$$

We have used that

$$\frac{d}{ds} = \frac{(w_2 - w_3)}{2(w_1-w_2)(w_1-w_3)}\frac{d}{dx}$$

and the differential equations (2.6). Furthermore, the Wronskian $W = v_1(v_2)_s - v_2(v_1)_s$ of the solutions to the differential equation (3.1) satisfies $W_s = 0$, so $W = c_0$ and we have

$$v_1^2 = 2c_0(w_1 - w_2)s$$

from the consideration that $s' = 2(w_1 - w_2)s = \frac{v_1^2}{W}$. We also obtain from the differential equation the Wronskian $W = \frac{v(s(x))^2}{2(w_1-w_2)s(x)}$ satisfies, that

$$v_x - v(w_1 - w_2 - w_3) = 0. \tag{3.3}$$

This equation implies the differential equation (3.2) for v above, by using the fact that the w_i's satisfy the differential equations (2.6).

Upon making the substitution $\rho = \frac{u(x)}{v(x)}$ into equation (1.2), and using equation (3.3), we obtain a differential equation for $u(x)$ of the form

$$\begin{aligned} &u_{xx} - 2(w_1 - w_2 - w_3)u_x - ((\tilde{\alpha}^2-1)w_1^2 + (\tilde{\beta}^2-1)w_2^2 + (\tilde{\gamma}^2-1)w_3^2)u \\ &+((\tilde{\alpha}^2+\tilde{\beta}^2-\tilde{\gamma}^2-1)w_1w_2 + (\tilde{\alpha}^2-\tilde{\beta}^2+\tilde{\gamma}^2-1)w_1w_3 - (\tilde{\alpha}^2-\tilde{\beta}^2-\tilde{\gamma}^2+1)w_2w_3)u = 0, \end{aligned}$$

which is the same differential equation for v with different constants $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$. We claim that this is the differential equation $u_{ss} + \frac{1}{4}\tilde{V}u = 0$ associated to the generalised Chazy equation with parameter $k = 3$, with

$$\tilde{V} = \frac{1-\tilde{\beta}^2}{s^2} + \frac{1-\tilde{\gamma}^2}{(s-1)^2} + \frac{\tilde{\beta}^2+\tilde{\gamma}^2-\tilde{\alpha}^2-1}{s(s-1)}$$

and $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ is one of the triples in (2.3), (2.4) or (2.5) with $k = 3$. We compute the triples $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ when $Q = P^2 - 6P'$ and P is the solution of the generalised Chazy equation with parameter $k=\frac{3}{2}$. Fixing (α, β, γ) to be one of the triples in (2.3), (2.4) or (2.5) determines the values $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ up to sign. Specialising to the case where $k=\frac{3}{2}$, we obtain the following:

For the solutions given by $P = -2w_1-2w_2-2w_3$, when $(\alpha,\beta,\gamma)=\left(\frac{4}{3},\frac{4}{3},\frac{4}{3}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})=\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$. When $(\alpha,\beta,\gamma)=\left(\frac{4}{3},\frac{1}{3},\frac{1}{3}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})=\left(\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)$.

For the solutions given by $P = -w_1-2w_2-3w_3$, when $(\alpha,\beta,\gamma)=\left(\frac{2}{3},\frac{1}{3},\frac{1}{2}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})=\left(\frac{1}{3},\frac{1}{3},\frac{1}{2}\right)$. When $(\alpha,\beta,\gamma)=\left(\frac{2}{3},\frac{4}{3},\frac{1}{2}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})=\left(\frac{1}{3},\frac{2}{3},\frac{1}{2}\right)$. When $(\alpha,\beta,\gamma)=\left(\frac{2}{3},\frac{1}{3},2\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})=\left(\frac{1}{3},\frac{1}{3},1\right)$.

Finally, for the solutions given by $P = -4w_1 - w_2 - w_3$, when $(\alpha,\beta,\gamma)=\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})=\left(\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)$. When $(\alpha,\beta,\gamma)=\left(\frac{8}{3},\frac{2}{3},\frac{2}{3}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})=\left(\frac{4}{3},\frac{1}{3},\frac{1}{3}\right)$.

The values of $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ are precisely the triples (2.3), (2.4) or (2.5) that show up in the solutions of the $k = 3$ generalised Chazy equation. See [13] for the list of $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ when $k = 3$.

The determination of solutions to equation (1.5) is similar to that of Theorem 3.1.

Theorem 3.2. *The solution to the differential equation*

$$\eta'' - \frac{1}{40}Q\eta = 0, \tag{3.4}$$

where $Q = P^2 - 6P'$ and P is a solution of the $k=\frac{2}{3}$ generalised Chazy equation, is given by $\eta = \frac{u}{v}$, where v is a solution to the second-order differential equation associated to the $k=\frac{2}{3}$ generalised Chazy equation and u is a solution to the second-order differential equation associated to the $k = 2$ generalised Chazy equation.

Proof. The proof of the claim is similar to the proof of the previous theorem. From the differential equation of the form $v_{ss}+\frac{1}{4}Vv=0$ associated to the $k=\frac{2}{3}$ generalised Chazy equation, where V is the function given by

$$V = \frac{1-\beta^2}{s^2}+\frac{1-\gamma^2}{(s-1)^2}+\frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)},$$

and (α, β, γ) is one of the triples in (2.3), (2.4) or (2.5) with $k=\frac{2}{3}$, we find that $v = v(s(x))$ as a function of x satisfies

$$\begin{aligned} &v_{xx} - 2(w_1 - w_2 - w_3)v_x - ((\alpha^2-1)w_1^2 + (\beta^2-1)w_2^2 + (\gamma^2-1)w_3^2)v \\ &+((\alpha^2+\beta^2-\gamma^2-1)w_1w_2 + (\alpha^2-\beta^2+\gamma^2-1)w_1w_3 - (\alpha^2-\beta^2-\gamma^2+1)w_2w_3)v = 0. \end{aligned} \tag{3.5}$$

Like in the proof of Theorem 3.1, it can also be deduced that (3.3) holds for v, i.e.

$$v_x - v(w_1 - w_2 - w_3) = 0, \tag{3.6}$$

which again implies the differential equation (3.5) for v above, by using the fact that the w_i's satisfy the differential equations (2.6).

Upon making the substitution $\eta = \frac{u(x)}{v(x)}$ into equation (3.4), and using equation (3.6), we obtain a differential equation for $u(x)$ again given by

$$\begin{aligned} &u_{xx} - 2(w_1 - w_2 - w_3)u_x - ((\tilde{\alpha}^2 - 1)w_1^2 + (\tilde{\beta}^2 - 1)w_2^2 + (\tilde{\gamma}^2 - 1)w_3^2)u \\ &+((\tilde{\alpha}^2 + \tilde{\beta}^2 - \tilde{\gamma}^2 - 1)w_1w_2 + (\tilde{\alpha}^2 - \tilde{\beta}^2 + \tilde{\gamma}^2 - 1)w_1w_3 - (\tilde{\alpha}^2 - \tilde{\beta}^2 - \tilde{\gamma}^2 + 1)w_2w_3)u = 0, \end{aligned} \tag{3.7}$$

which is the same differential equation for v but with different constants $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$. Equation (3.7) corresponds to the second-order differential equation $u_{ss} + \frac{1}{4}\tilde{V}u = 0$ associated to the $k = 2$ generalised Chazy equation, with

$$\tilde{V} = \frac{1-\tilde{\beta}^2}{s^2} + \frac{1-\tilde{\gamma}^2}{(s-1)^2} + \frac{\tilde{\beta}^2 + \tilde{\gamma}^2 - \tilde{\alpha}^2 - 1}{s(s-1)}$$

and $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ is one of the triples in (2.3), (2.4) or (2.5) with $k = 2$. To see this, we shall compute these constants when $Q = P^2 - 6P'$ and P is the solution of the generalised Chazy equation with parameter $k = \frac{2}{3}$. Specialising to the case where $k = \frac{2}{3}$, we obtain the following:

For the solutions given by $P = -2w_1 - 2w_2 - 2w_3$, when $(\alpha, \beta, \gamma) = (3, 3, 3)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}) = (1, 1, 1)$. When $(\alpha,\beta,\gamma) = \left(3, \frac{1}{3}, \frac{1}{3}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}) = \left(1, \frac{1}{3}, \frac{1}{3}\right)$.

For the solutions given by $P = -w_1 - 2w_2 - 3w_3$, when $(\alpha,\beta,\gamma) = \left(\frac{3}{2}, \frac{1}{3}, \frac{1}{2}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{2}\right)$. When $(\alpha,\beta,\gamma) = \left(\frac{3}{2}, 3, \frac{1}{2}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}) = \left(\frac{1}{2}, 1, \frac{1}{2}\right)$. When $(\alpha,\beta,\gamma) = \left(\frac{3}{2}, \frac{1}{3}, \frac{9}{2}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}) = \left(\frac{1}{2}, \frac{1}{3}, \frac{3}{2}\right)$.

Finally, for the solutions given by $P = -4w_1 - w_2 - w_3$, when $(\alpha,\beta,\gamma) = \left(\frac{2}{3}, \frac{3}{2}, \frac{3}{2}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}) = \left(\frac{2}{3}, \frac{1}{2}, \frac{1}{2}\right)$. When $(\alpha,\beta,\gamma) = \left(6, \frac{3}{2}, \frac{3}{2}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}) = \left(2, \frac{1}{2}, \frac{1}{2}\right)$.

The values of $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ are again precisely the triples (2.3), (2.4) or (2.5) that show up in the solutions of the $k = 2$ generalised Chazy equation. See also [13] for the list of $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ when $k = 2$.

4. SOLUTION TO THE EQUATION FOR RICCI-FLATNESS FOR GENERAL CHAZY PARAMETER

More generally, when P is a solution to the generalised Chazy equation with parameter k, the metric g is no longer conformally flat but we can still find the conformal scale for which the Ricci tensor vanishes.

In the case of (1.2) with solutions given by $\rho = \frac{u}{v}$ where v is the second-order differential equation associated to the generalised Chazy equation with parameter k, we find that u is a solution to the second-order differential equation associated to the generalised Chazy equation with parameter $\tilde{k}$ with

$$\frac{45}{\tilde{k}^2} - \frac{9}{k^2} = 1. \tag{4.1}$$

The values (α, β, γ) appearing in V in the differential equation $v_{ss} + \frac{1}{4}Vv = 0$ are related to the values $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ appearing in $\tilde{V}$ in the differential equation $u_{ss} + \frac{1}{4}\tilde{V}u = 0$ by the following. For the solutions given by $P = -2w_1 - 2w_2 - 2w_3$, when $(\alpha,\beta,\gamma) = \left(\frac{2}{k}, \frac{2}{k}, \frac{2}{k}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ with

$$\frac{45}{4}\tilde{\alpha}^2 - \left(\frac{3}{k}\right)^2 = 1, \quad \frac{45}{4}\tilde{\beta}^2 - \left(\frac{3}{k}\right)^2 = 1, \quad \frac{45}{4}\tilde{\gamma}^2 - \left(\frac{3}{k}\right)^2 = 1.$$

When $(\alpha,\beta,\gamma)=\left(\frac{2}{k},\frac{1}{3},\frac{1}{3}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ with

$$\frac{45}{4}\tilde{\alpha}^2-\left(\frac{3}{k}\right)^2=1$$

and $\tilde{\beta}=\frac{1}{3}$, $\tilde{\gamma}=\frac{1}{3}$. Here and subsequently, we consider the positive square root that gives positive $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$.

For the solutions given by $P=-w_1-2w_2-3w_3$, when $(\alpha,\beta,\gamma)=\left(\frac{1}{k},\frac{1}{3},\frac{1}{2}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ with

$$45\tilde{\alpha}^2-\left(\frac{3}{k}\right)^2=1,\quad \tilde{\beta}=\frac{1}{3},\quad \tilde{\gamma}=\frac{1}{2}.$$

When $(\alpha,\beta,\gamma)=\left(\frac{1}{k},\frac{2}{k},\frac{1}{2}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ with

$$45\tilde{\alpha}^2-\left(\frac{3}{k}\right)^2=1,\quad \frac{45}{4}\tilde{\beta}^2-\left(\frac{3}{k}\right)^2=1,\quad \tilde{\gamma}=\frac{1}{2}.$$

When $(\alpha,\beta,\gamma)=\left(\frac{1}{k},\frac{1}{3},\frac{3}{k}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ with

$$45\tilde{\alpha}^2-\left(\frac{3}{k}\right)^2=1,\quad \tilde{\beta}=\frac{1}{3},\quad 5\tilde{\gamma}^2-\left(\frac{3}{k}\right)^2=1.$$

Finally for the solutions given by $P=-4w_1-w_2-w_3$, when $(\alpha,\beta,\gamma)=\left(\frac{4}{k},\frac{1}{k},\frac{1}{k}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ with

$$\frac{45}{16}\tilde{\alpha}^2-\left(\frac{3}{k}\right)^2=1,\quad 45\tilde{\beta}^2-\left(\frac{3}{k}\right)^2=1,\quad 45\tilde{\gamma}^2-\left(\frac{3}{k}\right)^2=1.$$

When $(\alpha,\beta,\gamma)=\left(\frac{2}{3},\frac{1}{k},\frac{1}{k}\right)$, we find $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ with

$$\tilde{\alpha}=\frac{2}{3},\quad 45\tilde{\beta}^2-\left(\frac{3}{k}\right)^2=1,\quad 45\tilde{\gamma}^2-\left(\frac{3}{k}\right)^2=1.$$

In all cases the appropriate substitution of $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ in terms of the Chazy parameter $\tilde{k}$ gives equation (4.1), so it can be seen that the equation for u is the second-order differential equation associated to the generalised Chazy equation with parameter $\tilde{k}$, related to k by (4.1). The further substitution $k=\frac{3}{m}$ and $\tilde{k}=\frac{3}{\tilde{m}}$ into (4.1) gives

$$5\tilde{m}^2-m^2=1,$$

which has integer solutions when considered as a negative Pell equation. For integer solutions m and $\tilde{m}$ we obtain

$$m=\pm\left(\frac{1}{2}\left(2+\sqrt{5}\right)^{2n+1}+\frac{1}{2}\left(2-\sqrt{5}\right)^{2n+1}\right),$$
$$\tilde{m}=\pm\left(\frac{\sqrt{5}}{10}\left(2+\sqrt{5}\right)^{2n+1}-\frac{\sqrt{5}}{10}\left(2-\sqrt{5}\right)^{2n+1}\right).$$

They take on values $(m,\tilde{m})$ = (2, 1), (38, 17), (682, 305), (12238, 5473) and so on for $n\in\mathbb{N}\cup\{0\}$. They also give the corresponding pairs of Chazy parameters $(k,\tilde{k})=\left(\frac{3}{2},3\right),\left(\frac{3}{38},\frac{3}{17}\right)$ and so on, with the fundamental solution (n = 0) agreeing with the result of Theorem 3.1 in the conformally flat case.

In the case of (1.5) with solutions given by $\eta = \frac{u}{v}$ where v is the second-order differential equation associated to the generalised Chazy equation with parameter k, we find that u is a solution to the second-order differential equation associated to the generalised Chazy equation with parameter $\tilde{k}$ with

$$\frac{40}{\tilde{k}^2} - \frac{4}{k^2} = 1. \tag{4.2}$$

In this case we obtain the relationship between the values (α, β, γ) and $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ as follows. For $P = -2w_1 - 2w_2 - 2w_3$, when $(\alpha, \beta, \gamma) = \left(\frac{2}{k}, \frac{2}{k}, \frac{2}{k}\right)$, we find $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ with

$$10\tilde{\alpha}^2 - \left(\frac{2}{k}\right)^2 = 1, \quad 10\tilde{\beta}^2 - \left(\frac{2}{k}\right)^2 = 1, \quad 10\tilde{\gamma}^2 - \left(\frac{2}{k}\right)^2 = 1.$$

Considering integer solutions α and $\tilde{\alpha}$ to the negative Pell equation $10\tilde{\alpha}^2 - \alpha^2 = 1$ (and also β, $\tilde{\beta}$ and γ, $\tilde{\gamma}$ respectively), we find

$$\alpha = \pm\left(\frac{1}{2}\left(3+\sqrt{10}\right)^{2n+1} + \frac{1}{2}\left(3-\sqrt{10}\right)^{2n+1}\right),$$
$$\tilde{\alpha} = \pm\left(\frac{\sqrt{10}}{20}\left(3+\sqrt{10}\right)^{2n+1} - \frac{\sqrt{10}}{20}\left(3-\sqrt{10}\right)^{2n+1}\right),$$

where $n \in \mathbb{Z}$. Positive integer solutions are given by $(\alpha, \tilde{\alpha})$ = (3, 1), (117, 37), (4443, 1405), (168717, 53353) and so on for $n \in \mathbb{N} \cup \{0\}$. They give the relationship between the pairs of Chazy parameters $k = \frac{2}{\alpha}$ and $\tilde{k} = \frac{2}{\tilde{\alpha}}$, with $(k, \tilde{k}) = \left(\frac{2}{3}, 2\right), \left(\frac{2}{117}, \frac{2}{37}\right)$ and so on for $n \in \mathbb{N} \cup \{0\}$. For these parameters, the associated hypergeometric functions are algebraic. Again the fundamental solution ($n = 0$) agrees with the result of Theorem 3.2 in the conformally flat case.

The determination of the other values of (α, β, γ) and $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ is as follows. For the same P, when $(\alpha, \beta, \gamma) = \left(\frac{2}{k}, \frac{1}{3}, \frac{1}{3}\right)$, we find $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ with

$$10\tilde{\alpha}^2 - \left(\frac{2}{k}\right)^2 = 1$$

and $\tilde{\beta} = \frac{1}{3}$, $\tilde{\gamma} = \frac{1}{3}$.

For the solutions given by $P = -w_1 - 2w_2 - 3w_3$, when $(\alpha, \beta, \gamma) = \left(\frac{1}{k}, \frac{1}{3}, \frac{1}{2}\right)$, we find $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ with

$$40\tilde{\alpha}^2 - \left(\frac{2}{k}\right)^2 = 1, \quad \tilde{\beta} = \frac{1}{3}, \quad \tilde{\gamma} = \frac{1}{2}.$$

When $(\alpha, \beta, \gamma) = \left(\frac{1}{k}, \frac{2}{k}, \frac{1}{2}\right)$, we find $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ with

$$40\tilde{\alpha}^2 - \left(\frac{2}{k}\right)^2 = 1, \quad 10\tilde{\beta}^2 - \left(\frac{2}{k}\right)^2 = 1, \quad \tilde{\gamma} = \frac{1}{2}.$$

When $(\alpha, \beta, \gamma) = \left(\frac{1}{k}, \frac{1}{3}, \frac{3}{k}\right)$, we find $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ with

$$40\tilde{\alpha}^2 - \left(\frac{2}{k}\right)^2 = 1, \quad \tilde{\beta} = \frac{1}{3}, \quad \frac{40}{9}\tilde{\gamma} - \left(\frac{2}{k}\right)^2 = 1.$$

Finally, for the solutions given by $P = -4w_1 - w_2 - w_3$, when $(\alpha,\beta,\gamma) = \left(\frac{4}{k},\frac{1}{k},\frac{1}{k}\right)$, we find $(\tilde\alpha,\tilde\beta,\tilde\gamma)$ with

$$\frac{5}{2}\tilde\alpha^2 - \left(\frac{2}{k}\right)^2 = 1, \quad 40\tilde\beta^2 - \left(\frac{2}{k}\right)^2 = 1, \quad 40\tilde\gamma^2 - \left(\frac{2}{k}\right)^2 = 1.$$

When $(\alpha,\beta,\gamma) = \left(\frac{2}{3},\frac{1}{k},\frac{1}{k}\right)$, we find $(\tilde\alpha,\tilde\beta,\tilde\gamma)$ with

$$\tilde\alpha = \frac{2}{3}, \quad 40\tilde\beta^2 - \left(\frac{2}{k}\right)^2 = 1, \quad 40\tilde\gamma^2 - \left(\frac{2}{k}\right)^2 = 1.$$

In all cases the appropriate substitution of $\tilde\alpha$, $\tilde\beta$ and $\tilde\gamma$ in terms of the Chazy parameter $\tilde k$ gives equation (4.2), and therefore the equation for u is the second-order differential equation associated to the generalised Chazy equation with parameter $\tilde k$, related to k by (4.2). Altogether, with the exception of the parameters $k = \frac{3}{2}$ and $k = \frac{2}{3}$ as mentioned above, they give Ricci-flat but non-conformally flat examples of Nurowski's metric.

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