ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics
Vol. **28**(**1**); March (2021), *pp.* 53–67
DOI: https://doi.org/10.2991/jnmp.k.200922.005; ISSN 1402-9251; eISSN 1776-0852
https://www.atlantis-press.com/journals/jnmp

Research Article

The Miura Links of the Symmetries in the q-Deformed Case

Siqi Jian[1], Jipeng Cheng[2,*]

[1]*School of Statistics, Capital University of Economics and Business, Beijing 100070, P. R. China*
[2]*School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221116, P. R. China*

ARTICLE INFO

Article History

Received 29 April 2019
Accepted 27 April 2020

Keywords

Squared eigenfunction symmetry
additional symmetry
Miura transformation
the q-KP hierarchies
the q-mKP hierarchies

2010 Mathematics Subject Classification

35Q51
37K10
37K40

ABSTRACT

In this paper, we first construct the squared eigenfunction symmetries for the q-deformed Kadomtsev–Petviashvili (KP) and q-deformed modified KP hierarchies, including the unconstrained and constrained cases. Then the Miura links of the squared eigenfunction symmetries are investigated. At last, we also discuss the Miura links of the additional symmetries, since the additional symmetries are closely related with the squared eigenfunction symmetries.

1. INTRODUCTION

The q-deformed integrable systems [1,5,9–13,16,18–20,28–32] play important roles in the theoretical physics and mathematics, which are usually defined by replacing the usual derivative ∂ with respect to the space variable x in the classical integrable systems with the q-derivative ∂_q [see (2) in Section 2]. Among them, the q-deformed Kadomtsev–Petviashvili (q-KP) [1,9,11,12,30,31] and q-deformed modified Kadomtsev–Petviashvili (q-mKP) hierarchies [5,19,28] are two typical ones, which are connected with each other by the Miura links [2]. Here, **Miura transformation** means the one from the solutions of the q-mKP hierarchy to the solutions of the q-KP hierarchy, while the reverse one from the q-KP hierarchy to the q-mKP hierarchy is called **reverse-Miura transformation**. There are many integrable properties, such as the Hamiltonian structures [15,25,27] and the solution structures [14,21], which can coincide with the Miura links. In this paper, we will invesigate the changes of the Squared Eigenfunction (SE) symmetries and the additional symmetries under the Miura links in the q-deformed case.

The SE symmetry [3,22–24], also called the "ghost" symmetry [3], is a kind of symmetry generated by the squared eigenfunctions, i.e. the product of the eigenfunctions and their adjoints. The SE symmetry plays an important role in various aspects of the integrable systems, and thus is extensively studied recently [4,6,7,17]. In this paper, we construct the SE symmetries for the q-KP and q-mKP hierarchies, including the unconstrained and constrained cases. And further, the Miura links of the SE symmetries are considered. We find that the Miura links can keep the structures of the SE symmetries in the q-deformed case. Another important kind of symmetries which are closely related with the SE symmetry is the additional symmetries [2,3,6,8,26], depending explicitly on the space and time variables. The additional symmetries of the q-KP and q-mKP hierarchies are investigated in Tian et al. [29], Tu [31]. In this paper, we also consider the Miura links of the additional symmetries by direct computations, which can confirm the results of the SE symmetry.

The paper is organized as follows. In Section 2, the Miura links between the q-KP and q-mKP hierarchies are reviewed. In Section 3, the SE symmetries are constructed and their Miura links are investigated in the unconstrained case. The corresponding constrained case is discussed in Section 4. In Section 5, we investigate the Miura links of the additional symmetries. Finally, Section 6 is devoted to the conclusions and discussions.

*Corresponding author. Email: chengjp@cumt.edu.cn

2. *q*-KP AND *q*-mKP HIERARCHIES

In this section, we will review and revise the results in Cheng [5] for the further study. Let us firstly review the q-pseudo difference operators [11–13]. Denote θ and ∂_q be the q-shift operator and q-difference operator acting on a function $f(x)$:

$$[\theta f](x)=f(qx),\quad [\partial_q f](x)=\frac{f(qx)-f(x)}{x(q-1)}.$$

The commutative law between ∂_q and θ is as follows,

$$[\partial_q\theta^k f]=q^k[\theta^k\partial_q f],\quad k\in\mathbb{Z}. \tag{2.1}$$

For any q-pseudo-differential operator $A=\sum_{i\ll\infty}A_i\partial_q^i$ and any function $f(x)$, the symbol Af denotes the operator product of A and f, which obeys the q-deformed Leibniz rule

$$\partial_q^n f=\sum_{k=0}^{\infty}\binom{n}{k}_q[\theta^{n-k}\partial_q^k f]\partial_q^{n-k},\quad n\in\mathbb{Z}, \tag{2.2}$$

where the q-number and the q-binomial are defined by

$$(n)_q=\frac{q^n-1}{q-1},\ \binom{n}{k}_q=\frac{(n)_q(n-1)_q\cdots(n-k+1)_q}{(1)_q(2)_q\cdots(k)_q},\ \binom{n}{0}_q=1. \tag{2.3}$$

The symbol $[Af]$ or f_A denotes the action of A on f. Some simple notations of operations on the operator A are defined as follows,

$$A_{\geq k}=\sum_{i\geq k}A_i\partial_q^i,\ A_{<k}=\sum_{i<k}A_i\partial_q^i,\ (A)_k=A_k\partial_q^k,\ res(A)=A_{-1}. \tag{2.4}$$

Define θ^* and ∂_q^* as the adjoints of the q-shift operator θ and the q-difference operator ∂_q respectively:

$$\theta^*=\theta^{-1},\ \partial_q^*=-\partial_q\theta^{-1}=-\frac{1}{q}\partial_{\frac{1}{q}}. \tag{2.5}$$

And the adjoint operator $*$ satisfies $(AB)^*=B^*A^*$ for any q-difference operators A and B. For the q-pseudo-differential operator $A=\sum_{i\ll\infty}A_i\partial_q^i$, we use the notation $A|_{x/q}=\sum_{i\ll\infty}[\theta^{-1}A_i]q^i\partial_q^i$.

Then the q-KP and q-mKP hierarchies [5,11,12,19] are defined as the following Lax equations,

$$\partial_{t_n}L=[(L^n)_{\geq k},L]. \tag{2.6}$$

Here the Lax operator L is a general q-pseudo difference operator,

$$L=\begin{cases}\partial_q+u+u_1\partial_q^{-1}+u_2\partial_q^{-2}+u_3\partial_q^{-3}+\cdots, & k=0\ (L=L_{qKP}),\\ v\partial_q+v_0+v_1\partial_q^{-1}+v_2\partial_q^{-2}+v_3\partial_q^{-3}+\cdots, & k=1\ (L=L_{qmKP}),\end{cases} \tag{2.7}$$

which can be expressed in a terms of a dressing operator

$$L=\begin{cases}W\partial_q W^{-1}, & k=0\ (L=L_{qKP}),\\ Z\partial_q Z^{-1}, & k=1\ (L=L_{qmKP}).\end{cases} \tag{2.8}$$

with

$$W=1+w_1\partial_q^{-1}+w_2\partial_q^{-2}+\cdots, \tag{2.9}$$

$$Z=z_0+z_1\partial_q^{-1}+z_2\partial_q^{-2}+\cdots\ (z_0^{-1}\text{ exists}). \tag{2.10}$$

Then the Lax equations are equivalent to

$$\partial_{t_n}W=-(L_{qKP}^n)_{<0}W=-(W\partial_q^n W^{-1})_{<0}W, \tag{2.11}$$

$$\partial_{t_n}Z=-(L_{qmKP}^n)_{\leq 0}Z=-(Z\partial_q^n Z^{-1})_{\leq 0}Z. \tag{2.12}$$

With the dressing operators W and Z, the q-wave function $\omega(x, t, z)$ and q-adjoint wave function $\omega^*(x, t, z)$ are defined by

$$\omega_q(x,t,z) = [We_q(xz)\exp\xi(t,z)] \quad (k=0), \tag{2.13}$$

$$\omega_q(x,t,z) = [Ze_q(xz)\exp\xi(t,z)] \quad (k=1), \tag{2.14}$$

and

$$\omega_q^*(x,t,z) = [(W^*)^{-1}|_{x/q}\, e_{1/q}(-xz)\exp(-\xi(t,z))] \quad (k=0), \tag{2.15}$$

$$\omega_q^*(x,t,z) = [(Z^*)^{-1}|_{x/q}\, e_{1/q}(-xz)\exp(-\xi(t,z))] \quad (k=1). \tag{2.16}$$

Here $e_q(x) = \sum_{i=0}^{\infty}\frac{x^i}{(i)_q!} = \exp\left[\sum_{k=0}^{\infty}\frac{(1-q)^k}{k(1-q^k)}x^k\right]$ and $\xi(t,z) = \sum_{i=1}^{\infty} t_i z^i$. It can be proved that

$$[L\omega_q] = z\omega_q, \quad \partial_{t_n}\omega_q = [(L^n)_{\geq k}\omega_q], \qquad (k=0,1), \tag{2.17}$$

$$[L^*|_{x/q}\,\omega_q^*] = z\omega_q^*, \quad \partial_{t_n}\omega_q^* = -[((L^n)_{\geq k}|_{x/q})^*\omega_q^*], \quad (k=0,1). \tag{2.18}$$

The construction of the Miura transformations depends on the eigenfunction ϕ and adjoint eigenfunction ψ, which are defined as follows

$$\phi_{t_n} = [(L^n)_{\geq k}\phi], \quad \psi_{t_n} = -[(\partial_q^k (L^n)_{\geq k}\partial_q^{-k})^*\psi], \quad k=0,1. \tag{2.19}$$

Note that in Cheng [5], the definition of the adjoint eigenfunction ψ (in the case of $k = 1$) is different from the one here. Another important objects are the SE potentials $\Omega(\psi^{(k)}, \phi)$ and $\widehat{\Omega}(\psi, \phi^{(k)})$,

$$\Omega(\psi^{(k)},\phi)_{\partial_q} = \psi^{(k)}\phi, \quad \Omega(\psi^{(k)},\phi)_{t_n} = res(\partial_q^{-1}\psi^{(k)}(L^n)_{\geq k}\phi\partial_q^{-1}), \quad k=0,1, \tag{2.20}$$

where, in turn for $k = 0, 1$, $\phi^{(k)}$ denotes ϕ or ϕ_{∂_q}, and $\psi^{(k)}$ denotes ψ or $\psi_{\partial_q^*}$. The potential $\widehat{\Omega}$ is defined by

$$\widehat{\Omega}(\psi,\phi^{(k)}) = \begin{cases} \Omega(\psi,\phi), & k=0, \\ \Omega(\psi^{(k)},\phi) + [\theta^{-1}\psi]\phi, & k=1. \end{cases} \tag{2.21}$$

From the q-KP hierarchy to the q-mKP hierarchy, there are two different types of reverse-Miura transformations

$$T_\alpha(\phi) = \phi^{-1}, \quad T_\beta(\psi) = \partial_q^{-1}\psi, \tag{2.22}$$

where ϕ and ψ are the eigenfunction and the adjoint eigenfunction of the q-KP hierarchy respectively. As for Miura transformations, there are also two different types from the q-mKP hierarchy to the q-KP hierarchy

$$T_\mu(z_0) = z_0^{-1}, \quad T_\nu(z_0) = [\theta z_0]^{-1}\partial_q. \tag{2.23}$$

Here z_0 is the coefficient of ∂_q^0-term in the dressing operator of the q-mKP hierarchy [see (2.10)].

Next, we review the results in Cheng [5] of the Miura links in the q-deformed cases without constraints.

Theorem 2.1. *(Reverse-Miura transformation: q-KP → q-mKP) [5] Let* $\Omega(.,.)$ *and* $\widehat{\Omega}(.,.)$ *be defined in (2.20) and (2.21). Let* $L = L_{qKP}$ *be the solution of the q-KP hierarchy* $L_{t_n} = [(L^n)_{\geq 0}, L]$ *with (adjoint) eigenfunctions* $\phi, \phi_1, \ldots, \phi_{m_1}, \psi, \psi_1, \ldots, \psi_{m_1}$ *defined in (2.19). Then under the reverse-Miura transformation* T_α *or* T_β,

(i) $L \xrightarrow{\alpha} \tilde{L} = \phi^{-1}L\phi,\ W \xrightarrow{\alpha} Z = \phi^{-1}W,\ \phi_i \xrightarrow{\alpha} \tilde{\phi}_i = \phi^{-1}\phi_i,\ \psi_i \xrightarrow{\alpha} \tilde{\psi}_i = -[\theta\Omega(\psi_i, \phi)]$;

(ii) $L \xrightarrow{\beta} \tilde{L} = \partial_q^{-1}\psi L \psi^{-1}\partial_q,\ W \xrightarrow{\beta} Z = \partial_q^{-1}\psi W,\ \phi_i \xrightarrow{\beta} \tilde{\phi}_i = \Omega(\psi,\phi_i),\ \psi_i \xrightarrow{\beta} \tilde{\psi}_i = \psi^{-1}\psi_i$,

$\tilde{L}$ *satisfies the q-mKP hierarchy* $\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 1}, \tilde{L}]$ *and* $\tilde{\phi}_i, \tilde{\psi}_i, i = 1, \cdots, m_1$ *are (adjoint) eigenfunctions of* $\tilde{L}$.

Theorem 2.2. *(Miura transformation: q-mKP → q-KP)* [5] *Let $L = L_{qmKP}$ be the solution of the q-mKP hierarchy $L_{t_n} = [(L^n)_{\geq 1}, L]$ with (adjoint) eigenfunctions $\phi_1, \ldots, \phi_{m_1}, \psi_1, \ldots, \psi_{m_1}$ defined in (2.19). Let z_0 be the zero term in the dressing operator Z such that $L = Z\partial_q Z^{-1}$. Then under the Miura transformation T_μ or T_ν,*

(i) $L \xrightarrow{\mu} \tilde{L} = z_0^{-1} L z_0,\ Z \xrightarrow{\mu} W = z_0^{-1} Z,\ \phi_i \xrightarrow{\mu} \tilde{\phi}_i = z_0^{-1}\phi_i,\ \psi_i \xrightarrow{\mu} \tilde{\psi}_i = z_0[\partial_q^* \psi_i];$

(ii) $L \xrightarrow{\nu} \tilde{L} = [\theta z_0]^{-1}\partial_q L \partial_q^{-1}[\theta z_0],\ Z \xrightarrow{\nu} W = [\theta z_0]^{-1}\partial_q Z,\ \phi_i \xrightarrow{\nu} \tilde{\phi}_i = [\theta z_0]^{-1}[\partial_q \phi_i],\ \psi_i \xrightarrow{\nu} \tilde{\psi}_i = [\theta z_0]\psi_i,$

$\tilde{L}$ satisfies the q-KP hierarchy $\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 0}, \tilde{L}]$ and $\tilde{\phi}_i$, $\tilde{\psi}_i$, $i = 1, \cdots, m_1$ are (adjoint) eigenfunctions of $\tilde{L}$.

Remark: Please note that in Cheng [5], for $k = 1$, the definition of the adjoint eigenfunction ψ is different from the one here. Thus, our results here are slightly different from the ones in Cheng [5].

It is also given in Cheng [5] the results of the constrained case. Here we will revise the corresponding results. The constrained case means the imposition of the constraints on the Lax operator L in (2.7),

$$(L^N)_{<k} = \sum_{j=1}^{m} q_j \partial_q^{-1} r_j \partial_q^k, \quad k = 0, 1, \tag{2.24}$$

for some $N \in \mathbb{N}$ and some functions $q_1, \ldots, q_m, r_1, \ldots, r_m$, where

$$L_{t_n} = [(L^n)_{\geq k}, L],\ q_{jt_n} = [(L_{t_n})_{\geq k} q_j],\ r_{jt_n} = -[\partial_q^{-k*}(L_{t_n})^*_{\geq k}\partial_q^{k*} r_j]. \tag{2.25}$$

Theorem 2.3. *(Reverse-Miura transformation: constrained q-KP → constrained q-mKP) Let $\Omega(.,.)$ and $\widehat{\Omega}(.,.)$ be defined in (2.20) and (2.21). Let $L = L_{qKP}$ satisfy the constraint $(L^N)_{<0} = \sum_{j=1}^m q_j \partial_q^{-1} r_j$. Then for any function ϕ or ψ, under the reverse-Miura transformation T_α or T_β,*

(i) $L \xrightarrow{\alpha} \tilde{L} = \phi^{-1} L \phi,\ q_j \xrightarrow{\alpha} \tilde{q}_j = \phi^{-1} q_j,\ r_j \xrightarrow{\alpha} \tilde{r}_j = -[\theta\widehat{\Omega}(r_j, \phi)],\ j = 1, \cdots, m,$ *and* $\tilde{q}_{m+1} = \phi^{-1}[(L^N)_{\geq 0}\phi] + \phi^{-1}\sum_{j=1}^m q_j \widehat{\Omega}(r_j, \phi) = \phi^{-1}[L^N \phi],$ $\tilde{r}_{m+1} = 1;$

(ii) $L \xrightarrow{\beta} \tilde{L} = \partial_q^{-1}\psi L \psi^{-1}\partial_q,\ q_j \xrightarrow{\beta} \tilde{q}_j = \Omega(\psi, q_j),\ r_j \xrightarrow{\beta} \tilde{r}_j = \psi^{-1} r_j,\ j = 1, \cdots, m$ *and* $\tilde{q}_{m+1} = 1,\ \tilde{r}_{m+1} = -\sum_{j=1}^m [\theta\Omega(\psi, q_j)] r_j \psi^{-1} + [(L^N)^*_{\geq 0}\psi]\psi^{-1} = \psi^{-1}[(L^N)^*\psi],$

the transformed operator $\tilde{L}$ satisfies the constraint,

$$(\tilde{L}^N)_{<1} = \sum_{j=1}^{m+1} \tilde{q}_j \partial_q^{-1} \tilde{r}_j \partial_q.$$

If L, q_j, r_j satisfy the constrained q-KP dynamics (2.25, $k = 0$) and ϕ or ψ is an (adjoint) eigenfunction (i.e. $\phi_{t_n} = [(L^n)_{\geq 0}\phi]$ or $\psi_{t_n} = -[(L^n)^*_{\geq 0}\phi]$), according to Theorem 2.1, the transformed Lax operator $\tilde{L}$ satisfies the q-mKP dynamics $\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 1}, \tilde{L}]$. Further, $\tilde{q}_j$, $\tilde{r}_j$ are the new (adjoint) eigenfunctions.

Theorem 2.4. *(Miura transformation: constrained q-mKP → constrained q-KP) Let $L = L_{qmKP}$ satisfy the constraint $(L^N)_{<1} = \sum_{j=1}^m q_j \partial_q^{-1} r_j \partial_q$. Then for any function z_0, under the Miura transformation T_μ or T_ν,*

(i) $L \xrightarrow{\mu} \tilde{L} = z_0^{-1} L z_0,\ q_j \xrightarrow{\mu} \tilde{q}_j = z_0^{-1} q_j,\ r_j \xrightarrow{\mu} \tilde{r}_j = -z_0[\theta^{-1}\partial_q r_j];$

(ii) $L \xrightarrow{\nu} \tilde{L} = [\theta z_0]^{-1}\partial_q L \partial_q^{-1}[\theta z_0],\ q_j \xrightarrow{\nu} \tilde{q}_j = [\theta z_0]^{-1}[\partial_q q_j],\ r_j \xrightarrow{\nu} \tilde{r}_j = [\theta z_0] r_j,$

the transformed operator $\tilde{L}$ satisfies the constraint,

$$(\tilde{L}^N)_{<0} = \sum_{j=1}^{m} \tilde{q}_j \partial_q^{-1} \tilde{r}_j.$$

If L, q_j, r_j satisfies the constrained q-mKP dynamics (2.25, $k = 1$) and z_0 is the zero term in the dressing operator Z such that $L = Z\partial_q Z^{-1}$, according to Theorem 2.2, the transformed Lax operator $\tilde{L}$ satisfies the q-KP dynamics $\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 0}, \tilde{L}]$. Further, $\tilde{q}_j$, $\tilde{r}_j$ are the new (adjoint) eigenfunctions.

3. SE SYMMETRIES AND MIURA LINKS: UNCONSTRAINED CASE

The SE symmetry flow [3,22–24] is a one parameter symmetry group of the Lax equations generated by the squared eigenfunctions, i.e. the product of the eigenfunctions and their adjoints. We denote the corresponding group parameter by s and prove that the flow is the symmetry of q-KP and q-mKP hierarchies via its infinitesimal generator.

Theorem 3.1. *For $k = 0$ or 1, let L satisfy the Lax equations $L_{t_n} = [(L^n)_{\geq k}, L]$, and $\phi_1,\cdots,\phi_{m_1}$ and $\psi_1,\cdots,\psi_{m_1}$ be the (adjoint) eigenfunctions defined in (2.19). Then the SE symmetry flow defined by*

$$L_s = \left[\sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i\partial_q^k, L\right] \tag{3.1}$$

can commute with the q-deformed Lax hierarchy, i.e. $[\partial_s, \partial_{t_n}]L = 0$, which is equivalent to the zero curvature equation

$$M_{ns} - M'_{t_n} = [M', M_n], \tag{3.2}$$

where $M_n = (L^n)_{\geq k}$ and $M' = \sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i\partial_q^k$.

Proof. The Lax equation $L_s = [M', L]$ leads to $L^n_s = [M', L^n]$, then $M_{ns} = (L^n_s)_{\geq k} = [M', L^n]_{\geq k} = [M', (L^n)_{\geq k}]_{\geq k} + [M', (L^n)_{<k}]_{\geq k} = [M', M_n]_{\geq k}$. We obtain

$$M_{ns} = [M', M_n]_{\geq k}. \tag{3.3}$$

On the other hand, from the formula

$$(\partial_q^{-1}A)_{<0} = \partial_q^{-1}(A^*)_0 + \partial_q^{-1}(A)_{<0}, \tag{3.4}$$

we have

$$\begin{aligned}
&[M_n, M']_{<k}\\
&= \sum_i (M_n\phi_i\partial_q^{-1}\psi_i\partial_q^k)_{<k} - \sum_i(\phi_i\partial_q^{-1}\psi_i\partial_q^k M_n)_{<k}\\
&= \sum_i (M_n\phi_i\partial_q^{-1})_{<0}\psi_i\partial_q^k - \sum_i\phi_i(\partial_q^{-1}\psi_i\partial_q^k M_n\partial_q^{-k})_{<0}\partial_q^k\\
&\overset{(3.4)}{=} \sum_i (M_n\phi_i)_0\partial_q^{-1}\psi_i\partial_q^k - \sum_i\phi_i\partial_q^{-1}((\psi_i\partial_q^k M_n\partial_q^{-k})^*)_0\partial_q^k\\
&= \sum_i [M_n\phi_i]\partial_q^{-1}\psi_i\partial_q^k - \sum_i\phi_i\partial_q^{-1}[\partial_q^{-k^*}(L^n)^*_{\geq 1}\partial_q^{k^*}\psi_i]\partial_q^k\\
&= \sum_i (\phi_i\partial_q^{-1}\psi_i\partial_q^k)_{t_n},
\end{aligned}$$

which equals to

$$M'_{t_n} = [M_n, M']_{<k}. \tag{3.5}$$

Equations (3.3) and (3.5) lead to the zero curvature equation (3.2) and

$$L_{st_n} - L_{t_ns} = [M', L]_{t_n} - [M_n, L]_s = [M'_{t_n} - M_{ns} - [M', M_n], L] = 0.$$

As in Theorem 3.1, s is the group parameter of the SE symmetry flows. We will introduce conditions of ϕ and ψ corresponding to the parameter s,

$$\phi_s = \sum_i \phi_i\widehat{\Omega}(\psi_i, \phi^{(k)}), \qquad \psi_s = \sum_i \psi_i[\theta\Omega(\psi^{(k)}, \phi_i)], \quad k = 0, 1. \tag{3.6}$$

Further,

Lemma 3.1. *The SE symmetry in (3.1) is the compatible condition of linear problems:*

$$\phi_{t_n} = [(L^n)_{\geq k}\phi], \quad \phi_s = \sum_i \phi_i\widehat{\Omega}(\psi_i, \phi^{(k)}), \tag{3.7}$$

and

$$\psi_{t_n} = -[\partial_q^{*(-k)}(L^n)^*_{\geq k}\partial_q^{*k}\psi], \quad \psi_s = \sum_i \psi_i[\theta\Omega(\psi^{(k)}, \phi_i)], \quad k = 0, 1, \tag{3.8}$$

i.e. $\phi_{t_n s} = \phi_{st_n}$ *and* $\psi_{t_n s} = \psi_{st_n}$.

When $k = 1$, one can deduce by $L_s = \left[\sum_i \phi_i \partial_q^{-1} \psi_i \partial_q, L\right]$,

$$z_{0s} = \sum_i \phi_i [\theta^{-1} \psi_i] z_0, \tag{3.9}$$

where z_0 is the zero order term of the dressing operator Z.

Next we discuss the Miura links of the SE symmetries, which are given in the next two theorems.

Theorem 3.2. *(Reverse-Miura transformation: q-KP → q-mKP) Let ϕ and ψ be the (adjoint) eigenfunctions also satisfying (3.6, $k = 0$), i.e. $\phi_s = \sum_i \phi_i \widehat{\Omega}(\psi_i, \phi)$ and $\psi_s = \sum_i \psi_i [\theta \Omega(\psi, \phi_i)]$. And the SE symmetry of the q-KP hierarchy is given by $L_s = \left[\sum_i \phi_i \partial_q^{-1} \psi_i, L\right]$. Then under the reverse-Miura transformation T_α or T_β in Theorem 2.1, $\tilde{L}$ also satisfies the SE symmetry*

$$\tilde{L}_s = \left[\sum_i \tilde{\phi}_i \partial_q^{-1} \tilde{\psi}_i \partial_q, \tilde{L}\right],$$

where $\tilde{L}$, $\tilde{\phi}_i$ and $\tilde{\psi}_i$ are defined in Theorem 2.1.

Proof. For the case T_α, by using Theorem 3.1, we have

$$\begin{aligned}
\tilde{L}_s &= [\phi^{-1} L \phi]_s = -[\phi^{-1}\phi_s, \tilde{L}] + \phi^{-1} L_s \phi \\
&= \left[\sum_i \phi^{-1}\phi_i [\theta^{-1}\tilde{\psi}_i], \tilde{L}\right] + \phi^{-1}\left[\sum_i \phi_i \partial_q^{-1}\psi_i, L\right]\phi \\
&= \left[\sum_i \phi^{-1}\phi_i[\theta^{-1}\tilde{\psi}_i] + \phi^{-1}\phi_i\partial_q^{-1}\psi_i\phi, \tilde{L}\right] \\
&= \left[\sum_i \tilde{\phi}_i[\theta^{-1}\tilde{\psi}_i] - \tilde{\phi}_i\partial_q^{-1}[\theta^{-1}\tilde{\psi}_i]_{\partial_q}, \tilde{L}\right] \\
&= \left[\sum_i \tilde{\phi}_i\partial_q^{-1}(\partial_q[\theta^{-1}\tilde{\psi}_i] - [\theta^{-1}\tilde{\psi}_i]_{\partial_q}), \tilde{L}\right] = \left[\sum_i \tilde{\phi}_i\partial_q^{-1}\tilde{\psi}_i\partial_q, \tilde{L}\right].
\end{aligned}$$

For the case T_β, also by using Theorem 3.1, we have

$$\begin{aligned}
\tilde{L}_s &= (\partial_q^{-1}\psi L^n \psi^{-1}\partial_q)_s = \partial_q^{-1}(\psi L \psi^{-1})_s \partial_q \\
&= \partial_q^{-1}[\psi^{-1}\psi_s, \psi L \psi^{-1}]\partial_q + \partial_q^{-1}\psi L_s \psi^{-1}\partial_q \\
&= \sum_i \partial_q^{-1}[\psi^{-1}\psi_i[\theta\Omega(\psi, \phi_i)], \psi L \psi^{-1}]\partial_q + \sum_i \partial_q^{-1}\psi[\phi_i\partial_q^{-1}\psi_i, L]\psi^{-1}\partial_q \\
&= \sum_i [\partial_q^{-1}\tilde{\psi}_i[\theta\tilde{\phi}_i]\partial_q, \tilde{L}] + \sum_i [\partial_q^{-1}\psi\phi\partial_q^{-1}\tilde{\psi}_i\partial_q, \tilde{L}] \\
&= \sum_i [\partial_q^{-1}([\theta\tilde{\phi}_i] + [\partial_q\tilde{\phi}_i]\partial_q^{-1})\tilde{\psi}_i\partial_q, \tilde{L}] \\
&= \sum_i [\partial_q^{-1}\partial_q\tilde{\phi}_i\partial_q^{-1}\tilde{\psi}_i\partial_q, \tilde{L}] = \sum_i [\tilde{\phi}_i\partial_q^{-1}\tilde{\psi}_i\partial_q, \tilde{L}].
\end{aligned}$$

Theorem 3.3. *(Miura transformation: q-mKP → q-KP) Let z_0 be a function satisfying (3.9), i.e., $z_{0s} = \sum_i \phi_i[\theta^{-1}\psi_i]z_0$. The SE symmetry of the q-mKP hierarchy is given by $L_s = \left[\sum_i \phi_i \partial_q^{-1}\psi_i\partial_q, L\right]$. Then under the Miura transformation T_μ or T_ν defined in Theorem 2.2, $\tilde{L}$ also satisfies the SE symmetry*

$$\tilde{L}_s = \left[\sum_i \tilde{\phi}_i\partial_q^{-1}\tilde{\psi}_i, \tilde{L}\right].$$

Here $\tilde{L}$, $\tilde{\phi}_i$ and $\tilde{\psi}_i$ are defined in Theorem 2.2.

Proof. For the case T_μ, by using Theorem 3.1, we have

$$\begin{aligned}
\tilde{L}_s &= (z_0^{-1} L z_0)_s = -[z_0^{-1} z_{0s}, \tilde{L}] + z_0^{-1} L_s z_0 \\
&= \sum_i [-z_0^{-1}\phi_i[\theta^{-1}\psi_i]z_0, \tilde{L}] + \sum_i z_0^{-1}[\phi_i\partial_q^{-1}\psi_i\partial_q, L]z_0 \\
&= \sum_i [z_0^{-1}\phi_i(-[\theta^{-1}\psi_i] + \partial_q^{-1}\psi_i\partial_q)z_0, \tilde{L}] \\
&= \sum_i [z_0^{-1}\phi_i\partial_q^{-1}[\partial_q^{*}\psi_i]z_0, \tilde{L}] = \sum_i [\tilde{\phi}_i\partial_q^{-1}\tilde{\psi}_i.\tilde{L}].
\end{aligned}$$

For the case T_v, also by using Theorem 3.1, we have

$$\begin{aligned}\tilde{L}_s &= ([\theta z_0]^{-1}\partial_q L\partial_q^{-1}[\theta z_0])_s = -[[\theta z_0]^{-1}[\theta z_0]_s, \tilde{L}] + [\theta z_0]^{-1}\partial_q L_s\partial_q^{-1}[\theta z_0]\\ &= [-[\theta z_0]^{-1}[\theta z_0]_s, \tilde{L}] + \sum_i [\theta z_0]^{-1}\partial_q[\phi_i\partial_q^{-1}\psi_i\partial_q, L]\partial_q^{-1}[\theta z_0]\\ &= \sum_i[-[\theta z_0]^{-1}[\theta\phi_i]\psi_i[\theta z_0], \tilde{L}] + \sum_i[[\theta z_0]^{-1}\partial_q\phi_i\partial_q^{-1}\psi_i[\theta z_0], \tilde{L}]\\ &= \sum_i[[\theta z_0]^{-1}(-[\theta\phi_i] + \partial_q\phi_i\partial_q^{-1})\psi_i[\theta z_0], \tilde{L}]\\ &= \sum_i[[\theta z_0]^{-1}[\partial_q\phi_i]\partial_q^{-1}\psi_i[\theta z_0], \tilde{L}] = \sum_i[\tilde{\phi}_i\partial_q^{-1}\tilde{\psi}_i, \tilde{L}].\end{aligned}$$

4. SE SYMMETRIES AND MIURA LINKS: CONSTRAINED CASE

The SE symmetries for the constrained q-KP and q-mKP hierarchies are given in the next theorem.

Theorem 4.1. *Let $\Omega(.,.)$ and $\widehat{\Omega}(.,.)$ be defined in (2.20) and (2.21). For $k = 0$ or 1, assuming the (adjoint) eigenfunctions ϕ_i, ψ_i, $i = 1, \ldots, m_1$ be defined in (2.19). If $\phi_1, \ldots, \phi_{m_1}$, and $\psi_1, \ldots, \psi_{m_1}$ satisfy*

$$\left[(L^N)_{\geq 0}\phi_i\right] + \sum_{j=1}^{m} q_j\widehat{\Omega}(r_j, \phi_i^{(k)}) = \lambda_i\phi_i, \quad \left[\partial_q^{-k^*}(L^N)^*_{\geq 0}\partial_q^{k^*}\psi_i\right] - \sum_{j=1}^{m}\left[\theta\Omega(\psi_i^{(k)}, q_j)\right]r_j = \lambda_i\psi_i, \tag{4.1}$$

where $\lambda_i \in \mathbb{C}$, and in turn for $k = 0, 1$, $\phi_i^{(k)}$ denotes ϕ_i or $\phi_{i\partial_q}$, $\psi_i^{(k)}$ denotes ψ_i or $\psi_{i\partial_q^*}$, $q_j^{(k)}$ denotes q_j or $q_{j\partial_q}$ and $r_j^{(k)}$ denotes r_j or $r_{j\partial_q^*}$. The constraint (2.24) is invariant under the SE symmetry flow,

$$L_s = \left[\sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i\partial_q^k, L\right], \quad q_{js} = \sum_{i=1}^{m_1}\phi_i\widehat{\Omega}(\psi_i, q_j^{(k)}), \quad r_{js} = \sum_{i=1}^{m_1}[\theta\Omega(r_j^{(k)}, \phi_i)]\psi_i. \tag{4.2}$$

Proof. By the direct computation,

$$\partial_q^{-1}\psi\partial_q^k\phi\partial_q^{-1} = \widehat{\Omega}(\psi, \phi^{(k)})\partial_q^{-1} - \partial_q^{-1}[\theta\Omega(\psi^{(k)}, \phi)], \quad k = 0, 1. \tag{4.3}$$

With

$$\begin{aligned}((L^N)_{<k})_s &= \left[\sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i\partial_q^k, L^N\right]_{<k}\\ &= \left[\sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i\partial_q^k, (L^N)_{\geq k}\right]_{<k} + \left[\sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i\partial_q^k, (L^N)_{<k}\right]_{<k},\end{aligned}$$

$$\begin{aligned}\left[\sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i\partial_q^k, (L^N)_{\geq k}\right]_{<k} &= \left(\sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i\partial_q^k(L^N)_{\geq k} - \sum_{i=1}^{m_1}(L^N)_{\geq k}\phi_i\partial_q^{-1}\psi_i\partial_q^k\right)_{<k}\\ &= \sum_{i=1}^{m_1}\phi_i\left(\partial_q^{-1}\psi_i\partial_q^k(L^N)_{\geq k}\partial_q^{-k}\right)_{<0}\partial_q^k - \sum_{i=1}^{m_1}\left((L^N)_{\geq k}\phi_i\partial_q^{-1}\psi_i\right)_{<0}\partial_q^k\\ &\overset{(3.4)}{=} \sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\left(\psi_i\partial_q^k(L^N)_{\geq k}\partial_q^{-k}\right)^*_0\partial_q^k - \sum_{i=1}^{m_1}\left[(L^N)_{\geq k}\phi_i\right]\partial_q^{-1}\psi_i\partial_q^k\\ &= \sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\left[\partial_q^{-k^*}(L^N)^*_{\geq k}\partial_q^{k^*}\psi_i\right]\partial_q^k - \sum_{i=1}^{m_1}\left[(L^N)_{\geq k}\phi_i\right]\partial_q^{-1}\psi_i\partial_q^k\end{aligned}$$

and

$$\begin{aligned}\left[\sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i\partial_q^k, (L^N)_{<k}\right]_{<k} &\overset{(2.24)}{=} \left[\sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i\partial_q^k, \sum_{j=1}^{m} q_j\partial_q^{-1}r_j\partial_q^k\right]_{<k}\\ &= \left(\sum_{i,j}\phi_i\partial_q^{-1}\psi_i\partial_q^k q_j\partial_q^{-1}r_j\partial_q^k - \sum_{i,j} q_j\partial_q^{-1}r_j\partial_q^k\phi_i\partial_q^{-1}\psi_i\partial_q^k\right)_{<k}\end{aligned}$$

$$\begin{aligned}
&\overset{(4.3)}{=}\left(\sum_{i,j}\phi_i\widehat{\Omega}(\psi_i,q_j^{(k)})\partial_q^{-1}r_j\partial_q^k-\sum_{i,j}\phi_i\partial_q^{-1}[\theta\Omega(\psi_i^{(k)},q_j)]r_j\partial_q^k\right)_{<k}\\
&-\left(\sum_{i,j}q_j\widehat{\Omega}(r_j,\phi_i^{(k)})\partial_q^{-1}\psi_i\partial_q^k-\sum_{i,j}q_j\partial_q^{-1}[\theta\Omega(r_j^{(k)},\phi_i)]\psi_i\partial_q^k\right)_{<k}\\
&=\sum_{i,j}\phi_i\widehat{\Omega}(\psi_i,q_j^{(k)})\partial_q^{-1}r_j\partial_q^k-\sum_{i,j}\phi_i\partial_q^{-1}[\theta\Omega(\psi_i^{(k)},q_j)]r_j\partial_q^k\\
&-\sum_{i,j}q_j\widehat{\Omega}(r_j,\phi_i^{(k)})\partial_q^{-1}\psi_i\partial_q^k+\sum_{i,j}q_j\partial_q^{-1}[\theta\Omega(r_j^{(k)},\phi_i)]\psi_i\partial_q^k,
\end{aligned}$$

by the Eq. (4.1), we have

$$\begin{aligned}
((L^N)_{<k})_s&=\sum_{i,j}\phi_i\widehat{\Omega}(\psi_i,q_j^{(k)})\partial_q^{-1}r_j\partial_q^k+\sum_{i,j}q_j\partial_q^{-1}[\theta\Omega(r_j^{(k)},\phi_i)]\psi_i\partial_q^k\\
&=\sum_j q_{js}\partial_q^{-1}r_j\partial_q^k+\sum_j q_j\partial_q^{-1}r_{js}\partial_q^k=\left(\sum_j q_j\partial_q^{-1}r_j\partial_q^k\right)_s.
\end{aligned}$$

The Miura links of the SE symmetries in the constrained case are listed in the two theorems below.

Theorem 4.2. *(Reverse-Miura transformation: constrained q-KP → constrained q-mKP) Let $L=L_{qKP}$ satisfy the constraint $(L^N)_{<0}=\sum_{j=1}^m q_j\partial_q^{-1}r_j$. Then for any functions ϕ and ψ satisfying (3.6, $k=0$), under the reverse-Miura transformation T_α or T_β defined in Theorems 2.1 and 2.3, if $\phi_1,\dots,\phi_{m_1}$, $\psi_1,\dots,\psi_{m_1}$, $q_1,\dots,q_m$ and $r_1,\dots,r_m$ satisfy*

$$\left[(L^N)_{\geq 0}\phi_i\right]+\sum_{j=1}^m q_j\widehat{\Omega}(r_j,\phi_i)=\lambda_i\phi_i,\ [(L^N)_{\geq 0}^*\psi_i]-\sum_{j=1}^m[\theta\Omega(\psi_i,q_j)]r_j=\lambda_i\psi_i, \tag{4.4}$$

$$L_s=\left[\sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i,L\right],\quad q_{js}=\sum_{i=1}^{m_1}\phi_i\widehat{\Omega}(\psi_i,q_j),\quad r_{js}=\sum_{i=1}^{m_1}[\theta\Omega(r_j,\phi_i)]\psi_i, \tag{4.5}$$

then these two conditions keep invariant,

$$\left[(\tilde{L}^N)_{\geq 1}\tilde{\phi}_i\right]+\sum_{j=1}^{m+1}\tilde{q}_j\widehat{\Omega}(\tilde{r}_j,\tilde{\phi}_{i\partial_q})=\lambda_i\tilde{\phi}_i,\ \left[\partial_q^{-1^*}(\tilde{L}^N)_{\geq 1}^*\partial_q^*\tilde{\psi}_i\right]-\sum_{j=1}^{m+1}[\theta\Omega(\tilde{\psi}_{i\partial_q^*},\tilde{q}_j)]\tilde{r}_j=\lambda_i\tilde{\psi}_i, \tag{4.6}$$

$$\tilde{L}_s=\left[\sum_{i=1}^{m_1}\tilde{\phi}_i\partial_q^{-1}\tilde{\psi}_i\partial_q,\tilde{L}\right],\quad \tilde{q}_{js}=\sum_{i=1}^{m_1}\tilde{\phi}_i\widehat{\Omega}(\tilde{\psi}_i,\tilde{q}_{j\partial_q}),\quad \tilde{r}_{js}=\sum_{i=1}^{m_1}[\theta\Omega(\tilde{r}_{j\partial_q^*},\tilde{\phi}_i)]\tilde{\psi}_i. \tag{4.7}$$

Proof. We give a proof of the case T_α, and the proof of the case T_β is similar. With

$$\begin{aligned}
\sum_{j=1}^m\tilde{q}_j\widehat{\Omega}(\tilde{r}_j,\tilde{\phi}_{i\partial_q})&=\sum_{j=1}^m\phi^{-1}q_j\widehat{\Omega}(-[\theta\widehat{\Omega}(r_j,\phi)],\phi_{i\partial_q})\\
&\overset{(2.21)}{=}-\sum_{j=1}^m\phi^{-1}q_j\Omega(r_j,\phi)\phi^{-1}\phi_i+\sum_{j=1}^m\phi^{-1}q_j\Omega(r_j\phi,\phi^{-1}\phi_i)\\
&=-\sum_{j=1}^m\phi^{-2}q_j\Omega(r_j,\phi)\phi_i+\sum_{j=1}^m\phi^{-1}q_j\Omega(r_j,\phi_i),
\end{aligned}$$

and

$$\tilde{q}_{m+1}\widehat{\Omega}(\tilde{r}_{m+1},\tilde{\phi}_{i\partial_q})=\phi^{-1}\left[(L^N)_{\geq 0}\phi\right]\phi^{-1}\phi_i+\phi^{-1}\sum_{j=1}^m q_j\widehat{\Omega}(r_j,\phi)\phi^{-1}\phi_i,$$

we have

$$\sum_{j=1}^{m+1}\tilde{q}_j\widehat{\Omega}(\tilde{r}_j,\tilde{\phi}_{i\partial_q})=\phi^{-1}\left[(L^N)_{\geq 0}\phi\right]\phi^{-1}\phi_i+\sum_{j=1}^m\phi^{-1}q_j\Omega(r_j,\phi_i).$$

On the other side,

$$\left[(\tilde{L}^N)_{\geq 1}\tilde{\phi}_i\right]=\left[(\phi^{-1}L^N\phi)_{\geq 1}\phi^{-1}\phi_i\right]=\phi^{-1}\left[(L^N)_{\geq 0}\phi_i\right]-\phi^{-1}\left[(L^N)_{\geq 0}\phi\right]\phi^{-1}\phi_i,$$

we have the first equation in (4.6)

$$\left[(\tilde{L}^N)_{\geq 1}\tilde{\phi}_i\right]+\sum_{j=1}^{m+1}\tilde{q}_j\widehat{\Omega}(\tilde{r}_j,\tilde{\phi}_{i\partial_q})=\phi^{-1}\left[(L^N)_{\geq 0}\phi_i\right]+\sum_{j=1}^{m}\phi^{-1}q_j\Omega(r_j,\phi_i)=\lambda_i\tilde{\phi}_i.$$

We now prove the second equation in (4.6). With

$$\begin{aligned}\left[\partial_q^{-1*}(\tilde{L}^N)^*_{\geq 1}\partial_q^*\tilde{\psi}_i\right] &=\left[\partial_q^{-1*}(\phi^{-1}L^N\phi)^*_{\geq 1}\phi\psi_i\right]\\ &=\left[\partial_q^{-1*}\phi(L^N)^*_{\geq 0}\phi^{-1}\phi\psi_i\right]-\left[\partial_q^{-1*}[(L^N)_{\geq 0}\phi]\phi^{-1}\phi\psi_i\right]\\ &=\left[\partial_q^{-1*}\phi\sum_{j=1}^{m}[\theta\Omega(\psi_i,q_j)]r_j\right]+\left[\partial_q^{-1*}\phi\lambda_i\psi_i\right]-\left[\partial_q^{-1*}[(L^N)_{\geq 0}\phi]\psi_i\right]\end{aligned}$$

and

$$\begin{aligned}&-\sum_{j=1}^{m+1}[\theta\Omega(\tilde{\psi}_{i\partial_q^*},\tilde{q}_j)]\tilde{r}_j\\ &=\sum_{j=1}^{m}[\theta\Omega(\psi_i\phi,\phi^{-1}q_j)][\theta\widehat{\Omega}(r_j,\phi)]-\left[\theta\Omega\left(\psi_i\phi,\phi^{-1}\left[(L^N)_{\geq 0}\phi\right]+\phi^{-1}\sum_{j=1}^{m}q_j\widehat{\Omega}(r_j,\phi)\right)\right]\\ &=\sum_{j=1}^{m}[\theta\Omega(\psi_i,q_j)][\theta\widehat{\Omega}(r_j,\phi)]-\left[\theta\Omega\left(\psi_i,\left[(L^N)_{\geq 0}\phi\right]\right)\right]-\left[\theta\Omega\left(\psi_i,\sum_{j=1}^{m}q_j\widehat{\Omega}(r_j,\phi)\right)\right],\end{aligned}$$

we have the second equation in (4.6)

$$\left[\partial_q^{-1*}(\tilde{L}^N)^*_{\geq 1}\partial_q^*\tilde{\psi}_i\right]-\sum_{j=1}^{m+1}[\theta\Omega(\tilde{\psi}_{i\partial_q^*},\tilde{q}_j)]\tilde{r}_j=\left[\partial_q^{-1*}\phi\lambda_i\psi_i\right]=\lambda_i\tilde{\psi}_i,$$

by using the equality

$$\sum_{j=1}^{m}[\theta\Omega(\psi_i,q_j)][\theta\widehat{\Omega}(r_j,\phi)]+\left[\partial_q^{-1*}\phi\sum_{j=1}^{m}[\theta\Omega(\psi_i,q_j)]r_j\right]-\left[\theta\Omega\left(\psi_i,\sum_{j=1}^{m}q_j\widehat{\Omega}(r_j,\phi)\right)\right]=0.$$

The first equation in (4.7) has been proved in Theorem 3.2. We now prove the second equation in (4.7). For $j=1,\ldots,m$, with

$$\begin{aligned}\tilde{q}_{js} &=-\phi^{-2}q_j\phi_s+\phi^{-1}q_{js}\\ &=-\sum_{i=1}^{m_1}\phi^{-2}q_j\phi_i\widehat{\Omega}(\psi_i,\phi)+\phi^{-1}\sum_{i=1}^{m_1}\phi_i\widehat{\Omega}(\psi_i,q_j)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{m_1}\tilde{\phi}_i\widehat{\Omega}(\tilde{\psi}_i,\tilde{q}_{j\partial_q}) &=\sum_{i=1}^{m_1}\phi^{-1}\phi_i\widehat{\Omega}(-[\theta\Omega(\psi_i,\phi)],(\phi^{-1}q_j)_{\partial_q})\\ &\overset{(2.21)}{=}\sum_{i=1}^{m_1}\phi^{-1}\phi_i\widehat{\Omega}(\psi_i,q_j)-\sum_{i=1}^{m_1}\phi^{-1}\phi_i\Omega(\psi_i,\phi)\phi^{-1}q_j,\end{aligned}$$

we have the second equation in (4.7).

For $j=m+1$, with

$$\begin{aligned}\tilde{q}_{(m+1)s} &=-\phi^{-2}\phi_s[L^n\phi]+\phi^{-1}[L^n_s\phi]+\phi^{-1}[L^n\phi_s]\\ &=-\phi^{-2}\sum_{i=1}^{m_1}\phi_i\widehat{\Omega}(\psi_i,\phi)[L^n\phi]+\phi^{-1}\sum_{i=1}^{m_1}[\phi_i\partial_q^{-1}\psi_iL^n\phi]\\ &-\phi^{-1}\sum_{i=1}^{m_1}[L^n\phi_i\partial_q^{-1}\psi_i\phi]+\phi^{-1}\sum_{i=1}^{m_1}[L^n\phi_i\widehat{\Omega}(\psi_i,\phi)]\\ &=-\phi^{-2}\sum_{i=1}^{m_1}\phi_i\widehat{\Omega}(\psi_i,\phi)[L^n\phi]+\phi^{-1}\sum_{i=1}^{m_1}[\phi_i\partial_q^{-1}\psi_iL^n\phi]\end{aligned}$$

and

$$\sum_{i=1}^{m_1}\tilde{\phi}_i\widehat{\Omega}(\tilde{\psi}_i,\tilde{q}_{(m+1)\partial_q}) = \sum_{i=1}^{m_1}\phi^{-1}\phi_i\widehat{\Omega}(-[\theta\Omega(\psi_i,\phi)],(\phi^{-1}[L^n\phi])_{\partial_q}) \overset{(2.21)}{=} \sum_{i=1}^{m_1}\phi^{-1}\phi_i\widehat{\Omega}(\psi_i,[L^n\phi]) - \sum_{i=1}^{m_1}\phi^{-1}\phi_i\Omega(\psi_i,\phi)\phi^{-1}[L^n\phi],$$

we have the second equation in (4.7).

For $j = 1, \ldots, m$, with

$$\tilde{r}_{js} = -[\theta\widehat{\Omega}(r_j,\phi)]_s = -[\theta\widehat{\Omega}(r_{js},\phi)] - [\theta\widehat{\Omega}(r_j,\phi_s)] = -\sum_{i=1}^{m_1}[\theta\widehat{\Omega}([\theta\Omega(r_j,\phi_i)]\psi_i,\phi)] - \sum_{i=1}^{m_1}[\theta\,\widehat{\Omega}(r_j,\phi_i\widehat{\Omega}(\psi_i,\phi))]$$

and

$$\sum_{i=1}^{m_1}[\theta\Omega(\tilde{r}_{j\partial_q^*},\tilde{\phi}_i)]\tilde{\psi}_i = -\sum_{i=1}^{m_1}[\theta\Omega(r_j\phi,\phi^{-1}\phi_i)][\theta\Omega(\psi_i,\phi)],$$

by taking the adjoint q-difference operator ∂_q^*, we have the third equation in (4.7).

Since $r_{m+1} = 1$, it is trivial that the third equation in (4.7) holds.

Theorem 4.3. *(Miura transformation: constrained q-KP → constrained q-mKP) Let $L = L_{qmKP}$ satisfy the constraint $(L^N)_{<1} = \sum_{j=1}^{m} q_j\partial_q^{-1}r_j\partial_q$. Then for any function z_0 satisfying (3.9), under the Miura transformation T_μ or T_ν defined in Theorems 2.2 and 2.4, if $\phi_1, \cdots, \phi_{m_1}$, $\psi_1, \cdots, \psi_{m_1}$, $q_1, \ldots, q_m$ and $r_1, \ldots, r_m$ satisfy*

$$[(L^N)_{\geq 1}\phi_i] + \sum_{j=1}^{m} q_j\widehat{\Omega}(r_j,\phi_{i\partial_q}) = \lambda_i\phi_i, \quad [\partial_q^{-1^*}(L^N)_{\geq 1}^*\partial_q^*\psi_i] - \sum_{j=1}^{m}[\theta\Omega(\psi_{i\partial_q^*},q_j)]r_j = \lambda_i\psi_i, \tag{4.8}$$

$$L_s = \left[\sum_{i=1}^{m_1}\phi_i\partial_q^{-1}\psi_i\partial_q,\, L\right], \quad q_{js} = \sum_{i=1}^{m_1}\phi_i\widehat{\Omega}(\psi_i, q_{j\partial_q}), \quad r_{js} = \sum_{i=1}^{m_1}[\theta\Omega(r_{j\partial_q^*},\phi_i)]\psi_i, \tag{4.9}$$

then these two conditions keep invariant,

$$\left[(\tilde{L}^N)_{\geq 0}\tilde{\phi}_i\right] + \sum_{j=1}^{m}\tilde{q}_j\widehat{\Omega}(\tilde{r}_j,\tilde{\phi}_i) = \lambda_i\tilde{\phi}_i, \quad [(\tilde{L}^N)_{\geq 0}^*\tilde{\psi}_i] - \sum_{j=1}^{m}[\theta\Omega(\tilde{\psi}_i,\tilde{q}_j)]\tilde{r}_j = \lambda_i\tilde{\psi}_i, \tag{4.10}$$

$$\tilde{L}_s = \left[\sum_{i=1}^{m_1}\tilde{\phi}_i\partial_q^{-1}\tilde{\psi}_i,\, \tilde{L}\right], \quad \tilde{q}_{js} = \sum_{i=1}^{m_1}\tilde{\phi}_i\widehat{\Omega}(\tilde{\psi}_i,\tilde{q}_j), \quad \tilde{r}_{js} = \sum_{i=1}^{m_1}[\theta\Omega(\tilde{r}_j,\tilde{\phi}_i)]\tilde{\psi}_i. \tag{4.11}$$

Proof. We give a proof of the case T_μ, and the proof of the case T_ν is similar. With

$$\sum_{j=1}^{m}\tilde{q}_j\widehat{\Omega}(\tilde{r}_j,\tilde{\phi}_i) = \sum_{j=1}^{m} z_0^{-1}q_j\widehat{\Omega}(z_0[\partial_q^* r_j], z_0^{-1}\phi_i) = \sum_{j=1}^{m} z_0^{-1}q_j\widehat{\Omega}([\partial_q^* r_j],\, \phi_i)$$

and

$$[(\tilde{L}^N)_{\geq 0}\tilde{\phi}_i] = [(z_0^{-1}L^N z_0)_{\geq 0}z_0^{-1}\phi_i] = [z_0^{-1}(L^N)_{\geq 0}z_0z_0^{-1}\phi_i] = [z_0^{-1}(L^N)_{\geq 1}\phi_i] + [z_0^{-1}(L^N)_0\phi_i] = -z_0^{-1}\sum_{j=1}^{m}q_j\widehat{\Omega}(r_j,\, \phi_{i\partial_q}) + \lambda_i z_0^{-1}\phi_i + z_0^{-1}\sum_{j=1}^{m}q_j[\theta^{-1}r_j]\phi_i,$$

we have the first equation in (4.10) by

$$([\theta^{-1}r_j]\phi_i)_{\partial_q} = r_j\phi_{i\partial_q} + [\theta^{-1}\partial_q r_j]\phi_i.$$

We now prove the second equation in (4.10). With

$$
\begin{aligned}
[(\tilde{L}^N)^*_{\geq 0}\tilde{\psi}_i] &= [(z_0^{-1}L^N z_0)^*_{\geq 0} z_0[\partial_q^*\psi_i]] = [(z_0(L^N)^*_{\geq 1} z_0^{-1} + z_0(L^N)^*_0 z_0^{-1}) z_0[\partial_q^*\psi_i]] \\
&= z_0[(L^N)^*_{\geq 1}[\partial_q^*\psi_i]] + z_0\sum_{j=1}^{m} q_j[\theta^{-1}r_j][\partial_q^*\psi_i] \\
&= z_0[\partial_q^*\partial_q^{-1*}(L^N)^*_{\geq 1}\partial_q^*\psi_i] + z_0\sum_{j=1}^{m} q_j[\theta^{-1}r_j][\partial_q^*\psi_i] \\
&= z_0\sum_{j=1}^{m}[\partial_q^*[\theta\Omega(\psi_{i\partial_q^*}, q_j)]r_j] + \lambda_i z_0[\partial_q^*\psi_i] + z_0\sum_{j=1}^{m} q_j[\theta^{-1}r_j][\partial_q^*\psi_i] \\
&= z_0\sum_{j=1}^{m}[\theta\Omega(\psi_{i\partial_q^*}, q_j)][\partial_q^* r_j] + \lambda_i z_0[\partial_q^*\psi_i]
\end{aligned}
$$

and

$$
\begin{aligned}
-\sum_{j=1}^{m}[\theta\Omega(\tilde{\psi}_i, \tilde{q}_j)]\tilde{r}_j &= -\sum_{j=1}^{m}[\theta\Omega(z_0[\partial_q^*\psi], z_0^{-1}q_j)] z_0[\partial_q^* r_j] \\
&= -z_0\sum_{j=1}^{m}[\theta\Omega([\partial_q^*\psi], q_j)][\partial_q^* r_j],
\end{aligned}
$$

we have the second equation in (4.10).

The first equation in (4.11) has been proved in Theorem 3.3. We now prove the second equation in (4.11). For $j = 1, \ldots, m$, with

$$
\begin{aligned}
\tilde{q}_{js} &= (z_0^{-1}q_j)_s = -z_0^{-2}q_j z_{0s} + z_0^{-1}q_{js} \\
&= -z_0^{-1}q_j\sum_{i=1}^{m_1}\phi_i[\theta^{-1}\psi_i] + z_0^{-1}\sum_{i=1}^{m_1}\phi_i\widehat{\Omega}(\psi_i, q_{j\partial_q})
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{m_1}\tilde{\phi}_i\widehat{\Omega}(\tilde{\psi}_i, \tilde{q}_j) &= \sum_{i=1}^{m_1} z_0^{-1}\phi_i\Omega(z_0[\partial_q^*\psi_i], z_0^{-1}q_j) \\
&\overset{(2.21)}{=} z_0^{-1}\sum_{i=1}^{m_1}\phi_i\Omega(\psi_i, q_{j\partial_q}) - z_0^{-1}\sum_{i=1}^{m_1}\phi_i[\theta^{-1}\psi_i]q_j,
\end{aligned}
$$

we have the second equation in (4.11).

For $j = 1, \ldots, m$, with

$$
\begin{aligned}
\tilde{r}_{js} &= (z_0[\partial_q^* r_j])_s = z_{0s}[\partial_q^* r_j] + z_0[\partial_q^* r_{js}] \\
&= \sum_{i=1}^{m_1}\phi_i[\theta^{-1}\psi_i] z_0[\partial_q^* r_j] + z_0\sum_{i=1}^{m_1}[\partial_q^*[\theta\Omega(r_{j\partial_q^*}, \phi_i)]\psi_i] \\
&= z_0\sum_{i=1}^{m_1}[\theta\Omega(r_{j\partial_q^*}, \phi_i)][\partial_q^*\psi_i]
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{m_1}[\theta\Omega(\tilde{r}_j, \tilde{\phi}_i)]\tilde{\psi}_i &= \sum_{i=1}^{m_1}[\theta\Omega(z_0[\partial_q^* r_j], z_0^{-1}\phi_i)] z_0[\partial_q^*\psi_i] \\
&= z_0\sum_{i=1}^{m_1}[\theta\Omega([\partial_q^* r_j], \phi_i)][\partial_q^*\psi_i],
\end{aligned}
$$

we have the third equation in (4.11).

Remark: Theorems 4.2 and 4.3 show that the additional conditions (4.1) and (4.2) that keep the constaint (2.24) invariant under the SE symmetries are consistent with the Miura and the reverse Miura transformations. By considering Theorems 3.2 and 3.3, we can further find that the SE symmetries are consistent with the Miura and reverse Miura transformations, for the constrained case. In the meanwhile, the conditions (3.6), (3.9), (4.1) and (4.2) are required in these constructions.

5. ADDITIONAL SYMMETRIES AND MIURA LINKS

It is proved [3,6] that the SE symmetries defined by the (adjoint) wave functions in the q-deformed case can be viewed as the generating functions of the additional symmetries. Thus in this section, we will compute the Miura links directly, which can confirm the results of Miura links of the SE symmetries.

The additional symmetries [29,31] of the q-KP and q-mKP hierarchies are defined as follows. For each pair integers m and l, the additional symmetry flows of q-KP and q-mKP hierarchies are defined respectively,

$$\begin{cases} \partial^*_{ml}W = -(M_q^m L^l)_{<0}W, & q\text{-KP}, \\ \partial^*_{ml}Z = -(M_q^m L^l)_{<1}Z, & q\text{-mKP}. \end{cases} \tag{5.1}$$

with some additional variable t^*_{ml} and its derivative $\partial^*_{ml} \equiv \frac{\partial}{\partial t^*_{ml}}$. Here the Orlov-Schulman operator M_q [26,29,31] of the q-KP and q-mKP hierarchies is

$$M_q = \begin{cases} W\Gamma_q W^{-1}, & q\text{-KP}, \\ Z\Gamma_q Z^{-1}, & q\text{-mKP}, \end{cases} \tag{5.2}$$

where the operator Γ_q is defined by

$$\Gamma_q = \sum_{i=1}^{\infty}\left[it_i + \frac{(1-q)^i}{1-q^i}x_i \right]. \tag{5.3}$$

Γ_q commutes with the operator $\partial_{t_m} - \partial_q^m$, i.e. $[\partial_{t_m} - \partial_q^m, \Gamma_q] = 0$. By dressing it in q-KP and q-mKP hierarchies respectively, we have the equation $[\partial_{t_m} - (L^m)_{\geq k}, M_q] = 0$ or

$$\partial_{t_m} M_q = [(L^m)_{\geq k}, M_q], \quad k = 0, 1. \tag{5.4}$$

The additional symmetry flow ∂^*_{ml} acts on the (adjoint) eigenfunctions of the q-KP hierarchy are defined as follows

$$[\partial^*_{ml}\phi] = [(M^m L^l)_{\geq 0}\phi], \quad [\partial^*_{ml}\psi] = -[(M^m L^l)^*_{\geq 0}\psi], \tag{5.5}$$

and for z_0 is the zero order term in the dressing operator Z of q-mKP hierarchy,

$$[\partial^*_{ml} z_0] = -(M_q^m L^l)_0 z_0. \tag{5.6}$$

Then the Miura links of the additional symmetries are given in the next two theorems.

Theorem 5.1. *(Reverse-Miura transformation with additional symmetries: q-KP → q-mKP) Let* $L = L_{qKP}$ *have the dressing form* $L = W\partial_q W^{-1}$. *Let* ϕ *and* ψ *be the (adjoint) eigenfuntions respectively. Then under the reverse-Miura transformation* T_α *or* T_β *defined in Theorem 2.1,*

(i) $W \xrightarrow{\alpha} Z = \phi^{-1}W,\ M_q \xrightarrow{\alpha} \tilde{M}_q = \phi^{-1}M_q\phi;$

(ii) $W \xrightarrow{\beta} Z = \partial_q^{-1}\psi W,\ M_q \xrightarrow{\beta} \tilde{M}_q = \partial_q^{-1}\psi M_q \psi^{-1}\partial_q,$

one can find

$$\partial_{t_m}\tilde{M}_q = [(\tilde{L}^m)_{\geq 1}, \tilde{M}_q].$$

If ϕ *or* ψ *satisfies (5.5), and the dressing operator W satisfies the additional flows (5.1, k = 0), i.e.* $\partial^*_{ml}W = -(M^m L^l)_{<0}W$, *then in any case* T_α *or* T_β, *the tranformed dressing operator Z also satisfies the additional flows (5.1, k = 1), i.e.,*

$$\partial^*_{ml}Z = -(\tilde{M}_q^m \tilde{L}^l)_{\leq 0}Z.$$

Proof. From the definition of $\tilde{M}_q$, it can be proved that $\partial_{t_m}\tilde{M}_q = [(\tilde{L}^m)_{\geq 1}, \tilde{M}_q]$ holds.

Now we prove the additional flows (5.1, $k = 1$) for the transformed operator Z.

Case T_α.

$$\begin{aligned}\partial^*_{ml} Z &= [\partial^*_{ml}\phi^{-1}W] = [\partial^*_{ml}\phi^{-1}]W + \phi^{-1}[\partial^*_{ml}W]\\ &= -\phi^{-1}[\partial^*_{ml}\phi]\phi^{-1}W - \phi^{-1}(M_q^m L^l)_{<0}W\\ &\overset{(5.5)}{=} -\phi^{-1}[(M_q^m L^l)_{\geq 0}\phi]\phi^{-1}W - \phi^{-1}(M_q^m L^l\phi)_{<0}\phi^{-1}W\\ &= -(\phi^{-1}M_q^m L^l\phi)_{\leq 0}\phi^{-1}W = -(\tilde{M}_q^m\tilde{L}^l)_{\leq 0}Z.\end{aligned}$$

Case T_β.

Note that

$$\begin{aligned}-(\tilde{M}_q^m\tilde{L}^l)_{\leq 0}Z &= -(\partial_q^{-1}\psi M_q^m L^l\psi^{-1}\partial_q)_{\leq 0}Z\\ &= -\partial_q^{-1}\psi(M_q^m L^l)_{<0}\psi^{-1}\partial_q Z - \partial_q^{-1}[(M_q^m L^l)^*_{\geq 0}\psi]\psi^{-1}\partial_q Z\\ &= -\partial_q^{-1}\psi(M_q^m L^l)_{<0}W - \partial_q^{-1}[(M_q^m L^l)^*_{\geq 0}\psi]W,\end{aligned}$$

then we have

$$\begin{aligned}\partial^*_{ml} Z &= \partial^*_{ml}(\partial_q^{-1}\psi W) = \partial_q^{-1}[\partial^*_{ml}\psi]W + \partial_q^{-1}\psi[\partial^*_{ml}W]\\ &= \partial_q^{-1}[\partial^*_{ml}\psi]W - \partial_q^{-1}\psi(M_q^m L^l)_{<0}W\\ &\overset{(5.5)}{=} -\partial_q^{-1}[(M_q^m L^l)^*_{\geq 0}\psi]W - \partial_q^{-1}\psi(M_q^m L^l)_{<0}W\\ &= -(\tilde{M}_q^m\tilde{L}^l)_{\leq 0}Z.\end{aligned}$$

Theorem 5.2. *(Miura transformation with additional symmetries: q-mKP → q-KP) Let $L = L_{qmKP}$ have the dressing form $L = \partial_q Z^{-1}$. Let z_0 be the zero term in the dressing operator Z. Then under the Miura transformation T_μ or T_ν defined in Theorem 2.2,*

(i) $Z \xrightarrow{\mu} W = z_0^{-1}Z,\ M_q \xrightarrow{\mu} \tilde{M}_q = z_0^{-1}M_q z_0;$

(ii) $Z \xrightarrow{\nu} W = [\theta z_0]^{-1}\partial_q Z,\ M_q \xrightarrow{\nu} \tilde{M}_q = [\theta z_0]^{-1}\partial_q M_q \partial_q^{-1}[\theta z_0],$

one can obtain

$$\partial_{t_m}\tilde{M}_q = [(\tilde{L}^m)_{\geq 0}, \tilde{M}_q].$$

If the dressing operator Z satisfies the additional flows (5.1, k = 1), i.e. $\partial^*_{ml}Z = -(M^m L^l)_{\leq 0}Z$, then in any case T_μ or T_ν, the tranformed dressing operator W also satisfies the additional flows (5.1, k = 0), i.e.,

$$\partial^*_{ml}W = -(\tilde{M}_q^m\tilde{L}^l)_{<0}W.$$

Proof. From the definition of $\tilde{M}_q$, we have $\partial_{t_m}\tilde{M}_q = [(\tilde{L}^m)_{\geq 0}, \tilde{M}_q]$.

Now we prove the additional flows (5.1, $k = 0$) for the transformed operator W.

Case T_μ.

$$\begin{aligned}\partial^*_{ml}W &= \partial^*_{ml}(z_0^{-1}Z) = [\partial^*_{ml}z_0^{-1}]Z + z_0^{-1}[\partial^*_{ml}Z]\\ &= -z_0^{-1}[\partial^*_{ml}z_0]z_0^{-1}Z - z_0^{-1}(M_q^m L^l)_{\leq 0}Z\\ &\overset{(5.6)}{=} z_0^{-1}(M_q^m L^l)_{=0}z_0 z_0^{-1}Z - z_0^{-1}(M_q^m L^l)_{\leq 0}Z\\ &= -z_0^{-1}(M_q^m L^l)_{<0}Z\\ &= -(z_0^{-1}M_q^m L^l z_0)_{<0}z_0^{-1}Z = -(\tilde{M}_q^m\tilde{L}^l)_{<0}W.\end{aligned}$$

Case T_ν.

Note that

$$\begin{aligned}-(\tilde{M}_q^m\tilde{L}^l)_{<0}W &= -([\theta z_0]^{-1}\partial_q M_q^m L^l\partial_q^{-1}[\theta z_0])_{<0}[\theta z_0]^{-1}\partial_q Z\\ &= -[\theta z_0]^{-1}(\partial_q M_q^m L^l\partial_q^{-1})_{<0}\partial_q Z\\ &= -[\theta z_0]^{-1}[\partial_q(M_q^m L^l)_{=0}]Z - [\theta z_0]^{-1}\partial_q(M_q^m L^l)_{<0}Z,\end{aligned}$$

then we have

$$\begin{aligned}\partial^*_{ml}W &= \partial^*_{ml}([\theta z_0]^{-1}\partial_q Z) = \partial^*_{ml}([\theta z_0]^{-1})\partial_q Z + [\theta z_0]^{-1}\partial_q[\partial^*_{ml}Z]\\ &= -[\theta z_0]^{-1}[\partial^*_{ml}\theta z_0][\theta z_0]^{-1}\partial_q Z - [\theta z_0]^{-1}\partial_q(M_q^m L^l)_{\le 0}Z\\ &\overset{(5.6)}{=} [\theta z_0]^{-1}[\theta(M_q^m L^l)_{=0}][\theta z_0][\theta z_0]^{-1}\partial_q Z - [\theta z_0]^{-1}\partial_q(M_q^m L^l)_{=0}Z - [\theta z_0]^{-1}\partial_q(M_q^m L^l)_{<0}Z\\ &= -[\theta z_0]^{-1}[\partial_q(M_q^m L^l)_{=0}]Z - [\theta z_0]^{-1}\partial_q(M_q^m L^l)_{<0}Z = -(\tilde{M}_q^m \tilde{L}^l)_{<0}W.\end{aligned}$$

6. CONCLUSION AND DISCUSSION

We summarize the main results of the paper. In Theorem 3.1, we construct the SE symmetry for the q-KP and q-mKP hierarchies. In Theorems 3.2 and 3.3, with the conditions (3.6) and (3.9), we investigate the Miura links of the SE symmetries for the unconstrained case. For the constrained q-KP and q-mKP hierarchies with the condition (2.24) on the Lax operators, the conditions (4.1) and (4.2) are required to ensure the SE flows are consistent with the constraint (2.24) in Theorem 4.1. Theorems 4.2 and 4.3 reveal that with the previous conditions (3.6) and (3.9), the additional conditions (4.1) and (4.2) are maintained under the Miura links in the constrained case. Thus for the constrained case, as in the unconstrained case, the SE symmetries are consistent with the Miura and reverse-Miura transformations. Finally in Theorems 5.1 and 5.2, we show that additional symmetries are invariant under the Miura and reverse-Miura transformations, which confirm the previous results of the Miura links of the SE symmetries.

Our results show that for the q-KP and q-mKP hierarchies, there exists the SE symmetries and the Miura links of the SE symmetries, both in the unconstrained and constrained cases. The additional symmetries also coincide with the Miura links. These results indicate that the structures of the Miura links are closely related to the SE symmetries and additional symmetries in the integrable systems.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

ACKNOWLEDGMENT

The first author is supported by Scientific Research Level Improvement Quota Project of Capital University of Economics and Business.

REFERENCES

[1] M. Adler, E. Horozov, P. van Moerbeke, The solution to the q-KdV equation, Phys. Lett. A 242 (1998), 139–151.
[2] M. Adler, T. Shiota, P. van Moerbeke, A Lax representation for the vertex operator and the central extension, Commun. Math. Phys. 171 (1995), 547–588.
[3] H. Aratyn, E. Nissimov, S. Pacheva, Method of squared eigenfunction potentials in integrable hierarchies of KP type, Commun. Math. Phys. 193 (1998), 493–525.
[4] K. Chen, X. Deng, D.j. Zhang, Symmetry constraint of the differential-difference KP hierarchy and a second discretization of the ZS-AKNS system, J. Nonlin. Math. Phys. 24 (2017), 18–35.
[5] J. Cheng, Miura and auto-Backlund transformations for the q-deformed KP and q-deformed modified KP hierarchies, J. Nonlin. Math. Phys. 24 (2017), 7–19.
[6] J. Cheng, J. He, On the squared eigenfunction symmetry of the Toda lattice hierarchy, J. Math. Phys. 54 (2013), 023511.
[7] J. Cheng, J. He, S. Hu, The "ghost" symmetry of the BKP hierarchy, J. Math. Phys. 51 (2010), 053514.
[8] L.A. Dickey, On additional symmetries of the KP hierarchy and Sato's Backlund transformation, Commun. Math. Phys. 167 (1995), 227–233.
[9] E. Frenkel, Deformations of the KdV hierarchy and related soliton equations, Int. Math. Res. Not. 1996 (1996), 55–76.
[10] L. Geng, H. Chen, N. Li, J. Cheng, The gauge transformations of the constrained q-deformed KP hierarchy, Mod. Phys. Lett. B 32 (2018), 1850176.
[11] J. He, Y. Li, Y. Cheng, q-deformed KP hierarchy and q-deformed constrained KP hierarchy, SIGMA 2 (2006), 1–32.
[12] P. Iliev, Tau function solutions to a q-deformation of the KP hierarchy, Lett. Math. Phys. 44 (1998), 187–200.
[13] V. Kac, P. Cheung, Quantum Calculus, Springer-Verlag, New York, 2002.
[14] B.G. Konopelchenko, On the gauge-invariant description of the evolution equations integrable by Gelfand-Dikij spectral problems, Phys. Lett. A 92 (1982), 323–327.
[15] B.A. Kupershmidt, Canonical property of the Miura maps between the MKP and KP hierarchies, continuous and discrete, Commun. Math. Phys. 167 (1995), 351–371.
[16] C. Li, Sato theory on the q-Toda hierarchy and its extension, Chaos Solitons Fractals 76 (2015), 10–23.
[17] C. Li, J. Cheng, K. Tian, M. Li, J. He, Ghost symmetry of the discrete KP hierarchy, Monatsh. Math. 180 (2016), 815–832.
[18] R. Lin, X. Liu, Y. Zeng, A new extended q-deformed KP hierarchy, J. Nonlin. Math. Phys. 15 (2008), 333–347.

[19] R. Lin, H. Peng, M. Mañas, The q-deformed mKP hierarchy with self-consistent sources, Wronskian solutions and solitons, J. Phys. A Math. Theor. 43 (2010), 434022.
[20] S. Majid, Foundations of Quantum Group Theory, Cambridge University Press, Cambridge, 1995.
[21] R.M. Miura, Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation, J. Math. Phys. 9 (1968), 1202–1204.
[22] W. Oevel, Darboux theorems and Wronskian formulas for integrable systems: I. constrained KP flows, Phys. A 195 (1993), 533–576.
[23] W. Oevel, S. Carillo, Squared eigenfunction symmetries for soliton equations: part I, J. Math. Anal. Appl. 217 (1998), 161–178.
[24] W. Oevel, S. Carillo, Squared eigenfunction symmetries for soliton equations: part II, J. Math. Anal. Appl. 217 (1998), 179–199.
[25] W. Oevel, W. Strampp, Constrained KP hierarchy and bi-Hamiltonian structures, Commun. Math. Phys. 157 (1993), 51–81.
[26] A.Y. Orlov, E.I. Schulman, Additional symmetries for integrable equations and conformal algebra representation, Lett. Math. Phys. 12 (1986), 171–179.
[27] J.C. Shaw, M.H. Tu, Miura and auto-Bäcklund transformations for the cKP and cmKP hierarchies, J. Math. Phys. 38 (1997), 5756–5773.
[28] K. Takasaki, q-Analogue of modified KP hierarchy and its quasi-classical Limit, Lett. Math. Phys. 72 (2005), 165–181.
[29] K. Tian, Y. Ge, X. Zhu, On the q-deformed modified Kadomtsev-Petviashvili hierarchy and its additional symmetries, Roman. Rep. Phys. 69 (2017), 110.
[30] K. Tian, J. He, Y. Su, Symmetric q-deformed KP hierarchy, Chin. Ann. Math. Ser. B 36 (2015), 1–10.
[31] M.H. Tu, q-Deformed KP hierarchy: its additional symmetries and infinitesimal Bäcklund transformations, Lett. Math. Phys. 49 (1999), 95–103.
[32] S. Wang, K. Wu, X. Wu, D. Yu, The q-deformation of AKNS-D hierarchy, J. Phys. A: Math. Gen. 34 (2001), 9641–9651.