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Research Article

The Complex Hamiltonian Systems and Quasi-periodic Solutions in the Hirota Equation

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ABSTRACT

The Hirota equation is reduced to a pair of complex Finite-dimensional Hamiltonian Systems (FDHSs) with real-valued Hamiltonians, which are proven to be completely integrable in the Liouville sense. It turns out that involutive solutions of the complex FDHSs yield finite parametric solutions of the Hirota equation. From a Lax matrix of the complex FDHSs, the Hirota flow is linearized to display its evolution behavior on the Jacobi variety of a Riemann surface. With the technique of Riemann–Jacobi inversion, the quasi-periodic solution of the Hirota equation is presented in the form of Riemann theta functions.

1. INTRODUCTION

The Hirota equation was introduced in 1973 as a generalization of the Nonlinear Schrödinger (NLS) equation and the Modified Korteweg–de Vries (mKdV) equation [23]

$$iv_t + \delta v|v|^2 + \alpha v_{xx} + 3i\gamma v_x|v|^2 + i\beta v_{xxx} = 0, \tag{1.1}$$

where v is a scalar function depending on x and t ($(x, t) \in \mathbb{R}^2$), i is the imaginary unit, and α, β, δ, and γ are real positive constants satisfying $\alpha\gamma = \beta\delta$. Let δ be 2α and γ be 2β. The Hirota equation (1.1) can be rewritten as

$$iv_t + \alpha(2v|v|^2 + v_{xx}) + i\beta(6v_x|v|^2 + v_{xxx}) = 0. \tag{1.2}$$

Obviously, as $\alpha = 0$, $\beta = 1$, and v being real, the Hirota equation becomes the focusing mKdV equation; whereas $\alpha = 1$, $\beta = 0$, and v being complex it is reduced to the focusing NLS equation.

The NLS equation is a universal model with various physical applications ranging from nonlinear optics and hydrodynamics to Bose–Einstein condensates due to a simple balance between nonlinear and dispersive effects. Thanks to the significant complexity of ocean waves, the third-order dispersion v_{xxx} and a time-delay correction to the cubic term $v_x|v|^2$ are added to the NLS equation for a more precise description [35], similar to those high-order equations related to water waves considered by Osborne [33]. Under the Hasimoto map, it has been shown the relevance of the Hirota equation (1.2) in the modelling of the vortex string motion for a three dimensional Euler incompressible fluid [16,25]. As for the wave propagation of picosecond pulses in optical fibers [29], one needs to bring in the high-order dispersion and some other nonlinear effects for the simulation. Therefore, such an integrable extension of the NLS equation is relevant to the physical contexts in the high-intensity and short pulse picosecond regime [20,28].

The Hirota equation is of also mathematical interests, since it can be identified as an integrable PT-symmetric extension of the NLS equation [7]. The N envelope-soliton solution has been derived by the Hirota's bilinear method [23]. A more general soliton solution formula was obtained through the inverse scattering transformation, which includes the N-soliton solution, the breather solution, and a class of multipole soliton solutions [14]. With the nonlinear steepest descent method, the long-time asymptotic was analysed for the Hirota equation

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[24], as well as that of initial and boundary value problems on the half line [22]. Remarkably, by modifying the Darboux transformation method, it is found that the second-order rational solution of the Hirota equation (1.2) can be used to describe high-order rogue waves under random initial conditions with a given small amplitude of chaotic perturbations [2].

From the isospectral nature of Lax representations [26], a linear spectral problem usually results in a hierarchy of soliton equations, including both the positive and negative directions in view of bidirectional Lenard gradients [8]. It has been confirmed that the integrable couplings of arbitrary two commutable flows lying in the same soliton hierarchy are integrable in the sense of Lax compatibility [38]. Seen from the profile of equation (1.2), the Hirota equation can be regarded as an integrable coupling of NLS and mKdV flows in reference to the Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem [1]. However, this kind of combination does not automatically give us explicit solutions to the integrable equation.

It is necessary to know not only soliton solutions, but also quasi-periodic (finite-gap, or algebro-geometric) solutions of integrable Nonlinear Evolution Equations (NLEEs) in a number of physical problems. The quasi-periodic solutions to the NLS and mKdV equations have been obtained using either by the algebro-geometric method or by the combination of commutation methods and Hirota's τ-function approach in Belokolos et al. [4], Gesztesy [19] and some others, but the quasi-periodic solutions are still missing for the Hirota equation. Using the nonlinearization of Lax pair [5], the rogue periodic waves to the NLS and mKdV equations have been presented in Chen and Pelinovsky [9], Chen and Pelinovsky [10], Chen et al. [11], and the rogue waves on the periodic background have been given to the Hirota equation in Gao and Zhang [17], Peng et al. [34]. In the present work, the complex Finite-dimensional Hamiltonian Systems (FDHSs) with real-valued Hamiltonians are generalized to deduce some quasi-periodic solutions of the Hirota equation in view of finite-dimensional integrable reductions.

The real FDHSs have been used to derive soliton solutions, quasi-periodic solutions, and rogue periodic waves for NLEEs [6,8–11,13,17,18,34]. A natural issue is whether the complex FDHSs can be adapted to deduce solutions for complex NLEEs. To obtain solutions of integrable NLEEs, no matter N-solitons or quasi-periodic solutions, one key step is to specify a finite-dimensional invariant subspace associated with the phase flows [27,32]. It was known that the solution space of Novikov equation is a finite-dimensional invariant set of infinite-dimensional integrable systems. Recently, it is found that integration constants appearing in the Novikov equation are determined by eigenvalues and conserved quantities of FDHSs, from which the branch points of spectral bands are figured out in view of the symmetric constraint [5]. As a result, some interesting exact solution, such as the algebraically decaying solitons and the rogue periodic waves, are obtained by means of the Darboux transformation [9–11,17,34]. In this study, we reduce the Hirota equation to two complex FDHSs and construct its quasi-periodic solution.

The purpose of this work is to develop an alternative algorithm for getting quasi-periodic solutions of the Hirota equation by virtue of complex FDHSs. Subject to the finite-dimensional integrable reduction, the Hirota equation is decomposed into a pair of complex FDHSs with real-valued Hamiltonians by separating temporal and spatial variables. The relation between the Hirota equation and the complex FDHSs is established in view of the commutability of complex Hamiltonian flows, which simplifies the process of getting explicit solutions. Also, the finite-gap potential to the complex Novikov (high-order stationary) equation is presented, which cuts out a finite-dimensional invariant subspace for the Hirota flow via the symmetric constraint. Followed by a set of elliptic variables of complex FDHSs, a systematic way is given to elaborate Abel–Jacobi variables that straighten out the complex Hamiltonian and Hirota flows on the Jacobi variety of a Riemann surface. By using the technique of Riemann–Jacobi inversion [21,30], the Abel–Jacobi solution of the Hirota flow is transformed to the potential represented by Riemann theta functions. Although our computations are reported in the context of Hirota equation, the constructing scheme can also be applied to some other complex integrable NLEEs [12].

This paper is organized as follows. Section 2 is to decompose the Hirota equation into two complex FDHSs. The connection between the Hirota equation and the complex FDHSs is established in Section 3. Section 4 exhibits the evolution behavior of various flows on the Jacobi variety of a Riemann surface. Finally, in Section 5 the algebraic geometrical datum are processed to deduce quasi-periodic solutions for the Hirota equation.

2. REDUCTION TO THE HIROTA EQUATION

To reduce the Hirota equation, we first reformulate it into the Lenard scheme. Let us begin with the AKNS spectral problem [1]

$$\varphi_x = U\varphi, \quad U = i\lambda\sigma_1 + \bar{v}\sigma_2 - v\sigma_3, \quad \varphi = (\varphi_1, \varphi_2)^T, \tag{2.1}$$

where $\bar{v}$ is the complex conjugate of v, and

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Solve the stationary zero-curvature equation of the AKNS spectral problem (2.1)

$$V_x = [U, V], \quad V = a\sigma_1 + b\sigma_2 + c\sigma_3 = \sum_{j\ge 0}(a_j\sigma_1 + b_j\sigma_2 + c_j\sigma_3)\lambda^{-j}, \tag{2.2}$$

which coincides with

$$a_{jx} = vb_j + \bar{v}c_j, \quad b_{jx} = 2ib_{j+1} - 2\bar{v}a_j, \quad c_{jx} = -2ic_{j+1} - 2va_j, \quad j \geq 0. \tag{2.3}$$

Let $a_0 = 2i$ and $b_0 = c_0 = 0$ be the initial values. Up to constants of integration, a_j, b_j and c_j can be uniquely determined by means of the recursive formula (2.3), for example

$$\begin{aligned}
&a_1 = 0, \quad b_1 = 2\bar{v}, && c_1 = -2v, && a_2 = -i|v|^2, \quad b_2 = -i\bar{v}_x, \quad c_2 = -iv_x,\\
&a_3 = \frac{1}{2}(\bar{v}v_x - \bar{v}_x v), && b_3 = -\frac{1}{2}(\bar{v}_{xx} + 2\bar{v}|v|^2), && c_3 = \frac{1}{2}(v_{xx} + 2v|v|^2),\\
&a_4 = \frac{i}{4}(v\bar{v}_{xx} + \bar{v}v_{xx} - v_x\bar{v}_x + 3|v|^4), && b_4 = \frac{i}{4}(\bar{v}_{xxx} + 6|v|^2\bar{v}_x), && c_4 = \frac{i}{4}(v_{xxx} + 6|v|^2 v_x).
\end{aligned} \tag{2.4}$$

Based on the recurrence chain (2.3), we introduce the Lenard gradients $\{g_j\}$ and the Lenard operator pair K and J:

$$Kg_j = Jg_{j+1}, \qquad g_j = (ic_{j+1}, -ib_{j+1})^T, \qquad j \geq -1, \tag{2.5}$$

where

$$K = \begin{pmatrix} -2i\bar{v}\partial_x^{-1}\bar{v} & i(\partial_x + 2\bar{v}\partial_x^{-1}v) \\ i(\partial_x + 2v\partial_x^{-1}\bar{v}) & -2iv\partial_x^{-1}v \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \tag{2.6}$$

are two skew-symmetric operators, and ∂_x^{-1} is to denote the inverse operator of $\partial x = \partial/\partial x$ under the condition $\partial_x\partial_x^{-1} = \partial_x\partial_x^{-1} = 1$. Recalling (2.4), it is clear to see that

$$\begin{aligned}
&g_{-1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad g_0 = \begin{pmatrix} -2iv \\ -2iv \end{pmatrix}, \qquad g_1 = \begin{pmatrix} v_x \\ -\bar{v}_x \end{pmatrix},\\
&g_2 = \begin{pmatrix} \frac{i}{2}v_{xx} + iv|v|^2 \\ \frac{i}{2}\bar{v}_{xx} + i\bar{v}|v|^2 \end{pmatrix}, \quad g_3 = \begin{pmatrix} -\frac{1}{4}(v_{xxx} + 6|v|^2 v_x) \\ \frac{1}{4}(\bar{v}_{xxx} + 6|v|^2\bar{v}_x) \end{pmatrix}.
\end{aligned} \tag{2.7}$$

It is assumed that φ satisfies a spectral problem determined by the Lenard gradients $\{g_j\}$

$$\varphi_{t_n} = V^{(n)}\varphi, \quad V^{(n)} = \partial_x^{-1}(vg^{(2)} - \bar{v}g^{(1)})\sigma_1 + g^{(2)}\sigma_2 - g^{(1)}\sigma_3, \qquad n \geq 0, \tag{2.8}$$

where

$$g = (g^{(1)}, g^{(2)})^T = i\sum_{j=0}^{n} g_{j-1}\lambda^{n-j}.$$

The zero-curvature equation of spectral problems (2.1) and (2.8), i.e. $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$, gives the focusing NLS hierarchy

$$(\bar{v}_{t_n}, v_{t_n})^T = Jg_n =: X_n, \qquad n \geq 0, \tag{2.9}$$

together with a fundamental identity

$$V_x^{(n)} - [U, V^{(n)}] = U_*[-i(K - \lambda J)g], \tag{2.10}$$

where

$$U_*[\Xi] = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} U(\bar{v} + \varepsilon\Xi_1, v + \varepsilon\Xi_2), \quad \Xi = (\Xi_1, \Xi_2)^T.$$

It is found that the Hirota equation (1.2) is the compatibility condition of Lax pair (2.1) and

$$\varphi_t = V^{(2,3)}\varphi, \qquad V^{(2,3)} = \alpha V^{(2)} + 2\beta V^{(3)}, \tag{2.11}$$

where

$$\begin{aligned} V^{(2)} &= (2i\lambda^2 - i|v|^2)\sigma_1 + (2\lambda\bar{v} - i\bar{v}_x)\sigma_2 - (2\lambda v + iv_x)\sigma_3, \\ V^{(3)} &= [2i\lambda^3 - i\lambda|v|^2 + \frac{1}{2}(v_x\bar{v} - v\bar{v}_x)]\sigma_1 + [2\lambda^2\bar{v} - i\lambda\bar{v}_x - \frac{1}{2}(\bar{v}_{xx} + 2|v|^2\bar{v})]\sigma_2 + [-2\lambda^2 v - i\lambda v_x + \frac{1}{2}(v_{xx} + 2|v|^2 v)]\sigma_3. \end{aligned} \tag{2.12}$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be N arbitrary distinct nonzero complex eigenvalues, namely, $(\lambda_i \neq \bar{\lambda}_j, 1 \le i, j \le N)$, $(\psi_{1j}, \psi_{2j})^T$ be the vector eigenfunction pertinent to λ_j. Due to the symmetry of (2.1), $(\bar{\psi}_{2j}, -\bar{\psi}_{1j})^T$ corresponds to the eigenvalue $\bar{\lambda}_j$. Only for the convenience, we make the conventions $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$, $\psi_1 = (\psi_{11}, \psi_{12}, \cdots, \psi_{1N})^T$, and $\psi_2 = (\psi_{21}, \psi_{22}, \cdots, \psi_{2N})^T$. The diamond bracket $\langle .,. \rangle$ stands for the vector product: $\langle \xi, \eta \rangle = \sum_{j=1}^{N} \xi_j \eta_j$, where $\xi = (\xi_1, \xi_2, \cdots, \xi_N)^T$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_N)^T$. According to the nonlinearization of Lax pair [5], we consider N copies of spectral problem (2.1)

$$\begin{cases} \psi_{1jx} = i\lambda_j\psi_{1j} + \bar{v}\psi_{2j}, \\ \psi_{2jx} = -v\psi_{1j} - i\lambda_j\psi_{2j}, \\ \bar{\psi}_{1jx} = -i\bar{\lambda}_j\bar{\psi}_{1j} + v\bar{\psi}_{2j}, \\ \bar{\psi}_{2jx} = -\bar{v}\bar{\psi}_{1j} + i\bar{\lambda}_j\bar{\psi}_{2j}. \end{cases} \tag{2.13}$$

It follows from [37,39] that the functional gradients of λ_j and $\bar{\lambda}_j$ with respect to $\bar{v}$ and v are

$$\nabla\lambda_j = \begin{pmatrix} \dfrac{\delta\lambda_j}{\delta\bar{v}} \\ \dfrac{\delta\lambda_j}{\delta v} \end{pmatrix} = \begin{pmatrix} -2i\psi_{2j}^2 \\ -2i\psi_{1j}^2 \end{pmatrix}, \quad \nabla\bar{\lambda}_j = \begin{pmatrix} -\dfrac{\delta\bar{\lambda}_j}{\delta\bar{v}} \\ \dfrac{\delta\bar{\lambda}_j}{\delta v} \end{pmatrix} = \begin{pmatrix} -2i\bar{\psi}_{1j}^2 \\ -2i\bar{\psi}_{2j}^2 \end{pmatrix}, \tag{2.14}$$

which satisfy the Lenard eigenvalue equations

$$(K - \lambda_j J)\nabla\lambda_j = 0, \qquad (K - \bar{\lambda}_j J)\nabla\bar{\lambda}_j = 0. \tag{2.15}$$

Recall the Bargmann (symmetric) constraint

$$g_0 = \sum_{j=1}^{N}(\nabla\lambda_j + \nabla\bar{\lambda}_j), \tag{2.16}$$

which gives a Bargmann map to connect the potential v with the eigenfunctions (ψ_1, ψ_2)

$$v = \langle\bar{\psi}_1, \bar{\psi}_1\rangle + \langle\psi_2, \psi_2\rangle. \tag{2.17}$$

On $\mathbb{C}^{2N}$, we define the symplectic structure [3]

$$\omega^2 = \sum_{j=1}^{N}(d\psi_{1j} \wedge d\psi_{2j} + d\bar{\psi}_{1j} \wedge d\bar{\psi}_{2j}), \tag{2.18}$$

and the Poisson bracket

$$\{f, g\} = \sum_{j=1}^{N}\left(\frac{\partial f}{\partial\psi_{2j}}\frac{\partial g}{\partial\psi_{1j}} + \frac{\partial f}{\partial\bar{\psi}_{2j}}\frac{\partial g}{\partial\bar{\psi}_{1j}} - \frac{\partial f}{\partial\psi_{1j}}\frac{\partial g}{\partial\psi_{2j}} - \frac{\partial f}{\partial\bar{\psi}_{1j}}\frac{\partial g}{\partial\bar{\psi}_{2j}}\right). \tag{2.19}$$

Substituting (2.17) back into (2.1) and (2.11), we arrive at two complex FDHSs with real-valued Hamiltonians

$$\psi_{1x} = \{\psi_1, H_1\}, \quad \psi_{2x} = \{\psi_2, H_1\}, \quad \bar{\psi}_{1x} = \{\bar{\psi}_1, H_1\}, \quad \bar{\psi}_{2x} = \{\bar{\psi}_2, H_1\}, \tag{2.20}$$

where

$$H_1 = -i\langle\Lambda\psi_1, \psi_2\rangle + i\langle\bar{\Lambda}\bar{\psi}_1, \bar{\psi}_2\rangle - \frac{1}{2}|\langle\psi_1, \psi_1\rangle + \langle\bar{\psi}_2, \bar{\psi}_2\rangle|^2, \tag{2.21}$$

and

$$\psi_{1t} = \{\psi_1, H_{(2,3)}\}, \quad \psi_{2t} = \{\psi_2, H_{(2,3)}\}, \quad \bar{\psi}_{1t} = \{\bar{\psi}_1, H_{(2,3)}\}, \quad \bar{\psi}_{2t} = \{\bar{\psi}_2, H_{(2,3)}\}, \tag{2.22}$$

where $H_{(2,3)} = \alpha H_2 + 2\beta H_3$ together with

$$\begin{aligned} H_2 = & -2i(\langle \Lambda^2\psi_1,\psi_2\rangle - \langle \bar\Lambda^2\bar\psi_1,\bar\psi_2\rangle) + i|\langle\psi_1,\psi_1\rangle + \langle\bar\psi_2,\bar\psi_2\rangle|^2(\langle\psi_1,\psi_2\rangle - \langle\bar\psi_1,\bar\psi_2\rangle) \\ & -(\langle\psi_1,\psi_1\rangle + \langle\bar\psi_2,\bar\psi_2\rangle)(\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar\Lambda\bar\psi_1,\bar\psi_1\rangle) - (\langle\psi_2,\psi_2\rangle + \langle\bar\psi_1,\bar\psi_1\rangle)(\langle\Lambda\psi_1,\psi_1\rangle + \langle\bar\Lambda\bar\psi_2,\bar\psi_2\rangle). \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} H_3 = & -2i(\langle \Lambda^3\psi_1,\psi_2\rangle - \langle \bar\Lambda^3\bar\psi_1,\bar\psi_2\rangle) + i|\langle\psi_1,\psi_1\rangle + \langle\bar\psi_2,\bar\psi_2\rangle|^2(\langle\Lambda\psi_1,\psi_2\rangle - \langle\bar\Lambda\bar\psi_1,\bar\psi_2\rangle) + i(\langle\psi_1,\psi_2\rangle - \langle\bar\psi_1,\bar\psi_2\rangle) \\ & \times[(\langle\psi_1,\psi_1\rangle + \langle\bar\psi_2,\bar\psi_2\rangle)(\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar\Lambda\bar\psi_1,\bar\psi_1\rangle) + (\langle\psi_2,\psi_2\rangle + \langle\bar\psi_1,\bar\psi_1\rangle)(\langle\Lambda\psi_1,\psi_1\rangle + \langle\bar\Lambda\bar\psi_2,\bar\psi_2\rangle)] \\ & + |\langle\psi_1,\psi_1\rangle + \langle\bar\psi_2,\bar\psi_2\rangle|^2(\langle\psi_1,\psi_2\rangle - \langle\bar\psi_1,\bar\psi_2\rangle)^2 - (\langle\psi_1,\psi_1\rangle + \langle\bar\psi_2,\bar\psi_2\rangle)(\langle\Lambda^2\psi_2,\psi_2\rangle + \langle\bar\Lambda^2\bar\psi_1,\bar\psi_1\rangle) \\ & - (\langle\Lambda^2\psi_1,\psi_1\rangle + \langle\bar\Lambda^2\bar\psi_2,\bar\psi_2\rangle)(\langle\psi_2,\psi_2\rangle + \langle\bar\psi_1,\bar\psi_1\rangle) - |\langle\Lambda\psi_1,\psi_1\rangle + \langle\bar\Lambda\bar\psi_2,\bar\psi_2\rangle|^2 + \frac{1}{4}|\langle\psi_1,\psi_1\rangle + \langle\bar\psi_2,\bar\psi_2\rangle|^4. \end{aligned} \tag{2.24}$$

It is noted that the Hirota equation (1.2) can be represented as the compatibility condition of spectral problems (2.1) and (2.11). The Hirota equation (1.2) is indeed reduced to two complex FDHSs separating its temporal and spatial variables over $(\mathbb{C}^{2N}, \omega^2)$.

3. RELATION BETWEEN THE HIROTA EQUATION AND THE COMPLEX FDHSs

In order to establish the relation between the Hirota equation (1.2) and the complex FDHSs (2.20) and (2.22), it is necessary for us to prove the Liouville integrability of the complex FDHSs. The Liouville's definition of integrability is based on the notion of integrals of motion [3]. We need to construct a sufficient number of involutive integrals of motion for the complex FDHSs (2.20) and (2.22). Firstly, let us bring in a bilinear generating function

$$G_\lambda = \frac{i}{2}\sum_{j=1}^{N}\left(\frac{\nabla\lambda_j}{\lambda-\lambda_j} + \frac{\nabla\bar\lambda_j}{\lambda-\bar\lambda_j}\right) = \begin{pmatrix} Q_\lambda(\psi_2,\psi_2) + Q_{\bar\lambda}(\bar\psi_1,\bar\psi_1) \\ Q_\lambda(\psi_1,\psi_1) + Q_{\bar\lambda}(\bar\psi_2,\bar\psi_2) \end{pmatrix}, \tag{3.1}$$

where

$$Q_\lambda(\xi,\eta) = \sum_{j=1}^{N}\frac{\xi_j\eta_j}{\lambda-\lambda_j}, \qquad Q_{\bar\lambda}(\xi,\eta) = \sum_{j=1}^{N}\frac{\xi_j\eta_j}{\lambda-\bar\lambda_j}.$$

It follows from (2.15) that

$$(K-\lambda J)G_\lambda = 0. \tag{3.2}$$

Substituting G_λ back into the expression of $V^{(n)}$ gives rise to a Lax matrix

$$V_\lambda = \begin{pmatrix} i - Q_\lambda(\psi_1,\psi_2) + Q_{\bar\lambda}(\bar\psi_1,\bar\psi_2) & Q_\lambda(\psi_1,\psi_1) + Q_{\bar\lambda}(\bar\psi_2,\bar\psi_2) \\ -Q_\lambda(\psi_2,\psi_2) - Q_{\bar\lambda}(\bar\psi_1,\bar\psi_1) & -i + Q_\lambda(\psi_1,\psi_2) - Q_{\bar\lambda}(\bar\psi_1,\bar\psi_2) \end{pmatrix}, \tag{3.3}$$

which satisfies the Lax equation

$$(V_\lambda)_x - [U, V_\lambda] = 0, \tag{3.4}$$

in view of (2.10) and (3.2). It follows from (3.4) that $\det V_\lambda$ is a generating function of integrals of motion for the complex FDHSs (2.20) [36]. With $|\lambda| > \max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_N|\}$, we come to

$$\begin{aligned} F_\lambda = \det V_\lambda = & 1 + 2iQ_\lambda(\psi_1,\psi_2) + Q_\lambda(\psi_1,\psi_1)Q_\lambda(\psi_2,\psi_2) - Q_\lambda^2(\psi_1,\psi_2) - 2iQ_{\bar\lambda}(\bar\psi_1,\bar\psi_2) + Q_{\bar\lambda}(\bar\psi_1,\bar\psi_1)Q_{\bar\lambda}(\bar\psi_2,\bar\psi_2) - Q_{\bar\lambda}^2(\bar\psi_1,\bar\psi_2) \\ & + Q_\lambda(\psi_1,\psi_1)Q_{\bar\lambda}(\bar\psi_1,\bar\psi_1) + 2Q_\lambda(\psi_1,\psi_2)Q_{\bar\lambda}(\bar\psi_1,\bar\psi_2) + Q_\lambda(\psi_2,\psi_2)Q_{\bar\lambda}(\bar\psi_2,\bar\psi_2) \\ = & 1 + \sum_{j=1}^{N}\frac{E_j}{\lambda-\lambda_j} + \sum_{j=1}^{N}\frac{\bar E_j}{\lambda-\bar\lambda_j} = 1 + \sum_{k=0}^{\infty}F_k\lambda^{-k-1}, \end{aligned} \tag{3.5}$$

where

$$E_j = 2i\psi_{1j}\psi_{2j} + \sum_{k=1,k\neq j}^{N}\frac{(\psi_{1j}\psi_{2k} - \psi_{1k}\psi_{2j})^2}{\lambda_j-\lambda_k} + \sum_{k=1}^{N}\frac{(\psi_{1j}\bar\psi_{1k} + \psi_{2j}\bar\psi_{2k})^2}{\lambda_j-\bar\lambda_k}, \tag{3.6}$$

$$F_0 = 2i(\langle\psi_1,\psi_2\rangle - \langle\bar\psi_1,\bar\psi_2\rangle), \tag{3.7}$$

$$\begin{aligned}F_1 &= 2i(\langle\Lambda\psi_1,\psi_2\rangle - \langle\bar\Lambda\bar\psi_1,\bar\psi_2\rangle) + \langle\psi_1,\psi_1\rangle\langle\psi_2,\psi_2\rangle - \langle\psi_1,\psi_2\rangle^2 + \langle\bar\psi_1,\bar\psi_1\rangle\langle\bar\psi_2,\bar\psi_2\rangle\\ &\quad - \langle\bar\psi_1,\bar\psi_2\rangle^2 + \langle\psi_1,\psi_1\rangle\langle\bar\psi_1,\bar\psi_1\rangle + 2\langle\psi_1,\psi_2\rangle\langle\bar\psi_1,\bar\psi_2\rangle + \langle\psi_2,\psi_2\rangle\langle\bar\psi_2,\bar\psi_2\rangle,\end{aligned} \tag{3.8}$$

$$\begin{aligned}F_2 &= 2i(\langle\Lambda^2\psi_1,\psi_2\rangle - \langle\bar\Lambda^2\bar\psi_1,\bar\psi_2\rangle) + (\langle\Lambda\psi_1,\psi_1\rangle + \langle\bar\Lambda\bar\psi_2,\bar\psi_2\rangle)(\langle\psi_2,\psi_2\rangle + \langle\bar\psi_1,\bar\psi_1\rangle)\\ &\quad + (\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar\Lambda\bar\psi_1,\bar\psi_1\rangle)(\langle\psi_1,\psi_1\rangle + \langle\bar\psi_2,\bar\psi_2\rangle) - 2(\langle\Lambda\psi_1,\psi_2\rangle - \langle\bar\Lambda\bar\psi_1,\bar\psi_2\rangle)(\langle\psi_1,\psi_2\rangle - \langle\bar\psi_1,\bar\psi_2\rangle),\end{aligned} \tag{3.9}$$

$$F_k = 2i(\langle\Lambda^k\psi_1,\psi_2\rangle - \langle\bar\Lambda^k\bar\psi_1,\bar\psi_2\rangle) + \sum_{j=0}^{k-1}\begin{vmatrix}\langle\Lambda^j\psi_1,\psi_1\rangle + \langle\bar\Lambda^j\bar\psi_2,\bar\psi_2\rangle & \langle\Lambda^{k-j-1}\psi_1,\psi_2\rangle - \langle\bar\Lambda^{k-j-1}\bar\psi_1,\bar\psi_2\rangle\\ \langle\Lambda^j\psi_1,\psi_2\rangle - \langle\bar\Lambda^j\bar\psi_1,\bar\psi_2\rangle & \langle\Lambda^{k-j-1}\psi_2,\psi_2\rangle + \langle\bar\Lambda^{k-j-1}\bar\psi_1,\bar\psi_1\rangle\end{vmatrix}, \quad k \ge 3. \tag{3.10}$$

Let F_λ be a real-valued Hamiltonian on $(\mathbb{C}^{2N}, \omega^2)$, and τ_λ be the flow variable of F_λ. From the Poisson bracket, a direct calculation results in two canonical Hamiltonian equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix} = \begin{pmatrix}\{\psi_{1j},F_\lambda\}\\ \{\psi_{2j},F_\lambda\}\end{pmatrix} = W(\lambda,\lambda_j)\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}, \tag{3.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\begin{pmatrix}\bar\psi_{2j}\\ -\bar\psi_{1j}\end{pmatrix} = \begin{pmatrix}\{\bar\psi_{2j},F_\lambda\}\\ \{-\bar\psi_{1j},F_\lambda\}\end{pmatrix} = W(\lambda,\bar\lambda_j)\begin{pmatrix}\bar\psi_{2j}\\ -\bar\psi_{1j}\end{pmatrix}, \tag{3.12}$$

where

$$W(\lambda,\mu) = -\frac{2}{\lambda-\mu}V_\lambda. \tag{3.13}$$

Lemma 3.1. *On $(\mathbb{C}^{2N}, \omega^2)$, the Lax matrix V_μ satisfies a Lax equation*

$$\frac{\mathrm{d}V_\mu}{\mathrm{d}\tau_\lambda} = [W(\lambda,\mu), V_\mu], \qquad \forall\lambda,\mu\in\mathbb{C},\ \lambda\ne\mu. \tag{3.14}$$

Besides,

$$\{F_\mu, F_\lambda\} = 0, \qquad \forall\lambda,\mu\in\mathbb{C},\ \lambda\ne\mu, \tag{3.15}$$

$$\{F_j, F_k\} = 0, \qquad j,k = 0,1,2,\cdots. \tag{3.16}$$

Proof. Only for simplifying the description, we denote

$$\varepsilon_{1j} = -\psi_{1j}\psi_{2j}\sigma_1 + \psi_{1j}^2\sigma_2 - \psi_{2j}^2\sigma_3, \quad \varepsilon_{2j} = \bar\psi_{1j}\bar\psi_{2j}\sigma_1 + \bar\psi_{2j}^2\sigma_2 - \bar\psi_{1j}^2\sigma_3.$$

It follows from (3.11) and (3.12) that

$$\frac{\mathrm{d}\varepsilon_{1j}}{\mathrm{d}\tau_\lambda} = [W(\lambda,\lambda_j),\varepsilon_{1j}], \qquad \frac{\mathrm{d}\varepsilon_{2j}}{\mathrm{d}\tau_\lambda} = [W(\lambda,\bar\lambda_j),\varepsilon_{2j}]. \tag{3.17}$$

Resorting to (3.3), (3.13) and (3.17), a direct calculation yields

$$\begin{aligned}\frac{\mathrm{d}V_\mu}{\mathrm{d}\tau_\lambda} &= \sum_{j=1}^N\frac{1}{\mu-\lambda_j}\frac{\mathrm{d}\varepsilon_{1j}}{\mathrm{d}\tau_\lambda} + \sum_{j=1}^N\frac{1}{\mu-\bar\lambda_j}\frac{\mathrm{d}\varepsilon_{2j}}{\mathrm{d}\tau_\lambda}\\ &= \sum_{j=1}^N\frac{1}{\mu-\lambda_j}[W(\lambda,\lambda_j),\varepsilon_{1j}] + \sum_{j=1}^N\frac{1}{\mu-\bar\lambda_j}[W(\lambda,\bar\lambda_j),\varepsilon_{2j}]\\ &= -\frac{2}{\lambda-\mu}\sum_{j=1}^N\left(\left(\frac{1}{\mu-\lambda_j}-\frac{1}{\lambda-\lambda_j}\right)[V_\lambda,\varepsilon_{1j}] + \left(\frac{1}{\mu-\bar\lambda_j}-\frac{1}{\lambda-\bar\lambda_j}\right)[V_\lambda,\varepsilon_{2j}]\right)\\ &= -\frac{2}{\lambda-\mu}\left[V_\lambda, \sum_{j=1}^N\left(\frac{\varepsilon_{1j}}{\mu-\lambda_j}+\frac{\varepsilon_{2j}}{\mu-\bar\lambda_j}-\frac{\varepsilon_{1j}}{\lambda-\lambda_j}-\frac{\varepsilon_{2j}}{\lambda-\bar\lambda_j}\right)\right]\\ &= -\frac{2}{\lambda-\mu}[V_\lambda, V_\mu - V_\lambda] = [W(\lambda,\mu), V_\mu].\end{aligned}$$

Furthermore, from (3.14) we arrive at

$$\{F_\mu, F_\lambda\} = \frac{\mathrm{d}F_\mu}{\mathrm{d}\tau_\lambda} = \frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}(-\frac{1}{2}trV_\mu^2) = -trV_\mu tr[W(\lambda,\mu),V_\mu] = 0. \tag{3.18}$$

Substituting (3.5) into (3.18) leads to the identity (3.16), which completes the proof.

Apart from the involutivity of integrals of motion, the other essential element to the Liouville integrability of FDHSs is the functional independence, which means that solutions of the FDHSs can be obtained by solving a finite number of algebraic equations and computing a finite number of integrals. Below, we turn to the functional independence of F_k $(0 \le k \le 2N-1)$.

Lemma 3.2. *The integrals of motion* $\{F_0, F_1, \cdots, F_{2N-1}\}$ *given by (3.7)–(3.10) are functionally independent in a dense open subset of* $(\mathbb{C}^{2N}, \omega^2)$.

Proof. It is known from (3.5) that

$$F_k = \sum_{j=1}^{N}(\lambda_j^k E_j + \bar{\lambda}_j^k \bar{E}_j), \qquad 0 \le k \le 2N-1. \tag{3.19}$$

Let $P_0 = (\psi_{11}, \cdots, \psi_{1N}, \bar{\psi}_{21}, \cdots, \bar{\psi}_{2N}; \psi_{21}, \cdots, \psi_{2N}, -\bar{\psi}_{11}, \cdots, -\bar{\psi}_{1N})^T$ be a fixed point in $\mathbb{C}^{2N}$ with $\psi_{1j} = 0,\ \psi_{2j} \ne 0,\ (1 \le j \le N)$. And then,

$$\left.\frac{\partial E_j}{\partial \psi_{1k}}\right|_{P_0} = 2i\delta_{jk}\psi_{2j}, \quad \left.\frac{\partial \bar{E}_j}{\partial \bar{\psi}_{1k}}\right|_{P_0} = -2i\delta_{jk}\bar{\psi}_{2j}, \qquad 1 \le j,k \le N. \tag{3.20}$$

By (3.20), we arrive at the Jacobi determinant of $\{E_j, \bar{E}_j\}$ associated with $\{\psi_{1j}, \bar{\psi}_{1j}\}$ at P_0

$$\left.\frac{\partial(E_1,\cdots,E_N,\bar{E}_1,\cdots,\bar{E}_N)}{\partial(\psi_{11},\cdots,\psi_{1N},\bar{\psi}_{11},\cdots,\bar{\psi}_{1N})}\right|_{P_0} = 2^{2N}\prod_{j=1}^{N}|\psi_{2j}|^2, \tag{3.21}$$

which signifies the linear independence of $\{dE_1, \cdots, dE_N, d\bar{E}_1, \cdots, d\bar{E}_N\}$ over a dense open subset of $\mathbb{C}^{2N}$ [3]. It is supposed that there are $2N$ constants $\gamma_0, \gamma_1, \cdots, \gamma_{2N-1}$ such that

$$\sum_{k=0}^{2N-1}\gamma_k dF_k = 0, \tag{3.22}$$

which is in agreement with

$$\sum_{k=0}^{2N-1}\gamma_k\lambda_j^k = 0, \qquad \sum_{k=0}^{2N-1}\gamma_k\bar{\lambda}_j^k = 0, \qquad 1 \le j \le N, \tag{3.23}$$

in view of (3.19) and the linear independence of $\{dE_j, d\bar{E}_j\}$. It is noted that the determinant of coefficients of γ_k is the Vandermonde determinant. Namely, $\gamma_0 = \cdots = \gamma_{2N-1} = 0$, which means that $\{F_k\}(0 \le k \le 2N-1)$ are functionally independent in a dense open subset of $(\mathbb{C}^{2N}, \omega^2)$.

On one hand, it is seen from (2.21), (2.23), (2.24), and (3.7)–(3.10) that

$$\begin{aligned} &H_1 = -\frac{1}{2}F_1 + \frac{1}{8}F_0^2, \qquad H_2 = -F_2 + \frac{1}{2}F_1F_0 - \frac{1}{8}F_0^3, \\ &H_3 = -F_3 + \frac{1}{2}F_0F_2 + \frac{1}{4}F_1^2 - \frac{1}{8}F_0^2F_1 + \frac{1}{64}F_0^4, \end{aligned} \tag{3.24}$$

on the other hand, by (3.16), (3.24) and the Leibniz rule of Poisson bracket we obtain

$$\frac{\mathrm{d}F_\lambda}{\mathrm{d}t} = \{F_\lambda, H_{(2,3)}\} = \{F_\lambda, \alpha H_2 + 2\beta H_3\} = 0, \tag{3.25}$$

which indicates that $\{F_k\}$ are also integrals of motion for the complex FDHSs (2.22). We attain the Liouville integrability to the complex FDHSs (2.20) and (2.22).

Proposition 3.1. *The complex FDHSs* $(H_1, \omega^2, \mathbb{C}^{2N})$ *and* $(H_{(2,3)}, \omega^2, \mathbb{C}^{2N})$ *are completely integrable in the Liouville sense.*

Based on Proposition 3.1, it is known that two complex FDHSs (2.20) and (2.22) reduced from the Hirota equation (1.2) are compatible over $(\mathbb{C}^{2N}, \omega^2)$. This means that there exists a smooth function in x and t giving an involutive solution for complex FDHSs $(H_1, \omega^2, \mathbb{C}^{2N})$ and

$(H_{(2,3)}, \omega^2, \mathbb{C}^{2N})$. To progress further, from the commutability of Hamiltonian flows, we are in a position to establish the relation between the Hirota equation and the complex FDHSs, and then to confirm the existence of a finite number of spectral bands for the eigenvalue problem (2.1).

Proposition 3.2. *Let $(\psi_1(x,t), \psi_2(x,t))^T$ be an involutive solution of integrable complex FDHSs (2.20) and (2.22). Then*

$$v(x,t) = \langle \bar{\psi}_1(x,t), \bar{\psi}_1(x,t)\rangle + \langle \psi_2(x,t), \psi_2(x,t)\rangle, \tag{3.26}$$

is a finite parametric solution of the Hirota equation (1.2).

Proof. Resorting to the complex FDHSs (2.20) and (2.22), we compute

$$v_x = -2i(\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle) - 2(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle), \tag{3.27}$$

$$\begin{aligned} v_{xx} &= 4i(\langle\Lambda\psi_1,\psi_2\rangle - \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_2\rangle)(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle) - 4(\langle\Lambda^2\psi_2,\psi_2\rangle + \langle\bar{\Lambda}^2\bar{\psi}_1,\bar{\psi}_1\rangle) \\ &\quad + 4(i(\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle) + (\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle))(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle), \end{aligned} \tag{3.28}$$

$$\begin{aligned} v_{xxx} &= 8i(\langle\Lambda^3\psi_2,\psi_2\rangle + \langle\bar{\Lambda}^3\bar{\psi}_1,\bar{\psi}_1\rangle) - 8(\langle\psi_2,\psi_2\rangle + \langle\bar{\psi}_1,\bar{\psi}_1\rangle)(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle)^3 \\ &\quad - 8i(\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle)(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle)^2 + 8(\langle\psi_2,\psi_2\rangle + \langle\bar{\psi}_1,\bar{\psi}_1\rangle)(\langle\Lambda^2\psi_1,\psi_2\rangle - \langle\bar{\Lambda}^2\bar{\psi}_1,\bar{\psi}_2\rangle) \\ &\quad + 4i\,|\langle\psi_1,\psi_1\rangle + \langle\bar{\psi}_2,\bar{\psi}_2\rangle|^2\,(\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle) + 8(\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle)(\langle\Lambda\psi_1,\psi_2\rangle - \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_2\rangle) \\ &\quad - 16i(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle)(\langle\psi_2,\psi_2\rangle + \langle\bar{\psi}_1,\bar{\psi}_1\rangle)(\langle\Lambda\psi_1,\psi_2\rangle - \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_2\rangle) - 4i(\langle\psi_2,\psi_2\rangle + \langle\bar{\psi}_1,\bar{\psi}_1\rangle)^2(\langle\Lambda\psi_1,\psi_1\rangle \\ &\quad + \langle\bar{\Lambda}\bar{\psi}_2,\bar{\psi}_2\rangle) + 8(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle)(\langle\Lambda^2\psi_2,\psi_2\rangle + \langle\bar{\Lambda}^2\bar{\psi}_1,\bar{\psi}_1\rangle), \end{aligned} \tag{3.29}$$

$$\begin{aligned} v_t &= \alpha[-4i(\langle\Lambda^2\psi_2,\psi_2\rangle + \langle\bar{\Lambda}^2\bar{\psi}_1,\bar{\psi}_1\rangle) + 2i(\langle\psi_1,\psi_1\rangle + \langle\bar{\psi}_2,\bar{\psi}_2\rangle)(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)^2 - 4(\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle)(\langle\bar{\psi}_1,\bar{\psi}_2\rangle \\ &\quad - \langle\bar{\psi}_1,\bar{\psi}_2\rangle) + 4i(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle)^2 - 4(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)(\langle\Lambda\psi_1,\psi_2\rangle - \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_2\rangle)] \\ &\quad + \beta[-8(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)(\langle\Lambda^2\psi_1,\psi_2\rangle - \langle\bar{\Lambda}^2\bar{\psi}_1,\bar{\psi}_2\rangle) - 8(\langle\Lambda\psi_1,\psi_2\rangle - \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_2\rangle)(\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle) \\ &\quad + 16i(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle)(\langle\Lambda\psi_1,\psi_2\rangle - \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_2\rangle) - 8(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle)(\langle\Lambda^2\psi_2,\psi_2\rangle + \langle\bar{\Lambda}^2\bar{\psi}_1,\bar{\psi}_1\rangle) \\ &\quad + 8i(\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle)(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle)^2 + 8(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle)^3 - 8i(\langle\Lambda^3\psi_2,\psi_2\rangle + \langle\bar{\Lambda}^3\bar{\psi}_1,\bar{\psi}_1\rangle) \\ &\quad + 12(\langle\psi_1,\psi_1\rangle + \langle\bar{\psi}_2,\bar{\psi}_2\rangle)(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle)(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)^2 + 8i\,|\langle\psi_1,\psi_1\rangle + \langle\bar{\psi}_2,\bar{\psi}_2\rangle|^2\,(\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle) \\ &\quad + 4i(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)^2(\langle\Lambda\psi_1,\psi_1\rangle + \langle\bar{\Lambda}\bar{\psi}_2,\bar{\psi}_2\rangle)]. \end{aligned} \tag{3.30}$$

Substituting (3.27)–(3.30) back into the Hirota equation (1.2), it is shown that the expression (3.26) exactly solves the Hirota equation (1.2).

Remark 3.1. As a concrete application of Proposition 3.2, the derivation of explicit solutions to the Hirota equation is transformed to the problem of solving two complex FDHSs.

Proposition 3.3. *Let $(\psi_1(x), \psi_2(x))^T$ be a solution of the complex FDHSs (2.20). Then*

$$v = \langle\bar{\psi}_1(x),\bar{\psi}_1(x)\rangle + \langle\psi_2(x),\psi_2(x)\rangle, \tag{3.31}$$

is a finite-gap solution to the complex Novikov (high-order stationary NLS) equation

$$X_{2N} + \tilde{a}_1 X_{2N-1} + \tilde{c}_2 X_{2N-2} + \cdots + \tilde{c}_{2N} X_0 = 0, \qquad N \geq 2, \tag{3.32}$$

where

$$\tilde{c}_j = \tilde{a}_j + \sum_{k=0}^{j-2} \tilde{a}_k \hat{c}_{j-k}, \qquad j = 2,3,\cdots,2N,$$

$$\tilde{a}_0 = 1, \quad \tilde{a}_1 = -\sum_{j=1}^{N}(\lambda_j + \bar{\lambda}_j), \quad \tilde{a}_2 = \sum_{1\leq i<j\leq N}(\lambda_i\lambda_j + \bar{\lambda}_i\bar{\lambda}_j) + \sum_{i,j=1}^{N}\lambda_i\bar{\lambda}_j,$$

$$\tilde{a}_j = (-1)^j \sum_{1\leq i_1<\cdots<i_j\leq N}(\lambda_{i_1}\cdots\lambda_{i_j} + \bar{\lambda}_{i_1}\cdots\bar{\lambda}_{i_j}) + \sum_{\substack{1\leq s\leq N-1 \\ 1\leq k_1<\cdots<k_s\leq N \\ 1\leq l_1<\cdots<l_{j-s}\leq N}} \lambda_{k_1}\cdots\lambda_{k_s}\bar{\lambda}_{l_1}\cdots\bar{\lambda}_{l_{j-s}}, \quad 3\leq j\leq N,$$

$$\tilde{a}_j = (-1)^j \sum_{\substack{j-N\leq s\leq N \\ 1\leq k_1<\cdots<k_s\leq N \\ 1\leq l_1<\cdots<l_{j-s}\leq N}} \lambda_{k_1}\cdots\lambda_{k_s}\bar{\lambda}_{l_1}\cdots\bar{\lambda}_{l_{j-s}}, \quad N+1\leq j\leq 2N,$$

and $\hat{c}_2, \hat{c}_3, \cdots, \hat{c}_{2N}$ are some constants of integration.

Proof. One one hand, take into account an auxiliary polynomial in λ

$$a(\lambda)=\prod_{j=1}^{N}(\lambda-\lambda_j)(\lambda-\bar{\lambda}_j)=\tilde{a}_0\lambda^{2N}+\tilde{a}_1\lambda^{2N-1}+\cdots+\tilde{a}_{2N}. \tag{3.33}$$

Applying the operator $J^{-1}K$ on the symmetric constraint (2.16) k times, we derive

$$\sum_{j=1}^{N}(\lambda_j^k\nabla\lambda_j+\bar{\lambda}_j^k\nabla\bar{\lambda}_j)=g_k+\hat{c}_2 g_{k-2}+\cdots+\hat{c}_k g_0,\qquad k\geq 3, \tag{3.34}$$

in view of the Lenard eigenvalue equations (2.15). On the other hand, it follows from (2.16), (3.33) and (3.34) that

$$\begin{aligned}0&=\sum_{j=1}^{N}[a(\lambda_j)\nabla\lambda_j+a(\bar{\lambda}_j)\nabla\bar{\lambda}_j]\\&=\sum_{j=1}^{N}[(\lambda_j^{2N}\nabla\lambda_j+\bar{\lambda}_j^{2N}\nabla\bar{\lambda}_j)+\tilde{a}_1(\lambda_j^{2N-1}\nabla\lambda_j+\bar{\lambda}_j^{2N-1}\nabla\bar{\lambda}_j)+\cdots+\tilde{a}_{2N}(\nabla\lambda_j+\nabla\bar{\lambda}_j)]\\&=(g_{2N}+\hat{c}_2 g_{2N-2}+\cdots+\hat{c}_{2N}g_0)+\tilde{a}_1(g_{2N-1}+\hat{c}_2 g_{2N-3}+\cdots+\hat{c}_{2N-1}g_0)+\cdots+\tilde{a}_{2N}g_0\\&=g_{2N}+\tilde{a}_1 g_{2N-1}+\tilde{c}_2 g_{2N-2}+\cdots+\tilde{c}_{2N}g_0,\end{aligned} \tag{3.35}$$

which immediately becomes the complex Novikov equation (3.32) after being acted with the Lenard operator J. This completes the proof.

4. STRAIGHTENING OUT OF HIROTA FLOW

It is shown that the Hirota equation (1.2) has been reduced to two complex FDHSs with real-valued Hamiltonians on $(\mathbb{C}^{2N},\omega^2)$. And further, the Bargmann map (2.17) results in a finite-gap potential to the complex Novikov (high-order stationary NLS) equation (3.32). In this section, the complex FDHSs (2.20) and (2.22) serve as a basis to display the evolution picture of Hirota flow on the Jacobi variety of a Riemann surface.

For the sake of succinctness in writing, let us make the notation

$$V_\lambda=:V_\lambda^{11}\sigma_1+V_\lambda^{12}\sigma_2+V_\lambda^{21}\sigma_3.$$

From Lemma 3.1, we know that the Lax matrix V_μ satisfies a Lax equation along with τ_λ-flow. In particular, after a direct but tedious calculation, the Lax matrix V_λ also satisfies two Lax equations associated with the variables of x and t, respectively.

Lemma 4.1.

$$\partial_x V_\lambda=[\tilde{U},V_\lambda],\quad \tilde{U}=i\lambda\sigma_1+(\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle)\sigma_2-(\langle\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\psi_2,\psi_2\rangle)\sigma_3, \tag{4.1}$$

$$\partial_t V_\lambda=[\tilde{V}^{(2,3)},V_\lambda],\quad \tilde{V}^{(2,3)}=\tilde{V}_{11}^{(2,3)}\sigma_1+\tilde{V}_{12}^{(2,3)}\sigma_2+\tilde{V}_{21}^{(2,3)}\sigma_3,\quad \partial_t=\partial/\partial_t, \tag{4.2}$$

where

$$\begin{aligned}\tilde{V}_{11}^{(2,3)}&=\alpha[2i\lambda^2-i\,|\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle|^2]+\beta[4i\lambda^3-2i\lambda\,|\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle|^2-2(\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle)(((\langle\bar{\psi}_1,\bar{\psi}_1\rangle\\&\quad+\langle\psi_2,\psi_2\rangle)(\langle\psi_1,\psi_2\rangle-\langle\bar{\psi}_1,\bar{\psi}_2\rangle)+i(\langle\Lambda\psi_2,\psi_2\rangle+\langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle))-2(\langle\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\psi_2,\psi_2\rangle)\\&\quad\times(i(\langle\Lambda\psi_1,\psi_1\rangle+\langle\bar{\Lambda}\bar{\psi}_2,\bar{\psi}_2\rangle)+(\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle)(\langle\psi_1,\psi_2\rangle-\langle\bar{\psi}_1,\bar{\psi}_2\rangle)))],\\\tilde{V}_{12}^{(2,3)}&=\alpha[2\lambda(\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle)+2(\langle\Lambda\psi_1,\psi_1\rangle+\langle\bar{\Lambda}\bar{\psi}_2,\bar{\psi}_2\rangle)-2i(\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle)(\langle\psi_1,\psi_2\rangle-\langle\bar{\psi}_1,\bar{\psi}_2\rangle)]\\&\quad+\beta[4\lambda^2(\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle)+4\lambda((\langle\Lambda\psi_1,\psi_1\rangle+\langle\bar{\Lambda}\bar{\psi}_2,\bar{\psi}_2\rangle)-i(\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle)(\langle\psi_1,\psi_2\rangle-\langle\bar{\psi}_1,\bar{\psi}_2\rangle))\\&\quad-2(\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle)^2(\langle\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\psi_2,\psi_2\rangle)+4(\langle\Lambda^2\psi_1,\psi_1\rangle+\langle\bar{\Lambda}^2\bar{\psi}_2,\bar{\psi}_2\rangle)-4i(\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle)(\langle\Lambda\psi_1,\psi_2\rangle-\langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_2\rangle)\\&\quad-4i(\langle\Lambda\psi_1,\psi_1\rangle+\langle\bar{\Lambda}\bar{\psi}_2,\bar{\psi}_2\rangle)(\langle\psi_1,\psi_2\rangle-\langle\bar{\psi}_1,\bar{\psi}_2\rangle)-4(\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle)(\langle\psi_1,\psi_2\rangle-\langle\bar{\psi}_1,\bar{\psi}_2\rangle)^2],\\\tilde{V}_{21}^{(2,3)}&=\alpha[-2\lambda(\langle\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\psi_2,\psi_2\rangle)-2(\langle\Lambda\psi_2,\psi_2\rangle+\langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle)+2i(\langle\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\psi_2,\psi_2\rangle)(\langle\psi_1,\psi_2\rangle-\langle\bar{\psi}_1,\bar{\psi}_2\rangle)]\\&\quad-\beta[4\lambda^2(\langle\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\psi_2,\psi_2\rangle)+4\lambda((\langle\Lambda\psi_2,\psi_2\rangle+\langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle)-i(\langle\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\psi_2,\psi_2\rangle)(\langle\psi_1,\psi_2\rangle-\langle\bar{\psi}_1,\bar{\psi}_2\rangle))\\&\quad-2(\langle\psi_1,\psi_1\rangle+\langle\bar{\psi}_2,\bar{\psi}_2\rangle)(\langle\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\psi_2,\psi_2\rangle)^2+4(\langle\Lambda^2\psi_2,\psi_2\rangle+\langle\bar{\Lambda}^2\bar{\psi}_1,\bar{\psi}_1\rangle)-4i(\langle\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\psi_2,\psi_2\rangle)(\langle\Lambda\psi_1,\psi_2\rangle-\langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_2\rangle)\\&\quad-4i(\langle\Lambda\psi_2,\psi_2\rangle+\langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle)(\langle\psi_1,\psi_2\rangle-\langle\bar{\psi}_1,\bar{\psi}_2\rangle)-4(\langle\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\psi_2,\psi_2\rangle)(\langle\psi_1,\psi_2\rangle-\langle\bar{\psi}_1,\bar{\psi}_2\rangle)^2].\end{aligned} \tag{4.3}$$

It follows from (3.3) and (3.5) that F_λ and V_λ^{21} are the rational polynomial functions of λ with simple poles at $\{\lambda_j, \bar{\lambda}_j\}(j=1,2,\cdots,N)$. As a result, we define

$$F_\lambda = -V_\lambda^{12}V_\lambda^{21} - (V_\lambda^{11})^2 = \frac{b(\lambda)}{a(\lambda)} = \frac{a(\lambda)b(\lambda)}{a^2(\lambda)} = \frac{R(\lambda)}{a^2(\lambda)}, \tag{4.4}$$

$$V_\lambda^{21} = -Q_\lambda(\psi_2,\psi_2) - Q_{\bar{\lambda}}(\bar{\psi}_1,\bar{\psi}_1) = -(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)\frac{n(\lambda)}{a(\lambda)}, \tag{4.5}$$

where

$$\begin{aligned} &a(\lambda) = \prod_{k=1}^{N}(\lambda-\lambda_k)(\lambda-\bar{\lambda}_k), \quad b(\lambda) = \prod_{k=1}^{2N}(\lambda-\lambda_{N+k}), \\ &R(\lambda) = \prod_{k=1}^{N}(\lambda-\lambda_k)(\lambda-\bar{\lambda}_k)\prod_{k=1}^{2N}(\lambda-\lambda_{N+k}), \quad n(\lambda) = \prod_{k=1}^{2N-1}(\lambda-\nu_k), \end{aligned} \tag{4.6}$$

and $\nu_1, \nu_2, \ldots, \nu_{2N-1}$ are a set of elliptic variables for the complex FDHSs (2.20) and (2.22).

Lemma 4.2.

$$\frac{\langle\Lambda\psi_2,\psi_2\rangle + \langle\bar{\Lambda}\bar{\psi}_1,\bar{\psi}_1\rangle}{\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle} = \sum_{k=1}^{N}(\lambda_k+\bar{\lambda}_k) - \sum_{k=1}^{2N-1}\nu_k, \tag{4.7}$$

$$\frac{\langle\Lambda^2\psi_2,\psi_2\rangle + \langle\bar{\Lambda}^2\bar{\psi}_1,\bar{\psi}_1\rangle}{\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle} = \sum_{i<j}\nu_i\nu_j + \sum_{k=1}^{N}(\lambda_k+\bar{\lambda}_k)\left(\sum_{k=1}^{N}(\lambda_k+\bar{\lambda}_k) - \sum_{k=1}^{2N-1}\nu_k\right) - \sum_{i<j}(\lambda_i\lambda_j + \bar{\lambda}_i\bar{\lambda}_j) - \sum_{i=1}^{N}\lambda_i\sum_{j=1}^{N}\bar{\lambda}_j, \tag{4.8}$$

$$2i(\langle\psi_1,\psi_2\rangle - \langle\bar{\psi}_1,\bar{\psi}_2\rangle) = \sum_{k=1}^{N}(\lambda_k+\bar{\lambda}_k) - \sum_{k=1}^{2N}\lambda_{k+N}. \tag{4.9}$$

Proof. Multiplied by $-a(\lambda)$ on both sides of (4.5), the Right-hand Side (RHS) of (4.5) can be rewritten as

$$\text{RHS} = (\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)\left(\lambda^{2N-1} - \lambda^{2N-2}\sum_{j=1}^{N}\nu_j + \lambda^{2N-3}\sum_{i<j}\nu_i\nu_j + \cdots - \prod_{j=1}^{2N-1}\nu_j\right), \tag{4.10}$$

simultaneously, the Left-hand Side (LHS) of (4.5) can be expanded as

$$\begin{aligned} \text{LHS} = &\sum_{l=1}^{N}\psi_{2l}^2\left(\lambda^{2N-1} - \lambda^{2N-2}\left(\sum_{i=1,i\neq l}^{N}\lambda_i + \sum_{j=1}^{N}\bar{\lambda}_j\right) + \lambda^{2N-3}\left(\sum_{i<j;i,j\neq l}\lambda_i\lambda_j + \sum_{i<j}\bar{\lambda}_i\bar{\lambda}_j + \sum_{i=1,i\neq l}^{N}\lambda_i\sum_{j=1}^{N}\bar{\lambda}_j\right) + \cdots - \prod_{i=1,i\neq l}^{N}\lambda_i\prod_{j=1}^{N}\bar{\lambda}_j\right) \\ &+\sum_{l=1}^{N}\bar{\psi}_{1l}^2\left(\lambda^{2N-1} - \lambda^{2N-2}\left(\sum_{i=1}^{N}\lambda_i + \sum_{j=1,j\neq l}^{N}\bar{\lambda}_j\right) + \lambda^{2N-3}\left(\sum_{i<j}\lambda_i\lambda_j + \sum_{i<j;i,j\neq l}\bar{\lambda}_i\bar{\lambda}_j + \sum_{i=1}^{N}\lambda_i\sum_{j=1,j\neq l}^{N}\bar{\lambda}_j\right) + \cdots - \prod_{i=1}^{N}\lambda_i\prod_{j=1,j\neq l}^{N}\bar{\lambda}_j\right), \end{aligned} \tag{4.11}$$

By comparing the coefficient of λ^{2N-2} and λ^{2N-3} in (4.10) and (4.11), we have

$$(\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)\sum_{j=1}^{N}\nu_j = \sum_{l=1}^{N}\psi_{2l}^2\left(\sum_{k=1}^{N}(\lambda_k+\bar{\lambda}_k) - \lambda_l\right) + \sum_{l=1}^{N}\bar{\psi}_{1l}^2\left(\sum_{k=1}^{N}(\lambda_k+\bar{\lambda}_k) - \bar{\lambda}_l\right),$$

$$\begin{aligned} (\langle\bar{\psi}_1,\bar{\psi}_1\rangle + \langle\psi_2,\psi_2\rangle)\sum_{i<j}\nu_i\nu_j = &\sum_{l=1}^{N}\psi_{2l}^2\left(\lambda_l^2 - \lambda_l\sum_{k=1}^{N}(\lambda_k+\bar{\lambda}_k) + \sum_{i<j}(\lambda_i\lambda_j + \bar{\lambda}_i\bar{\lambda}_j) + \sum_{i=1}^{N}\lambda_i\sum_{j=1}^{N}\bar{\lambda}_j\right) \\ &+\sum_{l=1}^{N}\bar{\psi}_{1l}^2\left(\bar{\lambda}_l^2 - \bar{\lambda}_l\sum_{k=1}^{N}(\lambda_k+\bar{\lambda}_k) + \sum_{i<j}(\lambda_i\lambda_j + \bar{\lambda}_i\bar{\lambda}_j) + \sum_{i=1}^{N}\lambda_i\sum_{j=1}^{N}\bar{\lambda}_j\right), \end{aligned}$$

which leads to the formulas (4.7) and (4.8). It is seen from (3.4), (3.7) and (3.25) that the LHS of (4.9) is the constant of motion F_0 both in x and t. Similar to the treatment as (4.10) and (4.11), the coefficient of λ^{2N-1} in the expansion of (4.4) reads

$$\sum_{k=1}^{N}(\lambda_k+\bar{\lambda}_k) - \sum_{l=1}^{N}(E_l+\bar{E}_l) = \sum_{k=1}^{2N}\lambda_{k+N},$$

which gives the formula (4.9) since E_j are described by (3.6).

Replacing λ with ν_k in (4.4) gives rise to

$$V_\lambda^{11}\Big|_{\lambda=\nu_k} = \frac{\sqrt{-R(\nu_k)}}{a(\nu_k)}, \qquad 1 \le k \le 2N-1. \tag{4.12}$$

Considering the (2, 1)-entry of Lax equations (4.1) and (4.2), we derive

$$\begin{aligned} &\partial_x V_\lambda^{21} = -2(\langle \bar{\psi}_1, \bar{\psi}_1 \rangle + \langle \psi_2, \psi_2 \rangle) V_\lambda^{11} - 2i\lambda V_\lambda^{21}, \\ &\partial_t V_\lambda^{21} = 2V_\lambda^{11} \tilde{V}_{21}^{(2,3)} - 2V_\lambda^{21} \tilde{V}_{11}^{(2,3)}. \end{aligned} \tag{4.13}$$

By combining (4.5), (4.12), (4.13) and Lemma 4.2, we attain the Dubrovin type equations

$$\frac{\mathrm{d}\nu_k}{\mathrm{d}x} = -\frac{2\sqrt{-R(\nu_k)}}{\prod_{j=1, j\neq k}^{2N-1} (\nu_k - \nu_j)}, \qquad 1 \le k \le 2N-1, \tag{4.14}$$

$$\begin{aligned} \frac{\mathrm{d}\nu_k}{\mathrm{d}t} = \frac{-2\sqrt{-R(\nu_k)}}{\prod_{j=1, j\neq k}^{2N-1} (\nu_k - \nu_j)} \Bigg[& \alpha \left(2\nu_k - 2\sum_{j=1}^{2N-1} \nu_j + \sum_{j=1}^{N} (\lambda_j + \bar{\lambda}_j) + \sum_{j=1}^{2N} \lambda_{j+2N} \right) \\ & + 2\beta \left(2\nu_k^2 - 2\nu_k \sum_{j=1}^{2N-1} \nu_j + 2\sum_{i<j} \nu_i \nu_j + \left(\sum_{j=1}^{N} (\lambda_j + \bar{\lambda}_j) + \sum_{j=1}^{2N} \lambda_{j+N} \right) \right. \\ & \left. \times \left(\nu_k - \sum_{j=1}^{2N-1} \nu_j \right) + \frac{1}{2} \sum_{j=1}^{N} (\lambda_j^2 + \bar{\lambda}_j^2) + \frac{1}{2} \sum_{j=1}^{2N} \lambda_{j+N}^2 + \frac{1}{4} \left(\sum_{j=1}^{N} (\lambda_j + \bar{\lambda}_j) + \sum_{j=1}^{2N} \lambda_{j+N} \right)^2 \right) \Bigg], \qquad 1 \le k \le 2N-1, \end{aligned} \tag{4.15}$$

which control the dynamics of elliptic variables $\{\nu_k\}$.

To solve the Dubrovin type equations (4.14) and (4.15), the subsequent attention in this section is instructed to the theory of algebraic curves. From the generating function of integrals of motion, we define a hyperelliptic curve of Riemann surface Γ: $\xi^2 + R(\lambda) = 0$, which allows with $2N - 1$ linearly independent holomorphic differentials

$$\tilde{\omega}_l = \frac{\lambda^{l-1} d\lambda}{2\sqrt{-R(\lambda)}}, \qquad 1 \le l \le 2N-1.$$

Thanks to deg $R(\lambda) = 4N$ by Eq. (4.6), the genus of Γ is $2N - 1$ that coincides with the number of elliptic variables $\{\nu_k\}$. For any $\lambda\, (\neq \lambda_j, \bar{\lambda}_j\, (1 \le j \le N); \lambda_{N+k}\, (1 \le k \le 2N)) \in \mathbb{C}$, there exist two points $P_+(\lambda) = (\lambda, \sqrt{-R(\lambda)})$ and $P_-(\lambda) = (\lambda, -\sqrt{-R(\lambda)})$ on the upper and lower sheets of Γ. In particular, there are two infinite points ∞_1 and ∞_2 as $\lambda = \infty$, which are not the branch points and can be expressed as $(0, -1)$ and $(0, 1)$ in the local coordinate $\lambda = z^{-1}$.

Introduce a set of canonical basis of cycles $\{a_j, b_j\}_{j=1}^{2N-1}$ on Γ, which are independent with the intersection numbers $a_i \circ a_j = b_i \circ b_j = 0$, $a_i \circ b_j = \delta_{ij}$, $(i, j = 1, 2, \cdots, 2N-1)$. By the canonical basis of cycles, let us bring in the integral $A_{ij} = \int_{a_j} \tilde{\omega}_i, (1 \le i, j \le 2N-1),$ which yields a $(2N - 1)$ by $(2N - 1)$ nondegenerate matrix $C = (C_{ij}) = (A_{ij})^{-1}$ [21,30]. And then, the holomorphic differential $\tilde{\omega}_l$ can be converted into a normalized one

$$\omega_j = \sum_{l=1}^{2N-1} C_{jl} \tilde{\omega}_l, \qquad 1 \le j \le 2N-1, \tag{4.16}$$

with the property

$$\int_{a_i} \omega_j = \sum_{l=1}^{2N-1} C_{jl} \int_{a_i} \tilde{\omega}_l = \sum_{l=1}^{2N-1} C_{jl} A_{li} = \delta_{ji} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4.17}$$

Write $\omega = (\omega_1, \omega_2, \cdots \omega_{2N-1})^T$ for short, and define

$$\delta_j = \int_{a_j} \omega, \qquad B_j = \int_{b_j} \omega, \qquad 1 \le j \le 2N-1. \tag{4.18}$$

It is found that $\delta = (\delta_{ij})_{2N-1\times 2N-1}$ is a unit matrix, and $B = (B_{ij})_{2N-1\times 2N-1}$ is a symmetric matrix ($B_{ij} = B_{ji}$) with positive-definite imaginary part [21,30]. Moreover, the $4N-2$ periodic vectors $\{\delta_j, B_j\}$ span a lattice $\mathcal{T}$ in $\mathbb{C}^{2N-1}$ that specifies the Jacobi variety $J(\Gamma) = \mathbb{C}^{2N-1}/\mathcal{T}$ of Riemann surface Γ.

After the above preparations, we suitably select out the Abel–Jacobi variable with a fixed point p_0 ($\neq \infty_i$ $(i=1,2)$; $\lambda_j, \bar{\lambda}_j$ $(1\le j\le N)$; λ_{N+k} $(1\le k\le 2N)$) on Γ

$$\rho_j(x,t) = \sum_{k=1}^{2N-1}\int_{p_0}^{\nu_k(x,t)}\omega_j = \sum_{k=1}^{2N-1}\sum_{l=1}^{2N-1}C_{jl}\int_{p_0}^{\nu_k(x,t)}\frac{\lambda^{l-1}d\lambda}{2\sqrt{-R(\lambda)}}. \tag{4.19}$$

By using (4.14) and (4.15), a direct calculation results in

$$\partial_x\rho_j(x,t) = -\sum_{k=1}^{2N-1}\sum_{l=1}^{2N-1}\frac{C_{jl}\nu_k^{l-1}}{\prod_{j=1,j\neq k}^{N}(\nu_k-\nu_j)}, \tag{4.20}$$

$$\begin{aligned}\partial_t\rho_j(x,t) = \sum_{k=1}^{2N-1}\sum_{l=1}^{2N-1}\frac{-C_{jl}\nu_k^{l-1}}{\prod_{j=1,j\neq k}^{N}(\nu_k-\nu_j)}\Bigg[&\alpha\Bigg(2\nu_k - 2\sum_{j=1}^{2N-1}\nu_j + \sum_{j=1}^{N}(\lambda_j+\bar{\lambda}_j)\\ &+\sum_{j=1}^{2N}\lambda_{j+N}\Bigg) + 2\beta\Bigg(2\nu_k^2 - 2\nu_k\sum_{j=1}^{2N-1}\nu_j + 2\sum_{i<j}\nu_i\nu_j\\ &+\left(\sum_{j=1}^{N}(\lambda_j+\bar{\lambda}_j)+\sum_{j=1}^{2N}\lambda_{j+N}\right)\left(\nu_k-\sum_{j=1}^{2N-1}\nu_j\right)+\frac{1}{2}\sum_{j=1}^{N}(\lambda_j^2+\bar{\lambda}_j^2)\\ &+\frac{1}{2}\sum_{j=1}^{2N}\lambda_{j+N}^2 + \frac{1}{4}\left(\sum_{j=1}^{N}(\lambda_j+\bar{\lambda}_j)+\sum_{j=1}^{2N}\lambda_{j+N}\right)^2\Bigg)\Bigg].\end{aligned} \tag{4.21}$$

With the aid of the algebraic formulas [31]

$$\begin{aligned}&I_s = \sum_{k=1}^{2N-1}\frac{\nu_k^s}{\prod_{j=1,j\neq k}^{2N-1}(\nu_k-\nu_j)} = \delta_{s,2N-2}, \quad 1\le s\le 2N-2,\\ &I_{2N-1} = I_{2N-2}\sum_{j=1}^{2N-1}\nu_j, \quad I_{2N} = I_{2N-1}\sum_{j=1}^{2N-1}\nu_j - I_{2N-2}\sum_{i<j;i,j=1}^{2N-1}\nu_i\nu_j,\end{aligned} \tag{4.22}$$

we arrive at

$$\begin{aligned}&\partial_x\rho_j(x,t) = -C_{j2N-1} =: \Omega_j^{(1)},\\ &\partial_t\rho_j(x,t) = -\alpha\left[2C_{j2N-2} + C_{j2N-1}\left(\sum_{j=1}^{N}(\lambda_j+\bar{\lambda}_j)+\sum_{j=1}^{2N}\lambda_{j+N}\right)\right]\\ &\quad -\beta\Bigg[4C_{j2N-3} + 2C_{j2N-2}\left(\sum_{j=1}^{N}(\lambda_j+\bar{\lambda}_j)+\sum_{j=1}^{2N}\lambda_{j+N}\right) + C_{j2N-1}\times\\ &\quad\left(\sum_{j=1}^{N}(\lambda_j^2+\bar{\lambda}_j^2)+\sum_{j=1}^{2N}\lambda_{j+N}^2+\frac{1}{2}\left(\sum_{j=1}^{N}(\lambda_j+\bar{\lambda}_j)+\sum_{j=1}^{2N}\lambda_{j+N}\right)^2\right)\Bigg] =: \Omega_j^{(2)}.\end{aligned} \tag{4.23}$$

By using (4.23), the H_1-, $H_{(2,3)}$- and the Hirota-flows are represented as

$$\begin{aligned}&H_1-\text{flow}: \quad \rho_j(x) = \Omega_j^{(1)}x + \rho_{0j},\\ &H_2-\text{flow}: \quad \rho_j(t) = \Omega_j^{(2)}t + \rho_{0j},\\ &\text{Hirota}-\text{flow}: \quad \rho_j(x,t) = \Omega_j^{(1)}x + \Omega_j^{(2)}t + \rho_{0j},\end{aligned} \tag{4.24}$$

where $\rho_{0j} = \sum_{k=1}^{2N-1}\int_{p_0}^{\nu_k(0,0)}\omega_j$ is a constant of integration.

It has been shown that the evolution velocities $\Omega_j^{(1)}$ and $\Omega_j^{(2)}$ are the combination of constants of motion and constants of integration. Having a look at the shape of (4.24), the Abel–Jacobi variable $\rho_j(x,t)$ can be understood as the angle variable, which exhibit the linearity of Hirota flow on the Jacobi variety $J(\Gamma)$ of a Riemann surface.

5. QUASI-PERIODIC SOLUTIONS

Followed by the Bargmann map (2.17) and Lemma 4.2, we bridge the gap between the Hirota equation (1.2) and the complex FDHSs (2.20) and (2.22), and further connect the eigenfunctions with the symmetric functions of elliptic variables. It is noted from (4.24) that the Hirota equation has been integrated with the Abel–Jacobi solution over $J(\Gamma)$, which stimulates us to discuss the Riemann–Jacobi inversion from $\rho_j(x, t)$ to $\{v_k\}$.

We turn to the Abel map from the divisor group to the Jacobi variety $\mathcal{A}: \mathrm{Div}(\Gamma) \to J(\mathcal{T})$

$$\mathcal{A}(p) = \int_{p_0}^{p} \omega, \qquad \mathcal{A}\left(\sum_{k=1}^{2N-1} n_k p_k\right) = \sum_{k=1}^{2N-1} n_k \mathcal{A}(p_k).$$

Let us choose a special divisor $p = \sum_{k=1}^{2N-1} p(v_k)$, where $p(v_k) = (v_k, \xi(v_k))$. Denote $\rho = (\rho_1, \rho_2, \cdots, \rho_{2N-1})$ for short. The Abel–Jacobi variable can be rewritten as

$$\rho = \sum_{k=1}^{2N-1} \int_{p_0}^{p(v_k)} \omega = \mathcal{A}\left(\sum_{k=1}^{2N-1} p(v_k)\right) = \sum_{k=1}^{2N-1} \mathcal{A}(p(v_k)). \tag{5.1}$$

By the symmetric matrix B, we introduce the Riemann theta function of Γ [21,30]

$$\theta(\zeta) = \sum_{z \in \mathbb{Z}^{2N-1}} \exp \pi i(\langle Bz, z\rangle + 2\langle \zeta, z\rangle), \qquad \zeta \in \mathbb{C}^{2N-1}.$$

According to the Riemann theorem [21], it is known from the Abel–Jacobi variable (5.1) that there exists a vector of Riemann constant $M = (M_1, M_2, \cdots, M_{2N-1})^T \in \mathbb{C}^{2N-1}$ such that $f(\lambda) = \theta(A(p(\lambda)) - \rho - M)$ has $2N - 1$ simple zeros at $v_1, v_2, \cdots, v_{2N-1}$. To make the function $f(\lambda)$ single value, the Riemann surface Γ should be suitably cut along with the contours a_j and b_j to form a simply connected region with the boundary γ, which is consisted of $8N - 4$ edges in the order $a_1^+ b_1^+ a_1^- b_1^- a_2^+ b_2^+ a_2^- b_2^- \cdots\cdots a_{2N-1}^+ b_{2N-1}^+ a_{2N-1}^- b_{2N-1}^-$, where the symbols +, – denote the orientation. And then, the positive power sums of $\{v_j\}_{j=1}^{2N-1}$ can be figured out by the calculation of residues of $f(\lambda)$ at ∞_1 and ∞_2, namely

$$\sum_{j=1}^{2N-1} v_j^k = I_k(\Gamma) - \sum_{s=1}^{2} \operatorname*{Res}_{\lambda=\infty_s} \lambda^k d \ln f(\lambda), \tag{5.2}$$

where

$$I_k(\Gamma) = \frac{1}{2\pi i} \oint_\gamma \lambda^k d \ln f(\lambda) = \sum_{j=1}^{2N-1} \int_{a_j} \lambda^k \omega_j$$

is a constant independent of the Abel–Jacobi variable ρ [15].

Lemma 5.1. *Let* $S_k = \sum_{j=1}^{N} (\lambda_j^k + \bar{\lambda}_j^k) + \sum_{j=1}^{2N} \lambda_{j+N}^k$. *The coefficients in the expansion*

$$\frac{\lambda^{2N}}{\sqrt{R(\lambda)}} = \sum_{k=0}^{\infty} \Lambda_k \lambda^{-k}, \; |\lambda| > \max\{|\lambda_1|, \cdots, |\lambda_N|, |\lambda_{N+1}|, \cdots, |\lambda_{3N}|\}, \tag{5.3}$$

are given by the recursive formulae

$$\begin{aligned} &\Lambda_{-k} = 0\,(k \geq 1), \quad \Lambda_0 = 1, \quad \Lambda_1 = \frac{1}{2} S_1, \\ &\Lambda_2 = \frac{1}{4}(S_2 + S_1 \Lambda_1) = \frac{1}{4} S_2 + \frac{1}{8} S_1^2, \\ &\Lambda_k = \frac{1}{2k}\Big(S_k + \sum_{i+j=k, i, j \geq 1} S_i \Lambda_j\Big), \quad k \geq 3. \end{aligned} \tag{5.4}$$

Lemma 5.2. *Near* ∞_s $(s = 1, 2)$, *under the local coordinate* $z = \lambda^{-1}$ *the holomorphic differential* $\tilde{\omega}_l$ *can be described by*

$$\tilde{\omega}_l = \frac{i}{2} \sum_{k=0}^{\infty} \Lambda_k z^{2N-1-l+k} dz. \tag{5.5}$$

We denote the jth component of $f(\lambda)$ by ς_j, $\partial_j = \partial/\partial\varsigma_j$, $\partial_{jk}^2 = \partial^2/\partial\varsigma_j\partial\varsigma_k$, etc. With the Einstein summation convention, in the neighborhood of $\lambda = \infty_s (s = 1, 2)$ the Riemann theta function $f(\lambda)$ has the asymptotic expansion $(z = \lambda^{-1})$

$$f(\lambda)=\theta_s^{(\infty)}+\frac{(-1)^s}{2i}C_{j2N-1}z\partial_j\theta_s^{(\infty)}+\frac{z^2}{2}\left(-\frac{1}{4}C_{j2N-1}C_{k2N-1}\partial_{jk}^2\theta_s^{(\infty)}+\frac{(-1)^s}{2i}(C_{j2N-1}\Lambda_1+C_{j2N-2})\partial_j\theta_s^{(\infty)}\right)$$
$$+\frac{z^3}{6}\left(\frac{(-1)^{s+1}}{8i}C_{j2N-1}C_{k2N-1}C_{l2N-1}\partial_{jkl}^3\theta_s^{(\infty)}-\frac{3}{4}C_{j2N-1}(C_{k2N-1}\Lambda_1+C_{k2N-2})\partial_{jk}^2\theta_s^{(\infty)}\right.$$
$$\left.+\frac{(-1)^s}{i}(\Lambda_2C_{j2N-1}+\Lambda_1C_{j2N-2}+C_{j2N-3})\partial_j\theta_s^{(\infty)}\right)+O(z^4), \tag{5.6}$$

which together with (4.23) gives rise to

$$\frac{\mathrm{d}\ln f(\lambda)}{\mathrm{d}\lambda}=\frac{(-1)^{s+1}}{2i}\partial_x\ln\theta_s^{(\infty)}-\frac{1}{4}\left(\frac{2(-1)^{s+1}}{i}\frac{\tilde{\Omega}^{(2)}}{\Omega^{(2)}}\partial_t\ln\theta_s^{(\infty)}+\partial_x^2\ln\theta_s^{(\infty)}\right)z+O(z^2), \tag{5.7}$$

where $\tilde{\Omega}_j^{(2)}=C_{j2N-1}\Lambda_1+C_{j2N-2},\theta_s^{(\infty)}=\theta_s^{(\infty)}(\rho+M+\chi_s)$ and $\chi_s=\int_{\infty_s}^{P_0}\omega$. Resorting to (5.2) and (5.7), we attain the trace formulae

$$\sum_{k=1}^{2N-1}\nu_k=I_1(\Gamma)+\frac{1}{2i}\partial_x\ln\frac{\theta_2^{(\infty)}}{\theta_1^{(\infty)}},$$
$$\sum_{k=1}^{2N-1}\nu_k^2=I_2(\Gamma)+\frac{1}{2i}\frac{\tilde{\Omega}^{(2)}}{\Omega^{(2)}}\partial_t\ln\frac{\theta_2^{(\infty)}}{\theta_1^{(\infty)}}+\frac{1}{4}\partial_x^2\ln\theta_1^{(\infty)}\theta_2^{(\infty)}. \tag{5.8}$$

By using the Bargmann map (2.17) and the complex FDHS (2.20), a direct calculation yields

$$\partial_x\ln v=-2i\frac{\langle\Lambda\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\Lambda\psi_2,\psi_2\rangle}{\langle\bar{\psi}_1,\bar{\psi}_1\rangle+\langle\psi_2,\psi_2\rangle}-2(\langle\psi_1,\psi_2\rangle-\langle\bar{\psi}_1,\bar{\psi}_2\rangle), \tag{5.9}$$

which together with Lemma 4.2 and the trace formula (5.8) results in

$$\partial_x\ln v=N_1+\partial_x\ln\frac{\theta(\Omega^{(1)}x+\Omega^{(2)}t+\alpha_2)}{\theta(\Omega^{(1)}x+\Omega^{(2)}t+\alpha_1)}, \tag{5.10}$$

where

$$N_1=2iI_1(\Gamma)-i\left(\sum_{j=1}^{N}(\lambda_j+\bar{\lambda}_j)+\sum_{j=1}^{2N}\lambda_{j+2N}\right),\quad \alpha_s=\rho_0+M+\chi_s,\quad s=1,2.$$

Taking one integration on (5.10) with respect to x, we obtain

$$v=v_0e^{N_1x}\frac{\theta(\Omega^{(1)}x+\Omega^{(2)}t+\alpha_2)}{\theta(\Omega^{(1)}x+\Omega^{(2)}t+\alpha_1)}, \tag{5.11}$$

where v_0 is independent of x, but may depend on t. On the other hand, taking one partial derivative with respect to t on (5.11), we also have

$$\partial_t\ln v=N_2+\partial_t\ln\frac{\theta(\Omega^{(1)}x+\Omega^{(2)}t+\alpha_2)}{\theta(\Omega^{(1)}x+\Omega^{(2)}t+\alpha_1)},\qquad N_2=\partial_t\ln v_0. \tag{5.12}$$

Analogous to the treatment conducted as in Cao et al. [6] (see Theorem 11.1), it is found that N_2 is also a constant of motion with regards to t. Finally, based on the presentations (5.10) and (5.12), we obtain the quasi-periodic solution for the Hirota equation

$$v(x,t)=v(0,0)e^{N_1x+N_2t}\frac{\theta(\alpha_1)}{\theta(\alpha_2)}\frac{\theta(\Omega^{(1)}x+\Omega^{(2)}t+\alpha_2)}{\theta(\Omega^{(1)}x+\Omega^{(2)}t+\alpha_1)}. \tag{5.13}$$

Remark 5.1. It looks like that only the quasi-periodic solution of odd genus ($g=2N-1$) has been attained in the above constructing scheme. In fact, the solution in the case of even genus ($g=2N-2$) can be obtained by the degeneration procedure of $v_{2N-1}=0$, because the finite-genus solution can be embedded into an invariant torus with one more genus (for more details, see the subsection 2.4 in Chen and Pelinovsky [9]).

In conclusion, an explicit quasi-periodic solution has been constructed for the Hirota equation (1.2) with the aid of two complex FDHSs (2.20) and (2.22). In particular, as $\alpha=1$ and $\beta=0$, the quasi-periodic solution (5.13) becomes the exact solution of the focusing NLS equation that coincides with the one in the book [4] [see Eq. (4.1.22)]; whereas $\alpha=0$ and $\beta=1$, for real $v(x,t)$ the quasi-periodic solution (5.13) delivers a new solution for the focusing mKdV equation.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

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