Journal of Nonlinear Mathematical Physics
In Press, Corrected Proof
DOI: https://doi.org/10.2991/jnmp.k.201104.002; ISSN 1402-9251; eISSN 1776-0852
https://www.atlantis-press.com/journals/jnmp

Research Article

Symmetries of Kolmogorov Backward Equation

Roman Kozlov*

Department of Finance and Management Science, Norwegian School of Economics, Helleveien 30, Bergen 5045, Norway

ARTICLE INFO

Article History
Received 26 June 2020
Accepted 25 October 2020

Keywords
Lie symmetry analysis
stochastic differential equations
Kolmogorov backward equation

ABSTRACT

The note provides the relation between symmetries and first integrals of Itô stochastic differential equations and symmetries of the associated Kolmogorov Backward Equation (KBE). Relation between the symmetries of the KBE and the symmetries of the Kolmogorov forward equation is also given.

1. INTRODUCTION

Lie group theory of differential equations is well developed [16,27,28]. It studies transformations which take solutions of differential equations into other solutions of the same equations. This theory became a powerful tool for finding analytical solutions of differential equations.

Successful applications of Lie group theory to differential equations motivated its development for other equation types. Here we consider Stochastic Differential Equations (SDEs). First attempts were devoted to transformations which change only the dependent variables, i.e. transformations which do not change time [1,25,26]. After them fiber-preserving transformations were approached [14]. Later there were considered general point transformations in the space of the independent and dependent variables [9,10,35–37]. For them the transformation of the Brownian motion is induced by the random time change. We refer to a review paper [12] for symmetry development and to a chapter [24] for symmetry applications. More general framework includes transformations which depend on the Brownian motion [11,13,22,23].

One of the applications of the symmetries of SDEs was their relation to symmetries of the associated Kolmogorov Forward Equation (KFE), which is also known as the Fokker–Planck equation in physics [30]. First, this symmetry relation was treated in Gaeta and Quintero [14] for fiber-preserving symmetries. Later, it was considered for symmetries in the space of the independent and dependent variables in Ünal [36]. In Kozlov [20] (see also [21]) a more precise formulation of the symmetry relation was provided. There is also a relation between first integrals of the SDEs and symmetries of the KFE [20]. There are many papers devoted to symmetries of particular Fokker–Planck equations [3,4,8,31–34].

Symmetries of the Kolmogorov Backward Equation (KBE) received much less attention than symmetries of the KFE. The KBE is useful when one is interested whether at some future time the system will be in a target set, i.e. in a specified subset of states. In De Vecchi et al. [5] the authors considered symmetries of a KBE with diffusion matrix [matrix A_{ij} in Eq. (3.2)] of full rank. For such equations corresponding to autonomous SDEs and time changes restricted to scalings it was shown that symmetries of the SDEs are also symmetries of the KBE. The paper [6] examines more general stochastic transformations able to change the underlying probability measure. In this framework the weak extended symmetries of SDEs are more general than the Lie point symmetries of the KBE.

In the present note we consider Lie point symmetries of the KBE and examine how these symmetries can be related to the strong symmetries of the underlying SDEs without the restrictions which were imposed in De Vecchi et al. [5]. We also consider the Lie point symmetries of the KBE corresponding to first integrals of the underlying SDEs and show how the symmetries of KBEs are related to the symmetries of KFEs corresponding to the same underlying SDEs.

The paper is organized as follows. In the next section we recall basic results on Itô SDEs and their symmetries. In Section 3 we examine Lie point symmetries of the KBE and find out how they can be related to the symmetries of the SDEs and to the symmetries of the KFE. Finally, in Section 4 we consider scalar SDEs and $(1 + 1)$-dimensional Kolmogorov equations to illustrate the theoretical results of this paper. The last section also illustrates the theory on an example of geometric Brownian motion.

*Email: *Roman.Kozlov@nhh.no*

2. SDEs AND LIE POINT SYMMETRIES

Let us consider a system of stochastic differential equations in Itô form

$$dx_i = f_i(t, \mathrm{x})dt + g_{i\alpha}(t, \mathrm{x})dW_\alpha(t), \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m, \tag{2.1}$$

where $f_i(t, \mathrm{x})$ is a drift vector, $g_{i\alpha}(t, \mathrm{x})$ is a diffusion matrix and $W_\alpha(t)$ is a vector Wiener process (vector Brownian motion) [2,7,15,17,29]. We assume summation over repeated indexes and use notation $\mathrm{x} = (x_1, \ldots, x_n)$. Let us remark that $W_\alpha(t)$, $\alpha = 1, \ldots, m$ are independent one-dimensional Brownian motions.

2.1. Itô Formula

The transformation of the dependent variables in stochastic calculus is given by Itô formula (see, for example, [29]). For SDEs (2.1) we perform variable change $x \to y = y(t, \mathrm{x})$ according to

$$dy_i = \frac{\partial y_i}{\partial t}dt + \frac{\partial y_i}{\partial x_j}dx_j + \frac{1}{2}\frac{\partial^2 y_i}{\partial x_j \partial x_k}dx_j dx_k, \; i = 1, \ldots, n, \tag{2.2}$$

where $dx_j dx_k$ are found with the help of the substitution rules

$$dt \cdot dt = 0, \tag{2.3a}$$

$$dt \cdot dW_\alpha = dW_\alpha \cdot dt = 0, \tag{2.3b}$$

$$dW_\alpha \cdot dW_\beta = \delta_{\alpha\beta} dt. \tag{2.3c}$$

Thus, we obtain the formula for differentials in stochastic calculus

$$dF(t, \mathrm{x}) = D_0(F)dt + D_\alpha(F)dW_\alpha(t), \tag{2.4}$$

where

$$D_0 = \frac{\partial}{\partial t} + f_j \frac{\partial}{\partial x_j} + \frac{1}{2} g_{j\alpha} g_{k\alpha} \frac{\partial^2}{\partial x_j \partial x_k}, \; D_\alpha = g_{j\alpha} \frac{\partial}{\partial x_j}. \tag{2.5}$$

2.2. First Integrals

Stochastic differential equations can possess first integrals.

Definition 2.1.

A quantity $I(t, \mathrm{x})$ is a first integral of a system of SDEs (2.1) if it remains constant on the solutions of the SDEs.

Application of the Itô differential formula (2.4) to a first integral

$$dI(t, \mathrm{x}) = D_0(I)dt + D_\alpha(I)dW_\alpha(t) = 0$$

leads to a system of partial differential equations

$$D_0(I) = 0, \tag{2.6a}$$

$$D_\alpha(I) = 0. \tag{2.6b}$$

2.3. Determining Equations

We will be interested in infinitesimal group transformations (near identity changes of variables) in the space of the independent and dependent variables

$$\bar{t} = \bar{t}(t, \mathrm{x}, a) \approx t + \tau(t, \mathrm{x})a, \quad \bar{x}_i = \bar{x}_i(t, \mathrm{x}, a) \approx x_i + \xi_i(t, \mathrm{x})a, \tag{2.7}$$

which leave Eq. (2.1) and framework of Itô calculus invariant. Such transformations can be represented by generating operators of the form

$$X = \tau(t, \mathrm{x})\frac{\partial}{\partial t} + \xi_i(t, \mathrm{x})\frac{\partial}{\partial x_i}. \tag{2.8}$$

The determining equations for Lie point transformations (2.7) of Itô SDEs (2.1) were derived in Ünal [36]. It is convenient to present them with the help of the operators D_0 and D_α given in Eq. (2.5). The determining equations take a compact form

$$D_0(\xi_i) - X(f_i) - f_i D_0(\tau) = 0, \tag{2.9a}$$

$$D_\alpha(\xi_i) - X(g_{i\alpha}) - \frac{1}{2} g_{i\alpha} D_0(\tau) = 0, \tag{2.9b}$$

$$D_\alpha(\tau) = 0. \tag{2.9c}$$

In Kozlov [23] it was shown that one can also obtain these determining equations by restriction of more general transformations which involve Brownian motion. The Lie point symmetries (2.8) of Itô SDEs, which are given by the determining equations (2.9a)–(2.9c), form a Lie algebra [20].

3. SYMMETRIES OF KOLMOGOROV BACKWARD EQUATION

In this section we derive the determining equations for Lie point symmetries of the KBE and find out how these symmetries can be related to the symmetries and first integrals of SDEs. Later we show how these symmetries can be related to the symmetries of the KFE.

For SDEs (2.1) the associated KBE has the form

$$-\frac{\partial u}{\partial t} = f_i(t, \mathrm{x})\frac{\partial u}{\partial x_i} + \frac{1}{2} g_{i\alpha}(t, \mathrm{x}) g_{j\alpha}(t, \mathrm{x}) \frac{\partial^2 u}{\partial x_i \partial x_j}. \tag{3.1}$$

For symmetry analysis we rewrite it as

$$u_t + A_{ij} u_{x_i x_j} + B_k u_{x_k} = 0, \tag{3.2}$$

where

$$A_{ij} = \frac{1}{2} g_{i\alpha} g_{j\alpha}, \quad B_k = f_k.$$

In what follows we will assume that A_{ij} are not all zero.

3.1. Determining Equations

Let us find Lie point symmetries

$$X_{\mathrm{KB}} = \tau(t, \mathrm{x}, u)\frac{\partial}{\partial t} + \xi_i(t, \mathrm{x}, u)\frac{\partial}{\partial x_i} + \eta(t, \mathrm{x}, u)\frac{\partial}{\partial u} \tag{3.3}$$

which are admitted by the KBE. For our purpose we need the prolonged symmetry vector field

$$\mathbf{pr}^{(2)} X_{\mathrm{KB}} = \tau\frac{\partial}{\partial t} + \xi_i \frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_i \frac{\partial}{\partial u_{x_i}} + \zeta_{ij}\frac{\partial}{\partial u_{x_i x_j}}, \tag{3.4}$$

where the coefficients are computed according to the standard prolongation formulas

$$\zeta_t = D_t(\eta) - u_t D_t(\tau) - u_{x_j} D_i(\xi_j),$$
$$\zeta_i = D_i(\eta) - u_t D_i(\tau) - u_{x_j} D_i(\xi_j),$$
$$\zeta_{ij} = D_i(\zeta_j) - u_{tx_j} D_i(\tau) - u_{x_k x_j} D_i(\xi_k).$$

Here D_t and D_i are total differentiation operators with respect to t na x_i.

Infinitesimal invariance criteria [16,27,28] states that the application of the second prolongation of the operator X_{KB} to the second order PDE (3.2) should be zero on the solutions of this PDE:

$$\mathbf{pr}^{(2)} X_{\mathrm{KB}}(u_t + A_{ij} u_{x_i x_j} + B_k u_{x_k})\Big|_{(3.2)} = 0. \tag{3.5}$$

We review briefly the derivation of the determining equations for symmetries of the KBE. It is convenient to use notations

$$F_{,t}=\frac{\partial F}{\partial t},\ F_{,u}=\frac{\partial F}{\partial u}\ \text{and}\ F_{,i}=\frac{\partial F}{\partial x_i}$$

(the last notation uses indexes different from t and u).

Equation (3.5) splits for different spatial derivatives of u. We obtain

$$\tau_{,u}(A_{ij}A_{pq}u_{x_p x_q x_j}u_{x_i})=0$$

for products of third derivatives with first derivatives that leads to

$$\tau_{,u}=0$$

and

$$(A_{ij}\tau_{,i})(A_{pq}u_{x_p x_q x_j})=0$$

for third derivatives that gives

$$A_{ij}\tau_{,i}=0. \tag{3.6}$$

Then, for products of second derivatives with first derivatives we get the equations

$$\xi_{k,u}(A_{ij}u_{x_k x_i}u_{x_j})=0,$$

which give

$$\xi_{i,u}=0,$$

and as coefficients for the second derivatives we obtain equations

$$\tau A_{ij,t}+\xi_k A_{ij,k}+A_{ij}(\tau_{,t}+B_p\tau_{,p}+A_{pq}\tau_{,pq})-A_{ik}\xi_{j,k}-A_{kj}\xi_{i,k}=0. \tag{3.7}$$

For products of first derivatives we obtain

$$\eta_{,uu}(A_{ij}u_{x_i}u_{x_j})=0.$$

Therefore,

$$\eta=\varphi(t,\mathrm{x})u+\psi(t,\mathrm{x}).$$

Substituting it into the rest of Eq. (3.5), we get

$$\xi_{i,t}+B_p\xi_{i,p}+A_{pq}\xi_{i,pq}-\tau B_{i,t}-\xi_p B_{i,p}-2A_{ij}\varphi_{,j}-B_i(\tau_{,t}+B_p\tau_{,p}+A_{pq}\tau_{,pq})=0, \tag{3.8}$$

$$\varphi_{,t}+B_k\varphi_{,k}+A_{ij}\varphi_{,ij}=0 \tag{3.9}$$

and

$$\psi_{,t}+B_k\psi_{,k}+A_{ij}\psi_{,ij}=0 \tag{3.10}$$

for the terms with the first derivatives, the terms with u and the rest, respectively.

We can summarize the obtained results using the operators D_0 and D_α, which were given in Eq. (2.5).

Theorem 3.1.

Lie point symmetries of KBE (3.1) are given by

1. vector fields of the form

$$X_{\mathrm{KB}}=\tau(t,\mathrm{x})\frac{\partial}{\partial t}+\xi_i(t,\mathrm{x})\frac{\partial}{\partial x_i}+\varphi(t,\mathrm{x})u\frac{\partial}{\partial u} \tag{3.11}$$

with coefficients satisfying equations

$$g_{i\alpha} D_{\alpha}(\tau) = 0, \tag{3.12a}$$

$$g_{i\alpha}\left(D_{\alpha}(\xi_j) - X(g_{j\alpha}) - \frac{1}{2} g_{j\alpha} D_0(\tau)\right) + g_{j\alpha}\left(D_{\alpha}(\xi_i) - X(g_{i\alpha}) - \frac{1}{2} g_{i\alpha} D_0(\tau)\right) = 0, \tag{3.12b}$$

$$D_0(\xi_i) - X(f_i) - f_i D_0(\tau) = g_{i\alpha} D_{\alpha}(\varphi), \tag{3.12c}$$

$$D_0(\varphi) = 0 \tag{3.12d}$$

and

2. trivial symmetries

$$X^{*}_{\mathrm{KB}} = \psi(t, \mathrm{x}) \frac{\partial}{\partial u}, \tag{3.13}$$

where the coefficient is an arbitrary solution of the KBE, corresponding to the linear superposition principle.

The proof follows from the previous discussion of the equations for symmetry coefficients. In particular, Eqs. (3.12a)–(3.12d) for coefficients of the symmetry (3.11) represent Eqs. (3.6)–(3.9), which are rewritten with the help of the operators D_0 and D_α. The coefficient of the symmetry (3.13) satisfies Eq. (3.10).

Remark 3.2.

We see that the determining equations (3.12a)–(3.12d) always have a particular solution

$$\tau = 0, \quad \xi_1 = \ldots = \xi_n = 0, \quad \varphi = \text{const.}$$

It provides us with symmetry

$$X_0 = u \frac{\partial}{\partial u}, \tag{3.14}$$

corresponding to linearity of the KBE.

3.2. Symmetries of KBE and Symmetries of SDEs

Now we can relate symmetries of the SDEs to the symmetries of the associated KBE.

Theorem 3.3.

Let operator X of the form (2.8) be a symmetry of the SDEs (2.1), then X is also a symmetry of the associated KBE.

Proof. From the determining equations (2.9b) and (2.9c) it follows that Eqs. (3.12a) and (3.12b) hold. Choosing $\varphi \equiv 0$, which is always a solution of Eq. (3.12c) [if Eq. (2.9a) hold] and (3.12d), we get X as a symmetry of the KBE.

We can also relate some symmetries of the KBE to first integrals of the SDEs.

Theorem 3.4.

Let SDEs (2.1) possess a first integral I(t, x), then the associated KBE admits symmetry

$$Y = I(t, \mathrm{x}) u \frac{\partial}{\partial u}. \tag{3.15}$$

Proof. It follows from Eqs. (2.6a) and (2.6b) that the determining equations (3.12) for symmetries of the KBE are satisfied.

It is possible to state the converse results.

Theorem 3.5.

If KBE (3.1), which corresponds to SDEs (2.1), admits a symmetry X of the form (2.8) with coefficients satisfying equations (2.9b) and (2.9c), then the symmetry X is admitted by the SDEs.

Theorem 3.6.

If KBE (3.1), which correspond to SDEs (2.1), admits a symmetry of the form (3.15) and function I(t, x) satisfies the Eq. (2.6b), then I(t, x) is a first integral of the SDEs.

The additional requirements of Theorems 3.5 and 3.6 are not surprising. They specify the particular SDEs: the same KBE can correspond to different SDEs, which have the same drift coefficients f_i and diffusion matrix $A_{ij} = \frac{1}{2} g_{i\alpha} g_{j\alpha}$.

Finally, we summarize the results of this point by presenting four types of Lie point symmetries of the KBE. They are:

1. symmetries (3.13) and (3.14) corresponding to linearity of the KBE
2. symmetries (2.8) which are related to the symmetries of the SDEs
3. symmetries (3.15) which are related to the first integrals of the SDEs
4. the other symmetries, which are not related to the SDEs

3.3. Symmetries of KBE and Symmetries of KFE

For SDEs (2.1) the corresponding KFE, which is also called Fokker–Planck equation [30], takes the form

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x_i}(f_i(t, \mathrm{x})u) + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(g_{i\alpha}(t, \mathrm{x}) g_{j\alpha}(t, \mathrm{x}) u). \tag{3.16}$$

The relation of symmetries of KFE and the symmetries of the underlying SDEs was considered in several papers [14,20,36]. The most general results were established in Kozlov [20]. They were based on the following description of the symmetries of the KFE.

Theorem 3.7.

Lie point symmetries of KFE (3.16) are given by

1. vector fields of the form

$$X_{\mathrm{KF}} = \tau(t, \mathrm{x})\frac{\partial}{\partial t} + \xi_i(t, \mathrm{x})\frac{\partial}{\partial x_i} + \chi(t, \mathrm{x})u\frac{\partial}{\partial u} \tag{3.17}$$

with coefficients satisfying equations

$$g_{i\alpha} D_\alpha(\tau) = 0, \tag{3.18a}$$

$$g_{i\alpha}\left(D_\alpha(\xi_j) - X(g_{j\alpha}) - \frac{1}{2} g_{j\alpha} D_0(\tau)\right) + g_{j\alpha}\left(D_\alpha(\xi_i) - X(g_{i\alpha}) - \frac{1}{2} g_{i\alpha} D_0(\tau)\right) = 0, \tag{3.18b}$$

$$Q = \chi + \xi_{i,i} + \tau_{,t} - D_0(\tau), \tag{3.18c}$$

where function Q(t, x) is a solution of equations

$$D_0(\xi_i) - X(f_i) - f_i D_0(\tau) = -g_{i\alpha} D_\alpha(Q), \tag{3.19a}$$

$$D_0(Q) = 0, \tag{3.19b}$$

and

2. trivial symmetries

$$X_{\mathrm{KF}}^{*} = \psi(t, \mathrm{x})\frac{\partial}{\partial u}, \tag{3.20}$$

where the coefficient is an arbitrary solution of the KFE, corresponding to the linear superposition principle.

By direct comparison of the determining equations given in Theorems 3.1 and 3.7 we can establish the following result.

Theorem 3.8.

Let us consider KBE (3.1) and KFE (3.16) corresponding to the same SDEs (2.1). The KBE admits symmetry (3.11) if and only if the KFE admits symmetry (3.17) and

$$\varphi + \chi = D_0(\tau) - \tau_{,t} - \xi_{i,i}.$$

Proof. The result follows from the observation that the sets of variables $(\tau, \xi_1, ..., \xi_n, \varphi)$ and $(\tau, \xi_1, ..., \xi_n, -Q)$ satisfy the same equations.

Corollary 3.9.

Let us consider KBE (3.1) and KFE (3.16) corresponding to the same SDEs (2.1). The KBE admits symmetry (3.15) if and only if the KFE admits the same symmetry.

4. SCALAR SDEs AND (1 + 1)-DIMENSIONAL KOLMOGOROV EQUATIONS

Let us illustrate how one can use symmetries of the scalar SDEs

$$dx = f(t, x)dt + g(t, x)dW(t), \quad g(t, x) \not\equiv 0 \tag{4.1}$$

to find symmetries of the KBE

$$-\frac{\partial u}{\partial t} = f(t, x)\frac{\partial u}{\partial x} + \frac{1}{2}G(t, x)\frac{\partial^2 u}{\partial x^2}, \quad G(t, x) = g^2(t, x) \geq 0, \not\equiv 0. \tag{4.2}$$

Lie point symmetries of the KBE (4.2) are described by Theorem 3.1. They are symmetries

$$X_{\text{KB}} = \tau(t, x)\frac{\partial}{\partial t} + \xi(t, x)\frac{\partial}{\partial x} + \varphi(t, x)u\frac{\partial}{\partial u} \tag{4.3}$$

with coefficients satisfying equation

$$D_W(\tau) = 0, \tag{4.4a}$$

$$D_W(\xi) - X(g) - \frac{1}{2}gD_0(\tau) = 0, \tag{4.4b}$$

$$D_0(\xi) - X(f) - fD_0(\tau) = gD_W(\varphi), \tag{4.4c}$$

$$D_0(\varphi) = 0, \tag{4.4d}$$

where

$$D_0 = \frac{\partial}{\partial t} + f\frac{\partial}{\partial x} + \frac{1}{2}g^2\frac{\partial^2}{\partial x^2}, \quad D_W = g\frac{\partial}{\partial x}, \tag{4.5}$$

and trivial symmetries (3.13). Note that from (4.4a) we get $\tau = \tau(t)$.

In the general case the KBE (4.2) has only symmetries related to its linearity, namely

$$X^*_{\text{KB}} = \psi(t, x)\frac{\partial}{\partial u} \quad \text{and} \quad X_0 = u\frac{\partial}{\partial u}, \tag{4.6}$$

where $\psi(t, x)$ is an arbitrary solution of the KBE. For particular cases $f(t, x)$ and $G(t, x)$ there can be additional symmetries.

4.1. Symmetries of KBE via Symmetries of SDEs

Lie group classification of the scalar SDE (4.1) was carried out in Kozlov [18] by direct method. Alternatively, one can obtain this Lie group classification with the help of real Lie algebra realizations by vector fields. It was done in Kozlov [19].

In the general case the SDE (4.1) has no symmetries. Therefore, the KBE (4.2) admits only symmetries (4.6) corresponding to its linearity. We shall go through the cases of the Lie group classification of the scalar SDEs (4.1) and find the symmetries of the corresponding KBEs. It should be noted that we can always choose a representative SDE for each equivalence symmetry class in the form

$$dx = f(t, x)dt + dW(t) \tag{4.7}$$

because we can perform the variable change

$$x \rightarrow \int \frac{dx}{g(t, x)}.$$

The corresponding KBE is also simplified. It takes the form

$$-\frac{\partial u}{\partial t} = f(t, x)\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}. \tag{4.8}$$

4.1.1. SDE with one symmetry

The equivalence class of the SDEs admitting only one symmetry

$$X_1 = \frac{\partial}{\partial t} \tag{4.9}$$

can be represented by the equation

$$dx = f(x)dt + dW(t). \tag{4.10}$$

The corresponding KBE

$$-u_t = f(x)u_x + \frac{1}{2}u_{xx} \tag{4.11}$$

admits symmetries (4.6) and (4.9).

4.1.2. SDE with two symmetries

For the SDEs admitting two symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \tag{4.12}$$

one can chose the following representative equation

$$dx = \frac{A}{x}dt + dW(t), \quad A \neq 0. \tag{4.13}$$

For the KBE

$$-u_t = \frac{A}{x}u_x + \frac{1}{2}u_{xx} \tag{4.14}$$

we get two subcases

1. $A \neq 1$

In addition to symmetries (4.6) and (4.12) the KBE admits symmetry

$$Y_1 = 2t^2\frac{\partial}{\partial t} + 2tx\frac{\partial}{\partial x} + (x^2 - (1+2A)t)u\frac{\partial}{\partial u}. \tag{4.15}$$

2. $A = 1$

In this particular case the KBE possesses symmetries (4.6), (4.12), (4.15) and

$$Y_2 = t\frac{\partial}{\partial x} + \left(x - \frac{t}{x}\right)u\frac{\partial}{\partial u}, \quad Y_3 = \frac{\partial}{\partial x} - \frac{u}{x}\frac{\partial}{\partial u}. \tag{4.16}$$

4.1.3. SDE with three symmetries

Scalar SDEs can admit at most three symmetries. The equivalence class for SDEs with three symmetries can be represented by the equation

$$dx = dW(t), \tag{4.17}$$

which admits symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial x}. \tag{4.18}$$

In this case we get the KBE

$$-u_t = \frac{1}{2}u_{xx} \tag{4.19}$$

admits symmetries (4.6), (4.18) and

$$Y_1 = t\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \quad Y_2 = 2t^2\frac{\partial}{\partial t} + 2tx\frac{\partial}{\partial x} + (x^2 - t)u\frac{\partial}{\partial u}. \tag{4.20}$$

Remark 4.1.

The KBE (4.14) with $A = 1$, namely the equation

$$-u_t = \frac{1}{x}u_x + \frac{1}{2}u_{xx},$$

can be transformed into the KBE (4.19) by the change of the dependent variable

$$u = \frac{1}{x}\bar{u}.$$

However, the SDE (4.13) with $A = 1$ cannot be transformed into the SDE (4.17).

We cannot expect that Lie group classification of SDEs will provide us with Lie group classification of the associated KBE. Indeed, it gives only partial results on the symmetries of specified KBEs as we will see in the next point.

4.2. Lie Group Classification of (1 + 1)-Dimensional KBE

Lie group classification of the (1 + 1)-dimensional KFE

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(f(t, x)u) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(G(t, x)u) \tag{4.21}$$

is knows [34]. It can be used to obtain Lie group classification of the (1 + 1)-dimensional KBE with the help of Theorem 3.8, which relates symmetries of the KBE and KFE.

In addition to the symmetries

$$X^*_{\mathrm{KF}} = \psi(t, x)\frac{\partial}{\partial u} \quad \text{and} \quad X_0 = u\frac{\partial}{\partial u}, \tag{4.22}$$

where $\psi(t, x)$ is an arbitrary solution of the KFE, the KBE (4.21) can admit 0, 1, 3 or 5 symmetries. Due to Theorem 3.8 we get the same results for the KBE (4.2). It can admit 0, 1, 3 or 5 symmetries in addition to symmetries (4.6).

Lie group classification of the KBE (obtained with the help of Lie group classification of the KFE) can be compared with results of the previous point. We find out the following.

- Using Lie group classification of the scalar SDEs, we obtain correct description of the equivalence classes for KBEs admitting 0, 1 and 5 symmetries in addition to the symmetries (4.6). These equivalence classes are represented by Eqs. (4.8), (4.11) and (4.19), respectively.
- However, we do not obtain the correct description of the equivalence class for the KBEs admitting three additional symmetries. It is easy to see from the next theorem.

Theorem 4.2

([34]). KFEs (4.21) admitting three symmetries in addition to the linearity symmetries (4.22) can be transformed into the equation

$$u_t = (2k'(x)u)_x + u_{xx}, \tag{4.23}$$

where $k(x)$ is a solution of the equation

$$k'' - (k')^2 = \frac{\lambda}{x^2}, \ \lambda \neq 0. \tag{4.24}$$

In addition to the symmetries (4.22) Eq. (4.23) admits operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - xk'(x)u\frac{\partial}{\partial u},$$
$$X_3 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2 + 4t + 4txk'(x))u\frac{\partial}{\partial u},$$

where $k(x)$ is a solution of Eq. (4.24).

Using Theorem 3.8, which relates symmetries of the KBE and KFE, we state a similar result for the KBE.

Corollary 4.3.

KBEs (4.2) admitting three symmetries in addition to the linearity symmetries (4.6) can be transformed into the equation

$$-u_t = -2k'(x)u_x + u_{xx}, \tag{4.25}$$

where $k(x)$ is a solution of Eq. (4.24).

In addition to the symmetries (4.6) Eq. (4.25) admits operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + xk'(x)u\frac{\partial}{\partial u},$$
$$X_3 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + (x^2 + 4txk'(x))u\frac{\partial}{\partial u},$$

where $k(x)$ is a solution of Eq. (4.24). For

$$k'(x) \neq \frac{A}{x}$$

symmetry X_2 (up to factorization by X_0) is beyond the framework based on the symmetries of the underlying SDE. Thus, with the help of the scalar SDE classification we get only a subcase of the class of KBEs admitting three additional symmetries.

Therefore, using the Lie group classification of the scalar SDEs, we get *partial* results of the Lie group classification of the (1 + 1)-dimensional KBE. The same was observed for the (1 + 1)-dimensional KFE in Kozlov [21].

4.3. The Geometric Brownian Motion Equation

Let us examine the geometric Brownian motion [29]

$$dx = \alpha x dt + \sigma x dW(t), \quad \sigma \neq 0 \tag{4.26}$$

as a theory application. This SDE is an important model for stochastic prices in economics.

The SDE (4.26) admits three symmetries of the form (2.8), namely

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t\frac{\partial}{\partial t} + \left(\left(\alpha - \frac{\sigma^2}{2} \right) tx + x \ln x \right) \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial x} \tag{4.27}$$

and has no first integrals.

The associated KBE (4.2) takes the form

$$-\frac{\partial u}{\partial t} = \alpha x \frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}. \tag{4.28}$$

It admits the trivial symmetries (4.6) because of its linearity. Theorem 3.3 states that the symmetries (4.27) are also admitted by this KBE. Direct computation of the symmetries of the form (2.7) provides us with the two additional symmetries

$$Y_1 = tx\frac{\partial}{\partial x} - \frac{1}{\sigma^2}\left(\left(\alpha - \frac{\sigma^2}{2} \right) t - \ln x \right) u \frac{\partial}{\partial u}, \tag{4.29a}$$

$$Y_2 = 2t^2\frac{\partial}{\partial t} + 2tx\ln x\frac{\partial}{\partial x} + \left(\frac{1}{\sigma^2}\left(\left(\alpha - \frac{\sigma^2}{2} \right) t - \ln x \right)^2 - t \right) u \frac{\partial}{\partial u}. \tag{4.29b}$$

The KFE (4.21) for the geometric Brownian motion equation takes the form

$$\frac{\partial u}{\partial t} = -\alpha \frac{\partial}{\partial x}(xu) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}(x^2 u). \tag{4.30}$$

It is invariant with respect to symmetries (4.22). The other symmetries can be obtained with the help of Theorem 3.8 and the symmetries of the KBE (4.28). We find

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t\frac{\partial}{\partial t} + \left(\left(\alpha - \frac{\sigma^2}{2} \right) tx + x\ln x \right)\frac{\partial}{\partial x} - \left(\left(\alpha - \frac{\sigma^2}{2} \right) t + \ln x \right) u\frac{\partial}{\partial u}, \\
&X_3 = x\frac{\partial}{\partial x}, \quad Y_1 = tx\frac{\partial}{\partial x} + \left(\frac{1}{\sigma^2}\left(\left(\alpha - \frac{\sigma^2}{2} \right) t - \ln x \right) - t \right) u\frac{\partial}{\partial u}, \\
&Y_2 = 2t^2\frac{\partial}{\partial t} + 2tx\ln x\frac{\partial}{\partial x} - \left(\frac{1}{\sigma^2}\left(\left(\alpha - \frac{\sigma^2}{2} \right) t - \ln x \right)^2 + 2t\ln x + t \right) u\frac{\partial}{\partial u}.
\end{aligned}$$

Remark 4.4.

Let us note that symmetries of SDEs can be used to find symmetries of the associated KFE [20]. A symmetry (2.8) of the SDEs (2.1) provides with the symmetry

$$\bar{X} = X + (D_0(\tau) - \tau_{,t} - \xi_{i,i}) u \frac{\partial}{\partial u} \tag{4.31}$$

admitted by the associated KFE. This results can be used to find the symmetries X_1, X_2 and X_3 for the KFE (4.30).

CONFLICTS OF INTEREST

The author declares no conflicts of interest.

REFERENCES

[1] S. Albeverio, S.M. Fei, A remark on symmetry of stochastic dynamical systems and their conserved quantities, J. Phys. A 28 (1995), 6363–6371.
[2] L. Arnold, Stochastic differential equations: theory and applications, Wiley-Interscience, New York, London, Sydney, 1974.

[3] G. Cicogna, D. Vitali, Generalised symmetries of Fokker-Planck-type equations, J. Phys. A 22 (1989).
[4] G. Cicogna, D. Vitali, Classification of the extended symmetries of Fokker-Planck equations, J. Phys. A 23 (1990).
[5] F.C. De Vecchi, P. Morando, S. Ugolini, A note on symmetries of diffusions within a martingale problem approach, Stochast. Dyn. 19 (2019), 1950011.
[6] F.C. De Vecchi, P. Morando, S. Ugolini, Symmetries of stochastic differential equations using Girsanov transformations, J. Phys. A 53 (2020), 135204.
[7] L.C. Evans, An introduction to stochastic differential equations, American Mathematical Society, Rhode Island, US, 2013.
[8] F. Finkel, Symmetries of the Fokker-Planck equation with a constant diffusion matrix in 2 + 1 dimensions, J. Phys. A 32 (1999), 2671–2684.
[9] E. Fredericks, F.M. Mahomed, Symmetries of first-order stochastic ordinary differential equations revisited, Math. Methods Appl. Sci. 30 (2007), 2013–2025.
[10] E. Fredericks, F.M. Mahomed, A formal approach for handling Lie point symmetries of scalar first-order Itô stochastic ordinary differential equations, J. Nonlinear Math. Phys. 15 (2008), 44–59.
[11] G. Gaeta, Lie-point symmetries and stochastic differential equations: II, J. Phys. A 33 (2000), 4883.
[12] G. Gaeta, Symmetry of stochastic non-variational differential equations, Phys. Rep. 686 (2017), 1–62 [G. Gaeta, Erratum for "Symmetry of stochastic non-variational differential equations", Phys. Rep. 713 (2017) 18].
[13] G. Gaeta, F. Spadaro, Random Lie-point symmetries of stochastic differential equations, J. Math. Phys. 58 (2017), 053503 [G. Gaeta, F. Spadaro, Erratum: "Random Lie-point symmetries of stochastic differential equations", J. Math. Phys. 58 (2017) 129901].
[14] G. Gaeta, N.R. Quintero, Lie-point symmetries and stochastic differential equations, J. Phys. A 32 (1999), 8485–8505.
[15] T.C. Gard, Introduction to stochastic differential equations, Marcel Dekker, Inc., New York, 1988.
[16] N.H. Ibragimov, Transformation groups applied to mathematical physics, D. Reidel Publishing Co., Dordrecht, 1985.
[17] N. Ikeda, S. Watanabe, Stochastic differential equations and diffusion processes, North-Holland Publishing Co., Amsterdam, 1981.
[18] R. Kozlov, Group classification of a scalar stochastic differential equation, J. Phys. A 43 (2010), 055202.
[19] R. Kozlov, Lie point symmetries of Stratonovich stochastic differential equations, J. Phys. A Math. Theor. 51 (2018), 505201.
[20] R. Kozlov, On Lie group classification of a scalar stochastic differential equation, J. Nonlinear Math. Phys. 18 (2011), 177–187.
[21] R. Kozlov, On symmetries of stochastic differential equations, Commun. Nonlinear Sci. Numer. Simul. 17 (2012), 4947–4951.
[22] R. Kozlov, On symmetries of the Fokker-Planck equation, J. Eng. Math. 82 (2013), 39–57.
[23] R. Kozlov, Random Lie symmetries of Itô stochastic differential equations, J. Phys. A Math. Theor. 51 (2018), 305203.
[24] R. Kozlov, Symmetries of Itô stochastic differential equations and their applications, in: N. Euler, ed., Nonlinear Systems and their Remarkable Mathematical Structures, Boca Raton, Florida: CRC Press, 2018.
[25] T. Misawa, Conserved quantities and symmetry for stochastic dynamical systems, Phys. Lett. A 195 (1994), 185–189.
[26] T. Misawa, New conserved quantities derived from symmetry for stochastic dynamical systems, J. Phys. A Math. Gen. 27 (1994), L777.
[27] P.J. Olver, Applications of Lie groups to differential equations, Springer-Verlag, New York, 1993.
[28] L.V. Ovsiannikov, Group analysis of differential equations, Academic Press, Inc., New York, London, 1982.
[29] B. Øksendal, Stochastic differential equations, sixth ed., Springer-Verlag, Berlin, 2003.
[30] H. Risken, The Fokker-Planck equation: methods of solution and applications, Berlin Heidelberg, Springer-Verlag, 1996.
[31] P. Rudra, Symmetry classes of Fokker-Planck-type equations, J. Phys. A 23 (1990), 1663–1670.
[32] C.C.A. Sastri, K.A. Dunn, Lie symmetries of some equations of the Fokker–Planck type, J. Math. Phys. 26 (1985), 3042–3047.
[33] W.M. Shtelen, V.I. Stogny, Symmetry properties of one- and two-dimensional Fokker–Planck equations, J. Phys. A Math. Gen. 22 (1989), L539.
[34] S. Spichak, V. Stognii, Symmetry classification and exact solutions of the one-dimensional Fokker-Planck equation with arbitrary coefficients of drift and diffusion, J. Phys. A 32 (1999), 8341–8353.
[35] B. Srihirun, S.V. Meleshko, E. Schulz, On the definition of an admitted Lie group for stochastic differential equations, Commun. Nonlinear Sci. Numer. Simul. 12 (2007), 1379–1389.
[36] G. Ünal, Symmetries of Itô and Stratonovich dynamical systems and their conserved quantities, Nonlinear Dyn. 32 (2003), 417–426.
[37] C. Wafo Soh, F.M. Mahomed, Integration of stochastic ordinary differential equations from a symmetry standpoint, J. Phys. A 34 (2001), 177–194.