k-Metric Dimension of Generalized Fan Graph and $C_m \ast 2 K_n$ Graph

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Abstract

Let $G$ be a simple connected graph with the vertex set $V(G)$ and the set edge $E(G)$. The set $S \subseteq V(G)$ is called the k-metric generator on $G$ if and only if every two different vertices in $G$ are distinguished by at least $k$ elements in $S$, in other words for every two different vertices $u, v \in V(G)$, there are at least $k$ vertices $w_1, w_2, ..., w_k \in S$ such that $d(u, w_i) \neq d(v, w_i)$, for every $i \in \{1, ..., k\}$. The k-metric generator with the smallest cardinality is called the k-metric base and the cardinality of the k-metric base is on call the k-metric dimension of the graph $G$ denoted $\dim_k(G)$. In this study, the k-metric dimension was obtained in generalized fan $F_{m,n}$ graph with $m \geq 1$, $n \geq 3$ and $C_m \ast 2 K_n$ graph with $m \geq 3$, $n \geq 2$.

Keywords: k-metric dimension, k-metric generator, basis of k-metric, generalized fan $F_{m,n}$ graph, $C_m \ast 2 K_n$ graph

1. Introduction

Mathematics is a science that has developed and can be applied in various fields, one of which is graph theory. Various in this field include science, computing, and networking. Graph theory can be presenting real problems in the form of graphs. A graph $G$ according to Chartrand et al. [1] is a finite non-empty set $V(G) = \{v_1, v_2, ..., v_n\}$ called vertices and with a possibly empty set $E(G) = \{e_1, e_2, ..., e_n\}$ are non-consecutive pairings of members of $V(G)$ incall edge.

One concept in graph theory is the metric dimension. Metric dimension was first introduced by Slater [2] in 1975, which was later in continued by Harary and Melter [3] in 1976. Chartrand et al. [4] defining the differentiating set $W$ as the set of vertices in a graph $G$ such that each vertex in $G$ produces a different distance to a point in $W$. The metric dimension is the smallest cardinality of the differentiating set. Until now the concept of the metric dimension is still being studied and studied develop. Therefore, a new metric dimension concept was developed namely the k-metric dimension presented by Estrada-Moreno et al. [5].

Let $G$ be a connected and simple graph, the set $S \subseteq V(G)$ is called the k-metric generator on $G$ if and only if every two different vertices in $G$ are distinguished by at least $k$ elements in $S$, in other words for every two different vertices $u, v \in V(G)$ there are at least $k$ vertices $w_1, w_2, w_3, ..., w_k \in S$ such that $d(u, w_i) \neq d(v, w_i)$, for every $i \in \{1, ..., k\}$. The k-metric generator with the smallest cardinality is called the k-metric base and the cardinality of the k-metric base is called the k-metric dimension of the graph $G$ denoted $\dim_k(G)$.

In 2015, Estrada-Moreno et al. [5] on the path graph, cycle graph, tree graph, and graph the result of a join operation. In 2016, Estrada Moreno et al. [2] have found the k-metric dimension on the corona operation of two graphs. In 2017 Geetha and Sooryanarayana [6] have found the k-metric dimension in the graph of cartesian product operation result. In 2018 Rahmadi and Susanti [7] found the k-metric dimension in double fan graph, double cones graph, double fan snake graph, centralized double fan graph, generalized parachute graph, and graph parachute generalized with the upper path. The results of these studies can be a reference in finding dimensions k-metric in another graph class. This study
discusses the $k$-metric dimensions in generalized fan $(F_{m,n})$ graph and $C_m \ast_2 K_n$ graph.

2. Main Results

Before going to the main result, first will give the following lemma taken from Estrada-Moreno et al. [8].

Lemma 1. Let $G$ be a connected graph of order $n \geq 2$, $G$ is a 2-metric dimension if and only if $G$ has twin vertices.

A. $k$-metric dimension of $F_{m,n}$ graph

Intaja and Sithiwartham [9] define a generalized fan graph. Generalized fan graph $F_{m,n} \equiv \overline{K}_m + P_n$ is a graph with $V(F_{m,n}) = V(K_m) \cup V(P_n)$ and $E(F_{m,n}) = E(P_n) \cup \{ u,v \in V(K_m), v \in V(P_n) \}$ with $m \geq 1$ and $n \geq 2$. $F_{m,n}$ the graph is illustrated in Fig 1.

![Fig 1. Generalized Fan Graph](image)

From Fig 1, it can be seen that the graph $F_{m,n}$ has $m + n$ vertices. The following TABLE 1 gives the distance of every two vertices in the $F_{m,n}$ graph.

<table>
<thead>
<tr>
<th>Dist</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>...</th>
<th>$v_m$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>...</th>
<th>$u_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>0</td>
<td>2</td>
<td>...</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
</tr>
<tr>
<td>$v_2$</td>
<td>2</td>
<td>0</td>
<td>...</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
</tr>
<tr>
<td>$v_m$</td>
<td>2</td>
<td>2</td>
<td>...</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
</tr>
<tr>
<td>$u_1$</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>...</td>
<td>2</td>
</tr>
<tr>
<td>$u_2$</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>...</td>
<td>2</td>
</tr>
<tr>
<td>$u_n$</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>...</td>
<td>0</td>
</tr>
</tbody>
</table>

Following are the result regarding the $k$-metric dimensions in a graph $F_{m,n}$.

Lemma 2. Let $F_{m,n}$ be a generalized fan graph with $m \geq 2$ and $n \geq 3$, then $F_{m,n}$ is a 2-metric dimension graph.

Proof. It is known that $F_{m,n}$ is a generalized fan graph of order $m + n$. There are $N_{F_{m,n}}(v_1) = N_{F_{m,n}}(v_2) = \cdots = N_{F_{m,n}}(v_n)$, such that $(v_1, v_2, \ldots, v_n)$ is twin vertices. Then $F_{m,n}$ is a 2-metric dimension graph.

The following is given a lower bound for the 2-metric generator cardinality of $F_{m,n}$ graph.

Lemma 3. Let $F_{m,n}$ be a generalized fan graph with $m \geq 2$ and $n = 3, 4$. If $S$ is a 2-metric generator of $F_{m,n}$ then $|S| \geq (m + n) - 1$.

Proof. It is known that $S$ is a 2-metric generator, meaning that for each $u, v \in V(F_{m,n})$ has $W \subset S$ such that $r(u,W) \neq r(v,W)$ with $|W| = 2$. Suppose $S$ is a 2-metric generator with $|S| < (m + n) - 1$. Let $V_1 = \{ u_1, u_2, \ldots, u_n \}$ and $V_2 = \{ v_1, v_2, \ldots, v_m \}$. Write $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$, because $|S_1| + |S_2| < (m + n) - 1$, then there are $u, v \in V_1 \setminus S$ such that $r(u,W) = r(v,W)$ for each $W \subset S$ with $|W| = 2$. This contradicts with the statement that $S$ is a 2-metric generator. Presupposition wrong and must be denied, thus $S$ is not a 2-metric generator. Hence $|S| \geq (m + n) - 1$.

Lemma 4. Let $F_{m,n}$ be a generalized fan graph with $m \geq 2$ and $n \geq 5$. If $S$ is a 2-metric generator of $F_{m,n}$, then $|S| \geq (m + n) - 2$.

Proof. It is known that $S$ is a 2-metric generator, meaning that for each $u, v \in V(F_{m,n})$ has $W \subset S$ such that $r(u,W) \neq r(v,W)$ with $|W| = 2$. Suppose $S$ is a 2-metric generator with $|S| < (m + n) - 2$. Let $V_1 = \{ u_1, u_2, \ldots, u_n \}$ and $V_2 = \{ v_1, v_2, \ldots, v_m \}$. Write $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$, because $|S_1| + |S_2| < (m + n) - 2$, then there are $u, v \in V_1 \setminus S$ such that $r(u,W) = r(v,W)$ for each $W \subset S$ with $|W| = 2$. This contradicts with the statement that $S$ is a 2-metric generator. Presupposition wrong and must be denied, thus $S$ is not a 2-metric generator. Hence $|S| \geq (m + n) - 2$.

Theorem 1. Let $F_{m,n}$ be a graph with $m \geq 1$ and $n \geq 2$, then $\text{dim}_2(F_{m,n}) = \begin{cases} (m + n) - 1, & m \geq 2 \ n = 3, 4; \\ (m + n) - 2, & m \geq 2 \ n \geq 5. \end{cases}$

Proof. Let $F_{m,n}$ be a 2-metric dimension graph for $m \geq 2$ and $n \geq 3$, meaning that there is a 2-metric base on $F_{m,n}$. In this case, the evidence is divided into two cases, namely for $m \geq 2$ with $n = 3, 4$ and for $m \geq 2$ with $n \geq 5$.

Case 1. For $m \geq 2$ and $n = 3, 4$.

For $m \geq 2$ and $n = 3, 4$, take $S = \{ v_1, v_2, \ldots, v_m, u_1, u_2 \}$, will show that $S$ is a 2-metric basis. The following are given a representation of each vertex in $F_{m,3}$ with respect to $S$.

$r(v_1, S) = \{ 0, 2, \ldots, 2, 1, 1 \}$

$r(v_2, S) = \{ 2, 0, \ldots, 2, 1, 1 \}$

$r(v_m, S) = \{ 2, 2, \ldots, 0, 1, 1 \}$

$r(u_1, S) = \{ 1, 1, \ldots, 1, 0, 2 \}$

$r(u_2, S) = \{ 1, 1, \ldots, 1, 1, 1 \}$

$r(u_i, S) = \{ 1, 1, \ldots, 1, 2, 0 \}.$
Based on this representation, if taken a \( W \subset S \) with \(|W| = 2\), then for each \( u, v \in V(F_{m,3}) \) applies \( r(u|W) \neq r(v|W) \).

- For \( m \geq 2 \) and \( n = 3 \).

Take \( S = \{ v_1, v_2, \ldots, v_m, u_1, u_2, u_3 \} \), will show that \( S \) is a 2-metric basis. The following are given a representation of each vertex in \( F_{m,4} \) with respect to \( S \):

\[
\begin{align*}
    r(v_1|S) &= \{0,2,\ldots,2,1,1,1\}; \\
    r(v_2|S) &= \{2,0,\ldots,2,1,1,1\}; \\
    r(v_3|S) &= \{2,2,0,\ldots,0,1,1,1\}; \\
    r(u_1|S) &= \{1,1,\ldots,1,0,1,2\}; \\
    r(u_2|S) &= \{1,1,\ldots,1,1,0,1\}; \\
    r(u_3|S) &= \{1,1,\ldots,1,2,1,0\}; \\
    r(u_4|S) &= \{1,1,\ldots,1,2,2,1\}.
\end{align*}
\]

Based on this representation, if taken a \( W \subset S \) with \(|W| = 2\), then for each \( u, v \in V(F_{m,4}) \) applies \( r(u|W) \neq r(v|W) \).

Case 2. \( m \geq 2 \) and \( n \geq 5 \).

Take \( S_1 = \{ v_1, v_2, \ldots, v_m, u_2, u_3, u_4 \} \) or \( S_2 = \{ v_1, v_2, \ldots, v_m, u_2, u_3, u_4, u_5, u_6 \} \) will show that \( S_1 \) and \( S_2 \) is a 2-metric basis.

The following are given a representation of each vertex in \( F_{1,6} \) with respect to \( S_1 \):

\[
\begin{align*}
    r(v_1|S_1) &= \{0,2,\ldots,2,1,1,1\}; \\
    r(v_2|S_1) &= \{2,0,\ldots,2,1,1,1\}; \\
    r(v_3|S_1) &= \{2,2,0,\ldots,0,1,1,1\}; \\
    r(v_4|S_1) &= \{2,2,2,\ldots,2,1,1\}; \\
    r(u_1|S_1) &= \{1,1,\ldots,1,0,1,2\}; \\
    r(u_2|S_1) &= \{1,1,\ldots,1,1,0,1\}; \\
    r(u_3|S_1) &= \{1,1,\ldots,1,2,1,0\}; \\
    r(u_4|S_1) &= \{1,1,\ldots,1,2,2,1\}.
\end{align*}
\]

and

\[
\begin{align*}
    r(v_5|S_2) &= \{0,2,\ldots,2,1,1,1\}; \\
    r(v_6|S_2) &= \{2,0,\ldots,2,1,1,1\}; \\
    r(v_7|S_2) &= \{2,2,0,\ldots,0,1,1,1\}; \\
    r(v_8|S_2) &= \{2,2,2,\ldots,2,1,1\}; \\
    r(u_5|S_2) &= \{1,1,\ldots,1,0,1,2\}; \\
    r(u_6|S_2) &= \{1,1,\ldots,1,1,0,1\}; \\
    r(u_7|S_2) &= \{1,1,\ldots,1,2,1,0\}; \\
    r(u_8|S_2) &= \{1,1,\ldots,1,2,2,1\}.
\end{align*}
\]

Based on this representation, if taken a \( W \subset S \) with \(|W| = 2\), then for each \( u, v \in V(F_{1,6}) \) applies \( r(u|W) \neq r(v|W) \).

Thus it is obtained that \( S \) is a 2-metric generator. Furthermore, based on Lemma 3, it is obtained that \( S \) is a 2-metric basis, so \( \dim_2(F_{m,n}) = (m + n) - 1 \) and based on Lemma 4, it is obtained that \( S \) is a 2-metric basis, so \( \dim_2(F_{m,n}) = (m + n) - 2 \).

K-metric dimension of \( C_m \times K_n \) graph

Let \( C_m \times K_n \) be a graph with \( m \geq 3 \) and \( n \geq 3 \) is a graph resulting from operations edge amalgamation or combining one edge on \( C_m \) and one edge on \( K_n \) so that it becomes one edge that incident with vertex \( x \) and vertex \( y \). The graph of \( C_m \times K_n \) the graph is illustrated in Fig 2.

From Fig 2, it can be seen that the graph \( C_m \times K_n \) has \( m + n - 2 \) vertices. The following table is given a distance of every two vertices in the graph \( C_m \times K_n \) presented in Table 2.

<table>
<thead>
<tr>
<th>Dist</th>
<th>( x )</th>
<th>( y )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( \ldots )</th>
<th>( u_{m-2} )</th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( \ldots )</th>
<th>( v_{n-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
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<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>2</td>
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<td>1</td>
<td>0</td>
<td>4</td>
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<td>2</td>
<td>1</td>
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<td>1</td>
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<td>1</td>
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</tr>
<tr>
<td>1</td>
<td>7</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>2</td>
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<td>10</td>
<td>9</td>
<td>7</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Following are the result regarding the dimensions of \( K \)-metrics in \( C_m \times K_n \) graph.

**Lemma 5.** Let \( C_m \times K_n \) be a graph obtained from the results edge amalgamation operation between the \( C_m \) graph and the \( K_n \) graph with \( m \geq 3 \) and \( n \geq 3 \), then \( C_m \times K_n \) is a 2-metric dimension graph.

**Proof.** It is known that \( C_m \times K_n \) is a graph obtained from the operation results edge amalgamation between the \( C_m \) graph and \( K_n \) graph of order \( m + n + 2 \). Obtained that \( N_{c_m \times k_n}^2(v_1) = N_{c_m \times k_n}^2(v_2) = \cdots = N_{c_m \times k_n}^2(v_{n-2}) \), such that \( (v_1, v_2, \ldots, v_{n-2}) \) is twin vertices. Then \( C_m \times K_n \) is a 2-metric dimension graph.

The following is given a bound limit for the 2-metric generator cardinality of \( C_m \times K_n \) graph.

**Lemma 6.** Let \( C_m \times K_n \) be a graph obtained from the results edge amalgamation operation between \( C_m \) graph and \( K_n \)
graph with \( m \geq 3 \) and \( n \geq 3 \). If \( S \) is a 2-metric generator of \( C_m * _2 K_n \), then \( |S| \geq n \).

Proof. It is known that \( S \) is a 2-metric generator, meaning that for each \( u, v \in V(C_m * _2 K_n) \) has \( W \subseteq S \) such that \( r(u|W) \neq r(v|W) \) with \( |W| = 2 \). Suppose \( S \) is a 2-metric generator with \( |S| < n \), then there is the possible set of \( S \) that is,

1. \( S \subseteq \{x, y, v_k| 1 \leq i \leq n - 3 \} \) with \( k_j \in \{1, ..., n\} \).
2. \( S \subseteq \{x, v_k| 1 \leq i \leq n - 2 \} \) with \( k_j \in \{1, ..., n\} \).
3. \( S \subseteq \{y, v_k| 1 \leq i \leq n - 2 \} \) with \( k_j \in \{1, ..., n\} \).
4. \( S \subseteq \{u_h, u_j, v_k| 1 \leq i \leq n - 3 \} \) with \( k_j \in \{1, ..., n\} \) for some \( h, j \in \{1, m\} \).
5. \( S \subseteq \{u_i, v_k| 1 \leq i \leq n - 2 \} \) with \( k_j \in \{1, ..., n\} \) for some \( h, j \in \{1, m\} \).

Note that for each \( u, v \in V(C_m * _2 K_n) \), there is a set \( W \subseteq S \) with \( |W| = 2 \) which must satisfy \( r(u|W) \neq r(v|W) \) for \( S \) to be 2-metric generator. Based on Table 4.3 it is found that there are \( u, v \in V(C_m * _2 K_n) \), such that \( r(u|W) = r(v|W) \) for each \( W \subseteq S \) with \(|W| = 2\). This contradicts with the statement that \( S \) is a 2-metric generator. Assumption wrong and must be denied, thus \( S \) is not a 2-metric generator. So \(|S| \geq n\).

Theorem 2. Let \( C_m * _2 K_n \) be a graph with \( m \geq 3 \) and \( n \geq 3 \), then

\[
\dim_2(C_m * _2 K_n) = \begin{cases} 
4, & m = 3, n = 3; \\
3, & m \geq 3, n \geq 4.
\end{cases}
\]

Proof. It is known that \( C_m * _2 K_n \) is a 2-metric dimension graph for \( m \geq 3 \) and \( n \geq 3 \), meaning that there is a 2-metric base on \( C_m * _2 K_n \). The proof is divided into two cases, namely for \( m = 3 \) with \( n = 3 \) and \( m \geq 3 \) with \( n \geq 4 \).

Case 1. For \( m = 3 \) and \( n = 3 \).

Take \( S = \{x, y, u_{i_1}, v_1\} \), will show that \( S \) is a 2-metric basis. The following are given a representation of each vertex in \( C_3 * _2 K_3 \) with respect to \( S \).

\[
\begin{align*}
\text{r}(x|S) &= \{0, 1, 1, 1\}; \\
\text{r}(y|S) &= \{1, 0, 1, 1\}; \\
\text{r}(u_{i_1}|S) &= \{1, 1, 0, 2\}; \\
\text{r}(v_1|S) &= \{1, 1, 2, 0\}.
\end{align*}
\]

Based on the representation obtained, selected \( W_1 = \{x, y\} \).

\[
W_2 = \{x, u_{i_1}\}, \quad W_3 = \{y, u_{i_1}\} \text{ dan } W_4 = \{u_{i_1}, v_1\}.
\]

Obtained that \( \text{r}(x|W_1) \neq \text{r}(y|W_1) \);
\( \text{r}(x|W_2) \neq \text{r}(u_{i_1}|W_2) \);
\( \text{r}(x|W_3) \neq \text{r}(y|W_3) \);
\( \text{r}(y|W_3) \neq \text{r}(u_{i_1}|W_3) \);
\( \text{r}(y|W_4) \neq \text{r}(v_1|W_4) \);
\( \text{r}(u_{i_1}|W_4) \neq \text{r}(v_1|W_4) \).

It is thus obtained that for every two vertices in \( V(C_3 * _2 K_3) \) distinguished by two elements in \( S \), in other words, \( S \) is generator 2-metric.

It will be further shown that \( S \) is a 2-metric basis. Suppose \( S \) is not a 2-metric basis, meaning that there is \( S_1 < 4 \) which is a 2-metric generator. For example \(|S| = 3 \) obtained four possible \( S_1 \), that is,

1. \( S_1 = \{x, y, u_{i_1}\} \);
2. \( S_1 = \{x, y, v_1\} \);
3. \( S_1 = \{y, u_{i_1}, v_1\} \);
4. \( S_1 = \{x, u_{i_1}, v_1\} \).

From the possibilities (a) - (b), it can be seen that \( r(u_{i_1}|W) = r(v_1|W) \). Furthermore, from the possibility of (c) - (d) it can be seen that \( r(x|W) = r(y|W) \) with \( W \subseteq S_1 \) with \(|W| = 2 \), then for each \( W \subseteq S_1 \). Thus there is no 2-metric generator with a cardinality of 3.

Furthermore, if \(|S_1| = 2 \), then \( S_1 \) is not a 2-metric base for \( C_3 * _2 K_3 \), because \( C_3 * _2 K_3 \neq P \) for each \( n \geq 2 \). It is thus obtained that \( S = \{x, y, u_{i_1}, v_1\} \) is a 2-metric basis. So \( \dim_2(C_3 * _2 K_3) = 4 \).

Case 2. For \( m \geq 3 \) and \( n \geq 4 \).

Take \( S = \{x, y, v_{i_1}, ..., v_{m-2}\} \) will be indicated that \( S \) is a 2-metric base. The following are given a representation of each vertex at \( C_m * _2 K_n \) with respect to \( S \).

\[
\begin{align*}
\text{r}(x|S) &= \{0, 1, 1, ..., 1\}; \\
\text{r}(y|S) &= \{1, 0, 1, 1\}; \\
\text{r}(u_{i_1}|S) &= \{1, 2, 2, ..., 2\}; \\
\text{r}(v_1|S) &= \{1, 1, 0, ..., 1\}; \\
\text{r}(v_{m-2}|S) &= \{1, 1, 1, ..., 0\}.
\end{align*}
\]

Based on this representation, if taken a \( W \subseteq S \) with \(|W| = 2 \), then for each \( u, v \in V(C_m * _2 K_2) \) applies \( r(u|W) \neq r(v|W) \). Thus it was found that \( S \) is a 2-metric generator. Furthermore, based on Lemma 6, it is obtained that \( S \) is a 2-metric basis, so \( \dim_2(C_m * _2 K_n) = n \).

3. Conclusion

The k-metric generator with the smallest cardinality is called the k-metric base and the cardinality of the k-metric base is on call the k-metric dimension of the graph \( G \) denoted \( \dim_k(G) \). In this study, the k-metric dimension was obtained in generalized fan \( F_m(m,n) \) graph with \( m \geq 1, n \geq 3 \) and \( C_m * _2 K_n \) graph with \( m \geq 3, n \geq 2 \).

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References


