1. INTRODUCTION

Decision-making (DM) plays a vital role in the practical life activities of human beings as it refers to a process that lays out all the options according to the assessment data of the decision makers (DMrs) and then selects the excellent one, mostly happening in our everyday lives. In the early era of social development, DMrs utilized the real numbers as a rule to offer their assessment information. As the multi-attribute decision-making (MADM) problems are becoming complex, the experts cannot give exact real numbers to assess the alternatives. The ambiguities and imprecision of human judgements highlighted the deficiency of the crisp set theory. Therefore, Zadeh [50] laid the foundations of the fuzzy set (FS) theory for uncertain knowledge that permits the experts to describe their satisfaction level (membership and nonmembership degree) regarding performance of a member within the unit interval. To solve DM problems, many operators in fuzzy environment were introduced. Song et al. [38] studied few operators in fuzzy environment. Atanassov [9] introduced intuitionistic fuzzy set (IFS) which has both MD μ and nonmembership degree (NMD) ν with condition μ + ν ≤ 1. Xu [45] studied IF aggregation operators (AOs). Zhao et al. [51] developed generalized AOs for IFS. Tan et al. [39,40] developed generalized geometric AOs in IF environment. For aggregating the various alternatives, with the help of different AOs, many researchers gave more attention to IFS problems [1,11–13,28,37,46]. Cuong [10] introduced picture fuzzy set (PFS) as a generalization of IFS. PFS handles the situations when the expert’s judgment is of a type like yes, abstinence, no and rejection. PFS is denoted by triplet (μ, η, ν). Positive MD, neutral MD and negative MD, respectively with condition μ + η + ν ≤ 1. Garg [16] developed picture fuzzy AOs. Wei [43] studied PFAOs and discussed some of its applications in DM. The Einstein AOs under picture fuzzy (PF) environment were handled by Khan et al. [22]. Jana et al. [20] gave the theory of PF Dombi AOs. Many new operations on interval-valued PFS and interval-valued PF soft set were defined by Khalil et al. [21]. Lin et al. [23] discussed the MULTIMOORA-based MADM problem under PF environment.

Yager [48] introduced Pythagorean fuzzy set (PyFS). The main characteristic of this model is that it replaces the constraint of IFS with the condition 0 ≤ μ^2 + ν^2 ≤ 1. Yager [49] proposed q-rung orthopair fuzzy set (q-ROFS) in which sum of qth power of MD and NMD is less than or equal to 1. Liu and Wang [29] discussed q-rung orthopair fuzzy (q-ROF) weighted AOs. The idea of Bonferroni mean weighted AOs were extended to q-ROF information by Liu and Liu [27]. Liu et al. [25] developed multi-attribute group decision-making (MAGDM) technique using q-ROF power Maclaurin AOs. Dombi AOs for q-ROFS were defined by Jana et al. [19]. The neutrality AOs of q-ROFS were studied by Garg and Chen [14]. Garg et al. [15] proposed power AOs and VIKOR methods for complex q-ROFS. As an extension of PFS, Gundogdu and Kahraman [17] introduced spherical fuzzy set (SFS). SFS replaces the condition μ + η + ν ≤ 1 of PFS with μ^2 + η^2 + ν^2 ≤ 1.

Yager [48] introduced Pythagorean fuzzy set (PyFS). The main characteristic of this model is that it replaces the constraint of IFS with the condition 0 ≤ μ^2 + ν^2 ≤ 1. Yager [49] proposed q-rung orthopair fuzzy set (q-ROFS) in which sum of qth power of MD and NMD is less than or equal to 1. Liu and Wang [29] discussed q-rung orthopair fuzzy (q-ROF) weighted AOs. The idea of Bonferroni mean weighted AOs were extended to q-ROF information by Liu and Liu [27]. Liu et al. [25] developed multi-attribute group decision-making (MAGDM) technique using q-ROF power Maclaurin AOs. Dombi AOs for q-ROFS were defined by Jana et al. [19]. The neutrality AOs of q-ROFS were studied by Garg and Chen [14]. Garg et al. [15] proposed power AOs and VIKOR methods for complex q-ROFS. As an extension of PFS, Gundogdu and Kahraman [17] introduced spherical fuzzy set (SFS). SFS replaces the condition μ + η + ν ≤ 1 of PFS with μ^2 + η^2 + ν^2 ≤ 1.
Ashraf et al. [7] gave the notion of spherical fuzzy AOs. Li et al. [31] studied the conception of $q$-rung picture fuzzy set ($q$-RPFS) with constraint $μ^q + υ^q + ν^q ≤ 1$ in 2018. In DM problems, $q$-RPFS is more efficient than PFS and SFS. He et al. [18] explained the framework of $q$-RPF Dombi Hamy mean operators. For other terminologies not discussed in the paper, the readers are referred to [2–6,8,24,26,30,32–36,41,42,44,47].

The motivations of this article are summarized as follows:

1. $q$-RPFS is more flexible than PFS and SFS to study DM problems.
2. The assessment of the best alternative in a $q$-RPF environment is a very difficult MADM problem and has several imprecise factors. In the present MADM techniques, assessment data is simply portrayed by picture fuzzy and spherical fuzzy numbers which may prompt data mutilation. Therefore, we need a more general model to elaborate the potential of alternatives.
3. Taking into account that Yager AOs are a straightforward, however groundbreaking, approach for solving DM issues, this article, in general, aims to define Yager AOs in the $q$-RPF context to tackle difficult problems of choice.
4. Yager AOs make the decision results more precise and exact when applied to real-life MADM problems based on the $q$-RPF environment as compared to existing operators.
5. Yager AOs are very simplest and short approach for the evaluation of a single choice in the list of various choices.
6. The drawbacks and limitations of existing operators are run over by proposed operators as these operators are more general that work excellently not only for $q$-RPF information but also for picture fuzzy and spherical fuzzy data.

The contributions of this research are specified as follows:

1. The theory of Yager AOs is extended to $q$-rung picture fuzzy numbers ($q$-RPFNs) and some basic results related to them are discussed.
2. An algorithm is proposed to deal complex practical problems with $q$-RPF data. The proposed algorithm is supported by two MADM problems, one is the selection of suitable emerging technology enterprise and second is the selection of the suitable company for investment.
3. The effect of various values of parameter on DM results is discussed.
4. The importance of these operators is depicted through comparison analysis.

The structure of remaining paper is as follows: Section 2 provides basic definitions. In Section 3, Yager operations for $q$-RPFNs are developed. In Section 4, we study the $q$-rung picture fuzzy Yager weighted arithmetic ($q$-RPFYWA) operator, $q$-rung picture fuzzy Yager ordered weighted arithmetic ($q$-RPFYOWA) operator, $q$-rung picture fuzzy Yager weighted geometric ($q$-RPFYWG) operator, $q$-rung picture fuzzy Yager ordered weighted geometric ($q$-RPFYOWG) operator, $q$-rung picture fuzzy Yager hybrid weighted arithmetic ($q$-RPFYHWA) operator, $q$-rung picture fuzzy Yager hybrid weighted geometric ($q$-RPFYHWG) operator and some results of these operators. In Section 5, an algorithm is provided for MADM problems. In Section 6, we study two MADM problems under these operators, the effect of different data mutilation. Therefore, we need a more general model to elaborate the potential of alternatives.

Section 7 provides the conclusion about proposed theory.

2. PRELIMINARIES

**Definition 2.1.** [9] An IFS $I$ on nonempty set $V$ is defined as

$$I = \{(x, μ_I(x), γ_I(x))\},$$

where $μ_I : V \rightarrow [0, 1]$ and $γ_I : V \rightarrow [0, 1]$ specify MD and NMD of an element $x \in V$, respectively. $w_I(x) = 1 − μ_I(x) − γ_I(x)$ is indeterminacy degree (InD) of an element $x \in V$.

**Definition 2.2.** [48] A PyFS $P_y$ on nonempty set $V$ is defined as

$$P_y = \{(x, μ_{P_y}(x), γ_{P_y}(x))\},$$

where $μ_{P_y} : V \rightarrow [0, 1]$ and $γ_{P_y} : V \rightarrow [0, 1]$ specify MD and NMD of an element, respectively. $w_{P_y}(x) = \sqrt{1 − (μ_{P_y}(x))^2 − (γ_{P_y}(x))^2}$ is InD.

**Definition 2.3.** [49] A $q$-ROFS $F'$ on nonempty set $V$ is defined as

$$F' = \{(x, μ_{F'}(x), γ_{F'}(x))\},$$

where $μ_{F'} : V \rightarrow [0, 1]$ and $γ_{F'} : V \rightarrow [0, 1]$ indicate the MD and NMD of an element $x \in V$, respectively. $w_{F'} = \sqrt{1 − (μ_{F'}(x))^q + γ_{F'}^q}$ is InD.
Definition 2.4. [10] A PFS $P$ on nonempty set $V$ is represented by

$$P = \{(x, \mu_P(x), \eta_P(x), \nu_P(x))\},$$

where $\mu_P(x) : V \rightarrow [0, 1]$, $\eta_P(x) : V \rightarrow [0, 1]$ and $\nu_P(x) : V \rightarrow [0, 1]$ are positive, neutral and negative MD, respectively, of an element $x \in V$, respectively. $\pi_P(x) = 1 - (\mu_P(x) + \nu_P(x))$ is refusal MD.

Definition 2.5. [17] A SFS $S$ on nonempty set $V$ is represented by

$$S = \{(x, \mu_S(x), \eta_S(x), \nu_S(x))\},$$

where $\mu_S(x) : V \rightarrow [0, 1]$, $\eta_S(x) : V \rightarrow [0, 1]$ and $\nu_S(x) : V \rightarrow [0, 1]$ are positive, neutral and negative MD, respectively, of an element $x \in V$, respectively. $\pi_S(x) = \sqrt{1 - ((\mu_S(x))^2 + (\eta_S(x))^2 + (\nu_S(x))^2)}$ is refusal MD.

Definition 2.6. [31] A $q$-rung PFS $F$ on nonempty set $V$ is represented by

$$F = \{(x, \mu_F(x), \eta_F(x), \nu_F(x))\},$$

where $\mu_F(x) : V \rightarrow [0, 1]$, $\eta_F(x) : V \rightarrow [0, 1]$ and $\nu_F(x) : V \rightarrow [0, 1]$ are positive, neutral and negative MD, respectively, of an element $x \in V$, respectively. $\pi_F(x) = \sqrt{1 - ((\mu_F(x))^2 + (\eta_F(x))^2 + (\nu_F(x))^2)}$ is refusal MD. For easiness, $F = \{(x, \mu_F(x), \eta_F(x), \nu_F(x))\}$, called $q$-RPFN is represented by $F = (\mu_F, \eta_F, \nu_F)$.

3. YAGER OPERATIONS FOR $q$-RPFNS

Definition 3.1. Let $F_1 = (\mu_1, \eta_1, \nu_1)$ and $F_2 = (\mu_2, \eta_2, \nu_2)$ be two $q$-RPFNS, $\vartheta > 0$ and $\xi > 0$. Yager $t$-norm and $t$-conorm operations of $q$-RPFNS are

1. $F_1 \circledast F_2 = \sqrt[\vartheta]{\min \left(1, (\mu_1^\vartheta + \mu_2^\vartheta)^{\frac{1}{\vartheta}}\right)} \circledast \sqrt[\vartheta]{\min \left(1, ((1 - \mu_1^\vartheta)\vartheta + (1 - \mu_2^\vartheta)\vartheta)^{\frac{1}{\vartheta}}\right)} \circledast \sqrt[\vartheta]{\min \left(1, (\eta_1^\vartheta + \eta_2^\vartheta)^{\frac{1}{\vartheta}}\right)} \circledast \sqrt[\vartheta]{\min \left(1, ((1 - \eta_1^\vartheta)\vartheta + (1 - \eta_2^\vartheta)\vartheta)^{\frac{1}{\vartheta}}\right)} \circledast \sqrt[\vartheta]{\min \left(1, (\nu_1^\vartheta + \nu_2^\vartheta)^{\frac{1}{\vartheta}}\right)} \circledast \sqrt[\vartheta]{\min \left(1, ((1 - \nu_1^\vartheta)\vartheta + (1 - \nu_2^\vartheta)\vartheta)^{\frac{1}{\vartheta}}\right)}$.

2. $F_1 \otimes F_2 = \left(\min \left(1, (\mu_1^\vartheta + \mu_2^\vartheta)^{\frac{1}{\vartheta}}\right) \otimes \min \left(1, ((1 - \mu_1^\vartheta)\vartheta + (1 - \mu_2^\vartheta)\vartheta)^{\frac{1}{\vartheta}}\right) \otimes \min \left(1, (\eta_1^\vartheta + \eta_2^\vartheta)^{\frac{1}{\vartheta}}\right) \otimes \min \left(1, ((1 - \eta_1^\vartheta)\vartheta + (1 - \eta_2^\vartheta)\vartheta)^{\frac{1}{\vartheta}}\right) \otimes \min \left(1, (\nu_1^\vartheta + \nu_2^\vartheta)^{\frac{1}{\vartheta}}\right) \otimes \min \left(1, ((1 - \nu_1^\vartheta)\vartheta + (1 - \nu_2^\vartheta)\vartheta)^{\frac{1}{\vartheta}}\right)\right).$

3. $F_1^\xi = \left(\min \left(1, (\mu_1^\vartheta)^{\frac{1}{\vartheta}}\right) \otimes \min \left(1, ((1 - \mu_1^\vartheta)\vartheta)^{\frac{1}{\vartheta}}\right) \otimes \min \left(1, (\eta_1^\vartheta)^{\frac{1}{\vartheta}}\right) \otimes \min \left(1, ((1 - \eta_1^\vartheta)\vartheta)^{\frac{1}{\vartheta}}\right) \otimes \min \left(1, (\nu_1^\vartheta)^{\frac{1}{\vartheta}}\right) \otimes \min \left(1, ((1 - \nu_1^\vartheta)\vartheta)^{\frac{1}{\vartheta}}\right)\right).$

Example 3.1. Let $F_1 = (0.9, 0.3, 0.4)$, $F_2 = (0.5, 0.4, 0.6)$ be two $q$-RPFNS, then by Definition 3.1 for $q = 3$, $\vartheta = 4$, $\xi = 5$:

1. $F_1 \circledast F_2 = \sqrt[3]{\min \left(1, (0.9^{12} + 0.5^{12})^{\frac{1}{12}}\right)} \circledast \sqrt[3]{\min \left(1, ((1 - 0.3)^4 + (1 - 0.4)^4)^{\frac{1}{4}}\right)} \circledast \sqrt[3]{\min \left(1, (0.3^{12} + 0.4^{12})^{\frac{1}{12}}\right)} \circledast \sqrt[3]{\min \left(1, (0.4^{12} + 0.6^{12})^{\frac{1}{12}}\right)} = (0.90, 0.0)$.

2. $F_1 \otimes F_2 = \sqrt[3]{\min \left(1, (0.9^{12} + 0.5^{12})^{\frac{1}{12}}\right)} \otimes \sqrt[3]{\min \left(1, ((1 - 0.9)^4 + (1 - 0.5)^4)^{\frac{1}{4}}\right)} \otimes \sqrt[3]{\min \left(1, (0.3^{12} + 0.4^{12})^{\frac{1}{12}}\right)} \otimes \sqrt[3]{\min \left(1, (0.4^{12} + 0.6^{12})^{\frac{1}{12}}\right)} = (0.50, 0.40, 0.60).$

3. $F_1^5 = \sqrt[5]{\min \left(1, (5(0.9)^{12})^{\frac{1}{12}}\right)} \circledast \sqrt[5]{\min \left(1, (5(1 - 0.3)^4)^{\frac{1}{4}}\right)} \circledast \sqrt[5]{\min \left(1, (5(0.3)^{12})^{\frac{1}{12}}\right)} \circledast \sqrt[5]{\min \left(1, (5(0.4)^{12})^{\frac{1}{12}}\right)} = (1, 0, 0).$

4. $F_1^5 = \sqrt[5]{\min \left(1, (5(0.9)^{12})^{\frac{1}{12}}\right)} \otimes \sqrt[5]{\min \left(1, (5(1 - 0.9)^4)^{\frac{1}{4}}\right)} \otimes \sqrt[5]{\min \left(1, (5(0.3)^{12})^{\frac{1}{12}}\right)} \otimes \sqrt[5]{\min \left(1, (5(0.4)^{12})^{\frac{1}{12}}\right)} = (0.84, 0.34, 0.46).$
Theorem 3.1. Let \( F = \langle \mu, \eta, \nu \rangle, F_1 = \langle \mu_1, \eta_1, \nu_1 \rangle, F_2 = \langle \mu_2, \eta_2, \nu_2 \rangle \) be three q-RPFNs, then

1. \( F_1 \otimes F_2 = F_2 \otimes F_1 \),
2. \( F_1 \otimes F_2 = F_1 \otimes F_2 \),
3. \( \zeta(F_1 \otimes F_2) = \zeta F_1 \otimes \zeta F_2 \),
4. \( (\zeta_1 + \zeta_2)^F = \zeta_1 F \otimes \zeta_2 F \),
5. \( (F_1 \otimes F_2)^\zeta = F_1^\zeta \otimes F_2^\zeta, \zeta > 0 \),
6. \( F^{\zeta_1} \otimes F^{\zeta_2} = F^{(\zeta_1 + \zeta_2)}, \zeta_1, \zeta_2 > 0 \).

Proof. For three q-RPFNs \( F, F_1, F_2 \) and \( \zeta, \zeta_1, \zeta_2 > 0 \) by Definition 3.1,

1. \( F_1 \otimes F_2 = \left( \sqrt{\min \left( 1, \left( \mu_1^q \mu_2^q \right)^{\frac{1}{2}} \right)} \sqrt{1 - \min \left( 1, \left( (1 - \eta_1^q)(1 - \eta_2^q) \right)^{\frac{1}{2}} \right)} \right)^{\zeta} \left( \sqrt{\min \left( 1, \left( \mu_1^q + \mu_2^q \right)^{\frac{1}{2}} \right)} \sqrt{1 - \min \left( 1, \left( (1 - \eta_1^q)q + (1 - \eta_2^q)q \right)^{\frac{1}{2}} \right)} \right) \)

2. \( F_1 \otimes F_2 = \left( \sqrt{\min \left( 1, \left( \mu_1^q + \mu_2^q \right)^{\frac{1}{2}} \right)} \sqrt{1 - \min \left( 1, \left( (1 - \eta_1^q)q + (1 - \eta_2^q)q \right)^{\frac{1}{2}} \right)} \right)^{\zeta} \left( \sqrt{\min \left( 1, \left( \mu_1^q \mu_2^q \right)^{\frac{1}{2}} \right)} \sqrt{1 - \min \left( 1, \left( (1 - \eta_1^q)(1 - \eta_2^q) \right)^{\frac{1}{2}} \right)} \right) \)

3. \( \zeta(F_1 \otimes F_2) = \zeta \left( \sqrt{\min \left( 1, \left( \mu_1^q + \mu_2^q \right)^{\frac{1}{2}} \right)} \sqrt{1 - \min \left( 1, \left( (1 - \eta_1^q)q + (1 - \eta_2^q)q \right)^{\frac{1}{2}} \right)} \right)^{\zeta} \left( \sqrt{\min \left( 1, \left( \mu_1^q \mu_2^q \right)^{\frac{1}{2}} \right)} \sqrt{1 - \min \left( 1, \left( (1 - \eta_1^q)(1 - \eta_2^q) \right)^{\frac{1}{2}} \right)} \right) \)

4. \( \zeta_1 F \otimes \zeta_2 F = \left( \sqrt{\min \left( 1, \left( \zeta_1 \mu_1^q \right)^{\frac{1}{2}} \right)} \sqrt{1 - \min \left( 1, \left( \zeta_1 (1 - \eta_1)^q \right)^{\frac{1}{2}} \right)} \right)^{\zeta} \left( \sqrt{\min \left( 1, \left( \zeta_2 \mu_2^q \right)^{\frac{1}{2}} \right)} \sqrt{1 - \min \left( 1, \left( \zeta_2 (1 - \eta_2)^q \right)^{\frac{1}{2}} \right)} \right) \)

5. \( F^{\zeta_1} \otimes F^{\zeta_2} = F^{(\zeta_1 + \zeta_2)}, \zeta_1, \zeta_2 > 0 \).
Similarly, others can be verified.

**Definition 3.2.** Consider a q-RPFN $F = (\mu_F, \eta_F, \nu_F)$. The score $S(F)$ and accuracy functions $A(F)$ of $F$ are

$$ S(F) = \mu_F^q - \eta_F^q - \nu_F^q, \quad \text{where} \quad S(F) \in [-1, 1], $$

$$ A(F) = \mu_F^q + \eta_F^q + \nu_F^q, \quad \text{where} \quad A(F) \in [0, 1]. $$

**Definition 3.3.** Consider two q-RPFNs $F_1 = (\mu_{F_1}, \eta_{F_1}, \nu_{F_1})$ and $F_2 = (\mu_{F_2}, \eta_{F_2}, \nu_{F_2})$. Then

1. If $S(F_1) < S(F_2)$, then $F_1 < F_2$,
2. If $S(F_1) > S(F_2)$, then $F_1 > F_2$,
3. If $S(F_1) = S(F_2)$, then
   (a) If $A(F_1) < A(F_2)$, then $F_1 < F_2$,
   (b) If $A(F_1) > A(F_2)$, then $F_1 > F_2$,
   (c) If $A(F_1) = A(F_2)$, then $F_1 \sim F_2$.

### 4. q-RUNg PICTURE FUZZY YAGER AOs

#### 4.1. q-runG Picture Fuzzy Yager Hybrid-Weighted Arithmetic Operators

Here, we define Yager weighted arithmetic operators under q-RPF environment.

**Definition 4.1.** Let $F_i = (\mu_i, \eta_i, \nu_i)(i = 1, 2, \ldots, \delta)$ be a number of q-RPFNs. The q-RPFYWA operator is a function $Q^\delta \to Q$ s.t.

$$ q - \text{RPFYWA}_\chi(F_1, F_2, \ldots, F_\delta) = \bigoplus_{i=1}^\delta (\chi_i F_i), $$

where $\chi = (\chi_1, \chi_2, \ldots, \chi_\delta)^T$ is the weight vector (WV) of $F_i$ with $\chi_i > 0$ and $\sum_{i=1}^\delta \chi_i = 1$.

**Theorem 4.1.** Let $F_i = (\mu_i, \eta_i, \nu_i)$ be a number of q-RPFNs, then aggregated value of them by the q-RPFYWA operator is a q-RPFN and

$$ q - \text{RPFYWA}_\chi(F_1, F_2, \ldots, F_\delta) = \bigoplus_{i=1}^\delta (\chi_i F_i) \ \ (1) $$

**Proof.** The mathematical induction is used to prove the theorem.

(i) when $\delta = 2$,

As

$$ \chi_1 F_1 = \sqrt{\min(1, (\chi_1 \mu_1^q \eta_1^q \nu_1^q)^{\frac{1}{\delta}})}, \ \ \sqrt{1 - \min(1, (\chi_1 (1 - \eta_1^q) \nu_1^q)^{\frac{1}{\delta}})}, \ \ \sqrt{1 - \min(1, (\chi_1 (1 - \eta_1^q) \nu_1^q)^{\frac{1}{\delta}})}, $$

$$ \chi_2 F_2 = \sqrt{\min(1, (\chi_2 \mu_2^q \eta_2^q \nu_2^q)^{\frac{1}{\delta}})}, \ \ \sqrt{1 - \min(1, (\chi_2 (1 - \eta_2^q) \nu_2^q)^{\frac{1}{\delta}})}, \ \ \sqrt{1 - \min(1, (\chi_2 (1 - \eta_2^q) \nu_2^q)^{\frac{1}{\delta}})}.$$
Therefore,

\[
\chi_1 F_1 \oplus \chi_2 F_2 = \left( \sqrt{\min(1, (\chi_1 \mu_1^q)^{1/\bar{q}})}, \sqrt{1 - \min(1, (\chi_1 (1 - \eta_1^q)^{1/\bar{q}})}, \sqrt{1 - \min(1, (\chi_1 (1 - \nu_1^q)^{1/\bar{q}})} \right) \oplus \\
\left( \sqrt{\min(1, (\chi_2 \mu_2^q)^{1/\bar{q}})}, \sqrt{1 - \min(1, (\chi_2 (1 - \eta_2^q)^{1/\bar{q}})}, \sqrt{1 - \min(1, (\chi_2 (1 - \nu_2^q)^{1/\bar{q}})} \right)
\]

\[
= \left( \sqrt{\min(1, (\chi_1 \mu_1^q + \chi_2 \mu_2^q)^{1/\bar{q}})}, \sqrt{1 - \min(1, (\chi_1 (1 - \eta_1^q + \chi_2 (1 - \nu_2^q))^{1/\bar{q}})} \right)
\]

\[
= \left( \sqrt{\min\left(1, \left( \sum_{i=1}^{k+1} (\chi_i \mu_i^q) \right)^{1/\bar{q}} \right)}, \sqrt{1 - \min\left(1, \left( \sum_{i=1}^{k+1} (\chi_i (1 - \eta_i^q)) \right)^{1/\bar{q}} \right)}, \sqrt{1 - \min\left(1, \left( \sum_{i=1}^{k+1} (\chi_i (1 - \nu_i^q)) \right)^{1/\bar{q}} \right)} \right).
\]

Hence, Equation (1) is true for \( \bar{q} = 2 \).

(ii) Let Equation (1) holds for \( \bar{q} = k \),

\[
q - RPFYWA_{\chi}(F_1, F_2, \cdots, F_k) = \bigoplus_{i=1}^{k} (\chi_i F_i)
\]

\[
= \left( \sqrt{\min\left(1, \left( \sum_{i=1}^{k} (\chi_i \mu_i^q) \right)^{1/\bar{q}} \right)}, \sqrt{1 - \min\left(1, \left( \sum_{i=1}^{k} (\chi_i (1 - \eta_i^q)) \right)^{1/\bar{q}} \right)}, \sqrt{1 - \min\left(1, \left( \sum_{i=1}^{k} (\chi_i (1 - \nu_i^q)) \right)^{1/\bar{q}} \right)} \right).
\]

Now for \( \bar{q} = k + 1 \).

\[
q - RPFYWA_{\chi}(F_1, F_2, \cdots, F_{k+1})
\]

\[
= \left( \sqrt{\min\left(1, \left( \sum_{i=1}^{k+1} (\chi_i \mu_i^q) \right)^{1/\bar{q}} \right)}, \sqrt{1 - \min\left(1, \left( \sum_{i=1}^{k+1} (\chi_i (1 - \eta_i^q)) \right)^{1/\bar{q}} \right)}, \sqrt{1 - \min\left(1, \left( \sum_{i=1}^{k+1} (\chi_i (1 - \nu_i^q)) \right)^{1/\bar{q}} \right)} \right)
\]

Hence, Equation (1) is true for \( \bar{q} = k + 1 \). Thus, Equation (1) is true, \( \forall \bar{q} \).
Example 4.1. Let \( F_1 = (0.6, 0.2, 0.9), F_2 = (0.5, 0.4, 0.5), F_3 = (0.7, 0.4, 0.7) \) and \( F_4 = (0.9, 0.1, 0.6) \) be \( q \)-RPFNs with a WV \( \chi = (0.1, 0.4, 0.2, 0.3) \) and \( \vartheta = 2 \). By applying Theorem 4.1, the aggregated value of \( q \)-RPFNs for \( q = 3 \) is

\[
q - \text{RPFYWA}_\chi(F_1, F_2, F_3, F_4) = \frac{4}{4} (\chi, F_i)
\]

\[
= \left( 1 - \min \left( 1, \left( \sum_{i=1}^{4} (\chi_i \mu_i^q) \right) \right) \right) \left( 1 - \min \left( 1, \left( \sum_{i=1}^{4} (\chi_i (1 - \eta_i^q)) \right) \right) \right)
\]

\[
= \sqrt{\min \left( (0.1(0.6)^6 + 0.4(0.5)^6 + 0.2(0.7)^6 + 0.3(0.9)^6) \right)},
\]

\[
= \sqrt{\left( 1, (0.1(1 - 0.23)^2 + 0.4(1 - 0.43)^2 + 0.2(1 - 0.43)^2 + 0.3(1 - 0.13)^2) \right)}.
\]

\[
= \sqrt{\left( 1, (0.1(1 - 0.9)^2 + 0.4(1 - 0.53)^2 + 0.2(1 - 0.73)^2 + 0.3(1 - 0.63)^2) \right)}.
\]

\[
= (0.60, 0.34, 0.62).
\]

Theorem 4.2. (Idempotency). If all \( q \)-RPFNs are identical, i.e., \( F_i = F \) then

\[
q - \text{RPFYWA}(F_1, F_2, \ldots, F_q) = F.
\]

Proof. As \( F_i = (\mu_i, \eta_i, \nu_i) = F (i = 1, 2, \ldots, q) \). Then by Equation (1),

\[
q - \text{RPFYWA}_\chi(F_1, F_2, \ldots, F_q) = \bigoplus_{i=1}^{q} (\chi, F_i)
\]

\[
= \left( 1 - \min \left( 1, \left( \sum_{i=1}^{q} (\chi_i \mu_i^q) \right) \right) \right) \left( 1 - \min \left( 1, \left( \sum_{i=1}^{q} (\chi_i (1 - \eta_i^q)) \right) \right) \right)
\]

\[
= \sqrt{\left( \min \left( (\mu^q), (1 - \min \left( 1, ((1 - \eta_i^q)) \right) \right) \right) \right)}.
\]

Theorem 4.3. (Boundedness). Let \( F_i = (\mu_i, \eta_i, \nu_i) \) be a collection of \( q \)-RPFNs. Let \( F^- = \min(F_1, F_2, \ldots, F_q) \) and \( F^+ = \max(F_1, F_2, \ldots, F_q) \). Then

\[
F^- \leq q - \text{RPFYWA}(F_1, F_2, \ldots, F_q) \leq F^+.
\]

Proof. Suppose \( F^- = \min(F_1, F_2, \ldots, F_q) = (\mu^-, \eta^-, \nu^-) \) and \( F^+ = \max(F_1, F_2, \ldots, F_q) = (\mu^+, \eta^+, \nu^+) \), where \( \mu^- = \min(\mu_i), \eta^- = \max(\eta_i), \nu^- = \max(\nu_i), \mu^+ = \max(\mu_i), \nu^+ = \min(\nu_i) \). Thus

\[
\sqrt{\min \left( 1, \left( \sum_{i=1}^{q} (\chii \mu_i^{-q}) \right) \right)} \leq \sqrt{\min \left( 1, \left( \sum_{i=1}^{q} (\chii \mu_i^{-q}) \right) \right)} \leq \sqrt{\min \left( 1, \left( \sum_{i=1}^{q} (\chii \mu_i^{q}) \right) \right)}.
\]
Similarly,

\[
\sqrt{1 - \min \left( 1, \left( \sum_{i=1}^{\delta} (\chi_i(1 - \eta_i^{-\vartheta}))^{\frac{1}{\vartheta}} \right)^{\frac{1}{\vartheta}} \right)} \leq \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^{\delta} (\chi_i(1 - \eta_i^q))^{\frac{1}{q}} \right)^{\frac{1}{q}} \right)} \leq \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^{\delta} (\chi_i(1 - \eta_i^q))^{\frac{1}{q}} \right)^{\frac{1}{q}} \right)}.
\]

Similarly,

\[
\sqrt{1 - \min \left( 1, \left( \sum_{i=1}^{\delta} (\chi_i(1 - \eta_i^q))^{\frac{1}{q}} \right)^{\frac{1}{q}} \right)} \leq \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^{\delta} (\chi_i(1 - \eta_i^q))^{\frac{1}{q}} \right)^{\frac{1}{q}} \right)} \leq \sqrt{1 - \min \left( 1, \left( \sum_{i=1}^{\delta} (\chi_i(1 - \eta_i^q))^{\frac{1}{q}} \right)^{\frac{1}{q}} \right)}.
\]

Therefore, \( F^- \leq q - \text{RPFYWA}(F_1, F_2, \ldots, F_6) \leq F^+ \).

**Theorem 4.4.** (Monotonicity). Let \( F'_i = \{F'_1, F'_2, \ldots, F'_6\} \) and \( F_i = \{F_1, F_2, \ldots, F_6\} \) be two collections of \( q \)-RPFN. If \( \mu_i' \leq \mu_i, \eta_i' \geq \eta_i \) and \( \nu_i' \geq \nu_i, \forall i \). Then

\[
q - \text{RPFYWA}(F'_1, F'_2, \ldots, F'_6) \leq q - \text{RPFYWA}(F_1, F_2, \ldots, F_6).
\]

**Proof.** Let \( q - \text{RPFYWA}(F'_1, F'_2, \ldots, F'_6) = (G', U', K') \) and \( q - \text{RPFYWA}(F_1, F_2, \ldots, F_6) = (G, U, K) \). First, we show that \( G' \leq G \). As \( \mu_i' \leq \mu_i, \mu_i' \leq \mu_i \). Moreover,

\[
\left( \sum_{i=1}^{\delta} (\chi_i(\mu_i'^{q^\vartheta}))^{\frac{1}{q^\vartheta}} \right) \leq \left( \sum_{i=1}^{\delta} (\chi_i(\mu_i^{q^\vartheta}))^{\frac{1}{q^\vartheta}} \right)
\]

\[
\min \left( 1, \left( \sum_{i=1}^{\delta} (\chi_i(\mu_i'^{q^\vartheta}))^{\frac{1}{q^\vartheta}} \right)^{\frac{1}{q^\vartheta}} \right) \leq \min \left( 1, \left( \sum_{i=1}^{\delta} (\chi_i(\mu_i^{q^\vartheta}))^{\frac{1}{q^\vartheta}} \right)^{\frac{1}{q^\vartheta}} \right)
\]

\[
\sqrt{\min \left( 1, \left( \sum_{i=1}^{\delta} (\chi_i(\mu_i'^{q^\vartheta}))^{\frac{1}{q^\vartheta}} \right)^{\frac{1}{q^\vartheta}} \right)} \leq \sqrt{\min \left( 1, \left( \sum_{i=1}^{\delta} (\chi_i(\mu_i^{q^\vartheta}))^{\frac{1}{q^\vartheta}} \right)^{\frac{1}{q^\vartheta}} \right)}.
\]

Hence, \( G' \leq G \). Similarly, \( U' \geq U, K' \geq K \). Thus, \( (G', U', K') \leq (G, U, K) \), i.e.,

\[
q - \text{RPFYWA}(F'_1, F'_2, \ldots, F'_6) \leq q - \text{RPFYWA}(F_1, F_2, \ldots, F_6).
\]

**Theorem 4.5.** (Reducibility). Let \( F_i = (\mu_i, \eta_i, \nu_i) \) be a collection of \( q \)-RPFN with \( W^X = (\chi_1, \chi_2, \ldots, \chi_\delta)^T = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})^T \). Then, \( q \)-RPFYWA operator is

\[
q - \text{RPFYWA}_X(F_1, F_2, \ldots, F_6) = \sqrt{\min \left( 1, \left( \sum_{i=1}^{\delta} (\mu_i^{q^\vartheta})^{\frac{1}{q^\vartheta}} \right)^{\frac{1}{q^\vartheta}} \right)} \cdot \sqrt{\min \left( 1, \left( \sum_{i=1}^{\delta} (1 - \eta_i^q))^{\frac{1}{q}} \right)^{\frac{1}{q}} \right)}.
\]

We now define the \( q \)-RPFYOWA operators.

**Definition 4.2.** Let \( F_i = (\mu_i, \eta_i, \nu_i) \) be a family of \( q \)-RPFN with \( W^X = (\chi_1, \chi_2, \ldots, \chi_\delta)^T \) s.t. \( \chi_i > 0 \) and \( \sum_{i=1}^{\delta} \chi_i = 1 \). The \( q \)-RPFYOWA operator is a function \( Q^\delta \rightarrow Q \) s.t.

\[
q - \text{RPFYOWA}_{X,F}(F_1, F_2, \ldots, F_6) = \bigoplus_{i=1}^{\delta} (\chi_i(F_{\varphi(i)})^\varrho,
\]

where \( \varphi(1), \varphi(2), \ldots, \varphi(\delta) \) is the permutation of \( (i = 1, 2, \ldots, \delta) \) s.t. \( F_{\varphi(i-1)} \geq F_{\varphi(i)}, \forall i \).
Let $F_i = (\mu_i, \eta_i, \nu_i)$ be a collection of q-RPFNs with WV $\chi = (\chi_1, \chi_2, \cdots, \chi_s)^T$ s.t. $\chi_i > 0$ and $\sum_{i=1}^{s} \chi_i = 1$, then clumped value of them by q-RPFYOWA operator is a q-RPFN and

$$q - \text{RPFYOWA}_\chi(F_1, F_2, \cdots, F_s) = \bigoplus_{i=1}^{s} (\chi_i F_{\varphi(i)})$$

Thus by applying Theorem 4.6, the aggregated value of q-RPFNs with $q = 3$ is

$$S(F_1) = 0.8^3 - 0.3^3 - 0.7^3 = 0.14,$$

$$S(F_2) = 0.6^3 - 0.4^3 - 0.7^3 = -0.19,$$

$$S(F_3) = 0.6^3 - 0.4^3 - 0.8^3 = -0.36,$$

$$S(F_4) = 0.9^3 - 0.1^3 - 0.4^3 = 0.66.$$
Definition 4.3. A q-RPFYHWA is a function $Q^q \rightarrow Q$, with correlated WV $\chi = (\chi_1, \chi_2, \cdots, \chi_\delta)^T$ with $\chi_i > 0$ and $\sum_{i=1}^{\delta} \chi_i = 1$ s.t.

$$ q - \text{RPFYHWA}_\chi(F_1, F_2, \cdots, F_\delta) = \bigoplus_{i=1}^{\delta} \left( \chi_i \hat{F}_{\varphi(i)} \right) $$

where $\hat{F}_{\varphi(i)}$ is the $i$th biggest weighted $q$-rung picture fuzzy values $\hat{F}_{\varphi(i)} = \sum_{i=1}^{\delta} \chi_i F_i$, $i = 1, 2, \cdots, \delta$ and $\delta$ is the balancing coefficient.

Remark 4.2. For $\chi = (\frac{1}{\delta}, \frac{1}{\delta}, \cdots, \frac{1}{\delta})^T$, q-RPFYA and q-RPFYOWA operators are particular example of q-RPFYHWA operator. Thus, q-RPFYHWA operator is a generalization of them.

4.2. $q$-rung Picture Fuzzy Yager Hybrid-Weighted Geometric Operators

Here, we discuss Yager weighted geometric operators under $q$-RPF environment.

Definition 4.4. Let $F_i = \langle \mu_i, \eta_i, \nu_i \rangle$ be a number of $q$-RPFNs. The $q$-RPFWYG operator is a function $Q^q \rightarrow Q$ s.t.

$$ q - \text{RPFWYG}_\chi(F_1, F_2, \cdots, F_\delta) = \bigotimes_{i=1}^{\delta} F_i^{X_i} $$

where $X = (X_1, X_2, \cdots, X_\delta)^T$ is the WV of $F_i$ with $\chi_i > 0$ and $\sum_{i=1}^{\delta} \chi_i = 1$.

Theorem 4.7. Let $F_i = \langle \mu_i, \eta_i, \nu_i \rangle$ be a number of $q$-RPFNs, then clumped value of them by $q$-RPFWYG operator is a $q$-RPFN

$$ q - \text{RPFWYG}_\chi(F_1, F_2, \cdots, F_\delta) = \bigotimes_{i=1}^{\delta} F_i^{X_i} $$

Proof. It is similar to Theorem 4.1.
Example 4.3. Consider Example 4.1 and by Theorem 4.7, the clumped value for $q$-RPFNs is

$$q - RPFYWG_\chi(F_1, F_2, \ldots, F_4) = \bigotimes_{i=1}^4 (F_i)_{\chi_i}$$

$$= \left[ \sqrt[3]{1 - \min \left( 1, \left( \sum_{i=1}^4 (\chi_i(1 - \mu_i^q)^g) \right)^{\frac{1}{3}} \right)} \right] \sqrt[3]{\min \left( 1, \left( \sum_{i=1}^4 (\chi_i q_i^g) \right)^{\frac{1}{3}} \right)},$$

$$= \sqrt[3]{1 - \min (1, (0.2(1 - 0.6)^3 + 0.3(1 - 0.5)^3 + 0.1(1 - 0.7)^3 + 0.4(1 - 0.9)^3)^{\frac{1}{2}}),}$$

$$\sqrt[3]{\min (1, (0.2(0.2)^6 + 0.3(0.4)^6 + 0.1(0.4)^6 + 0.4(0.1)^6)^{\frac{1}{2}}),}$$

$$\sqrt[3]{\min (1, (0.2(0.9)^6 + 0.3(0.5)^6 + 0.1(0.7)^6 + 0.4(0.6)^6)^{\frac{1}{2}})} = (0.70, 0.34, 0.72).$$

Remark 4.3. $q$-RPFYWG operators satisfy the properties 4.2, 4.3, 4.4 and 4.5.

We now define $q$-RPFYOWG operators.

Definition 4.5. Let $F_i = (\mu_i, \eta_i, \nu_i)$ be a collection of $q$-RPFNs with WV $\chi = (\chi_1, \chi_2, \ldots, \chi_4)^T$ s.t. $X_i > 0$ and $\sum_{i=1}^4 X_i = 1$. The $q$-RPFYOWG operator is a function $Q^g \rightarrow Q$ s.t.

$$q - RPFYOWG_\chi(F_1, F_2, \ldots, F_4) = \bigotimes_{i=1}^4 (F_i)_{\chi_i},$$

where $\varphi(1), \varphi(2), \ldots, \varphi(4)$ is the permutation of $(1, 2, 3, 4)$ s.t. $F_{\varphi(i-1)} \geq F_{\varphi(i)}, \forall i$.

Theorem 4.8. Let $F_i = (\mu_i, \eta_i, \nu_i)$ be a number of $q$-RPFNs with WV $\chi = (\chi_1, \chi_2, \ldots, \chi_4)^T$ s.t. $X_i > 0$ and $\sum_{i=1}^4 X_i = 1$ then clumped value of them by $q$-RPFYOWG operator is a $q$-RPFN and

$$q - RPFYOWG_\chi(F_1, F_2, \ldots, F_4) = \bigotimes_{i=1}^4 (F_i)_{\chi_i},$$

$$= \left[ \sqrt[3]{1 - \min \left( 1, \left( \sum_{i=1}^4 (\chi_i(1 - \mu_i^q)^g) \right)^{\frac{1}{3}} \right)} \right] \sqrt[3]{\min \left( 1, \left( \sum_{i=1}^4 (\chi_i q_i^g) \right)^{\frac{1}{3}} \right)},$$

$$= \left[ \sqrt[3]{1 - \min \left( 1, \left( \sum_{i=1}^4 (\chi_i q_i^g) \right)^{\frac{1}{3}} \right)} \right] \sqrt[3]{\min \left( 1, \left( \sum_{i=1}^4 (\chi_i q_i^g) \right)^{\frac{1}{3}} \right)},$$

$$= (0.70, 0.34, 0.72).$$

Proof. It is similar to Theorem 4.1.
Example 4.4. Consider Example 4.2 and by Theorem 4.8, the clumped value for $q$-RPFNs is

$$q - \text{RPFYOWG}_q(P_1, P_2, P_3, P_4) = \bigotimes_{i=1}^{4} (P_{\tilde{q}(i)})^{\tilde{X}_i}$$

$$= \left\{ ^3 \min \left( 1 - \min \left( 1, \left( \sum_{i=1}^{4} (X_i(1 - \mu^q_{\tilde{q}(i)})^{\frac{1}{\tilde{q}}}) \right)^{\frac{1}{\tilde{q}}} \right) \right), \bigg| \frac{X_i}{\sum_{i=1}^{4} X_i} \right\}$$

$$= \left\{ ^3 \min \left( 1 - \min \left( 1, \left( \sum_{i=1}^{4} (X_i(1 - \mu^q_{\tilde{q}(i)})^{\frac{1}{\tilde{q}}}) \right)^{\frac{1}{\tilde{q}}} \right) \right), \bigg| \frac{\sum_{i=1}^{4} X_i}{\sum_{i=1}^{4} X_i} \right\}$$

Remark 4.4. $q$-RPFYOWG operators satisfy the properties 4.2, 4.3, 4.4 and 4.5.

Now, we define $q$-RPFYHWG operators.

Definition 4.6. A $q$-RPFYHWG operator is a function $Q^q : Q \rightarrow Q$, with correlated WV $X = (X_1, X_2, \ldots, X_4)^T$ with $X_i > 0$ and $\sum_{i=1}^{4} X_i = 1$ s.t.

$$q - \text{RPFYHWG}_q(F_1, F_2, \ldots, F_4) = \bigotimes_{i=1}^{4} (F_{\tilde{q}(i)})^{X_i}$$

(6)

where $F_{\tilde{q}(i)}$ is the $i$th biggest weighted $q$-rung picture fuzzy values $F_i(F_i = F^q_{\tilde{q}(i)}X_i, i = 1, 2, \ldots, 4)$.

5. MATHEMATICAL APPROACH FOR MADM UNDER $q$-RPF ENVIRONMENT

Here, we discuss the MADM problems with $q$-RPF information by using $q$-RPF Yager AOs proposed in the preceding sections. The following notations are used to show the MADM problem for the selection of alternative with $q$-RPF information. Let $L = \{ L_1, L_2, \ldots, L_m \}$ be a set of alternatives and $X = \{ X_1, X_2, \ldots, X_4 \}$ is the WV of the attributes $K = \{ K_1, K_2, \ldots, K_4 \}$, where $X_i > 0$ and $\sum_{i=1}^{4} X_i = 1$. Suppose that $\tilde{N} = (\mu_{\tilde{N}}, \eta_{\tilde{N}}, \nu_{\tilde{N}})$ is the $q$-RPF decision matrix (DMx), where $\mu_{\tilde{N}}$ represents the positive MD that the alternative $L_i$ satisfies the attribute $K_i$ given by the decision maker (DMr), $\eta_{\tilde{N}}$ represents the neutral MD that the alternative $L_i$ does not satisfy the attribute $K_i$, and $\nu_{\tilde{N}}$ represents the negative MD that the alternative $L_i$ does not satisfy the attribute $K_i$ given by the DMr, where $0 \leq \mu_{\tilde{N}}^q + \eta_{\tilde{N}}^q + \nu_{\tilde{N}}^q \leq 1$.

For solving a MADM problem, the Algorithm 1 is given as:
Algorithm 1: Steps to deal MADM problem by \( q \)-RPFYA (or \( q \)-RPFYW) operator

1. **Input:**
   - \( \mathcal{L} \): Set of alternatives,
   - \( \mathcal{K} \): Set of attributes,
   - \( \mathcal{X} \): WV for attributes.

2. Using the \( q \)-RPFYA (or \( q \)-RPFYW) operator to evaluate the information in \( q \)-RPFDMx, find preference values \( g_l, l = 1, 2, ..., m \) of the alternatives \( \mathcal{L}_l \).

3. Compute the score values.
4. Use the score values \( S(g_l) \), to rank the alternatives \( \mathcal{L}_l, l = 1, 2, ..., m \). For equal score, use the accuracy function for ranking of alternatives.

**Output:** The alternative with greatest score will be the decision.

6. NUMERICAL EXAMPLES

6.1. Selection of Emerging Technology Enterprise

In this section, we present a numerical result to build up the reasonable assessment of technology commercialization with \( q \)-RPF data in a such a way to get the desired result of our proposed approach in this article. There is a committee which takes four possible emerging technology enterprises \( \mathcal{L}_l(l = 1, 2, \cdots, 4) \). The attributes for the selection are as follows:

- \( \mathcal{K}_1 \): Technical advancement
- \( \mathcal{K}_2 \): Potential market and market risk
- \( \mathcal{K}_3 \): Industrialization framework, human resources and financial investments

1. The \( q \)-RPFDMx is shown in Table 1.
2. The weights assigned by the DMr are

\[
\chi_1 = 0.3, \chi_2 = 0.4, \chi_3 = 0.3 \text{ and } \sum_{i=1}^{3} \chi_i = 1.
\]

We proceed to select the suitable alternative by \( q \)-RPFYA operator. The steps are given as:

| Table 1 | \( q \)-rung picture fuzzy decision matrix (\( q \)-RPFDMx). |
| --- | --- | --- | --- |
| \( \mathcal{N} \) | \( \mathcal{K}_1 \) | \( \mathcal{K}_2 \) | \( \mathcal{K}_3 \) |
| \( \mathcal{L}_1 \) | (0.7, 0.06, 0.2) | (0.6, 0.01, 0.3) | (0.5, 0.04, 0.4) |
| \( \mathcal{L}_2 \) | (0.6, 0.09, 0.3) | (0.4, 0.02, 0.4) | (0.5, 0.03, 0.3) |
| \( \mathcal{L}_3 \) | (0.7, 0.01, 0.2) | (0.3, 0.01, 0.5) | (0.3, 0.05, 0.5) |
| \( \mathcal{L}_4 \) | (0.6, 0.08, 0.2) | (0.4, 0.05, 0.5) | (0.5, 0.05, 0.4) |
Step 1. The performance values $g_i$ of the alternatives by $q$-RPFYWA operator for $q = 3$ are

- $g_1 = (0.63, 0.04, 0.32)$,
- $g_2 = (0.54, 0.06, 0.35)$,
- $g_3 = (0.61, 0.03, 0.44)$,
- $g_4 = (0.54, 0.09, 0.41)$.

Step 2. The scores $S(g_i)$ of all $q$-RPFNs are

- $S(g_1) = 0.22$,
- $S(g_2) = 0.11$,
- $S(g_3) = 0.14$,
- $S(g_4) = 0.09$.

Step 3. Ranking of alternatives according to scores $S(g_i), 1 \leq i \leq 4$ is

$\mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_4$.

Step 4. $\mathcal{L}_1$ is best alternative.

Step 1. The performance values $g_i$ of the alternatives by $q$-RPFYWG operator for $q = 3$ are

- $g_1 = (0.60, 0.05, 0.35)$,
- $g_2 = (0.50, 0.08, 0.37)$,
- $g_3 = (0.46, 0.04, 0.48)$,
- $g_4 = (0.50, 0.07, 0.46)$.

Step 2. The scores $S(g_i)$ of all $q$-RPFNs are

- $S(g_1) = 0.22$,
- $S(g_2) = 0.11$,
- $S(g_3) = 0.14$,
- $S(g_4) = 0.09$.

Step 3. Ranking of alternatives according to scores $S(g_i), 1 \leq i \leq 4$ is

$\mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_4$.

Step 4. $\mathcal{L}_1$ is best alternative.

### 6.2. Selection of the Suitable Company for Investment

Let suppose another MADM problem in which a DMr wants to select a company for the investment of money. Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ be the possible companies for the investment where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{L}_4$ are the food, mobile, car and fabric companies, respectively. The DMr selects three components to assess companies which are given as follows:

- $\mathcal{K}_1$: Profit-loss ratio
- $\mathcal{K}_2$: Company managements
- $\mathcal{K}_3$: Competitive power

1. The $q$-RPFDnx is shown in Table 2.
2. The weights assigned by the DMr are $\chi_1 = 0.4, \chi_2 = 0.3, \chi_3 = 0.3$ and $\sum_{i=1}^{3} \chi_i = 1$. 
We proceed to select the suitable alternative by $q$-RPFWYA operator. The steps are given as:

**Step 1.**

The performance values $g_i$ of the alternatives by $q$-RPFWYA operator for $q = 3$ are

$$
\begin{align*}
g_1 &= (0.82, 0.02, 0.40), \\
g_2 &= (0.70, 0.26, 0.50), \\
g_3 &= (0.61, 0.27, 0.53), \\
g_4 &= (0.72, 0.06, 0.35).
\end{align*}
$$

**Step 2.**

The scores $S(g_i)$ of all $q$-RPFNs are

$$
\begin{align*}
S(g_1) &= 0.10, \\
S(g_2) &= 0.20, \\
S(g_3) &= 0.06, \\
S(g_4) &= 0.35.
\end{align*}
$$

**Step 3.**

Ranking of alternatives according to scores $S(g_i), 1 \leq i \leq 4$ is

$$
\mathcal{L}_4 > \mathcal{L}_1 > \mathcal{L}_2 > \mathcal{L}_3.
$$

**Step 4.** $\mathcal{L}_4$ is best alternative.

If the $q$-RPFWYG operator is used for selection, the best alternative can be chosen in a similar way. Now the steps are as follows:

**Step 1.**

The performance values $g_i$ of the alternatives by $q$-RPFWYA operator for $q = 3$ are

$$
\begin{align*}
g_1 &= (0.67, 0.03, 0.44), \\
g_2 &= (0.57, 0.35, 0.55), \\
g_3 &= (0.49, 0.35, 0.55), \\
g_4 &= (0.64, 0.06, 0.36).
\end{align*}
$$

**Step 2.**

The scores $S(g_i)$ of all $q$-RPFNs are

$$
\begin{align*}
S(g_1) &= 0.21, \\
S(g_2) &= -0.02, \\
S(g_3) &= -0.09, \\
S(g_4) &= 0.22.
\end{align*}
$$

**Step 3.**

Ranking of alternatives according to scores $S(g_i), 1 \leq i \leq 4$ is

$$
\mathcal{L}_4 > \mathcal{L}_1 > \mathcal{L}_2 > \mathcal{L}_3.
$$

**Step 4.** $\mathcal{L}_4$ is best alternative. The framework for the selection of investment company is shown in Figure 1.
6.3. Effect of Parameter $\vartheta$ on Decision-Making Results

To analyze the effect of parameter $\vartheta$ on results, we take distinct values of $\vartheta$ to handle the Application 6.1. The results for distinct values of $\vartheta$ for $q$-RPFYWA operator are shown in Table 3 and Figure 2, and for $q$-RPFYWG operator are shown in Table 4 and Figure 3.

From above tables and figures, it can be seen that by taking various values of $\vartheta$, the best alternative is same from both operators and best choice is $L_1$. However, the overall ranking is slightly different. It is clear from Table 3 and Figure 2 that by using $q$-RPFYWA operator for $\vartheta = 1, 2$, the ranking of alternatives is different and for $\vartheta = 3, 4, \cdots, 10$ the ranking is same. But for all values, $L_1$ is the best. The score values

Table 3 | Ranking order (RO) using $q$-rung picture fuzzy Yager weighted arithmetic ($q$-RPFYWA) operator.

<table>
<thead>
<tr>
<th>$\vartheta$</th>
<th>$S(q_1)$</th>
<th>$S(q_2)$</th>
<th>$S(q_3)$</th>
<th>$S(q_4)$</th>
<th>RO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.19</td>
<td>0.08</td>
<td>0.03</td>
<td>0.05</td>
<td>$L_1 &gt; L_2 &gt; L_4 &gt; L_3$</td>
</tr>
<tr>
<td>2</td>
<td>0.21</td>
<td>0.10</td>
<td>0.09</td>
<td>0.07</td>
<td>$L_1 &gt; L_2 &gt; L_3 &gt; L_4$</td>
</tr>
<tr>
<td>3</td>
<td>0.22</td>
<td>0.11</td>
<td>0.14</td>
<td>0.09</td>
<td>$L_1 &gt; L_3 &gt; L_2 &gt; L_4$</td>
</tr>
<tr>
<td>4</td>
<td>0.23</td>
<td>0.12</td>
<td>0.16</td>
<td>0.10</td>
<td>$L_1 &gt; L_3 &gt; L_2 &gt; L_4$</td>
</tr>
<tr>
<td>5</td>
<td>0.24</td>
<td>0.13</td>
<td>0.19</td>
<td>0.11</td>
<td>$L_1 &gt; L_3 &gt; L_2 &gt; L_4$</td>
</tr>
<tr>
<td>6</td>
<td>0.26</td>
<td>0.14</td>
<td>0.20</td>
<td>0.11</td>
<td>$L_1 &gt; L_3 &gt; L_2 &gt; L_4$</td>
</tr>
<tr>
<td>7</td>
<td>0.26</td>
<td>0.15</td>
<td>0.21</td>
<td>0.12</td>
<td>$L_1 &gt; L_3 &gt; L_2 &gt; L_4$</td>
</tr>
<tr>
<td>8</td>
<td>0.27</td>
<td>0.15</td>
<td>0.22</td>
<td>0.12</td>
<td>$L_1 &gt; L_3 &gt; L_2 &gt; L_4$</td>
</tr>
<tr>
<td>9</td>
<td>0.27</td>
<td>0.15</td>
<td>0.23</td>
<td>0.13</td>
<td>$L_1 &gt; L_3 &gt; L_2 &gt; L_4$</td>
</tr>
<tr>
<td>10</td>
<td>0.27</td>
<td>0.16</td>
<td>0.23</td>
<td>0.14</td>
<td>$L_1 &gt; L_3 &gt; L_2 &gt; L_4$</td>
</tr>
</tbody>
</table>

Figure 1 | Framework for the selection of investment company.
of alternatives increase if value of $\vartheta$ increased. From Table 4 and Figure 3, it is clear that by using $q$-RPF Yager W operator for $\vartheta = 1, 2, \cdots, 10$, the ranking of alternatives is same and best one is $\varnothing_1$.

6.4. Comparison Analysis and Discussion

To compute performance and validity of our proposed operators, here we aggregate the same data using different operators, namely, picture fuzzy Einstein weighted average (PFEWA) [22], spherical fuzzy weighted average (SFWA) [7], spherical fuzzy weighted geometric (SFWG) [7] and picture fuzzy Dombi average (PFDWA) [20] operators. The computed results by applying these operators are summarized in Table 5 and shown in Figure 4.

It is clear from Table 5 and Figure 4 that best alternative obtained by using PFEWA, SFWA, SFWG and PFDWA operators remains same as obtained from proposed operators. This implies that our proposed methods are authentic and can be applied in DM problems. The main logic behind our proposed approach is that PFS and SFS handle the situations like $\mu + \nu + \lambda \leq 1$ and $\mu^2 + \eta^2 + \pi^2 \leq 1$, respectively but fail in situations where $\mu^q + \eta^q + \pi^q \leq 1$, where $q \geq 3$. That’s why we need $q$-RPFs. The results from proposed theory are more accurate and closest to original results. However, in $q$-RPF Yager AOs, we can also discuss the effect of parameters.
Table 5 | Comparison analysis with PFEWA, SFWA, SFWG and PFDWA operators (suppose $q = 3$).

<table>
<thead>
<tr>
<th>Methods</th>
<th>$S(g_1)$</th>
<th>$S(g_2)$</th>
<th>$S(g_3)$</th>
<th>$S(g_4)$</th>
<th>Ranking Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$-RPFYWA</td>
<td>0.22</td>
<td>0.11</td>
<td>0.14</td>
<td>0.09</td>
<td>$L_1 &gt; L_3 &gt; L_2 &gt; L_4$</td>
</tr>
<tr>
<td>$q$-RPFYWG</td>
<td>0.17</td>
<td>0.07</td>
<td>-0.01</td>
<td>0.03</td>
<td>$L_1 &gt; L_2 &gt; L_4 &gt; L_3$</td>
</tr>
<tr>
<td>PFEWA</td>
<td>0.20</td>
<td>0.09</td>
<td>0.03</td>
<td>0.01</td>
<td>$L_1 &gt; L_2 &gt; L_3 &gt; L_4$</td>
</tr>
<tr>
<td>SFWA</td>
<td>0.20</td>
<td>0.09</td>
<td>0.06</td>
<td>0.01</td>
<td>$L_1 &gt; L_2 &gt; L_3 &gt; L_4$</td>
</tr>
<tr>
<td>SFWG</td>
<td>0.18</td>
<td>0.07</td>
<td>-0.03</td>
<td>0.04</td>
<td>$L_1 &gt; L_2 &gt; L_4 &gt; L_3$</td>
</tr>
<tr>
<td>PFDWA</td>
<td>0.22</td>
<td>0.10</td>
<td>0.05</td>
<td>0.04</td>
<td>$L_1 &gt; L_2 &gt; L_3 &gt; L_4$</td>
</tr>
</tbody>
</table>

$q$-RPFYWA, $q$-rung picture fuzzy Yager weighted arithmetic; PFEWA, picture fuzzy Einstein weighted average; SFWA, spherical fuzzy weighted average; SFWG, spherical fuzzy weighted geometric; PFDWA, picture fuzzy Dombi average.

Figure 4 | Comparison with existing operators.

7. CONCLUSIONS AND FUTURE DIRECTIONS

AOs are mathematical functions and imperative tools of unifying the many inputs into single valuable output. The $q$-RPFS, as a new generalization of PFS and SFS, allows to handle the situations with more generality than PFS and SFS, because it still works in the cases where $\mu + \eta + \nu \leq 1$ and $\mu^2 + \eta^2 + \nu^2 \leq 1$ but it satisfies $\mu^q + \eta^q + \nu^q \leq 1$, where $q \geq 3$. Here, we have discussed MADM problems using $q$-RPF information. We have studied arithmetic and geometric operations to develop some $q$-RPF Yager AOs from the motivation of Yager operators as $q$-RPFYWA, $q$-RPFWYWA, $q$-RFYHWA, $q$-RPFWYG, $q$-RPFWYG and $q$-RPFWHG operators. Different properties of these operators including idempotency, boundedness, monotonicity and reducibility are considered. Then, we have applied these operators to expand a few techniques to discuss MADM issues. Finally, practical example for emerging technology enterprise system selection is provided to develop a strategy. The proposed method is compared with the existing methods to show its advantages and appropriateness. Another example for the selection of a suitable company for investment under $q$-RPF data shows its importance. Moreover, we have discussed the DM results by taking distinct values of parameter. We will extend our work on the following DM problems:

- An in-depth study of the Yager AOs for $q$-rung picture fuzzy information such as induced $q$-rung picture fuzzy Yager AOs, $q$-rung picture fuzzy Yager interactive AOs will be a hot topic in the future.
- A MADM problem in medical diagnosis under $q$-rung picture fuzzy soft data using Yager AOs will be discussed.
- A MADM problem for the selection of a smartphone under $q$-rung picture fuzzy soft data using Yager AOs will be discussed.
- A robust DM model under $m$-polar fuzzy soft Yager AOs will be discussed.

CONFLICT OF INTEREST

The authors declare no conflicts of interest.

AUTHORS’ CONTRIBUTIONS

Conceptualization, Muhammad Akram; Methodology, Peide Liu; Investigation, Gulfam Shahzadi. Writing—original Draft Preparation, Gulfam Shahzadi and Muhammad Akram; and Writing—review and Editing, Peide Liu.
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