Bayesian Analysis of Misclassified Generalized Power Series Distributions Under Different Loss Functions

Peer Bilal Ahmad*

Department of Mathematical Sciences, Islamic University of Science & Technology, Awantipora, Pulwama, Jammu and Kashmir, India

ABSTRACT

In certain experimental investigations involving discrete distributions external factors may induce measurement error in the form of misclassification. For instance, a situation may arise where certain values are erroneously reported; such a situation termed as modified or misclassified has been studied by many researchers. Cohen (J. Am. Stat. Assoc. 55 (1960), 139–143; Ann. Inst. Stat. Math. 9 (1960), 189–193; Technometrics. 2 (1960), 109–113) studied misclassification in Poisson and the binomial random variables. In this paper, we discuss misclassification in the most general class of discrete distributions, the generalized power series distributions (GPSDs), where some of the observations corresponding to $x = c+1$; $c \geq 0$ are erroneously observed or at least reported as being $x = c$ with probability α. This class includes among others the binomial, negative binomial, logarithmic series and Poisson distributions. We derive the Bayes estimators of functions of parameters of the misclassified GPSD under different loss functions. The results obtained for misclassified GPSD are then applied to its particular cases like negative binomial, logarithmic series and Poisson distributions. Finally, few numerical examples are provided to illustrate the results.

1. INTRODUCTION

Binomial, negative binomial, Poisson and logarithmic series distributions, some well-known examples of generalized power series (GPS) distributions, are widely used for modeling count data. Modality and divisibility properties of these distributions are known in the literature. Stochastic ordering comparison between these distributions and their mixtures has also been recently of interest by Misra *et al.* [1], Alamatsaz and Abbasi [2], Aghababaei Jazi and Alamatsaz [3], Abbasi *et al.* [4] and Aghababaei Jazi *et al.* [5].

In certain experimental investigations involving discrete distributions external factors may induce measurement error in the form of misclassification. For instance, a situation may arise where certain values are erroneously reported; e.g., when defective item is inspected wrongly as nondefective item and vice versa. Such a situation termed as modified or misclassified has been studied by many researchers. Cohen [6–8] studied misclassification in Poisson and the binomial random variables. Cohen [6] with a suitable alteration of the data to reflect the misplacement of ones to zeroes, used Bortkiewicz's [9] classical example of deaths from the kick of a horse per army corps per year, for ten Prussian army corps for twenty years (1875–1894). For the purpose of this paper, Cohen assumed that twenty of the records which should have shown one death were in error by reporting no deaths.

Jani and Shah [10] studied misclassification in modified power series distributions (MPSDs) and Patel and Patel [11] in case of generalized power series distribution (GPSD), where some of the values of one are sometimes reported as zero. Hassan and Ahmad [12] studied misclassification in size-biased modified power series distributions (SBMPSDs), where some of the observations corresponding to $x = 2$ are misclassified as $x = 1$. Patel and Patel [13] also studied misclassification in MPSD and Hassan and Ahmad [14] in SBMPSD for a more general situation where sometimes the value $(c + 1)$ is reported erroneously to c. In all these five papers the authors studied the structural properties of the respective distributions.

Our aim is to give Bayes estimators of functions of parameters under squared error loss function (SELF) and weighted square error loss function (WSELF) of misclassified generalized power series distribution (MGPSD) where some of the observations corresponding to $x = c + 1$; $c \geq 0$ are erroneously observed or at least reported as being $x = c$ with probability α. This class includes among others the binomial, negative binomial, logarithmic series and Poisson distributions (PD).

---

*Email: peerbilal@yahoo.co.in; bilalahmadpz@gmail.com

A random variable X is said to have the MGPSD, if the resulting distribution is of the form

$$P[X = x] = \begin{cases} \dfrac{\theta^c \left(a_c + \alpha a_{c+1}\theta\right)}{f(\theta)}, & for \ x = c \\[3mm] (1 - \alpha)\dfrac{a_{c+1}\theta^{c+1}}{f(\theta)}, & for \ x = c + 1 \\[3mm] \dfrac{a_x\theta^x}{f(\theta)} & for \ x \in S \end{cases} \tag{1}$$

where $0 \le \alpha \le 1$, $f(\theta) = \sum_x a(x)\theta^x$ is positive, finite and differentiable and coefficients a_x are nonnegative and free of θ. S is the subset of the set x of nonnegative integers not containing c and $c + 1$.

It is interested to note that for $\alpha = 0$, the model (1) reduces to simple GPSD introduced by Patil [15] and is given by

$$P[X = x] = \frac{a_x\theta^x}{f(\theta)} \quad for \quad x \in T \tag{2}$$

where T is a subset of the set I of nonnegative intergers.

In Section 2, we obtained the Bayes estimators of functions of parameters under SELF and weighted square error loss function of MGPSD. In Sections 3, 4 and 5, the results of misclassified GPSD are used to obtain the Bayes estimators of functions of parameters of misclassified Poisson, misclassified negative binomial and misclassified logarithmic series distributions respectively under SELF and weighted square error loss function using different prior distributions. Finally, in Section 6, few numerical examples are provided to illustrate the results.

2. BAYESIAN ESTIMATION OF MGPSD

Let $X_1, X_2, \cdots, X_N$ represents a random sample of size N drawn from the misclassified GPSD (1), then the likelihood function of $X_1, X_2, \cdots, X_N$ is of the form

$$L\left(\theta, \alpha/\underline{x}\right) \propto \sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{a_c}{a_{c+1}}\right)^j \alpha^{N_c - j}(1 - \alpha)^{N_{c+1}}\theta^{y + N_c - j}\left[f(\theta)\right]^{-N} \tag{3}$$

where $\underline{x} = (x_1, x_2, \cdots, x_N)$, $y = \sum_{i=1}^{N} x_i$ and N_i is the number of observations in the ith class such that $\sum_{i \ge 0} N_i = N$.

As the parameter α represents the probability of misclassifying the observations corresponding to the class $x = c+1$ by reporting it as being corresponding to the class $x = c$, we may take Beta (u, v) prior for α with probability density function

$$g(\alpha) = \frac{\alpha^{u-1}(1 - \alpha)^{v-1}}{B(u, v)}, 0 < \alpha < 1, u, v > 0 \tag{4}$$

where, $B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$

and the prior distribution for θ is taken to be conjugate or nonconjugate prior distribution $h(\theta)$.

The Joint posterior p.d.f. of θ and α corresponding to the prior $h(\theta)$ and $g(\alpha)$ respectively is given by

$$\Pi\left(\theta, \alpha/\underline{x}\right) = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{a_c}{a_{c+1}}\right)^j \alpha^{N_c - j + u - 1}(1 - \alpha)^{N_{c+1} + v - 1}\theta^{y + N_c - j}\left[f(\theta)\right]^{-N} h(\theta)}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) \int_{\Theta} \theta^{y + N_c - j}\left[f(\theta)\right]^{-N} h(\theta) \, d\theta} \tag{5}$$

The marginal posterior distribution of θ and α are respectively given by

$$\Pi\left(\theta/\underline{x}\right) = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) \theta^{y + N_c - j}\left[f(\theta)\right]^{-N} h(\theta)}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) \int_{\Theta} \theta^{y + N_c - j}\left[f(\theta)\right]^{-N} h(\theta) \, d\theta} \tag{6}$$

$$\Pi\left(\alpha/\underline{x}\right) = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j \alpha^{N_c-j+u-1}(1-\alpha)^{N_{c+1}+v-1}\int_{\Theta}\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c-j+u, N_{c+1}+v\right)\int_{\Theta}\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta} \tag{7}$$

Under the SELF given by $L(\eta(\theta), d) = (\eta(\theta) - d)^2$ and $L(\gamma(\alpha), d) = (\gamma(\alpha) - d)^2$, where $\eta(\theta)$ and $\gamma(\alpha)$ are respectively the functions of θ and α, d is a decision, the Bayes estimates $\hat{\eta}(\theta)$ of $\eta(\theta)$ and $\hat{\gamma}(\alpha)$ of $\gamma(\alpha)$ are given by

$$\hat{\eta}_B = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c-j+u, N_{c+1}+v\right)\int_{\Theta}\eta(\theta)\,\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c-j+u, N_{c+1}+v\right)\int_{\Theta}\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta} \tag{8}$$

$$\hat{\gamma}_B = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j \int_0^1\int_{\Theta}\gamma(\alpha)\,\alpha^{N_c-j+u-1}(1-\alpha)^{N_{c+1}+v-1}\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta\,d\alpha}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c-j+u, N_{c+1}+v\right)\int_{\Theta}\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta} \tag{9}$$

Similarly, under the WSELF given by $L(\eta(\theta), d) = w(\theta)(\eta(\theta) - d)^2$ and $L(\gamma(\alpha), d) = z(\alpha)(\gamma(\alpha) - d)^2$ where $w(\theta)$ is a function of θ, and $z(\alpha)$ is a function of α, the Bayes estimate $\hat{\eta}_w$ of $\eta(\theta)$ and $\hat{\gamma}_w$ of $\gamma(\alpha)$ are given by

$$\hat{\eta}_w = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c-j+u, N_{c+1}+v\right)\int_{\Theta}w(\theta)\,\eta(\theta)\,\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c-j+u, N_{c+1}+v\right)\int_{\Theta}w(\theta)\,\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta} \tag{10}$$

$$\hat{\gamma}_w = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j \int_0^1\int_{\Theta}z(\alpha)\gamma(\alpha)\,\alpha^{N_c-j+u-1}(1-\alpha)^{N_{c+1}+v-1}\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta\,d\alpha}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j \int_0^1\int_{\Theta}z(\alpha)\,\alpha^{N_c-j+u-1}(1-\alpha)^{N_{c+1}+v-1}\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta\,d\alpha} \tag{11}$$

We consider two different forms of $w(\theta)$ and $z(\alpha)$ as given below:

i. Let $w(\theta) = \theta^{-2}$, $z(\alpha) = \alpha^{-2}$, the Bayes estimate $\hat{\eta}_M$ of $\eta(\theta)$ and $\hat{\gamma}_M$ of $\gamma(\alpha)$ known as the minimum expected loss (MEL) estimate are given by

$$\hat{\eta}_M = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c-j+u, N_{c+1}+v\right)\int_{\Theta}\eta(\theta)\,\theta^{y+N_c-j-2}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c-j+u, N_{c+1}+v\right)\int_{\Theta}\theta^{y+N_c-j-2}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta} \tag{12}$$

$$\hat{\gamma}_M = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j \int_0^1\int_{\Theta}\gamma(\alpha)\,\alpha^{(N_c-j+u-2)-1}(1-\alpha)^{N_{c+1}+v-1}\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta\,d\alpha}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{a_c}{a_{c+1}}\right)^j B\left(N_c-j+u-2, N_{c+1}+v\right)\int_{\Theta}\theta^{y+N_c-j}\left[f(\theta)\right]^{-N}h(\theta)\,d\theta} \tag{13}$$

This loss function was used by Tummala and Sathe [16] for estimating the reliability of certain life time distributions and by Zellner and Park [17] for estimating functions of parameters of some econometric models.

ii.   Let $w(\theta) = \theta^{-2}e^{-\delta\theta}; \delta > 0$ and $z(\alpha) = \alpha^{-2}e^{-\lambda\alpha}; \lambda > 0$. The Bayes estimate $\hat{\eta}_E$ of $\eta(\theta)$ and $\hat{\gamma}_E$ of $\gamma(\alpha)$ known as the exponentially weighted minimum expected loss (EWMEL) estimate are given by

$$\hat{\eta}_E = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{a_c}{a_{c+1}}\right)^j B(N_c - j + u, N_{c+1} + v) \int_\Theta \eta(\theta) \theta^{y + N_c - j - 2} [f(\theta)]^{-N} e^{-\delta\theta} h(\theta) d\theta}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{a_c}{a_{c+1}}\right)^j B(N_c - j + u, N_{c+1} + v) \int_\Theta \theta^{y + N_c - j - 2} [f(\theta)]^{-N} e^{-\delta\theta} h(\theta) d\theta} \qquad (14)$$

$$\hat{\gamma}_E = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{a_c}{a_{c+1}}\right)^j \int_0^1 \int_\Theta \gamma(\alpha) \alpha^{(N_c - j + u - 2) - 1} (1 - \alpha)^{N_{c+1} + v - 1} e^{-\lambda\alpha} \theta^{y + N_c - j} [f(\theta)]^{-N} h(\theta) d\theta d\alpha}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{a_c}{a_{c+1}}\right)^j B(N_c - j + u - 2, N_{c+1} + v) M(N_c - j + u - 2, N_c + N_{c+1} + u + v - J - 2; -\lambda) \int_\Theta \theta^{y + N_c - j} [f(\theta)]^{-N} h(\theta) d\theta} \qquad (15)$$

For solving (15), we have used the relation $\dfrac{\Gamma(b - a)\Gamma a\, M(a, b, z)}{\Gamma b} = \displaystyle\int_0^1 t^{b-1}(1 - t)^{b-a-1} e^{zt} dt$ given by Abramowitz and Stegum [18]. $M(a, b; z)$ is the confluent hypergeometric function and has a series representation given by

$$M(a, b; z) = \sum_{j=0}^\infty \frac{(a)_j z^j}{(b)_j j!} \qquad (16)$$

$$\text{where } (a)_0 = 1 \text{ and } (a)_j = a(a + 1)(a + 2)\ldots\ldots\ldots(a + j - 1) \qquad (17)$$

Atanasiu [19] used this loss function for estimating the premium for risks, containing a fraction (a part) of the variance of the risk as a loading on the net risk premium.

Now, we shall consider some special cases of the p.m.f. (1) and obtain the corresponding Bayesian estimation in each case.

3. BAYESIAN ESTIMATION OF MPD

A discrete random variable X is said to have misclassified Poisson distribution (MPD) if its probability mass function is given by

$$P[X = x] = \begin{cases} \dfrac{\theta^c}{c!} \left[1 + \alpha\theta\dfrac{1}{c + 1}\right] e^{-\theta}, & \text{for } x = c \\[3mm] (1 - \alpha)\dfrac{e^{-\theta}\theta^{c+1}}{(c + 1)!}, & \text{for } x = c + 1 \\[3mm] \dfrac{\theta^x e^{-\theta}}{x!} & \text{for } x \in S \end{cases} \qquad (18)$$

where $\theta > 0, 0 \le \alpha \le 1$ and $S = T - \{c, c + 1)$. For $c = 0$, it reduces to the modified PD defined by Cohen [6] and if $\alpha = 0$, the model (18) reduces to classical PD. It is a special case of misclassified GPSD (1) with

$$f(\theta) = e^\theta, a_x = \frac{1}{x!}, a_c = \frac{1}{c!}, a_{c+1} = \frac{1}{(c + 1)!}$$

In this case, the likelihood function $L(\theta, \alpha/\underline{x})$ is of the form

$$L(\theta, \alpha/\underline{x}) \propto \sum_{j=0}^{N_c} \binom{N_c}{j} (c + 1)^j \alpha^{N_c - j} (1 - \alpha)^{N_{c+1}} \theta^{y + N_c - j} e^{-\theta N} \qquad (19)$$

With the gamma prior for θ given by

$$h(\theta) = \frac{a^b}{\Gamma b} e^{-a\theta} \theta^{b-1}, \theta, a, b > 0 \qquad (20)$$

and beta prior for α given by (4), the joint Posterior probability density function of θ and α is given by

$$\Pi\left(\theta,\alpha/\underline{x}\right) = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j \alpha^{N_c-j+u-1} (1-\alpha)^{N_{c+1}+v-1} \theta^{y+N_c+b-j-1} e^{-\theta(N+a)}}{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j)}{(N+a)^{y+N_c+b-j}}} \tag{21}$$

The marginal posterior distribution of θ and α are respectively given by

$$\Pi\left(\theta/\underline{x}\right) = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u, N_{c+1}+v\right) \theta^{y+N_c+b-j-1} e^{-\theta(N+a)}}{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j)}{(N+a)^{y+N_c+b-j}}} \tag{22}$$

$$\Pi\left(\alpha/\underline{x}\right) = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j \alpha^{N_c-j+u-1} (1-\alpha)^{N_{c+1}+v-1} \frac{\Gamma(y+N_c+b-j)}{(N+a)^{y+N_c+b-j}}}{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j)}{(N+a)^{y+N_c+b-j}}} \tag{23}$$

Under the SELF, given by $L\left(\theta,d\right) = \left(\theta-d\right)^2$ and $L\left(\alpha,d\right) = \left(\alpha-d\right)^2$, where d is a decision, the Bayes estimate $\hat{\theta}_B^r$ of θ^r and $\hat{\alpha}_B^r$ of α^r are given by

$$\hat{\theta}_B^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j+r)}{(N+a)^{y+N_c+b-j+r}}}{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j)}{(N+a)^{y+N_c+b-j}}} \tag{24}$$

$$\hat{\alpha}_B^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u+r, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j)}{(N+a)^{y+N_c+b-j}}}{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j)}{(N+a)^{y+N_c+b-j}}} \tag{25}$$

Similarly, under the WSELF when $w\left(\theta\right) = \theta^{-2}, z\left(\alpha\right) = \alpha^{-2}$, the MEL estimate of $\eta\left(\theta\right) = \theta^r$ and $\gamma\left(\alpha\right) = \alpha^r$ are obtained as

$$\hat{\theta}_M^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j+r-2)}{(N+a)^{y+N_c+b-j+r-2}}}{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j-2)}{(N+a)^{y+N_c+b-j-2}}} \tag{26}$$

$$\hat{\alpha}_M^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u+r-2, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j)}{(N+a)^{y+N_c+b-j}}}{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u-2, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j)}{(N+a)^{y+N_c+b-j}}} \tag{27}$$

Finally, under WSELF, when $w\left(\theta\right) = \theta^{-2}e^{-\delta\theta}; \delta > 0$ and $z\left(\alpha\right) = \alpha^{-2}e^{-\lambda\alpha}; \lambda > 0$, the EWMEL estimate of $\eta\left(\theta\right) = \theta^r$ and $\gamma\left(\alpha\right) = \alpha^r$ are given by

$$\hat{\theta}_E^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j+r-2)}{(N+a+\delta)^{y+N_c+b-j+r-2}}}{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u, N_{c+1}+v\right) \frac{\Gamma(y+N_c+b-j-2)}{(N+a+\delta)^{y+N_c+b-j-2}}} \tag{28}$$

$$\hat{\alpha}_E^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u+r-2, N_{c+1}+v\right) M_1 \frac{\Gamma(y+N_c+b-j)}{(N+a)^{y+N_c+b-j}}}{\sum_{j=0}^{N_c} \binom{N_c}{j} (c+1)^j B\left(N_c-j+u-2, N_{c+1}+v\right) M_2 \frac{\Gamma(y+N_c+b-j)}{(N+a)^{y+N_c+b-j}}} \tag{29}$$

where $M_1 = \left[N_c - j + u + r - 2, N_c + N_{c+1} - j + u + v + r - 2; -\lambda\right]$

$$M_2 = \left[N_c - j + u - 2, N_c + N_{c+1} - j + u + v - 2; -\lambda\right]$$

4. BAYESIAN ESTIMATION OF MNBD

A discrete random variable X is said to have misclassified negative binomial distribution (MNBD) if its probability mass function is given by

$$
P[X = x] = \begin{cases}
\theta^c \left[\dfrac{(m + c - 1)!}{c!\,(m - 1)!} + \dfrac{\alpha\theta\,(m + c)!}{(c + 1)!\,(m - 1)!}\right](1 - \theta)^m, & \text{for } x = c \\[2ex]
(1 - \alpha)\dfrac{(m + c)!}{(c + 1)!\,(m - 1)!}\theta^{c+1}(1 - \theta)^m & \text{for } x = c + 1 \\[2ex]
\begin{pmatrix} m + x - 1 \\ x \end{pmatrix}\theta^x (1 - \theta)^m, & \text{for } x \in S
\end{cases}
\tag{30}
$$

where $0 < \theta < 1, 0 \le \alpha \le 1$ and S is the subset of the set x of nonnegative integers not containing c and $c + 1$. For $\alpha = 0$, (30) reduces to the negative binomial distribution.

It is a special case of (1) with $f(\theta) = (1 - \theta)^{-m}$ and $a_x = \begin{pmatrix} m + x - 1 \\ x \end{pmatrix}, a_c = \dfrac{(m + c - 1)!}{c!\,(m - 1)!}, a_{c+1} = \dfrac{(m + c)!}{(c + 1)!\,(m - 1)!}$.

The likelihood function $L\left(\theta, \alpha/\underline{x}\right)$ is of the form

$$
L\left(\theta, \alpha/\underline{x}\right) \propto \sum_{j=0}^{N_c} \begin{pmatrix} N_c \\ j \end{pmatrix} \left(\frac{c + 1}{m + c}\right)^j \alpha^{N_c - j} (1 - \alpha)^{N_{c+1}} \theta^{y + N_c - j} (1 - \theta)^{mN}
\tag{31}
$$

Since $0 < \theta < 1$, we have taken two different prior distributions for θ given below

$$
h_1(\theta) = \frac{\theta^{a-1} (1 - \theta)^{b-1}}{B(a, b)}, 0 < \theta \langle 1, a, b \rangle 0
\tag{32}
$$

where $B(a, b) = \dfrac{\Gamma a \Gamma b}{\Gamma(a + b)}$ and

$$
h_2(\theta) = \frac{e^{-k\theta}\theta^{a-1}(1 - \theta)^{b-1}}{B(a, b)\,M(a, a + b; -k)}, 0 < \theta \langle 1, a, b \rangle 0,
\tag{33}
$$

where $M(a, b; k)$ is the confluent hypergeometric function and has a series representation given by (16) and (17)

Both $h_1(\theta)$ and $h_2(\theta)$ are natural conjugate prior density. The prior density $h_2(\theta)$ is known as the generalized beta density considered by Holla [20] and Bhattacharya [21].

The joint posterior p.d.f of θ and α corresponding to the prior $h_1(\theta)$ and $g(\alpha)$ is given by

$$
\Pi_1\left(\theta, \alpha/\underline{x}\right) = \frac{\sum_{j=0}^{N_c} \begin{pmatrix} N_c \\ j \end{pmatrix} \left(\frac{c+1}{m+c}\right)^j \alpha^{N_c - j + u - 1} (1 - \alpha)^{N_{c+1} + v - 1} \theta^{y + N_c - j + a - 1}(1 - \theta)^{mN + b - 1}}{\sum_{j=0}^{N_c} \begin{pmatrix} N_c \\ j \end{pmatrix} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + a, mN + b\right)}
\tag{34}
$$

The marginal posterior distribution of θ and α are respectively given by

$$
\Pi_1\left(\theta/\underline{x}\right) = \frac{\sum_{j=0}^{N_c} \begin{pmatrix} N_c \\ j \end{pmatrix} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) \theta^{y + N_c - j + a - 1}(1 - \theta)^{mN + b - 1}}{\sum_{j=0}^{N_c} \begin{pmatrix} N_c \\ j \end{pmatrix} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + a, mN + b\right)}
\tag{35}
$$

$$
\Pi_1\left(\alpha/\underline{x}\right) = \frac{\sum_{j=0}^{N_c} \begin{pmatrix} N_c \\ j \end{pmatrix} \left(\frac{c+1}{m+c}\right)^j B\left(y + N_c - j + a, mN + b\right) \alpha^{N_c - j + u - 1}(1 - \alpha)^{N_{c+1} + v - 1}}{\sum_{j=0}^{N_c} \begin{pmatrix} N_c \\ j \end{pmatrix} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + a, mN + b\right)}
\tag{36}
$$

Similarly, the joint posterior p.d.f of θ and α corresponding to the prior $h_2(\theta)$ and $g(\alpha)$ is given by

$$\Pi_2(\theta, \alpha/\underline{x}) = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j \alpha^{N_c-j+u-1}(1-\alpha)^{N_{c+1}+v-1}\theta^{y+N_c-j+a-1}(1-\theta)^{mN+b-1}e^{-k\theta}}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u, N_{c+1}+v\right) B\left(y+N_c-j+a, mN+b\right) M_3} \tag{37}$$

where $M_3 = \left(y+N_c-j+a, y+N_c-j+mN+a+b; -k\right)$

The marginal posterior distribution of θ and α are respectively given by

$$\Pi_2(\theta/\underline{x}) = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u, N_{c+1}+v\right)\theta^{y+N_c-j+a-1}(1-\theta)^{mN+b-1}e^{-k\theta}}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u, N_{c+1}+v\right) B\left(y+N_c-j+a, mN+b\right) M_3} \tag{38}$$

$$\Pi_2(\alpha/\underline{x}) = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(y+N_c-j+a, mN+b\right) M_3\alpha^{N_c-j+u-1}(1-\alpha)^{N_{c+1}+v-1}}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u, N_{c+1}+v\right) B\left(y+N_c-j+a, mN+b\right) M_3} \tag{39}$$

Under the SELF, the Bayes estimate of θ^r and α^r corresponding to the posterior density (35) and (36) respectively, are given by

$$\hat{\theta}_{1B}^r = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u, N_{c+1}+v\right) B\left(y+N_c-j+a+r, mN+b\right)}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u, N_{c+1}+v\right) B\left(y+N_c-j+a, mN+b\right)} \tag{40}$$

$$\hat{\alpha}_{1B}^r = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(y+N_c-j+a, mN+b\right) B\left(N_c-j+u+r, N_{c+1}+v\right)}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u, N_{c+1}+v\right) B\left(y+N_c-j+a, mN+b\right)} \tag{41}$$

Under the WSELF when $w(\theta) = \theta^{-2}, z(\alpha) = \alpha^{-2}$, the MEL estimate of $\eta(\theta) = \theta^r$ and $\gamma(\alpha) = \alpha^r$ corresponding to the posterior density (35) and (36) respectively, are given by

$$\hat{\theta}_{1M}^r = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u, N_{c+1}+v\right) B\left(y+N_c-j+a+r-2, mN+b\right)}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u, N_{c+1}+v\right) B\left(y+N_c-j+a-2, mN+b\right)} \tag{42}$$

$$\hat{\alpha}_{1M}^r = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(y+N_c-j+a, mN+b\right) B\left(N_c-j+u+r-2, N_{c+1}+v\right)}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u-2, N_{c+1}+v\right) B\left(y+N_c-j+a, mN+b\right)} \tag{43}$$

Finally, under WSELF, when $w(\theta) = \theta^{-2}e^{-\delta\theta}; \delta > 0$ and $z(\alpha) = \alpha^{-2}e^{-\lambda\alpha}; \lambda > 0$, the EWMEL estimate of θ^r and α^r corresponding to the posterior density (35) and (36) respectively, are given by

$$\hat{\theta}_{1E}^r = \frac{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u, N_{c+1}+v\right) B\left(y+N_c-j+a+r-2, mN+b\right) M_4}{\sum_{j=0}^{N_c}\binom{N_c}{j}\left(\frac{c+1}{m+c}\right)^j B\left(N_c-j+u, N_{c+1}+v\right) B\left(y+N_c-j+a-2, mN+b\right) M_5} \tag{44}$$

where $M_4 = M\left(y+N_c-j+a+r-2, y+N_c-j+a+b+mN+r-2; -\delta\right)$

$$M_5 = M\left(y+N_c-j+a-2, y+N_c-j+a+b+mN-2; -\delta\right)$$

$$\hat{\alpha}_{1E}^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(y + N_c - j + a, mN + b\right) B\left(N_c - j + u + r - 2, N_{c+1} + v\right) M_1}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u - 2, N_{c+1} + v\right) B\left(y + N_c - j + a, mN + b\right) M_2}$$

(45)

Also, under the SELF, the Bayes estimate of θ^r and α^r corresponding to the posterior density (38) and (39) respectively, are given by

$$\hat{\theta}_{2B}^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + a + r, mN + b\right) M_6}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + a, mN + b\right) M_3}$$

(46)

where $M_6 = M\left(y + N_c - j + a + r, y + N_c - j + a + b + mN + r; -k\right)$

$$\hat{\alpha}_{2B}^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u + r, N_{c+1} + v\right) B\left(y + N_c - j + a, mN + b\right) M_3}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + a, mN + b\right) M_3}$$

(47)

Under the WSELF when $w(\theta) = \theta^{-2}, z(\alpha) = \alpha^{-2}$, the MEL estimate of θ^r and α^r corresponding to the posterior density (38) and (39) respectively, are given by

$$\hat{\theta}_{2M}^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + a + r - 2, mN + b\right) M_7}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + a - 2, mN + b\right) M_8}$$

(48)

where $M_7 = M\left(y + N_c - j + a + r - 2, y + N_c - j + a + b + mN + r - 2; -k\right)$

$$M_8 = M\left(y + N_c - j + a - 2, y + N_c - j + a + b + mN - 2; -k\right)$$

$$\hat{\alpha}_{2M}^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u + r - 2, N_{c+1} + v\right) B\left(y + N_c - j + a, mN + b\right) M_3}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u - 2, N_{c+1} + v\right) B\left(y + N_c - j + a, mN + b\right) M_3}$$

(49)

Finally, under WSELF, when $w(\theta) = \theta^{-2} e^{-\delta\theta}; \delta > 0$ and $z(\alpha) = \alpha^{-2} e^{-\lambda\alpha}, \lambda > 0$, the EWMEL estimate of θ^r and α^r corresponding to the posterior density (38) and (39) respectively, are given by

$$\hat{\theta}_{2E}^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + a + r - 2, mN + b\right) M_9}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + a - 2, mN + b\right) M_{10}}$$

(50)

where $M_9 = M(y + N_c - j + a + r - 2, y + N_c - j + a + b + mN + r - 2; -(\delta + k)$

$$M_{10} = M(y + N_c - j + a - 2, y + N_c - j + a + b + mN - 2; -(\delta + k)$$

$$\hat{\alpha}_{2M}^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u + r - 2, N_{c+1} + v\right) M_1 B\left(y + N_c - j + a, mN + b\right) M_3}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{m+c}\right)^j B\left(N_c - j + u - 2, N_{c+1} + v\right) M_2 B\left(y + N_c - j + a, mN + b\right) M_3}$$

(51)

5. BAYESIAN ESTIMATION OF MLSD

The probability mass function of misclassified logarithmic series distribution (MLSD) is given by

$$
P[X = x] = \begin{cases} \dfrac{\theta^c \left[\frac{1}{c} + \alpha\theta \frac{1}{c+1} \right]}{\left[-\log(1-\theta) \right]}, & \text{for } x = c \\[3ex] (1-\alpha)\dfrac{1}{c+1}\dfrac{\theta^{c+1}}{\left[-\log(1-\theta) \right]}, & \text{for } x = c+1 \\[3ex] \dfrac{\theta^x}{x\left[-\log(1-\theta) \right]}, & \text{for } x \in S \end{cases}
\tag{52}
$$

where $0 < \theta < 1, 0 \le \alpha \le 1$ and S is the subset of the set x of nonnegative integers not containing c and $c + 1$. For $\alpha = 0$, we note that (52) reduces to the usual logarithmic series distribution (LSD). It is also a special case of (1) with $f(\theta) = -\log(1-\theta), a_x = \frac{1}{x}, a_c = \frac{1}{c}$, $a_{c+1} = \dfrac{1}{c+1}$

The likelihood function in this case is given by

$$
L(\theta, \alpha/\underline{x}) \propto \sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c} \right)^j \alpha^{N_c - j}(1-\alpha)^{N_{c+1}} \theta^{y+N_c-j} \left[-\log(1-\theta) \right]^{-N}
\tag{53}
$$

Since $0 < \theta < 1$, we have taken the prior distribution for θ, as given below

$$
h(\theta) = \begin{cases} \dfrac{(k+1)^{N+1}(1-\theta)^k \left[-\log(1-\theta) \right]^N}{\Gamma(N+1)}; & 0 < \theta \langle 1, \quad k \rangle 0 \end{cases}
\tag{54}
$$

where N is a positive integer and is same as the size of the random sample. This is non-conjugate prior p.d.f. We can also take the conjugate priors given by (32) and (33).

The joint posterior p.d.f of θ and α corresponding to the prior $h(\theta)$ and $g(\alpha)$ is given by

$$
\Pi(\theta, \alpha/\underline{x}) = \dfrac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c} \right)^j \alpha^{N_c-j+u-1}(1-\alpha)^{N_{c+1}+v-1} \theta^{y+N_c-j}(1-\theta)^k}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c} \right)^j B\left(N_c - j + u, N_{c+1} + v \right) B\left(y + N_c - j + 1, k + 1 \right)}
\tag{55}
$$

The marginal posterior distribution of θ and α are respectively given by

$$
\Pi(\theta/\underline{x}) = \dfrac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c} \right)^j B\left(N_c - j + u, N_{c+1} + v \right) \theta^{y+N_c-j}(1-\theta)^k}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c} \right)^j B\left(N_c - j + u, N_{c+1} + v \right) B\left(y + N_c - j + 1, k + 1 \right)}
\tag{56}
$$

$$
\Pi(\alpha/\underline{x}) = \dfrac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c} \right)^j B\left(y + N_c - j, k + 1 \right) \alpha^{N_c-j+u-1}(1-\alpha)^{N_{c+1}+v-1}}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c} \right)^j B\left(N_c - j + u, N_{c+1} + v \right) B\left(y + N_c - j + 1, k + 1 \right)}
\tag{57}
$$

Under the SELF, the Bayes estimate of θ^r and α^r corresponding to the posterior density (56) and (57) respectively, are given by

$$
\hat{\theta}_B^r = \dfrac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c} \right)^j B\left(N_c - j + u, N_{c+1} + v \right) B\left(y + N_c - j + r + 1, k + 1 \right)}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c} \right)^j B\left(N_c - j + u, N_{c+1} + v \right) B\left(y + N_c - j + 1, k + 1 \right)}
\tag{58}
$$

$$
\hat{\alpha}_B^r = \dfrac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c} \right)^j B\left(N_c - j + u + r, N_{c+1} + v \right) B\left(y + N_c - j + 1, k + 1 \right)}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c} \right)^j B\left(N_c - j + u, N_{c+1} + v \right) B\left(y + N_c - j + 1, k + 1 \right)}
\tag{59}
$$

Under the WSELF when $w(\theta) = \theta^{-2}, z(\alpha) = \alpha^{-2}$, the MEL estimate of $\eta(\theta) = \theta^r$ and $\gamma(\alpha) = \alpha^r$ corresponding to the posterior density (56) and (57) respectively, are given by

$$\hat{\theta}_M^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + r - 1, k + 1\right)}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j - 1, k + 1\right)} \quad (60)$$

$$\hat{\alpha}_M^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c}\right)^j B\left(N_c - j + u + r - 2, N_{c+1} + v\right) B\left(y + N_c - j + 1, k + 1\right)}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c}\right)^j B\left(N_c - j + u - 2, N_{c+1} + v\right) B\left(y + N_c - j + 1, k + 1\right)} \quad (61)$$

Finally, under WSELF, when $w(\theta) = \theta^{-2}e^{-\delta\theta}; \delta > 0$ and $z(\alpha) = \alpha^{-2}e^{-\lambda\alpha}, \lambda > 0$, the EWMEL estimate of θ^r and α^r corresponding to the posterior density (56) and (57) respectively, are given by

$$\hat{\theta}_E^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j + r - 1, k + 1\right) M_{11}}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c}\right)^j B\left(N_c - j + u, N_{c+1} + v\right) B\left(y + N_c - j - 1, k + 1\right) M_{12}} \quad (62)$$

where, $M_{11} = M\left(y + N_c - j + r - 1, y + N_c - j + k + r; -\delta\right)$

$$M_{12} = M\left(y + N_c - j - 1, y + N_c - j + k; -\delta\right)$$

$$\hat{\alpha}_E^r = \frac{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c}\right)^j B\left(y + N_c - j + 1, k + 1\right) B\left(N_c - j + u + r - 2, N_{c+1} + v\right) M_1}{\sum_{j=0}^{N_c} \binom{N_c}{j} \left(\frac{c+1}{c}\right)^j B\left(y + N_c - j + 1, k + 1\right) B\left(N_c - j + u - 2, N_{c+1} + v\right) M_2} \quad (63)$$

6. SOME ILLUSTRATIVE EXAMPLES

In this section we will consider few numerical examples in order to apply and illustrate the foregoing results. We will compare the Bayesian estimates with the moment estimates and maximum likelihood estimates.

The data set presented in Tables 1–3 pertains to the number of strikes in 4-week period in three leading industries in the United Kingdom during 1948–1958. The same set of data was used by Kendall [22] for fitting of PD model. For our purpose of illustration, it has been assumed that ten of the observations, which correspond to the value one, are reported erroneously as value zero. Both the original and the altered frequencies are given in Tables 1–3. From the altered data we have derived the parameters of misclassified PD using moment method

Table 1 | Number of outbreaks of strike in transport industry in United Kingdom during 1948–1958 (Kendall [22]).

No. of Outbreaks of Strike	Frequencies		Estimation of Parameter				
	Original Data	Altered Data	Moments Estimates	Maximum Likelihood Estimates	Bayesian Estimates (Under Different Loss Functions)		
					SELF	MEL	EWMEL
0	114	124					
1	35	25	$\hat{\theta} = 0.4190$	$\hat{\theta} = 0.3767$	$\hat{\theta} = 0.3899$	$\hat{\theta} = 0.3678$	$\hat{\theta} = 0.3668$
2	4	4	$\hat{\alpha} = 0.4795$	$\hat{\alpha} = 0.3868$	$\hat{\alpha} = 0.4068$	$\hat{\alpha} = 0.3306$	$\hat{\alpha} = 0.3296$
3	2	2					
4	1	1					
Total	156	156					

SELF, squared error loss function; MEL, minimum expected loss; EWMEL, exponentially minimum expected loss.

Estimate of θ when the calculations are based on the original unaltered sample $= \hat{\theta} = \mathbf{0.3397}$.
Actual proportion of observations misclassified $= \mathbf{10/35 = 0.2857}$.

Table 2 | Number of outbreaks of strike in vehicle manufacturing industry in United Kingdom during 1948–1958 (Kendall [22]).

No. of Outbreaks of Strike	Frequencies		Estimation of Parameter				
	Original Data	Altered Data	Moments Estimates	Maximum Likelihood Estimates	Bayesian Estimates (Under Different Loss Functions)		
					SELF	MEL	EWMEL
0	110	120					
1	33	23	$\hat{\theta} = 0.5133$	$\hat{\theta} = 0.5073$	$\hat{\theta} = 0.5082$	$\hat{\theta} = 0.4874$	$\hat{\theta} = 0.4861$
2	9	9	$\hat{\alpha} = 0.5263$	$\hat{\alpha} = 0.5221$	$\hat{\alpha} = 0.5046$	$\hat{\alpha} = 0.4560$	$\hat{\alpha} = 0.4547$
3	3	3					
4	1	1					
Total	156	156					

SELF, squared error loss function; MEL, minimum expected loss; EWMEL, exponentially minimum expected loss.
Estimate of θ when the calculations are based on the original unaltered sample = $\hat{\theta}$ = **0.4103**.
Actual proportion of observations misclassified = **10/33 = 0.3030**.

Table 3 | Number of outbreaks of strike in ship building industry in United Kingdom during 1948–1958 (Kendall [22]).

No. of Outbreaks of Strike	Frequencies		Estimation of Parameter				
	Original Data	Altered Data	Moments Estimates	Maximum Likelihood Estimates	Bayesian Estimates (Under Different Loss Functions)		
					SELF	MEL	EWMEL
0	117	127					
1	29	19	$\hat{\theta} = 0.4165$	$\hat{\theta} = 0.4159$	$\hat{\theta} = 0.4151$	$\hat{\theta} = 0.3920$	$\hat{\theta} = 0.3909$
2	9	9	$\hat{\alpha} = 0.5613$	$\hat{\alpha} = 0.5570$	$\hat{\alpha} = 0.5298$	$\hat{\alpha} = 0.4735$	$\hat{\alpha} = 0.4724$
3	0	0					
4	1	1					
Total	156	156					

SELF, squared error loss function; MEL, minimum expected loss; EWMEL, exponentially minimum expected loss.
Estimate of θ when the calculations are based on the original unaltered sample = $\hat{\theta}$ = **0.3269**.
Actual proportion of observations misclassified = **10/29 = 0.3448**.

of estimation, maximum likelihood estimation and Bayesian estimation using different loss functions. The prior values used for the beta distribution (4) will be $u = v = 3$, while those used for the gamma distribution (20) will be $a = 0.25, b = 1$ and for EWMEL estimates (28) and (29) will be $\delta = \lambda = 0.25$. The values chosen for a and b in (20) provide a fairly flat form of the gamma distribution. Hassan [23] and Islam and Consul [24] while discussing the Bayesian estimation of generalized PD and generalized negative binomial distribution, respectively, has shown that the estimated Bayes frequencies were quite close to the simulated sample frequencies when u and v were equal. The values for the prior parameters $a, b, u, v, \delta, \lambda$ were chosen so that the posterior distribution would reflect the data as much, and the prior information as little, as possible.

The estimates of $\hat{\theta}$ obtained for the altered data is to be compared with $\hat{\theta}$ obtained when the calculations are based on the original unaltered sample. The estimates of $\hat{\alpha}$ is to be compared with the proportion of ones that were misclassified in the process of altering the original data for this illustration. It is clear from the tables that the Bayes estimates are closer than the maximum likelihood estimates as well as the moment estimates. It is also encouraging to observe from the tables that the Bayes estimates obtained under WSELF is much closer than the Bayes estimates obtained under SELF. Also, it is clear from the tables that EWMEL estimates are closer than the MEL estimates.

CONFLICT OF INTEREST

There is no potential conflict of Interest related to this study.

REFERENCES

1. N. Misra, H. Singh, E.J. Harner, Stat. Probab. Lett. 65 (2003), 279–290.
2. M.H. Alamatsaz, S. Abbasi, Stat. Probab. Lett. 78 (2008), 2234–2239.
3. M. Aghababaei Jazi, M.H. Alamatsaz, Commun. Stat. Theory Methods. 39 (2010), 3252–3263.
4. S. Abbasi, M. Aghababaei Jazi, M.H. Alamatsaz, Global J. Pure Appl. Math. 6 (2010), 305–316.
5. M. Aghababaei Jazi, M.H. Alamatsaz, S. Abbasi, Commun. Stat. Theory Methods. 40 (2010), 2591–2604.
6. A.C. Cohen, J. Am. Stat. Assoc. 55 (1960), 139–143.
7. A.C. Cohen, Ann. Inst. Stat. Math. 9 (1960), 189–193.

8. A.C. Cohen, Technometrics. 2 (1960), 109–113.
9. L. von Bortkiewicz, Das Gesetz der Kleinen Zahlen, Teubner, Leipzig, Germany, 1898.
10. P.N. Jani, S.M. Shah, Metron. 37 (1979), 121–136.
11. A.I. Patel, I.D. Patel, ASR. 10 (1996), 107–119.
12. A. Hassan, P.B. Ahmad, Math. Bohemica. 134 (2009), 1–17.
13. A.I. Patel, I.D. Patel, ASR. 15 (2001), 55–69.
14. A. Hassan, P.B. Ahmad, J. Korean Soc. Ind. Appl. Math. 13 (2009), 55–72.
15. G.P. Patil, Sankhya. 23 (1961), 269–280.
16. V.M. Tummala, P.T. Sathe, IEEE Trans. Reliab. 27 (1978), 283–285.
17. A. Zellner, S.B. Park, J. Am. Stat. Assoc. 74 (1979), 185–193.
18. M. Abramowitz, I.A. Stegum, Handbook of Mathematical Functions, Dover, New York, NY, USA, 1964.
19. V. Atanassiu, Revista Inf. Econ. 2 (2008), 12–17.
20. M.S. Holla, Ann. Inst. Stat. Math. 20 (1968), 151–157.
21. S.K. Bhattacharya, Ann. Inst. Stat. Math. 20 (1968), 375–381.
22. M.G. Kendall, J. Royal Stat. Soc. Ser. A. 124 (1961), 1–19.
23. A. Hassan, Problems of Estimation in Lagrangian Probability Distributions, PhD Thesis, Patna University, Patna, India, 1995.
24. M.N. Islam, P.C. Consul, Biometrics. 28 (1986), 250–256.