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m-Polar Picture Fuzzy Ideal of a BCK Algebra

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ABSTRACT

In this paper, the notions of m-polar picture fuzzy subalgebra (PFSA), m-polar picture fuzzy ideal (PFI) and m-polar picture fuzzy implicative ideal (PFII) of BCK algebra are introduced and some related basic results are presented. A relation between m-polar PFI and m-polar PFII is established. It is shown that an m-polar PFII of a BCK algebra is an m-polar PFI. But the converse of the proposition is not necessarily true. Converse is true only in implicative BCK algebra. The concept of m-polar picture fuzzy commutative ideal (PFCI) is also explored here and some related results are investigated.

1. INTRODUCTION

After the initiation of fuzzy set (FS) by Zadeh [1] in 1965, the notion of intuitionistic fuzzy set (IFS) was propounded by Atanassov [2] in 1986. IFS includes both the degree of membership (DMS) and the degree of non-membership (DNonMS), whereas fuzzy set includes only the DMS. The concept of BCI/BCK algebra was presented by Iseki and co-workers [3–5]. Merging the concepts of FS and BCK algebra, fuzzy BCK algebra was initiated by Xi [6]. In 1993, the idea of FS was connected with BCI algebra by Ahmad [7]. Later on a lot of works on BCK/BCI algebra and ideals in fuzzy set environment were done by several researchers [8–12]. Intuitionistic fuzzy subalgebra and intuitionistic fuzzy ideal (IFI) in BCK algebra were presented by Jun and Kim [13] in 2000 as an extension of FS concept in BCK algebra. As the time goes, BCK/BCI algebra and ideals were studied by Senapati *et al.* [14,15] in context of intuitionistic in various directions. Bipolar fuzzy set (BFS) [16] is the generalization of FS which involves the degree of positive membership (DPMS) and the degree of negative membership (DNegMS) of an element. Bipolar fuzzy environment can be realized by an example. The serials broadcasted in Television have both good effect and bad effect on young generation. Good effect can be treated as positive effect and bad effect can be treated as negative effect. Extension work on BFSwas given by Chen [17] in the form of m-polar FS. In 2013, including the measure of neutral membership and generalizing the notion of IFS, the concept of picture fuzzy set (PFS) was initiated by Cuong [18]. After the initiation of PFS, different types of research works in context of PFS were performed by several researchers [19–21]. In this paper, we introduce the concept of m-polar picture fuzzy subalgebra (PFSA), m-polar picture fuzzy ideal (PFI) and m-polar picture fuzzy implicative ideal (PFII), m-polar picture fuzzy commutative ideal (PFII) of BCK algebra and explore some results related to these. Also, we develop relationships of m-polar PFI with m-polar PFII and m-polar PFCI of BCK algebra.

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2. LIST OF ABBREVIATIONS

FS - Fuzzy set

IFS - Intuitionistic fuzzy set

BFS - Bipolar fuzzy Set

PFS - Picture fuzzy set

DMS - Degree of membership

DNonMS - Degree of non-membership

DPMS - Degree of positive membership

DNegMS - Degree of negative membership

DNeuMS - Degree of neutral membership-

FI - Fuzzy ideal

IFI - Intuitionistic fuzzy ideal

PFSA - Picture fuzzy subalgebra

PFI - Picture fuzzy ideal

PFII - Picture fuzzy implicative ideal

PFPII - Picture fuzzy positive implicative ideal

PFCI - Picture fuzzy commutative ideal

3. PRELIMINARIES

Here, we recapitulate some basic concepts of FS, IFS, BCK/BCI algebra, FI, IFI, BFS, m-polar FS and PFS. We define m-polar PFS, some basic operations on m-polar PFSs, (θ, ϕ, ψ)-cut of m-polar PFS, image and inverse of m-polar PFS.

Definition 1. Let A be the set of universe. Then a FS [1] P over A is defined as $P = \{(a, \mu_P(a)) : a \in A\}$, where $\mu_P : A \to [0,1]$. Here, $\mu_P(a)$ is DMS of a in P.

The DNonMS was missing in FS. Including this type of uncertainty, Atanassov defined IFS in 1986.

Definition 2. Let A be the set of universe. An IFS [2] P over A is defined as $P = \{(a, \mu_P(a), \nu_P(a)) : a \in A\}$, where $\mu_P(a) \in [0,1]$ is the DMS of a in P and $\nu_P(a) \in [0,1]$ is the DNonMS of a in P with the condition $0 \leqslant \mu_P(a) + \nu_P(a) \leqslant 1$ for all $a \in A$.

Here, $s_P(a) = 1 - (\mu_P(a) + \nu_P(a))$ is the measure of suspicion of a in P, which excludes the DMS and the DNonMS.

Iseki introduced a special type of algebra namely BCI algebra in 1980.

Definition 3. An algebra $(A, \diamondsuit, 0)$ is said to be BCI algebra [4] if for any $a, b, c \in A$, the below stated conditions are meet.

i. $[(a \diamondsuit b) \diamondsuit (a \diamondsuit c)] \diamondsuit (c \diamondsuit b) = 0$

ii. $[a \diamondsuit (a \diamondsuit b)] \diamondsuit b = 0$

iii. $a \diamondsuit a = 0$

iv. $a \diamondsuit b = 0$ and $b \diamondsuit a = 0 \Rightarrow a = b$

A BCI algebra with the condition $0 \diamondsuit a = 0$ for all $a \in A$ is called BCK algebra.

A relation "$\leqslant$" on A is defined as $a \leqslant b$ iff $a \diamondsuit b = 0$.

Proposition 1. *In a BCK algebra $(A, \diamondsuit, 0)$ the followings hold.*

i. $0 \diamondsuit a = 0$

ii. $a \diamondsuit 0 = a$

iii. $a \diamondsuit (a \diamondsuit b) \leqslant b$

iv. $a \diamondsuit b \leqslant a$

v. $(a \diamondsuit b) \diamondsuit c = (a \diamondsuit c) \diamondsuit b$

vi. $(a \diamondsuit (a \diamondsuit (a \diamondsuit b))) = a \diamondsuit b$ *for all* $a, b, c \in A$

Definition 4. Let $(A, \diamondsuit, 0)$ be a BCK algebra and $P = (\mu_P, \eta_P, \nu_P)$ be a FS in A. Then P is said to be FI [6] of A if

i. $\mu_P(0) \geqslant \mu_P(a)$

ii. $\mu_P(a) \geqslant \mu_P(a \diamondsuit b) \wedge \mu_P(b)$ for all $a, b \in A$ and for $l = 1, 2, \ldots, m$

Definition 5. Let $(A, \diamondsuit, 0)$ be a BCK algebra and $P = (\mu_P, \eta_P, \nu_P)$ be an IFS in A. Then P is said to IFI [13] of A if

i. $\mu_P(0) \geqslant \mu_P(a)$ and $\nu_P(0) \leqslant \nu_P(a)$

ii. $\mu_P(a) \geqslant \mu_P(a \diamondsuit b) \wedge \mu_P(b)$ and $\nu_P(a) \leqslant \nu_P(a \diamondsuit b) \vee \nu_P(b)$ for all $a, b \in A$

Definition 6. A BFS [16] P is defined as $P = (a, \mu_P(a), \nu_P(a)) : a \in A$, where $\mu_P(a) \in (0, 1]$ measures how much a particular property is satisfied by an element and $\nu_P(a) \in [-1, 0)$ measures how much its anti property is satisfied by that element. DMS 0 means the element has no relevancy to the property.

Definition 7. An m-polar FS [17] P over the set of universe A is an object of the form $P = \{(a, \mu_P(a)) : a \in A\}$, where $\mu_P : A \to [0,1]^m$ (m is a natural number). Here, $[0,1]^m$ is the poset with respect to partial order relation "$\leqslant$" which is defined as: $a \leqslant b$ iff $p_l(a) \leqslant p_l(b)$ for $l = 1, 2, \ldots, m$; where $p_l : [0,1]^m \to [0,1]$ is called l-th projection mapping.

Including more possible types of uncertainity, Cuong defined PFS in 2013 generalizing the concepts of FS and IFS.

Definition 8. Let A be the set of universe. Then a PFS [18] P over the universe A is defined as $P = \{(a, \mu_P(a), \eta_P(a), \nu_P(a)) : a \in A\}$, where $\mu_P(a) \in [0,1]$ is the DPMS of a in P, $\eta_P(a) \in [0,1]$ is the degree of neutral membership (DNeuMS) of a in P and $\nu_P(a) \in [0,1]$ is the DNegMS of a in P with the condition $0 \leqslant \mu_P(a) + \eta_P(a) + \nu_P(a) \leqslant 1$ for all $a \in A$. For all $a \in A, 1 - (\mu_P(a) + \eta_P(a) + \nu_P(a))$ is the measure of denial membership a in P. Sometimes, $(\mu_P(a), \eta_P(a), \nu_P(a))$ is called picture fuzzy value for $a \in A$.

Motivated by this definition, below we define m-polar PFS.

Definition 9. An m-polar PFS P over the set of universe A is an object of the form $P = \{(a, \mu_P(a), \eta_P(a), \nu_P(a)) : a \in A\}$, where $\mu_P : A \to [0,1]^m$, $\eta_P : A \to [0,1]^m$ and $\nu_P : A \to [0,1]^m$ (m is a natural number) with the condition $0 \leqslant p_l \circ \mu_P(a) + p_l \circ \eta_P(a) + p_l \circ \nu_P(a) \leqslant 1$ for all $a \in A$ and for $l = 1, 2, \ldots, m$. For $a \in A$, each of $\mu_P(a), \eta_P(a)$ and $\nu_P(a)$ is an m-tuple fuzzy value. Here, $p_l \circ \mu_P(a)$, $p_l \circ \eta_P(a)$ and $p_l \circ \nu_P(a)$ represent l-th components of $\mu_P(a)$, $\eta_P(a)$ and $\nu_P(a)$ respectively for $l = 1, 2, \ldots, m$.

The basic operations on m-polar PFSs consisting of equality, union and intersection are defined below.

Definition 10. Let $P = \{(a, \mu_P(a), \eta_P(a), \nu_P(a)) : a \in A\}$ and $Q = \{(a, \mu_Q(a), \eta_Q(a), \nu_Q(a)) : a \in A\}$ be two m-polar PFSs over the universe A. Then

i. $P \subseteq Q$ iff $p_l \circ \mu_P(a) \leqslant p_l \circ \mu_Q(a)$, $p_l \circ \eta_P(a) \leqslant p_l \circ \eta_Q(a)$ and $p_l \circ \nu_P(a) \geqslant p_l \circ \nu_Q(a)$ for all $a \in A$ and for $l = 1, 2, \ldots, m$.

ii. $P = Q$ iff $p_l \circ \mu_P(a) = p_l \circ \mu_Q(a)$, $p_l \circ \eta_P(a) = p_l \circ \eta_Q(a)$ and $p_l \circ \nu_P(a) = p_l \circ \nu_Q(a)$ for all $a \in A$ and for $l = 1, 2, \ldots, m$.

iii. $p_l \circ (P \cup Q) = \{(a, \max(p_l \circ \mu_P(a), p_l \circ \mu_Q(a)), \min(p_l \circ \eta_P(a), p_l \circ \eta_Q(a)), \min(p_l \circ \nu_P(a), p_l \circ \nu_Q(a))) : a \in A\}$ for $l = 1, 2, \ldots, m$.

iv. $p_l \circ (P \cap Q) = \{(a, \min(p_l \circ \mu_P(a), p_l \circ \mu_Q(a)), \min(p_l \circ \eta_P(a), p_l \circ \eta_Q(a)), \max(p_l \circ \nu_P(a), p_l \circ \nu_Q(a))) : a \in A\}$ for $l = 1, 2, \ldots, m$.

Definition 11. Let $P = \{(a, \mu_P, \eta_P, \nu_P) : a \in A\}$ be an m-polar PFS over the universe A. Then (θ, ϕ, ψ)-cut of P is the crisp set in

A denoted by $C_{\theta,\phi,\psi}(P)$ and is defined as $C_{\theta,\phi,\psi}(P) = \{a \in A : p_l \circ \mu_P(a) \geqslant p_l \circ \theta, p_l \circ \eta_P(a) \geqslant p_l \circ \phi, p_l \circ \nu_P(a) \leqslant p_l \circ \psi$ for $l = 1, 2, \ldots, m\}$, where $p_l \circ \theta \in [0,1], p_l \circ \phi \in [0,1], p_l \circ \psi \in [0,1]$ with the condition $0 \leqslant p_l \circ \theta + p_l \circ \phi + p_l \circ \psi \leqslant 1$ for $l = 1, 2, \ldots, m$. The mentionable fact is that each of θ, ϕ and ψ is an m-polar fuzzy value. Here, $p_l \circ \theta$, $p_l \circ \phi$ and $p_l \circ \psi$ represent l-th components of the m-polar fuzzy values θ, ϕ and ψ for $l = 1, 2, \ldots, m$

Definition 12. Let A_1 and A_2 be two sets of universe. Let $h : A_1 \to A_2$ be a surjective mapping and $P = \{(a_1, \mu_P(a_1), \eta_P(a_1), \nu_P(a_1)) : a_1 \in A_1\}$ be an m-polar PFS in A_1. Then the image of P under the map h is the m-polar PFS $h(P) = \{(a_2, \mu_{h(P)}(a_2), \eta_{h(P)}(a_2), \nu_{h(P)}(a_2)) : a_2 \in A_2\}$, where $p_l \circ \mu_{h(P)}(a_2) = \bigvee_{a_1 \in h^{-1}(a_2)} p_l \circ \mu_P(a_1)$, $p_l \circ \eta_{h(P)}(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} p_l \circ \eta_P(a_1)$ and $p_l \circ \nu_{h(P)}(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} p_l \circ \nu_P(a_1)$ for all $a_2 \in A_2$ and for $l = 1, 2, \ldots, m$.

Definition 13. Let A_1 and A_2 be two sets of universe. Let $h : A_1 \to A_2$ be a mapping and $Q = \{(a_2, \mu_Q(a_2), \eta_Q(a_2), \nu_Q(a_2)) : a_2 \in A_2\}$ be an m-polar PFS in A_2. Then the inverse image of Q under the map h is the m-polar PFS $h^{-1}(Q) = \{(a_1, \mu_{h^{-1}(Q)}(a_1), \eta_{h^{-1}(Q)}(a_1), \nu_{h^{-1}(Q)}(a_1)) : a_1 \in A_1\}$, where $p_l \circ \mu_{h^{-1}(Q)}(a_1) = p_l \circ \mu_Q(h(a_1))$, $p_l \circ \eta_{h^{-1}(Q)}(a_1) = p_l \circ \eta_Q(h(a_1))$ and $p_l \circ \nu_{h^{-1}(Q)}(a_1) = p_l \circ \nu_Q(h(a_1))$ for all $a_1 \in A_1$ and for $l = 1, 2, .., m$.

Definition 14. Let $P = \{(a_1, \mu_P(a_1), \eta_P(a_1), \nu_P(a_1)) : a_1 \in A_1\}$ and $Q = \{(a_2, \mu_Q(a_2), \eta_Q(a_2), \nu_Q(a_2)) : a_2 \in A_2\}$ be two m-polar PFSs over the sets of universe A_1 and A_2 respectively. Then the Cartesian product of P and Q is the m-polar PFS $P \times Q = \{((a,b), \mu_{P\times Q}((a,b)), \eta_{P\times Q}((a,b)), \nu_{P\times Q}((a,b))) : (a,b) \in A_1 \times A_2\}$, where $p_l \circ \mu_{P\times Q}((a,b)) = p_l \circ \mu_P(a) \wedge p_l \circ \mu_Q(b)$, $p_l \circ \eta_{P\times Q}((a,b)) = p_l \circ \eta_P(a) \wedge p_l \circ \eta_Q(b)$ and $p_l \circ \nu_{P\times Q}((a,b)) = p_l \circ \nu_P(a) \vee p_l \circ \nu_Q(b)$ for all $(a,b) \in A_1 \times A_2$ and for $l = 1, 2, \ldots, m$.

4. m-POLAR PFI

Let us first define m-polar PFSA of a BCK algebra.

Definition 15. Let $(A, \Diamond, 0)$ be a BCK algebra and $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFS in A. Then P is said to be m-polar PFSA of A if $p_l \circ \mu_P(a \Diamond b) \geqslant p_l \circ \mu_P(a) \wedge p_l \circ \mu_P(b)$, $p_l \circ \eta_P(a \Diamond b) \geqslant p_l \circ \eta_P(a) \wedge p_l \circ \eta_P(b)$ and $p_l \circ \nu_P(a \Diamond b) \leqslant p_l \circ \nu_P(a) \vee p_l \circ \nu_P(b)$ for all $a, b \in A$ and for $l = 1, 2, \ldots, m$.

Example 1. Consider a BCK algebra $(A, \Diamond, 0)$ defined in the following tabular form:

$\Diamond$	0	p	q	r
0	0	0	0	0
p	p	0	0	p
q	q	p	0	q
r	r	r	r	0

Now, let us consider a 3-polar PFS P as follows:

$$\mu_P(a) = \begin{cases} (0.25, 0.35, 0.4), & \text{if } a = 0 \\ (0.15, 0.25, 0.35), & \text{if } a = p \\ (0.1, 0.15, 0.25), & \text{if } a = q \\ (0.15, 0.25, 0.4), & \text{if } a = r \end{cases}$$

$$\eta_P(a) = \begin{cases} (0.2, 0.3, 0.4), & \text{if } a = 0 \\ (0.1, 0.2, 0.3), & \text{if } a = p \\ (0.05, 0.1, 0.2), & \text{if } a = q \\ (0.1, 0.2, 0.35), & \text{if } a = r \end{cases}$$

and

$$\nu_P(a) = \begin{cases} (0.1, 0.15, 0.2), & \text{if } a = 0 \\ (0.15, 0.2, 0.3), & \text{if } a = p \\ (0.2, 0.3, 0.4), & \text{if } a = q \\ (0.15, 0.2, 0.25), & \text{if } a = r \end{cases}$$

It is easy to show that P is a 3-polar PFSA of A.

Proposition 2. *Let $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFSA of a BCK algebra A. Then $p_l \circ \mu_P(0) \geqslant p_l \circ \mu_P(a)$, $p_l \circ \eta_P(0) \geqslant p_l \circ \eta_P(a)$ and $p_l \circ \nu_P(0) \leqslant p_l \circ \nu_P(a)$ for all $a \in A$ and for $l = 1, 2, \ldots, m$.*

Proof. It is observed that

$$\begin{aligned} p_l \circ \mu_P(0) &= p_l \circ \mu_P(a \Diamond a) \\ &\geqslant p_l \circ \mu_P(a) \wedge p_l \circ \mu_P(a) \\ &\quad \text{[because } P \text{ is an } m\text{-polar PFSA of } A] \\ &= p_l \circ \mu_P(a), \end{aligned}$$

$$\begin{aligned} p_l \circ \eta_P(0) &= p_l \circ \eta_P(a \Diamond a) \\ &\geqslant p_l \circ \eta_P(a) \wedge p_l \circ \eta_P(a) \\ &\quad \text{[because } P \text{ is an } m\text{-polar PFSA of } A] \\ &= p_l \circ \eta_P(a) \end{aligned}$$

$$\begin{aligned} \text{and } p_l \circ \nu_P(0) &= p_l \circ \nu_P(a \Diamond a) \\ &\leqslant p_l \circ \nu_P(a) \vee p_l \circ \nu_P(a) \\ &\quad \text{[because } P \text{ is an } m\text{-polar PFSA of } A] \\ &= p_l \circ \nu_P(a) \text{ for all } a \in A \\ &\quad \text{and for } l = 1, 2, \ldots, m. \end{aligned}$$

Thus, $p_l \circ \mu_P(0) \geqslant p_l \circ \mu_P(a)$, $p_l \circ \eta_P(0) \geqslant p_l \circ \eta_P(a)$ and $p_l \circ \nu_P(0) \leqslant p_l \circ \nu_P(a)$ for all $a \in A$ and for $l = 1, 2, \ldots, m$.

Now, let us define m-polar PFI of a BCK algebra.

Definition 16. Let $(A, \Diamond, 0)$ be a BCK algebra and $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFS in A. Then P is said to m-polar PFI of A if

i. $p_l \circ \mu_P(0) \geqslant p_l \circ \mu_P(a)$, $p_l \circ \eta_P(0) \geqslant p_l \circ \eta_P(a)$ and $p_l \circ \nu_P(0) \leqslant p_l \circ \nu_P(a)$
ii. $p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P(a \Diamond b) \wedge p_l \circ \mu_P(b)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P(a \Diamond b) \wedge p_l \circ \eta_P(b)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P(a \Diamond b) \vee p_l \circ \nu_P(b)$ for all $a, b \in A$ and for $l = 1, 2, \ldots, m$

Now, we are going to investigate some important results on m-polar PFI of a BCK algebra.

Proposition 3. *Let $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFI of a BCK algebra $(A, \Diamond, 0)$. Then $p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P(b)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P(b)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P(b)$ for $a, b \in A$ with $a \leqslant b$ and for $l = 1, 2, \ldots, m$.*

Proof. Let $a, b \in A$ such that $a \leqslant b$. Then $a \lozenge b = 0$.

$$\begin{aligned}
&\text{Now, } p_l \circ \mu_P(a) \\
&\geqslant p_l \circ \mu_P(a \lozenge b) \wedge p_l \circ \mu_P(b) \text{ [as } P \text{ is an } m\text{-polar PFI of } A] \\
&= p_l \circ \mu_P(0) \wedge p_l \circ \mu_P(b) \\
&= p_l \circ \mu_P(b) \text{ [as } P \text{ is an } m\text{-polar PFI of } A],
\end{aligned}$$

$$\begin{aligned}
&p_l \circ \eta_P(a) \\
&\geqslant p_l \circ \eta_P(a \lozenge b) \wedge p_l \circ \eta_P(b) \text{ [as } P \text{ is an } m\text{-polar PFI of } A] \\
&= p_l \circ \eta_P(0) \wedge p_l \circ \eta_P(b) \\
&= p_l \circ \eta_P(b) \text{ [as } P \text{ is an } m\text{-polar PFI of } A]
\end{aligned}$$

$$\begin{aligned}
&\text{and } p_l \circ \nu_P(a) \\
&\leqslant p_l \circ \nu_P(a \lozenge b) \vee p_l \circ \nu_P(b) \text{ [as } P \text{ is an } m\text{-polar PFI of } A] \\
&= p_l \circ \nu_P(0) \vee p_l \circ \nu_P(b) \\
&= p_l \circ \nu_P(b) \text{ [as } P \text{ is an } m\text{-polar PFI of } A] \\
&\text{for } l = 1, 2, \ldots, m.
\end{aligned}$$

Thus, $p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P(b)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P(b)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P(b)$ for $a, b \in A$ with $a \leqslant b$ and for $l = 1, 2, \ldots, m$.

Proposition 4. *Let $(A, \lozenge, 0)$ be a BCK algebra and $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFI of A. Then $p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P(b) \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P(b) \wedge p_l \circ \eta_P(c)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P(b) \vee p_l \circ \nu_P(c)$ for $a, b, c \in A$ with $a \lozenge b \leqslant c$.*

Proof. Let $a, b, c \in A$ with $a \lozenge b \leqslant c$. Then $(a \lozenge b) \lozenge c = 0$.

$$\begin{aligned}
&\text{Now, } p_l \circ \mu_P(a) \\
&\geqslant p_l \circ \mu_P(a \lozenge b) \wedge p_l \circ \mu_P(b) \\
&\text{[because } P \text{ is an } m\text{-polar PFI of } A] \\
&\geqslant p_l \circ \mu_P((a \lozenge b) \lozenge c) \wedge p_l \circ \mu_P(c) \wedge p_l \circ \mu_P(b) \\
&\text{[because } P \text{ is an } m\text{-polar PFI of } A] \\
&= p_l \circ \mu_P(0) \wedge p_l \circ \mu_P(c) \wedge p_l \circ \mu_P(b) \\
&= p_l \circ \mu_P(b) \wedge p_l \circ \mu_P(c) \\
&\text{[because } P \text{ is an } m\text{-polar PFI of } A],
\end{aligned}$$

$$\begin{aligned}
&p_l \circ \eta_P(a) \\
&\geqslant p_l \circ \eta_P(a \lozenge b) \wedge p_l \circ \eta_P(b) \\
&\text{[because } P \text{ is an } m\text{-polar PFI of } A] \\
&\geqslant p_l \circ \eta_P((a \lozenge b) \lozenge c) \wedge p_l \circ \eta_P(c) \wedge p_l \circ \eta_P(b) \\
&\text{[because } P \text{ is an } m\text{-polar PFI of } A] \\
&= p_l \circ \eta_P(0) \wedge p_l \circ \eta_P(c) \wedge p_l \circ \eta_P(b) \\
&= p_l \circ \eta_P(b) \wedge p_l \circ \eta_P(c) \\
&\text{[because } P \text{ is an } m\text{-polar PFI of } A]
\end{aligned}$$

$$\begin{aligned}
&\text{and } p_l \circ \nu_P(a) \\
&\leqslant p_l \circ \nu_P(a \lozenge b) \vee p_l \circ \nu_P(b) \\
&\text{[because } P \text{ is an } m\text{-polar PFI of } A] \\
&\leqslant p_l \circ \nu_P((a \lozenge b) \lozenge c) \vee p_l \circ \nu_P(c) \vee p_l \circ \nu_P(b) \\
&\text{[because } P \text{ is an } m\text{-polar PFI of } A] \\
&= p_l \circ \nu_P(0) \vee p_l \circ \nu_P(c) \vee p_l \circ \nu_P(b) \\
&= p_l \circ \nu_P(b) \vee p_l \circ \nu_P(c) \\
&\text{[because } P \text{ is an } m\text{-polar PFI of } A] \\
&\text{for } l = 1, 2, \ldots, m.
\end{aligned}$$

Thus, it is obtained that $p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P(b) \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P(b) \wedge p_l \circ \eta_P(c)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P(b) \vee p_l \circ \nu_P(c)$ for $a, b, c \in A$ with $a \lozenge b \leqslant c$.

Proposition 5. *Every m-polar PFI of a BCK algebra is an m-polar PFSA.*

Proof. Let $(A, \lozenge, 0)$ be a BCK algebra and A is an m-PFI of A. Since P is an m-polar PFI, therefore,

$$\begin{aligned}
&p_l \circ \mu_P(a \lozenge b) \\
&\geqslant p_l \circ \mu_P((a \lozenge b) \lozenge a) \wedge p_l \circ \mu_P(a) \\
&= p_l \circ \mu_P((a \lozenge a) \lozenge b) \wedge p_l \circ \mu_P(a) \text{ [by Proposition 1]} \\
&= p_l \circ \mu_P(0 \lozenge b) \wedge p_l \circ \mu_P(a) \\
&= p_l \circ \mu_P(0) \lozenge p_l \circ \mu_P(a) \text{ [by Proposition 1]} \\
&\geqslant p_l \circ \mu_P(a) \lozenge p_l \circ \mu_P(b),
\end{aligned}$$

$$\begin{aligned}
&p_l \circ \eta_P(a \lozenge b) \\
&\geqslant p_l \circ \eta_P((a \lozenge b) \lozenge a) \wedge p_l \circ \eta_P(a) \\
&= p_l \circ \eta_P((a \lozenge a) \lozenge b) \wedge p_l \circ \eta_P(a) \text{ [by Proposition 1]} \\
&= p_l \circ \eta_P(0 \lozenge b) \wedge p_l \circ \eta_P(a) \\
&= p_l \circ \eta_P(0) \wedge p_l \circ \eta_P(a) \text{ [by Proposition 1]} \\
&\geqslant p_l \circ \eta_P(a) \wedge p_l \circ \eta_P(b)
\end{aligned}$$

$$\begin{aligned}
&\text{and } p_l \circ \nu_P(a \lozenge b) \\
&\leqslant p_l \circ \nu_P((a \lozenge b) \lozenge a) \vee p_l \circ \nu_P(a) \\
&= p_l \circ \nu_P((a \lozenge a) \lozenge b) \vee p_l \circ \nu_P(a) \text{ [by Proposition 1]} \\
&= p_l \circ \nu_P(0 \lozenge b) \vee p_l \circ \nu_P(a) \\
&= p_l \circ \nu_P(0) \vee p_l \circ \nu_P(a) \text{ [by Proposition 1]} \\
&\leqslant p_l \circ \nu_P(a) \vee p_l \circ \nu_P(b), \\
&\text{for all } a, b \in A \text{ and for } l = 1, 2, \ldots, m.
\end{aligned}$$

Hence, P is an m-polar PFSA of A.

But, the converse of the above proposition is not true in general which is shown in following example. Proposition 6 states under which condition an m-polar PFSA is an m-polar PFI.

Example 2. Let us suppose the BCK algebra given in Example 1 and a 3-polar PFS P as follows:

$$\mu_P(a) = \begin{cases} (0.2, 0.3, 0.4), & \text{if } a = 0, q \\ (0.1, 0.2, 0.3), & \text{if } a = p, r \end{cases}$$

$$\eta_P(a) = \begin{cases} (0.25, 0.35, 0.45), & \text{if } a = 0, q \\ (0.15, 0.25, 0.3), & \text{if } a = p, r \end{cases}$$

and

$$\nu_P(a) = \begin{cases} (0.3, 0.2, 0.1), & \text{if } a = 0, q \\ (0.4, 0.3, 0.2), & \text{if } a = p, r \end{cases}$$

Here, $(0.1, 0.2, 0.3) = \mu_P(p) \ngeq \mu_P(p \lozenge q) \wedge \mu_P(q) = (0.2, 0.3, 0.4)$, $(0.15, 0.25, 0.3) = \eta_P(p) \ngeq \eta_P(p \lozenge q) \wedge \eta_P(q) = (0.25, 0.35, 0.45)$ and $(0.4, 0.3, 0.2) = \nu_P(p) \nleq \nu_P(p \lozenge q) \vee \nu_P(q) = (0.3, 0.2, 0.1)$. So, P is not a 3-polar PFI of A although it is a 3-polar PFSA.

Proposition 6. *Let $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFSA of a BCK algebra $(A, \lozenge, 0)$. Then P is an m-polar PFI of A if for all $a, b, c \in A$, $a \lozenge b \leqslant c \Rightarrow p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P(b) \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P(b) \wedge p_l \circ \eta_P(c)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P(b) \vee p_l \circ \nu_P(c)$ for $l = 1, 2, \ldots, m$.*

Proof. By given conditions, for all $a, b, c \in A$, $a \lozenge b \leqslant c \Rightarrow p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P(b) \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P(b) \wedge p_l \circ \eta_P(c)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P(b) \vee p_l \circ \nu_P(c)$. Since A is a BCK algebra therefore by Proposition 1, $a \lozenge (a \lozenge b) \leqslant b$. So, it is obtained that

$$\begin{aligned}
p_l \circ \mu_P(a) &\geqslant p_l \circ \mu_P(a \lozenge b) \wedge p_l \circ \mu_P(b) \\
p_l \circ \eta_P(a) &\geqslant p_l \circ \mu_P(a \lozenge b) \wedge p_l \circ \mu_P(b) \\
\text{and } p_l \circ \nu_P(a) &\leqslant p_l \circ \nu_P(a \lozenge b) \vee p_l \circ \nu_P(b) \\
&\text{for } l = 1, 2, \ldots, m
\end{aligned}$$

Thus, P is an m-polar PFI of A.

Proposition 7. *Let $(A, \diamondsuit, 0)$ a BCK algebra and $P = (\mu_p, \eta_P, \nu_P)$, $Q = (\mu_Q, \eta_Q, \nu_Q)$ be two m-polar PFIs of A. Then $P \cap Q$ is an m-polar PFI of A.*

Proof. Let $P \cap Q = R = (\mu_R, \eta_R, \nu_R)$. Then $p_l \circ \mu_R(a) = p_l \circ \mu_P(a) \wedge p_l \circ \mu_Q(a)$, $p_l \circ \eta_R(a) = p_l \circ \eta_P(a) \wedge p_l \circ \eta_Q(a)$ and $p_l \circ \nu_R(a) = p_l \circ \nu_P(a) \vee p_l \circ \nu_Q(a), \forall a \in A$ and for $l = 1, 2, \dots, m$.

$$\begin{aligned}
&\text{Now, } p_l \circ \mu_R(0)\\
&= p_l \circ \mu_P(0) \wedge p_l \circ \mu_Q(0)\\
&\geqslant p_l \circ \mu_P(a) \wedge p_l \circ \mu_Q(a)\\
&\ [\text{as } P, Q \text{ are } m\text{-polar PFIs of } A]\\
&= p_l \circ \mu_R(a)
\end{aligned}$$

$$\begin{aligned}
&p_l \circ \eta_R(0)\\
&= p_l \circ \eta_P(0) \wedge p_l \circ \eta_Q(0)\\
&\geqslant p_l \circ \eta_P(a) \wedge p_l \circ \eta_Q(a)\\
&\ [\text{as } P, Q \text{ are } m\text{-polar PFIs of } A]\\
&= p_l \circ \eta_R(a)
\end{aligned}$$

$$\begin{aligned}
&\text{and } p_l \circ \nu_R(0)\\
&= p_l \circ \nu_P(0) \vee p_l \circ \nu_Q(0)\\
&\leqslant p_l \circ \nu_P(a) \vee p_l \circ \nu_Q(a)\\
&\ [\text{as } P, Q \text{ are } m\text{-polar PFIs of } A]\\
&= p_l \circ \nu_R(a), \forall a \in A \text{ and } l = 1, 2, \dots, m.
\end{aligned}$$

$$\begin{aligned}
&\text{Also, } p_l \circ \mu_R(a)\\
&= p_l \circ \mu_P(a) \wedge p_l \circ \mu_Q(a)\\
&\geqslant (p_l \circ \mu_P(a \diamondsuit b) \wedge p_l \circ \mu_P(b)) \wedge (p_l \circ \mu_Q(a \diamondsuit b) \wedge p_l \circ \mu_Q(b))\\
&\ [\text{as } P, Q \text{ are } m\text{-polar PFIs of } A]\\
&= (p_l \circ \mu_P(a \diamondsuit b) \wedge p_l \circ \mu_Q(a \diamondsuit b)) \wedge (p_l \circ \mu_P(b) \wedge p_l \circ \mu_Q(b))\\
&= p_l \circ \mu_R(a \diamondsuit b) \wedge p_l \circ \mu_R(b),
\end{aligned}$$

$$\begin{aligned}
&p_l \circ \eta_R(a)\\
&= p_l \circ \eta_P(a) \wedge p_l \circ \eta_Q(a)\\
&\geqslant (p_l \circ \eta_P(a \diamondsuit b) \wedge p_l \circ \eta_P(b)) \wedge (p_l \circ \eta_Q(a \diamondsuit b) \wedge p_l \circ \eta_Q(b))\\
&\ [\text{as } P, Q \text{ are } m\text{-polar PFIs of } A]\\
&= (p_l \circ \eta_P(a \diamondsuit b) \wedge p_l \circ \eta_Q(a \diamondsuit b)) \wedge (p_l \circ \eta_P(b) \wedge p_l \circ \eta_Q(b))\\
&= p_l \circ \eta_R(a \diamondsuit b) \wedge p_l \circ \eta_R(b)
\end{aligned}$$

$$\begin{aligned}
&\text{and } p_l \circ \nu_R(a)\\
&= p_l \circ \nu_P(a) \vee p_l \circ \nu_Q(a)\\
&\leqslant (p_l \circ \nu_P(a \diamondsuit b) \vee p_l \circ \nu_P(b)) \vee (p_l \circ \nu_Q(a \diamondsuit b) \vee p_l \circ \nu_Q(b))\\
&\ [\text{as } P, Q \text{ are } m\text{-polar PFIs of } A]\\
&= (p_l \circ \nu_P(a \diamondsuit b) \vee p_l \circ \nu_Q(a \diamondsuit b)) \vee (p_l \circ \nu_P(b) \vee p_l \circ \nu_Q(b))\\
&= p_l \circ \nu_R(a \diamondsuit b) \vee p_l \circ \nu_R(b), \forall a, b \in A \text{ and } l = 1, 2, \dots, m.
\end{aligned}$$

Thus, $p_l \circ \mu_R(a) \geqslant p_l \circ \mu_R(a \diamondsuit b) \wedge p_l \circ \mu_R(b)$, $p_l \circ \eta_R(a) \geqslant p_l \circ \eta_R(a \diamondsuit b) \wedge p_l \circ \eta_R(b)$ and $p_l \circ \nu_R(a \diamondsuit b) \vee p_l \circ \nu_R(b), \forall a, b \in A$ and for $l = 1, 2, \dots, m$. Consequently, $R = P \cap Q$ is an m-polar PFI of A.

Proposition 8. *Let $P = (\mu_P, \eta_P, \nu_P)$ and $Q = (\mu_Q, \eta_Q, \nu_Q)$ be two m-polar PFIs of a BCK algebra $(A, \diamondsuit, 0)$. Then $P \times Q$ is an m-polar PFI of $A \times A$.*

Proof. Proof is same as Proposition 7. So, it is omitted.

Proposition 9. *Let $(A, \diamondsuit, 0)$ be a BCK algebra and $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFI of A. Then $C_{\theta,\phi,\psi}(P)$ is a crisp ideal of A, provided that $P_l \circ \mu_P(0) \geqslant p_l \circ \theta$, $p_l \circ \eta_P(0) \geqslant p_l \circ \phi$ and $p_l \circ \nu_P(0) \leqslant p_l \circ \psi$ for $l = 1, 2, \dots, m$.*

Proof. Clearly, $C_{\theta,\phi,\psi}(P)$ contains at least one element. Let $a \diamondsuit b$, $b \in C_{\theta,\phi,\psi}(P)$. Then $p_l \circ \mu_P(a \diamondsuit b) \geqslant p_l \circ \theta$, $p_l \circ \eta_P(a \diamondsuit b) \geqslant p_l \circ \phi$, $p_l \circ \nu_P(a \diamondsuit b) \leqslant \psi$ and $p_l \circ \mu_P(b) \geqslant p_l \circ \theta$, $p_l \circ \eta_P(b) \geqslant p_l \circ \phi$, $p_l \circ \nu_P(b) \leqslant p_l \circ \psi$ for $l = 1, 2, \dots, m$.

$$\begin{aligned}
&\text{Now, } p_l \circ \mu_P(a)\\
&\geqslant p_l \circ \mu_P(a \diamondsuit b) \wedge p_l \circ \mu_P(b)\\
&\ [\text{because } P \text{ is an } m\text{-polar PFI of } A]\\
&\geqslant p_l \circ \theta \wedge p_l \circ \theta = p_l \circ \theta,
\end{aligned}$$

$$\begin{aligned}
&p_l \circ \eta_P(a)\\
&\geqslant p_l \circ \eta_P(a \diamondsuit b) \wedge p_l \circ \eta_P(b)\\
&\ [\text{because } P \text{ is an } m\text{-polar PFI of } A]\\
&\geqslant p_l \circ \phi \wedge p_l \circ \phi = p_l \circ \phi
\end{aligned}$$

$$\begin{aligned}
&\text{and } p_l \circ \nu_P(a)\\
&\leqslant p_l \circ \nu_P(a \diamondsuit b) \vee p_l \circ \nu_P(b)\\
&\ [\text{because } P \text{ is an } m\text{-polar PFI of } A]\\
&\leqslant p_l \circ \psi \vee p_l \circ \psi = p_l \circ \psi \text{ for } l = 1, 2, \dots .m
\end{aligned}$$

Thus, $a \diamondsuit b$, $b \in C_{\theta,\phi,\psi}(P) \Rightarrow a \in C_{\theta,\phi,\psi}(P)$. So, $C_{\theta,\phi,\psi}(P)$ is a crisp ideal of A.

Proposition 10. *Let $(A, \diamondsuit, 0)$ be a BCK algebra and $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFS in A. Then P is an m-polar PFI of A if all (θ, ϕ, ψ)-cuts of P are crisp ideals of A.*

Proof. Let $a, b \in A$. Let $p_l \circ \mu_P(a \diamondsuit b) \wedge p_l \circ \mu_P(b) = p_l \circ \theta$, $p_l \circ \eta_P(a \diamondsuit b) \wedge p_l \circ \eta_P(b) = p_l \circ \phi$ and $p_l \circ \nu_P(a \diamondsuit b) \vee p_l \circ \nu_P(b) = p_l \circ \psi$ for $l = 1, 2, \dots, m$. Clearly, $p_l \circ \theta \in [0, 1]$, $p_l \circ \phi \in [0, 1]$ and $p_l \circ \psi \in [0, 1]$ with $0 \leqslant p_l \circ \theta + p_l \circ \phi + p_l \circ \psi \leqslant 1$ for $l = 1, 2, \dots, m$.

$$\begin{aligned}
\text{Now, } p_l \circ \mu_P(a \diamondsuit b) &\geqslant p_l \circ \mu_P(a \diamondsuit b) \wedge p_l \circ \mu_P(b) = p_l \circ \theta,\\
p_l \circ \eta_P(a \diamondsuit b) &\geqslant p_l \circ \mu_P(a \diamondsuit b) \wedge p_l \circ \mu_P(b) = p_l \circ \phi\\
\text{and } p_l \circ \nu_P(a \diamondsuit b) &\leqslant p_l \circ \nu_P(a \diamondsuit b) \vee p_l \circ \nu_P(b) = p_l \circ \psi\\
&\text{for } l = 1, 2, \dots, m.
\end{aligned}$$

$$\begin{aligned}
\text{Also, } p_l \circ \mu_P(b) &\geqslant p_l \circ \mu_P(a \diamondsuit b) \wedge p_l \circ \mu_P(b) = p_l \circ \theta,\\
p_l \circ \eta_P(b) &\geqslant p_l \circ \mu_P(a \diamondsuit b) \wedge p_l \circ \mu_P(b) = p_l \circ \phi\\
\text{and } p_l \circ \nu_P(b) &\leqslant p_l \circ \nu_P(a \diamondsuit b) \vee p_l \circ \nu_P(b) = p_l \circ \psi\\
&\text{for } l = 1, 2, \dots, m.
\end{aligned}$$

Thus, $a \diamondsuit b$ and $b \in C_{\theta,\phi,\psi}(P)$. Since $C_{\theta,\phi,\psi}(P)$ is a crisp ideal of A therefore $a \diamondsuit b \in C_{\theta,\phi,\psi}(P)$ and $b \in C_{\theta,\phi,\psi}(P) \Rightarrow a \in C_{\theta,\phi,\psi}(P)$.

Therefore, $p_l \circ \mu_P(a) \geqslant p_l \circ \theta = p_l \circ \mu_P(a \diamondsuit b) \wedge p_l \circ \mu_P(b)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \phi = p_l \circ \eta_P(a \diamondsuit b) \wedge p_l \circ \eta_P(b)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \psi = p_l \circ \nu_P(a \diamondsuit b) \vee p_l \circ \nu_P(b)$ for $l = 1, 2, .., m$.

Since a, b are arbitrary elements of A therefore $p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P(a \diamondsuit b) \wedge p_l \circ \mu_P(b)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P(a \diamondsuit b) \wedge p_l \circ \eta_P(b)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P(a \diamondsuit b) \vee p_l \circ \nu_P(b)$ for all $a, b \in A$ and for $l = 1, 2, \dots, m$. Hence, P is an m-polar PFI of A.

5. PRE-IMAGE AND IMAGE PFI UNDER HOMOMORPHISM OF BCK ALGEBRA

In the current section, we explore some properties of m-polar PFI of BCK algebra under homomorphism of BCK algebra.

Definition 17. Let $(A_1, \diamondsuit, 0)$ and $(A_2, *, 0)$ be two BCK algebras. Then a mapping $h : A_1 \rightarrow A_2$ is said to be homomorphism if $h(a \diamondsuit b) = h(a) * h(b)$ for all $a, b \in A_1$.

It is observed that $h(a) * h(a) = 0$ i.e. $h(a \diamondsuit a) = 0$ i.e. $h(0) = 0$.

Proposition 11. *Let $(A_1, \diamondsuit, 0)$ and $(A_2, *, 0)$ be two BCK algebras and $Q = (\mu_Q, \eta_Q, \nu_Q)$ be an m-polar PFI of A_2. Then for a BCK algebra homomorphism $h : A_1 \to A_2$, $h^{-1}(Q)$ is an m-polar PFI of A_1.*

Proof. Let $h^{-1}(Q) = (\mu_{h^{-1}(Q)}, \eta_{h^{-1}(Q)}, \nu_{h^{-1}(Q)})$, where $\mu_{h^{-1}(Q)} = \mu_Q(h(a))$, $\eta_{h^{-1}(Q)}(a) = \eta_Q(h(a))$ and $\nu_{h^{-1}(Q)}(a) = \nu_Q(h(a))$ for all $a \in A_1$.

Now, $p_l \circ \mu_{h^{-1}(Q)}(0)$
$= p_l \circ \mu_Q(h(0))$
$= p_l \circ \mu_Q(0)$ [as $h(0) = 0$]
$\geqslant p_l \circ \mu_Q(h(a))$ [because Q is an m-polar PFI of A_2]
$= p_l \circ \mu_{h^{-1}(Q)}(a)$,

$p_l \circ \eta_{h^{-1}(Q)}(0)$
$= p_l \circ \eta_Q(h(0))$
$= p_l \circ \eta_Q(0)$ [as $h(0) = 0$]
$\geqslant p_l \circ \eta_Q(h(a))$ [because Q is an m-polar PFI of A_2]
$= p_l \circ \eta_{h^{-1}(Q)}(a)$

and $p_l \circ \nu_{h^{-1}(Q)}(0)$
$= p_l \circ \nu_Q(h(0))$
$= p_l \circ \nu_Q(0)$ [as $h(0) = 0$]
$\leqslant p_l \circ \nu_Q(h(a))$ [because Q is an m-polar PFI of A_2]
$= p_l \circ \nu_{h^{-1}(Q)}(a)$ for all $a \in A_1$ and for $l = 1, 2, \dots, m$.

Thus, $p_l \circ \mu_{h^{-1}(Q)}(0) \geqslant p_l \circ \mu_{h^{-1}(Q)}(a)$, $p_l \circ \eta_{h^{-1}(Q)}(0) \geqslant p_l \circ \eta_{h^{-1}(Q)}(a)$ and $p_l \circ \nu_{h^{-1}(Q)}(0) \leqslant p_l \circ \nu_{h^{-1}(Q)}(a)$ for all $a \in A_1$ and for $l = 1, 2, \dots, m$.

Also, $p_l \circ \mu_{h^{-1}(Q)}(a)$
$= p_l \circ \mu_Q(h(a))$
$\geqslant p_l \circ \mu_Q(h(a) * h(b)) \wedge p_l \circ \mu_Q(h(b))$
[because Q is an m-polar PFI of A_2]
$= p_l \circ \mu_Q(h(a \diamondsuit b)) \wedge p_l \circ \mu_Q(h(b))$
[because h is a homomorphism]
$= p_l \circ \mu_{h^{-1}(Q)}(a \diamondsuit b) \wedge p_l \circ \mu_{h^{-1}(Q)}(b)$,

$p_l \circ \eta_{h^{-1}(Q)}(a)$
$= p_l \circ \eta_Q(h(a))$
$\geqslant p_l \circ \eta_Q(h(a) * h(b)) \wedge p_l \circ \eta_Q(h(b))$
[because Q is an m-polar PFI of A_2]
$= p_l \circ \eta_Q(h(a \diamondsuit b)) \wedge p_l \circ \eta_Q(h(b))$
[because h is a homomorphism]
$= p_l \circ \eta_{h^{-1}(Q)}(a \diamondsuit b) \wedge p_l \circ \eta_{h^{-1}(Q)}(b)$

and $p_l \circ \nu_{h^{-1}(Q)}(a)$
$= p_l \circ \nu_Q(h(a))$
$\leqslant p_l \circ \nu_Q(h(a) * h(b)) \vee p_l \circ \nu_Q(h(b))$
[because Q is an m-polar PFI of A_2]
$= p_l \circ \nu_Q(h(a \diamondsuit b)) \vee p_l \circ \nu_Q(h(b))$
[because h is a homomorphism]
$= p_l \circ \nu_{h^{-1}(Q)}(a \diamondsuit b) \vee p_l \circ \nu_{h^{-1}(Q)}(b)$ for all $a, b \in A_1$ and for $l = 1, 2, \dots, m$.

Thus, $p_l \circ \mu_{h^{-1}(Q)}(a) \geqslant p_l \circ \mu_{h^{-1}(Q)}(a \diamondsuit b) \wedge p_l \circ \mu_{h^{-1}(Q)}(b)$, $p_l \circ \eta_{h^{-1}(Q)}(a) \geqslant p_l \circ \eta_{h^{-1}}(a \diamondsuit b) \wedge p_l \circ \eta_{h^{-1}(Q)}(b)$ and $p_l \circ \nu_{h^{-1}(Q)}(a) \leqslant p_l \circ \nu_{h^{-1}(Q)}(a \diamondsuit b) \vee p_l \circ \nu_{h^{-1}(Q)}(b)$ for all $a, b \in A_1$ and for $l = 1, 2, \dots, m$. Hence, $h^{-1}(Q)$ is an m-polar PFI of A_1.

Proposition 12. *Let $(A_1, \diamondsuit)$ and $(A_2, *)$ be two BCK algebras and $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFI of A_1. Then for a bijective homomorphism $h : A_1 \to A_2$, $h(P)$ is an m-polar PFI of A_2.*

Proof. Let $h(P) = (\mu_{h(P)}, \eta_{h(P)}, \nu_{h(P)})$. Now, let $b \in A_2$.

$$\text{Then } p_l \circ \mu_{h(P)}(b) = \bigvee_{a \in h^{-1}(b)} p_l \circ \mu_P(a),$$
$$p_l \circ \eta_{h(P)}(b) = \bigwedge_{a \in h^{-1}(b)} p_l \circ \eta_P(a)$$
$$\text{and } p_l \circ \nu_{h(P)}(b) = \bigwedge_{a \in h^{-1}(b)} p_l \circ \nu_P(a) \text{ for } l = 1, 2, .., m.$$

Since h is bijective therefore $h^{-1}(b)$ must be a singleton set. So, for $b \in A_2$, there exists an unique $a \in A_1$ such that $a = h^{-1}(b)$ i.e. $h(a) = b$. Thus, in this case, $p_l \circ \mu_{h(P)}(b) = p_l \circ \mu_{h(P)}(h(a)) = p_l \circ \mu_P(a)$, $p_l \circ \eta_{h(P)}(b) = p_l \circ \eta_{h(P)}(h(a)) = p_l \circ \eta_P(a)$ and $p_l \circ \nu_{h(P)}(b) = p_l \circ \nu_{h(P)}(h(a)) = p_l \circ \nu_P(a)$ for $l = 1, 2, \dots, m$.

Now, $p_l \circ \mu_{h(P)}(0)$
$= p_l \circ \mu_{h(P)}(h(0))$ [as $h(0) = 0$]
$= p_l \circ \mu_P(0)$
$\geqslant p_l \circ \mu_P(a)$
$= p_l \circ \mu_{h(P)}(h(a))$
$= p_l \circ \mu_{h(P)}(b)$,

$p_l \circ \eta_{h(P)}(0)$
$= p_l \circ \eta_{h(P)}(h(0))$ [as $h(0) = 0$]
$= p_l \circ \eta_P(0)$
$\geqslant p_l \circ \eta_P(a)$
$= p_l \circ \eta_{h(P)}(h(a))$
$= p_l \circ \eta_{h(P)}(b)$

and $p_l \circ \nu_{h(P)}(0)$
$= p_l \circ \nu_{h(P)}(h(0))$ [as $h(0) = 0$]
$= p_l \circ \nu_P(0)$
$\leqslant p_l \circ \nu_P(a)$
$= p_l \circ \nu_{h(P)}(h(a))$
$= p_l \circ \nu_{h(P)}(b)$ for $l = 1, 2, \dots, m$.

Since b is an arbitrary element of A_2 therefore $p_l \circ \mu_{h(P)}(0) \geqslant p_l \circ \mu_{h(P)}(b)$, $p_l \circ \eta_{h(P)}(0) \geqslant p_l \circ \eta_{h(P)}(b)$ and $p_l \circ \nu_{h(P)}(0) \leqslant p_l \circ \nu_{h(P)}(b)$ for all $b \in A_2$ and for $l = 1, 2, \dots, m$.

Also, $p_l \circ \mu_{h(P)}(b)$
$= p_l \circ \mu_{h(P)}(h(a))$ [where $b = h(a)$ for unique $a \in A_1$]
$= p_l \circ \mu_P(a)$
$\geqslant p_l \circ \mu_P(a \diamondsuit c) \wedge p_l \circ \mu_P(c)$
[as P is an m-polar PFI of A_1]
$= p_l \circ \mu_{h(P)}(h(a \diamondsuit c)) \wedge p_l \circ \mu_{h(P)}(h(c))$
$= p_l \circ \mu_{h(P)}(h(a) * h(c)) \wedge p_l \circ \mu_{h(P)}(h(c))$
[as h is a homomorphism]
$= p_l \circ \mu_{h(P)}(b * h(c)) \wedge p_l \circ \mu_{h(P)}(h(c))$,

$p_l \circ \eta_{h(P)}(b)$
$= p_l \circ \eta_{h(P)}(h(a))$ [where $b = h(a)$ for unique $a \in A_1$]
$= p_l \circ \eta_P(a)$
$\geqslant p_l \circ \eta_P(a \diamondsuit c) \wedge p_l \circ \eta_P(c)$
[as P is an m-polar PFI of A_1]
$= p_l \circ \eta_{h(P)}(h(a \diamondsuit c)) \wedge p_l \circ \eta_{h(P)}(h(c))$
$= p_l \circ \eta_{h(P)}(h(a) * h(c)) \wedge p_l \circ \eta_{h(P)}(h(c))$
[as h is a homomorphism]
$= p_l \circ \eta_{h(P)}(b * h(c)) \wedge p_l \circ \eta_{h(P)}(h(c))$

and $p_l \circ \nu_{h(P)}(b)$
$= p_l \circ \nu_{h(P)}(h(a))$ [where $b = h(a)$ for unique $a \in A_1$]
$= p_l \circ \nu_P(a)$
$\leqslant p_l \circ \nu_P(a \diamondsuit c) \vee p_l \circ \nu_P(c)$
[as P is an m-polar PFI of A_1]
$= p_l \circ \nu_{h(P)}(h(a \diamondsuit c)) \vee p_l \circ \nu_{h(P)}(h(c))$
$= p_l \circ \nu_{h(P)}(h(a) * h(c)) \vee p_l \circ \nu_{h(P)}(h(c))$
[as h is a homomorphism]
$= p_l \circ \nu_{h(P)}(b * h(c)) \vee p_l \circ \nu_{h(P)}(h(c))$ for all $c \in A_1$
and for $l = 1, 2, \ldots, m$.

Thus, $p_l \circ \mu_{h(P)}(b) \geqslant p_l \circ \mu_{h(P)}(b * h(c)) \wedge p_l \circ \mu_{h(P)}(h(c))$, $p_l \circ \eta_{h(P)}(b) \geqslant p_l \circ \eta_{h(P)}(b * h(c)) \wedge p_l \circ \eta_{h(P)}(h(c))$ and $p_l \circ \nu_{h(P)}(b) \leqslant p_l \circ \nu_{h(P)}(b * h(c)) \vee p_l \circ \nu_{h(P)}(h(c))$ for all $c \in A_1$ and for $l = 1, 2, \ldots, m$. Since h is bijective therefore $h(A_1) = A_2$. So, for all $c \in A_1$, $h(c)$ can capture all the elements of A_2. Letting $d = h(c)$, it is observed that the inequalities hold for all $d \in A_2$. Since b is arbitrary therefore we obtain that $p_l \circ \mu_{h(P)}(b) \geqslant p_l \circ \mu_{h(P)}(b \diamondsuit d) \wedge p_l \circ \mu_{h(P)}(d)$, $p_l \circ \eta_{h(P)}(b) \geqslant p_l \circ \eta_{h(P)}(b \diamondsuit d) \wedge p_l \circ \eta_{h(P)}(d)$ and $p_l \circ \nu_{h(P)}(b) \leqslant p_l \circ \nu_{h(P)}(b \diamondsuit d) \vee p_l \circ \nu_{h(P)}(d)$ for all $b, d \in A_2$ and for $l = 1, 2, \ldots, m$. Hence, $h(P)$ is an m-polar PFI of A_2.

6. m-POLAR PFII

The current section introduces the concept of implicative BCK algebra, m-polar PFII of a BCK algebra and studies some properties related to these. We also investigate a relationship between m-polar PFI and m-polar PFII of a BCK algebra.

Definition 18. A BCK algebra $(A, \diamondsuit, 0)$ is said to be implicative if $a = (a \diamondsuit b) \diamondsuit a$ for all $a, b \in A$.

Proposition 13. *An m-polar PFS $P = (\mu_P, \eta_P, \nu_P)$ in a BCK algebra $(A, \diamondsuit, 0)$ is said to be m-polar PFII of A if the below stated conditions are meet.*

i. $p_l \circ \mu_P(0) \geqslant p_l \circ \mu_P(a)$, $p_l \circ \eta_P(0) \geqslant p_l \circ \eta_P(a)$ *and* $p_l \circ \nu_P(0) \leqslant p_l \circ \nu_P(a)$

ii. $p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P\{(a \diamondsuit (b \diamondsuit a)) \diamondsuit c\} \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P\{(a \diamondsuit (b \diamondsuit a)) \diamondsuit c\} \wedge p_l \circ \eta_P(c)$ *and* $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P\{(a \diamondsuit (b \diamondsuit a)) \diamondsuit c\} \vee p_l \circ \nu_P(c)$ *for all* $a, b, c \in A$ *and for* $l = 1, 2, \ldots, m$

Example 3. Let us consider the BCK algebra $(A, \diamondsuit)$ as follows:

$\diamondsuit$	0	p	q	r	s
0	0	0	0	0	0
p	p	0	p	0	0
q	q	q	0	0	0
r	r	r	r	0	0
s	s	r	s	p	0

Let us consider a 3-polar PFS $P = (\mu_P, \eta_P, \nu_P)$ as follows:

$$\mu_P(a) = \begin{cases} (0.39, 0.41, 0.42), & \text{if } a = 0, p, q \\ (0.25, 0.27, 0.3), & \text{if } a = r, s \end{cases}$$

$$\eta_P(a) = \begin{cases} (0.37, 0.39, 0.4), & \text{if } a = 0, p, q \\ (0.29, 0.33, 0.35), & \text{if } a = r, s \end{cases}$$

and

$$\nu_P(a) = \begin{cases} (0.14, 0.17, 0.18), & \text{if } a = 0, p, q \\ (0.3, 0.32, 0.35), & \text{if } a = r, s \end{cases}$$

It can be easily shown that P is a 3-polar PFII of A.

Proposition 14. *Every m-polar PFII of a BCK algebra $(A, \diamondsuit, 0)$ is an m-polar PFI of A.*

Proof. Let $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFII of A.

Then $p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P\{(a \diamondsuit (b \diamondsuit a)) \diamondsuit c\} \wedge p_l \circ \mu_P(c)$,
$p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P\{(a \diamondsuit (b \diamondsuit a)) \diamondsuit c\} \wedge p_l \circ \eta_P(c)$
and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P\{(a \diamondsuit (b \diamondsuit a)) \diamondsuit c\} \vee p_l \circ \nu_P(c)$
for all $a, b, c \in A$ and for $l = 1, 2, \ldots, m$.

Setting $b = a$, it is obtained that

$$\begin{aligned} p_l \circ \mu_P(a) &\geqslant p_l \circ \mu_P\{(a \diamondsuit (a \diamondsuit a)) \diamondsuit c\} \wedge p_l \circ \mu_P(c) \\ &= p_l \circ \mu_P\{(a \diamondsuit 0) \diamondsuit c\} \wedge p_l \circ \mu_P(c) \\ &= p_l \circ \mu_P(a \diamondsuit c) \wedge p_l \circ \mu_P(c) \\ &\quad \text{[by Proposition 1]} \end{aligned}$$

$$\begin{aligned} p_l \circ \eta_P(a) &\geqslant p_l \circ \eta_P\{(a \diamondsuit (a \diamondsuit a)) \diamondsuit c\} \wedge p_l \circ \eta_P(c) \\ &= p_l \circ \eta_P\{(a \diamondsuit 0) \diamondsuit c\} \wedge p_l \circ \eta_P(c) \\ &= p_l \circ \eta_P(a \diamondsuit c) \wedge p_l \circ \eta_P(c) \\ &\quad \text{[by Proposition 1]} \end{aligned}$$

$$\begin{aligned} \text{and } p_l \circ \nu_P(a) &\leqslant p_l \circ \nu_P\{(a \diamondsuit (a \diamondsuit a)) \diamondsuit c\} \vee p_l \circ \nu_P(c) \\ &= p_l \circ \nu_P\{(a \diamondsuit 0) \diamondsuit c\} \vee p_l \circ \nu_P(c) \\ &= p_l \circ \nu_P(a \diamondsuit c) \vee p_l \circ \nu_P(c) \\ &\quad \text{[by Proposition 1]} \\ &\quad \text{for all } a, c \in A \text{ and for } l = 1, 2, \ldots, m. \end{aligned}$$

Therefore, P is an m-polar PFI of A.

The above proposition does not hold in reverse direction i.e. an m-polar PFI of a BCK algebra is not necessarily m-polar PFII which is clear from the following example. It is necessary to mention that in an implicative BCK algebra, the converse of the above proposition holds which is shown through Proposition 15.

Example 4. Now, let us consider a 3-polar PFS $P = (\mu_P, \eta_P, \nu_P)$ in BCK algebra A given in Example 3 as follows:

$$\mu_P(a) = \begin{cases} (0.42, 0.43, 0.45), & \text{if } a = 0, q \\ (0.25, 0.27, 0.3), & \text{if } a = p, r, s \end{cases}$$

$$\eta_P(a) = \begin{cases} (0.3, 0.33, 0.35), & \text{if } a = 0, q \\ (0.15, 0.18, 0.2), & \text{if } a = p, r, s \end{cases}$$

and

$$\nu_P(a) = \begin{cases} (0.14, 0.16, 0.2), & \text{if } a = 0, q \\ (0.45, 0.48, 0.5), & \text{if } a = p, r, s \end{cases}$$

It is clear that $(0.25, 0.27, 0.3) = \mu_P(p) \ngeq \mu_P\{(p \Diamond (r \Diamond p)) \Diamond q\} \wedge \mu_P(q) = (0.42, 0.43, 0.45) \wedge (0.42, 0.43, 0.45) = (0.42, 0.43, 0.45)$, $(0.15, 0.18, 0.2) = \eta_P(p) \ngeq \eta_P\{(p \Diamond (r \Diamond p)) \Diamond q\} \wedge \eta_P(q) = (0.3, 0.33, 0.35) \wedge (0.3, 0.33, 0.35) = (0.3, 0.33, 0.35)$ and $(0.45, 0.48, 0.5) = \nu_P(p) \nleq \nu_P\{(p \Diamond (q \Diamond p)) \Diamond q\} \wedge \nu_P(q) = (0.14, 0.16, 0.2) \vee (0.14, 0.16, 0.2) = (0.14, 0.16, 0.2)$. Thus, P is not 3-polar PFII although it is a 3-polar PFI of A.

Proposition 15. *In an implicative BCK algebra, every m-polar PFI is m-polar PFII.*

Proof. Let $(A, \Diamond, 0)$ be an implicative BCK algebra. Therefore, $a = (a \Diamond b) \Diamond a$ for all $a, b \in A$. Let $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFI of A. Then

$$\begin{aligned} p_l \circ \mu_P(a) &\geqslant p_l \circ \mu_P(a \Diamond c) \wedge p_l \circ \mu_P(c) \\ &= p_l \circ \mu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \wedge p_l \circ \mu_P(c), \\ p_l \circ \eta_P(a) &\geqslant p_l \circ \eta_P(a \Diamond c) \wedge p_l \circ \eta_P(c) \\ &= p_l \circ \eta_P\{((a \Diamond b) \Diamond a) \Diamond c\} \wedge p_l \circ \eta_P(c) \\ \text{and } p_l \circ \nu_P(a) &\leqslant p_l \circ \nu_P(a \Diamond c) \vee p_l \circ \nu_P(c) \\ &= p_l \circ \nu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \vee p_l \circ \nu_P(c) \\ &\quad \text{for all } a, b, c \in A \text{ and for } l = 1, 2, \ldots, m. \end{aligned}$$

Thus, P is an m-polar PFII of A.

Proposition 16. *Let $(A, \Diamond, 0)$ be a BCK algebra and $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFII of A. Then $C_{\theta,\phi,\psi}(P)$ is an implicative ideal of A, provided that $p_l \circ \mu_P(0) \geqslant p_l \circ \theta$, $p_l \circ \eta_P(0) \geqslant p_l \circ \phi$ and $p_l \circ \nu_P(0) \leqslant p_l \circ \psi$ for $l = 1, 2, \ldots m$.*

Proof. Clearly, $C_{\theta,\phi,\psi}(P)$ contains at least one element. Let $((a \Diamond b) \Diamond a) \Diamond c$, $c \in C_{\theta,\phi,\psi}(P)$. Then $p_l \circ \mu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \geqslant p_l \circ \theta$, $p_l \circ \eta_P\{((a \Diamond b) \Diamond a) \Diamond c\} \geqslant p_l \circ \phi$, $p_l \circ \nu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \leqslant p_l \circ \psi$ and $p_l \circ \mu_P(c) \geqslant p_l \circ \theta$, $p_l \circ \eta_P(c) \geqslant p_l \circ \phi$, $p_l \circ \nu_P(c) \leqslant p_l \circ \psi$ for $l = 1, 2, \ldots, m$.

$$\begin{aligned} \text{Now, } p_l \circ \mu_P(a) &\geqslant p_l \circ \mu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \wedge p_l \circ \mu_P(c) \\ &\quad [\text{because } P \text{ is an } m\text{-polar PFII of } A] \\ &\geqslant p_l \circ \theta \wedge p_l \circ \theta = p_l \circ \theta, \end{aligned}$$

$$\begin{aligned} p_l \circ \eta_P(a) &\geqslant p_l \circ \eta_P\{((a \Diamond b) \Diamond a) \Diamond c\} \wedge p_l \circ \eta_P(c) \\ &\quad [\text{because } P \text{ is an } m\text{-polar PFII of } A] \\ &\geqslant p_l \circ \phi \wedge p_l \circ \phi = p_l \circ \phi \end{aligned}$$

$$\begin{aligned} \text{and } p_l \circ \nu_P(a) &\leqslant p_l \circ \nu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \vee p_l \circ \nu_P(c) \\ &\quad [\text{because } P \text{ is an } m\text{-polar PFII of } A] \\ &\leqslant p_l \circ \psi \vee p_l \circ \psi = p_l \circ \psi \\ &\quad \text{for } l = 1, 2, \ldots, m. \end{aligned}$$

Thus, it is observed that $\{((a \Diamond b) \Diamond a) \Diamond c\}$, $c \in C_{\theta,\phi,\psi}(P) \Rightarrow a \in C_{\theta,\phi,\psi}(P)$. So, $C_{\theta,\phi,\psi}(P)$ is an implicative ideal of A.

Proposition 17. *Let $(A, \Diamond, 0)$ be a BCK algebra and $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFS in A. Then P is an m-polar PFII of A if all (θ, ϕ, ψ)-cuts of P are implicative ideals of A.*

Proof. Let $a, b \in A$. Let $p_l \circ \mu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \wedge p_l \circ \mu_P(c) = p_l \circ \theta$, $p_l \circ \eta_P\{((a \Diamond b) \Diamond a) \Diamond c\} \wedge p_l \circ \eta_P(c) = p_l \circ \phi$ and $p_l \circ \nu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \vee p_l \circ \nu_P(c) = p_l \circ \psi$ for $l = 1, 2, \ldots, m$. Clearly, $p_l \circ \theta \in [0, 1]$, $p_l \circ \phi \in [0, 1]$ and $p_l \circ \psi \in [0, 1]$ with $0 \leqslant p_l \circ \theta + p_l \circ \phi + p_l \circ \psi \leqslant 1$ for $l = 1, 2, \ldots, m$.

$$\begin{aligned} \text{Now, } p_l \circ \mu_P\{((a \Diamond b) \Diamond a) \Diamond c\} &\geqslant p_l \circ \mu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \\ &\quad \wedge p_l \circ \mu_P(c) \\ &= p_l \circ \theta, \\ p_l \circ \eta_P\{((a \Diamond b) \Diamond a) \Diamond c\} &\geqslant p_l \circ \eta_P\{((a \Diamond b) \Diamond a) \Diamond c\} \\ &\quad \wedge p_l \circ \eta_P(c) \\ &= p_l \circ \phi \\ \text{and } p_l \circ \nu_P\{((a \Diamond b) \Diamond a) \Diamond c\} &\leqslant p_l \circ \nu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \\ &\quad \vee p_l \circ \nu_P(c) \\ &= p_l \circ \psi \text{ for } l = 1, 2, \ldots, m. \end{aligned}$$

$$\begin{aligned} \text{Also, } p_l \circ \mu_P(c) &\geqslant p_l \circ \mu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \\ &\quad \wedge p_l \circ \mu_P(c) = p_l \circ \theta, \\ p_l \circ \eta_P(c) &\geqslant p_l \circ \eta_P\{((a \Diamond b) \Diamond a) \Diamond c\} \\ &\quad \wedge p_l \circ \eta_P(c) = p_l \circ \phi \\ \text{and } p_l \circ \nu_P(c) &\leqslant p_l \circ \nu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \\ &\quad \vee p_l \circ \nu_P(c) = p_l \circ \psi \text{ for } l = 1, 2, \ldots, m. \end{aligned}$$

Thus, $((a \Diamond b) \Diamond a) \Diamond c$ and $c \in C_{\theta,\phi,\psi}(P)$. Since $C_{\theta,\phi,\psi}(P)$ is an implicative ideal of A therefore $a \in C_{\theta,\phi,\psi}(P)$.

Therefore, $p_l \circ \mu_P(a) \geqslant p_l \circ \theta = p_l \circ \mu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \phi = p_l \circ \eta_P\{((a \Diamond b) \Diamond a) \Diamond c\} \wedge p_l \circ \eta_P(c)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \psi = p_l \circ \nu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \vee p_l \circ \nu_P(c)$ for $l = 1, 2, \ldots, m$.

Since a, b, c are arbitrary elements of A therefore $p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P\{((a \Diamond b) \Diamond a) \Diamond c\} \wedge p_l \circ \eta_P(c)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P\{((a \Diamond b) \Diamond a) \Diamond c\} \vee p_l \circ \nu_P(c)$ for all $a, b, c \in A$ and for $l = 1, 2, \ldots, m$. Hence, P is an m-polar PFII of A.

Proposition 18. *Let S_1 and S_2 be two ideals of a BCK algebra $(A, \Diamond, 0)$ such that $S_1 \subseteq S_2$. If S_1 is implicative then S_2 also.*

Proposition 19. *Let P_1 and P_2 be two m-polar PFIs of a BCK algebra $(A, \Diamond, 0)$ with $P_1 \subseteq P_2$. If P_1 is m-polar PFII of A then P_2 also.*

Proof. Let $a \in C_{\theta,\phi,\psi}(P_1)$. Then $p_l \circ \mu_{P_1}(a) \geqslant p_l \circ \theta$, $p_l \circ \eta_{P_1}(a) \geqslant p_l \circ \phi$ and $p_l \circ \nu_{P_1}(a) \leqslant p_l \circ \psi$ for $l = 1, 2, \ldots, m$. Now, $P_1 \subseteq P_2 \Rightarrow p_l \circ \mu_{P_1}(a) \leqslant p_l \circ \mu_{P_2}(a)$, $p_l \circ \eta_{P_1}(a) \leqslant p_l \circ \eta_{P_2}(a)$ and $p_l \circ \nu_{P_1}(a) \geqslant p_l \circ \nu_{P_2}(a)$ for $l = 1, 2, \ldots, m$. It follows that $p_l \circ \mu_{P_2}(a) \geqslant p_l \circ \theta$, $p_l \circ \eta_{P_2}(a) \geqslant p_l \circ \phi$ and $p_l \circ \nu_{P_2}(a) \leqslant p_l \circ \psi$ for $l = 1, 2, \ldots, m$. Thus, $a \in C_{\theta,\phi,\psi}(P_2)$. As a result, $C_{\theta,\phi,\psi}(P_1) \subseteq C_{\theta,\phi,\psi}(P_2)$. Since P_1 is an m-polar PFII of A therefore $C_{\theta,\phi,\psi}(P_1)$ is implicative ideal of A by Proposition 16. By Proposition 18, $C_{\theta,\phi,\psi}(P_2)$ is implicative ideal of A. Therefore, by Proposition 17, P_2 is an m-polar PFII of A.

Proposition 20. *Let $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFI of a BCK algebra A. Then the below stated statements are equivalent.*

i. *P is m-polar PFII.*

ii. *$p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P(a \Diamond (b \Diamond a))$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P(a \Diamond (b \Diamond a))$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P(a \Diamond (b \Diamond a))$ for all $a, b \in A$ and for $l = 1, 2, \ldots, m$.*

iii. *$p_l \circ \mu_P(a) = p_l \circ \mu_P(a \Diamond (b \Diamond a))$, $p_l \circ \eta_P(a) = p_l \circ \eta_P(a \Diamond (b \Diamond a))$ and $p_l \circ \nu_P(a) = p_l \circ \nu_P(a \Diamond (b \Diamond a))$ for all $a, b \in A$ and for $l = 1, 2, \ldots, m$.*

Proof. $(i) \Rightarrow (ii)$: Since P is an m-polar PFII of A, therefore,

$$\begin{aligned} p_l \circ \mu_P(a) &\geqslant p_l \circ \mu_P\{(a \diamondsuit (b \diamondsuit a)) \diamondsuit 0\} \wedge p_l \circ \mu_P(0) \\ &= p_l \circ \mu_P(a \diamondsuit (b \diamondsuit a)) \wedge p_l \circ \mu_P(0) \\ &\quad \text{[by Proposition 1]} \\ &= p_l \circ \mu_P(a \diamondsuit (b \diamondsuit a)) \end{aligned}$$

$$\begin{aligned} p_l \circ \eta_P(a) &\geqslant p_l \circ \eta_P\{(a \diamondsuit (b \diamondsuit a)) \diamondsuit 0\} \wedge p_l \circ \eta_P(0) \\ &= p_l \circ \eta_P(a \diamondsuit (b \diamondsuit a)) \wedge p_l \circ \eta_P(0) \\ &\quad \text{[by Proposition 1]} \\ &= p_l \circ \eta_P(a \diamondsuit (b \diamondsuit a)) \\ \text{and } p_l \circ \nu_P(a) &\leqslant p_l \circ \nu_P\{(a \diamondsuit (b \diamondsuit a)) \diamondsuit 0\} \vee p_l \circ \nu_P(0) \\ &= p_l \circ \nu_P(a \diamondsuit (b \diamondsuit a)) \vee p_l \circ \nu_P(0) \\ &\quad \text{[by Proposition 1]} \\ &= p_l \circ \nu_P(a \diamondsuit (b \diamondsuit a)) \text{ for all } a, b \in A \\ &\quad \text{and for } l = 1, 2, \dots, m. \end{aligned}$$

$(ii) \Rightarrow (iii)$: It is known by Proposition 1 that $a \diamondsuit (b \diamondsuit a) \leqslant a$. Then by Proposition 3, $p_l \circ \mu_P(a) \leqslant p_l \circ \mu_P(a \diamondsuit (b \diamondsuit a))$, $p_l \circ \eta_P(a) \leqslant p_l \circ \eta_P(a \diamondsuit (b \diamondsuit a))$ and $p_l \circ \nu_P(a) \geqslant p_l \circ \nu_P(a \diamondsuit (b \diamondsuit a))$ for all $a, b \in A$ and for $l = 1, 2, \dots, m$. By (ii), $p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P(a \diamondsuit (b \diamondsuit a))$, $p_l \circ \eta_P(a) \geqslant p_l \circ \mu_P(a \diamondsuit (b \diamondsuit a))$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P(a \diamondsuit (b \diamondsuit a))$ for all $a, b \in A$ and $l = 1, 2, \dots, m$. As a result, $p_l \circ \mu_P(a) = p_l \circ \mu_P(a \diamondsuit (b \diamondsuit a))$, $p_l \circ \eta_P(a) = p_l \circ \mu_P(a \diamondsuit (b \diamondsuit a))$ and $p_l \circ \nu_P(a) = p_l \circ \nu_P(a \diamondsuit (b \diamondsuit a))$ for all $a, b \in A$ and for $l = 1, 2, \dots, m$.

$(iii) \Rightarrow (i)$: Since P is an m-polar PFI of A therefore $p_l \circ \mu_P(a \diamondsuit (b \diamondsuit a)) \geqslant p_l \circ \mu_P\{a \diamondsuit (b \diamondsuit a) \diamondsuit c\} \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P(a \diamondsuit (b \diamondsuit a)) \geqslant p_l \circ \eta_P\{a \diamondsuit (b \diamondsuit a) \diamondsuit c\} \wedge p_l \circ \eta_P(c)$ and $p_l \circ \nu_P(a \diamondsuit (b \diamondsuit a)) \leqslant p_l \circ \nu_P\{a \diamondsuit (b \diamondsuit a) \diamondsuit c\} \vee p_l \circ \nu_P(c)$ for all $a, b, c \in A$ and for $l = 1, 2, \dots, m$. By (iii), we have, $p_l \circ \mu_P(a) \geqslant p_l \circ \mu_P\{a \diamondsuit (b \diamondsuit a) \diamondsuit c\} \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P(a) \geqslant p_l \circ \eta_P\{a \diamondsuit (b \diamondsuit a) \diamondsuit c\} \wedge p_l \circ \eta_P(c)$ and $p_l \circ \nu_P(a) \leqslant p_l \circ \nu_P\{a \diamondsuit (b \diamondsuit a) \diamondsuit c\} \vee p_l \circ \nu_P(c)$ for all $a, b, c \in A$ and for $l = 1, 2, \dots, m$. Thus, P is an m-polar PFII of A.

Definition 19. Let $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFS in a BCK algebra $(A, \diamondsuit, 0)$. Then P is said to be an m-polar picture fuzzy positive implicative ideal (PFPII) if the below stated conditions are meet.

i. $p_l \circ \eta_P(0) \geqslant p_l \circ \eta_P(a)$ and $p_l \circ \nu_P(0) \leqslant p_l \circ \nu_P(a)$

ii. $p_l \circ \eta_P(a \diamondsuit c) \geqslant p_l \circ \eta_P((a \diamondsuit b) \diamondsuit c) \wedge p_l \circ \eta_P(b \diamondsuit c)$ and $p_l \circ \nu_P(a \diamondsuit c) \leqslant p_l \circ \nu_P((a \diamondsuit b) \diamondsuit c) \wedge p_l \circ \nu_P(b \diamondsuit c)$, $\forall a, b \in A$ and $l = 1, 2, \dots, m$

Proposition 21. *An m-polar PFI $P = (\mu_P, \eta_P, \nu_P)$ of a BCK algebra $(A, \diamondsuit, 0)$ is an m-polar PFPII iff $p_l \circ \mu_P(a \diamondsuit b) \geqslant p_l \circ \mu_P((a \diamondsuit b) \diamondsuit b)$, $p_l \circ \eta_P(a \diamondsuit b) \geqslant p_l \circ \eta_P((a \diamondsuit b) \diamondsuit b)$ and $p_l \circ \nu_P(a \diamondsuit b) \leqslant p_l \circ \nu_P((a \diamondsuit b) \diamondsuit b)$, $\forall a, b \in A$ and for $l = 1, 2, \dots, m$.*

Proof. The proof is easy. So, it is omitted here.

Since $(a \diamondsuit b) \diamondsuit b \leqslant a \diamondsuit b$, it follows from Proposition 3 that $p_l \circ \mu_P(a \diamondsuit b) \leqslant p_l \circ \mu_P((a \diamondsuit b) \diamondsuit b)$, $p_l \circ \eta_P(a \diamondsuit b) \leqslant p_l \circ \eta_P((a \diamondsuit b) \diamondsuit b)$ and $p_l \circ \nu_P(a \diamondsuit b) \geqslant p_l \circ \nu_P((a \diamondsuit b) \diamondsuit b)$, $\forall a, b \in A$ and $l = 1, 2, \dots, m$. So, the above Proposition can be modified in the following way:

Proposition 22. *An m-polar PFI $P = (\mu_P, \eta_P, \nu_P)$ of a BCK algebra $(A, \diamondsuit, 0)$ is a m-polar PFPII iff $p_l \circ \mu_P(a \diamondsuit b) = p_l \circ \mu_P((a \diamondsuit b) \diamondsuit b)$, $p_l \circ \eta_P(a \diamondsuit b) = p_l \circ \eta_P((a \diamondsuit b) \diamondsuit b)$ and $p_l \circ \nu_P(a \diamondsuit b) = p_l \circ \nu_P((a \diamondsuit b) \diamondsuit b)$, $\forall a, b \in A$ and $l = 1, 2, \dots, m$.*

7. *m*-POLAR PFCI

Definition 20. Let $(A, \diamondsuit, 0)$ be a BCK algebra and $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFS in A. Then P is said to be m-polar PFCI of A if the following conditions are met:

i. $p_l \circ \mu_P(0) \geqslant p_l \circ \mu_P(a)$, $p_l \circ \eta_P(0) \geqslant p_l \circ \eta_P(a)$ and $p_l \circ \nu_P(0) \leqslant p_l \circ \nu_P(a)$

ii. $p_l \circ \mu_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \geqslant p_l \circ \mu_P((a \diamondsuit b) \diamondsuit c) \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \geqslant p_l \circ \eta_P((a \diamondsuit b) \diamondsuit c) \wedge p_l \circ \eta_P(c)$ and $p_l \circ \nu_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \leqslant p_l \circ \nu_P((a \diamondsuit b) \diamondsuit c) \vee p_l \circ \nu_P(c)$ for all $a, b \in A$ and for $l = 1, 2, \dots, m$

Example 5. Let us consider the BCK algebra $(A, \diamondsuit)$ as follows:

$\diamondsuit$	0	p	q	r
0	0	0	0	0
p	p	0	0	p
q	q	p	0	q
r	r	r	r	0

Now, let us suppose a 3-polar PFS $P = (\mu_P, \eta_P, \nu_P)$ defined by

$$\mu_P(a) = \begin{cases} (0.34, 0.36, 0.37), & \text{if } a = 0 \\ (0.28, 0.3, 0.32), & \text{if } a = p \\ (0.17, 0.18, 0.18), & \text{if } a = q, r \end{cases}$$

$$\eta_P(a) = \begin{cases} (0.35, 0.36, 0.39), & \text{if } a = 0 \\ (0.25, 0.27, 0.3), & \text{if } a = p \\ (0.2, 0.23, 0.27), & \text{if } a = q, r \end{cases}$$

and

$$\eta_P(a) = \begin{cases} (0.1, 0.15, 0.17), & \text{if } a = 0 \\ (0.2, 0.27, 0.31), & \text{if } a = p \\ (0.55, 0.57, 0.58), & \text{if } a = q, r \end{cases}$$

Clearly, P is a 3-polar PFCI of A.

Definition 21. A BCK algebra $(A, \diamondsuit, 0)$ is said to be commutative if $b \diamondsuit (b \diamondsuit a) = a \diamondsuit (a \diamondsuit b)$ for all $a, b \in A$.

Proposition 23. *Every m-polar PFCI of a BCK algebra is an m-polar PFI.*

Proof. Let $P = (\mu_P, \eta_P, \nu_P)$ is an m-polar PFCI of a BCK algebra $(A, \diamondsuit, 0)$.

Now, $(a \diamondsuit (0 \diamondsuit (0 \diamondsuit a)))$

$= (a \diamondsuit 0)$ [by Proposition 1]

$= a$ [by Proposition 1]

Now, $p_l \circ \mu_P(a) = p_l \circ \mu_P(a \diamondsuit (0 \diamondsuit (0 \diamondsuit a))) \geqslant p_l \circ \mu_P((a \diamondsuit 0) \diamondsuit c) \wedge p_l \circ \mu_P(c) = p_l \circ \mu_P(a \diamondsuit c) \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P(a) = p_l \circ \eta_P(a \diamondsuit (0 \diamondsuit (0 \diamondsuit a))) \geqslant p_l \circ \eta_P((a \diamondsuit 0) \diamondsuit c) \wedge p_l \circ \eta_P(c) = p_l \circ \eta_P(a \diamondsuit c) \wedge p_l \circ \eta_P(c)$ and $p_l \circ \nu_P(a) = p_l \circ \nu_P(a \diamondsuit (0 \diamondsuit (0 \diamondsuit a))) \leqslant p_l \circ \nu_P((a \diamondsuit 0) \diamondsuit c) \vee p_l \circ \nu_P(c) = p_l \circ \nu_P(a \diamondsuit c) \vee p_l \circ \nu_P(c)$ for all $a, c \in A$ and for $l = 1, 2, \dots, m$. Consequently, P is an m-polar PFI of A.

The above proposition is not true in reverse direction which is clear from following example. But the converse of the above proposition holds in commutative BCK algebra which is highlighted through Proposition 24.

Example 6. Let us consider a BCK algebra $(A, \diamondsuit)$ as follows:

$\diamondsuit$	0	p	q	r	s
0	0	0	0	0	0
p	p	0	p	0	0
q	q	q	0	0	0
r	r	r	r	0	0
s	s	s	s	r	0

Now, let us suppose a 3-polar PFS $P = (\mu_P, \eta_P, \nu_P)$ defined by

$$\mu_P(a) = \begin{cases} (0.4, 0.41, 0.43), & \text{if } a = 0 \\ (0.3, 0.32, 0.33), & \text{if } a = p \\ (0.2, 0.24, 0.27), & \text{if } a = q, r, s \end{cases}$$

$$\eta_P(a) = \begin{cases} (0.43, 0.45, 0.47), & \text{if } a = 0 \\ (0.35, 0.36, 0.37), & \text{if } a = p \\ (0.21, 0.22, 0.23), & \text{if } a = q, r, s \end{cases}$$

$$\nu_P(a) = \begin{cases} (0.08, 0.09, 0.1), & \text{if } a = 0 \\ (0.27, 0.28, 0.3), & \text{if } a = p \\ (0.45, 0.47, 0.5), & \text{if } a = q, r, s \end{cases}$$

Clearly, P is a 3-polar PFI of A.

It is observed that

$\mu_P((q \diamondsuit (r \diamondsuit (r \diamondsuit q)))) = \mu_P(q) = (0.2, 0.24, 0.27)$,

$\mu_P((q \diamondsuit r) \diamondsuit 0) \wedge \mu_P(0) = (0.4, 0.41, 0.43)$

$\eta_P((q \diamondsuit (r \diamondsuit (r \diamondsuit q)))) = \eta_P(q) = (0.21, 0.22, 0.23)$,

$\eta_P((q \diamondsuit r) \diamondsuit 0) \wedge \eta_P(0) = (0.43, 0.45, 0.47)$

$\nu_P((q \diamondsuit (r \diamondsuit (r \diamondsuit q)))) = \nu_P(q) = (0.45, 0.47, 0.5)$,

$\nu_P((q \diamondsuit r) \diamondsuit 0) \vee \nu_P(0) = (0.08, 0.09, 0.1)$.

Here, $\mu_P((q \diamondsuit (r \diamondsuit (r \diamondsuit q)))) \ngeqslant \mu_P((q \diamondsuit r) \diamondsuit 0) \wedge \mu_P(0)$, $\eta_P((q\diamondsuit(r\diamondsuit(r\diamondsuit q)))) \ngeqslant \eta_P((q\diamondsuit r)\diamondsuit 0) \wedge \eta_P(0)$ and $\nu_P((q \diamondsuit (r \diamondsuit (r \diamondsuit q)))) \nleqslant \nu_P((q \diamondsuit r) \diamondsuit 0) \vee \nu_P(0)$. Clearly, P is not a 3-polar PFCI of A.

Proposition 24. *In a commutative BCK algebra, every m-polar PFI is an m-polar PFCI.*

Proof. Let $P = (\mu_P, \eta_P, \nu_P)$ be an m-polar PFI of a commutative BCK algebra $(A, \diamondsuit, 0)$. We have, $[((a\diamondsuit(b\diamondsuit(b\diamondsuit a)))\diamondsuit((a\diamondsuit b)\diamondsuit c))]\diamondsuit c = ((a\diamondsuit(b\diamondsuit(b\diamondsuit a)))\diamondsuit c)\diamondsuit((a\diamondsuit b)\diamondsuit c)$ [by Proposition 1]

$\leqslant (a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \diamondsuit (a \diamondsuit b)$ [by Proposition 1]

$= (a \diamondsuit (a \diamondsuit (a \diamondsuit b))) \diamondsuit (a \diamondsuit b)$ [as A is commutative therefore $(a \diamondsuit (a \diamondsuit b)) = (b \diamondsuit (b \diamondsuit a))$ for all $a, b \in A$]

$= (a \diamondsuit b) \diamondsuit (a \diamondsuit b)$ [by Proposition 1]

$= 0$

i.e. $(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \diamondsuit ((a \diamondsuit b) \diamondsuit c) \leqslant c$.

Thus, by Proposition 4, it is obtained that $p_l \circ \mu_P((a \diamondsuit (b \diamondsuit (b \diamondsuit a)))) \geqslant p_l \circ \mu_P(((a \diamondsuit b) \diamondsuit c)) \wedge p_l \circ \mu_P(c)$, $p_l \circ \eta_P((a \diamondsuit (b \diamondsuit (b \diamondsuit a)))) \geqslant p_l \circ \eta_P(((a \diamondsuit b) \diamondsuit c)) \wedge p_l \circ \eta_P(c)$ and $p_l \circ \nu_P((a \diamondsuit (b \diamondsuit (b \diamondsuit a)))) \leqslant p_l \circ \nu_P(((a \diamondsuit b) \diamondsuit c)) \vee p_l \circ \nu_P(c)$ for all $a, b, c \in A$ and for $l = 1, 2, \ldots, m$. Consequently, P is an m-polar PFCI of A.

Now, we are interested to develop a relationship between m-polar PFII and m-polar PFCI. Before that we state some propositions which are necessary in this regard.

Meng *et al.* [10] stated the following proposition:

Proposition 25. *The followings hold in a BCK algebra $(A, \diamondsuit, 0)$.*

i. $((a \diamondsuit c) \diamondsuit c) \diamondsuit (b \diamondsuit c) \leqslant (a \diamondsuit b) \diamondsuit c$.

ii. $(a \diamondsuit c) \diamondsuit (a \diamondsuit (a \diamondsuit c)) = (a \diamondsuit c) \diamondsuit c$.

iii. $(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \diamondsuit (b \diamondsuit (a \diamondsuit (b \diamondsuit (b \diamondsuit a)))) \leqslant a \diamondsuit b$.

Proposition 26. *An m-polar PFI $P = (\mu_P, \eta_P, \nu_P)$ of a BCK algebra $(A, \diamondsuit, 0)$ is an m-polar PFCI iff $p_l \circ \mu_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \geqslant p_l \circ \mu_P(a \diamondsuit b)$, $p_l \circ \eta_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \geqslant p_l \circ \eta_P(a \diamondsuit b)$ and $p_l \circ \nu_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \leqslant p_l \circ \nu_P(a \diamondsuit b)$, $\forall a, b \in A$ and $l = 1, 2, \ldots, m$.*

Proof. The proof is easy. So, it is omitted here.

It is observed that $a \diamondsuit b \leqslant a \diamondsuit (b \diamondsuit (b \diamondsuit a))$ and using Proposition 3 we get, $p_l \circ \mu_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \leqslant p_l \circ \mu_P(a \diamondsuit b)$, $p_l \circ \eta_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \leqslant p_l \circ \eta_P(a \diamondsuit b)$ and $p_l \circ \nu_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) \geqslant p_l \circ \nu_P(a \diamondsuit b)$, $\forall a, b \in A$ and $l = 1, 2, \ldots, m$. So, above Proposition can be modified in the following way:

Proposition 27. *An m-polar PFI $P = (\mu_P, \eta_P, \nu_P)$ of a BCK algebra $(A, \diamondsuit, 0)$ is an m-polar PFCI iff $p_l \circ \mu_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) = p_l \circ \mu_P(a \diamondsuit b)$, $p_l \circ \eta_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) = p_l \circ \eta_P(a \diamondsuit b)$ and $p_l \circ \nu_P(a \diamondsuit (b \diamondsuit (b \diamondsuit a))) = p_l \circ \nu_P(a \diamondsuit b)$, $\forall a, b \in A$ and $l = 1, 2, \ldots, m$.*

Proposition 28. *An m-polar PFI $P = (\mu_P, \eta_P, \nu_P)$ is an m-polar PFII iff P is both m-polar PFCI and m-polar PFPII.*

Proof. Suppose that P is m-polar PFII. Then by Proposition 25 (i) and Proposition 4,

$$\begin{aligned}
&p_l \circ \mu_P((a \diamondsuit b) \diamondsuit c) \wedge p_l \circ \mu_P(b \diamondsuit c) \\
&\leqslant p_l \circ \mu_P((a \diamondsuit c) \diamondsuit c) \\
&= p_l \circ \mu_P((a \diamondsuit c) \diamondsuit (a \diamondsuit (a \diamondsuit c))) \text{ [by Proposition 25 (ii)]} \\
&= p_l \circ \mu_P(a \diamondsuit c) \text{ [by Proposition 20 (iii)]}, \\
&p_l \circ \eta_P((a \diamondsuit b) \diamondsuit c) \wedge p_l \circ \eta_P(b \diamondsuit c) \\
&\leqslant p_l \circ \eta_P((a \diamondsuit c) \diamondsuit c) \\
&= p_l \circ \eta_P((a \diamondsuit c) \diamondsuit (a \diamondsuit (a \diamondsuit c))) \text{ [by Proposition 25 (ii)]} \\
&= p_l \circ \eta_P(a \diamondsuit c) \text{ [by Proposition 20 (iii)]} \\
&\text{and } p_l \circ \nu_P((a \diamondsuit b) \diamondsuit c) \wedge p_l \circ \nu_P(b \diamondsuit c) \\
&\geqslant p_l \circ \nu_P((a \diamondsuit c) \diamondsuit c) \\
&= p_l \circ \nu_P((a \diamondsuit c) \diamondsuit (a \diamondsuit (a \diamondsuit c))) \text{ [by Proposition 25 (ii)]} \\
&= p_l \circ \nu_P(a \diamondsuit c) \text{ [by Proposition 20 (iii)]}
\end{aligned}$$

Therefore, P is an m-polar PFPII.

By Proposition 25 (iii) and Proposition 3 we get,

$$\begin{aligned}
&p_l \circ \mu_P(a \diamond b)\\
&\leqslant p_l \circ \mu_P(a \diamond (b \diamond (b \diamond a))) \diamond (b \diamond (a \diamond (b \diamond (b \diamond a)))))\\
&= p_l \circ \mu_P((a \diamond (b \diamond (b \diamond a))) \text{ [by Proposition 20 (iii)]},\\
&p_l \circ \eta_P(a \diamond b)\\
&\leqslant p_l \circ \eta_P(a \diamond (b \diamond (b \diamond a))) \diamond (b \diamond (a \diamond (b \diamond (b \diamond a)))))\\
&= p_l \circ \eta_P((a \diamond (b \diamond (b \diamond a))) \text{ [by Proposition 20 (iii)]}\\
&\text{and } p_l \circ \nu_P(a \diamond b)\\
&\geqslant p_l \circ \nu_P(a \diamond (b \diamond (b \diamond a))) \diamond (b \diamond (a \diamond (b \diamond (b \diamond a)))))\\
&= p_l \circ \nu_P((a \diamond (b \diamond (b \diamond a))) \text{ [by Proposition 20 (iii)]}.
\end{aligned}$$

Therefore, P is an m-polar PFCI of A.

Conversely, let P be both m-polar PFPII and m-polar PFCI of A. Since $(b \diamond (b \diamond a)) \diamond (b \diamond a) \leqslant a \diamond (b \diamond a)$, by Proposition 3,

$$\begin{aligned}
&p_l \circ \mu_P(a \diamond (b \diamond a))\\
&\leqslant p_l \circ \mu_P((b \diamond (b \diamond a)) \diamond (b \diamond a)),\\
&p_l \circ \eta_P(a \diamond (b \diamond a))\\
&\leqslant p_l \circ \eta_P((b \diamond (b \diamond a)) \diamond (b \diamond a))\\
&\text{and } p_l \circ \nu_P(a \diamond (b \diamond a))\\
&\geqslant p_l \circ \nu_P((b \diamond (b \diamond a)) \diamond (b \diamond a))
\end{aligned}$$

By Proposition 22,

$$\begin{aligned}
p_l \circ \mu_P((b \diamond (b \diamond a)) \diamond (b \diamond a)) &= p_l \circ \mu_P(b \diamond (b \diamond a)),\\
p_l \circ \eta_P((b \diamond (b \diamond a)) \diamond (b \diamond a)) &= p_l \circ \eta_P(b \diamond (b \diamond a))\\
\text{and } p_l \circ \nu_P((b \diamond (b \diamond a)) \diamond (b \diamond a)) &= p_l \circ \nu_P(b \diamond (b \diamond a))
\end{aligned}$$

therefore it is obtained that

$$p_l \circ \mu_P(a \diamond (b \diamond a)) \leqslant p_l \circ \mu_P(b \diamond (b \diamond a)), \tag{1}$$

$$p_l \circ \eta_P(a \diamond (b \diamond a)) \leqslant p_l \circ \eta_P(b \diamond (b \diamond a)) \tag{2}$$

$$\text{and } p_l \circ \nu_P(a \diamond (b \diamond a)) \geqslant p_l \circ \nu_P(b \diamond (b \diamond a)) \tag{3}$$

Also, $a \diamond b \leqslant a \diamond (b \diamond a)$. Therefore, by Proposition 3,

$$\begin{aligned}
p_l \circ \mu_P(a \diamond (b \diamond a)) &\leqslant p_l \circ \mu_P(a \diamond b),\\
p_l \circ \eta_P(a \diamond (b \diamond a)) &\leqslant p_l \circ \eta_P(a \diamond b)\\
\text{and } p_l \circ \nu_P(a \diamond (b \diamond a)) &\geqslant p_l \circ \nu_P(a \diamond b)
\end{aligned}$$

Since P is an m-polar PFCI therefore by Proposition 27,

$$\begin{aligned}
p_l \circ \mu_P(a \diamond b) &= p_l \circ \mu_P(a \diamond (b \diamond (b \diamond a))),\\
p_l \circ \eta_P(a \diamond b) &= p_l \circ \eta_P(a \diamond (b \diamond (b \diamond a)))\\
\text{and } p_l \circ \nu_P(a \diamond b) &= p_l \circ \nu_P(a \diamond (b \diamond (b \diamond a)))
\end{aligned}$$

Hence it is obtained that

$$p_l \circ \mu_P(a \diamond (b \diamond a)) \leqslant p_l \circ \mu_P(a \diamond (b \diamond (b \diamond a))), \tag{4}$$

$$p_l \circ \eta_P(a \diamond (b \diamond a)) \leqslant p_l \circ \eta_P(a \diamond (b \diamond (b \diamond a))) \tag{5}$$

$$\text{and } p_l \circ \nu_P(a \diamond (b \diamond a)) \geqslant p_l \circ \nu_P(a \diamond (b \diamond (b \diamond a))) \tag{6}$$

Combining (1) and (4), (2) and (5), (3) and (6) it is obtained that

$$\begin{aligned}
&p_l \circ \mu_P(a \diamond (b \diamond a))\\
&\leqslant p_l \circ \mu_P(a \diamond (b \diamond (b \diamond a))) \wedge p_l \circ \mu_P(b \diamond (b \diamond a))\\
&\leqslant p_l \circ \mu_P(a),\\
&p_l \circ \eta_P(a \diamond (b \diamond a))\\
&\leqslant p_l \circ \eta_P(a \diamond (b \diamond (b \diamond a))) \wedge p_l \circ \eta_P(b \diamond (b \diamond a))\\
&\leqslant p_l \circ \eta_P(a)\\
&\text{and } p_l \circ \nu_P(a \diamond (b \diamond a))\\
&\geqslant p_l \circ \nu_P(a \diamond (b \diamond (b \diamond a))) \vee p_l \circ \nu_P(b \diamond (b \diamond a))\\
&\geqslant p_l \circ \nu_P(a).
\end{aligned}$$

So, by Proposition 20 (ii), P is an m-polar PFII of A.

8. CONCLUSION

In this paper, we have initiated the notion of m-polar PFI and m-polar PFII of BCK algebra. We have studied some basic results related to them. We have established a relationship between m-polar PFI and m-polar PFII of a BCK algebra. We have also investigated a relationship between m-polar PFI and m-polar PFCI. We have studied some properties of m-polar PFI under homomorphism of BCK algebra. It is our hope that our works will help the researchers to study some other types of algebraic structures in context of m-polar PFS.

CONFLICT OF INTEREST

Authors declare that they have no conflict of interest.

AUTHORS' CONTRIBUTIONS

S. Dogra Writing, reviewing and editing. M. Pal reviewing, editing and supervision.

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