Second Order Necessary Condition in Major Constraints Multi-Objective Programming

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Abstract—Second order necessary optimality condition in multi-objective programming with major constraints is studied. With the aid of the major constraints set and its structure representation, a second order necessary condition for major constraints local Pareto weakly efficient solution is given under a weakly convex inclusion condition and some constraint qualifications. The result develops the theory of multi-objective programming and is useful in the design of numerical methods in the major constraints multi-objective programming.

Keywords—major constraints multi-objective programming; major constraints Pareto weakly efficient solution; second order necessary condition

I. INTRODUCTION

Multi-objective programming (MOP) is an important part of mathematical programming, and its theory and methods have become practical tool in modern quantitative decision-making. In view of the fact that the decision-makers usually give various constraint conditions under their different needs in the establishment of a practical multi-objective programming model, so the inequality constraints in the given model are often incompatible. For this reason, we consider a kind of MOP problem called as major constraints (MC) MOP. Its major inequality constraints are compatible.

In Ref.[1], the concepts of major cone and major order were introduced, and a kind of MC problem was presented. In Ref.[2], the MC programming problem was studied and the optimality conditions of MC optimal solutions were proved by using the major constraints set and its structure representation and some constraint qualification conditions. Then, the result of Ref.[2] was extended to MCMOP in Ref.[3].

The study of optimality condition (OC) is always a central subject of mathematical programming theory, which is the basis of numerical method design and is indispensable to other theories such as sensitivity analysis and stability analysis. Since the 1940s, due to the success of mathematical programming theory and methods in solving various practical problems, the study of its first-order and second-order as well as higher-order optimality conditions have received extensive attention (see Ref. [4]-[17]).

In paper introduce the concept of weakly convex inclusion. With the aid of the structure representation of MC set, a second order necessary OC for the MC local Pareto weakly efficient solution is obtained when the objective vector function satisfies the weakly convex inclusion condition.

II. SECOND ORDER NECESSARY CONDITION

Let $f$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $g$ from $\mathbb{R}^n$ to $\mathbb{R}^o$ be nonlinear vector functions. The general nonlinear MOP is:

$$\begin{align*}
\min & \quad f(x), \\
\text{s.t.} & \quad g(x) \leq 0.
\end{align*}$$

(NMOP)

Denote $g(x) = (g_1(x), \ldots, g_p(x))^T$, then the constraint conditions in (NMOP) are

$$g_i(x) \leq 0 \quad (i \text{ from 1 to } p).$$

In practical application, the above each inequality $g_i(x) \leq 0 \quad (i \text{ from 1 to } p)$ is often given from decision-makers independently. It may sometimes occur that the inequality system is not mutually compatible. In this regard, the author (see Ref.[1]) proposed MC nonlinear programming problem under the help of major cone. Based on the results of Ref.[1]-[3], this paper considers the following problem called major constraints multi-objective programming (MCMOP):

$$\begin{align*}
\min & \quad f(x) = (f_1(x), \ldots, f_m(x)) \\
\text{s.t.} & \quad g(x) = (g_1(x), \ldots, g_p(x)) \leq 0.
\end{align*}$$

(MCMOP)

Where the major constraints condition $(g_1(x), \ldots, g_p(x)) \leq 0$ of problem (MCMOP) means that there exist at least $\left\lfloor \frac{p+1}{2} \right\rfloor$ numbers $i_1, \ldots, i_{\left\lfloor \frac{p+1}{2} \right\rfloor}$ belong to $\{1, \ldots, p\}$ such the condition that $x$ satisfies the inequality...
system $g_i(x) \leq 0, \ldots, g_{i_{\frac{p+1}{2}}}^i(x) \leq 0$ is true. The MC set $X_M$ is denoted as $X_M = \{x \mid g(x) \leq 0\}$.

**Definition 1** Let $f$ from $R^n$ to $R^m$ and $g$ from $R^n$ to $R^p$ be nonlinear vector functions. If $x^* \in X_M$ and there exists an $\delta > 0$ such that there is not $x \in X_M \cap N_\delta(x^*)$ satisfying

$$f_k(x) < f_k(x^*)(k \text{ from 1 to } m),$$

where $N_\delta(x^*) = \{x \in R^n \mid \|x - x^*\| < \delta\}$ is a neighborhood of $x^*$, then $x^*$ is said to be a MC local Pareto weakly efficient solution (WES). The set of all MC local Pareto WES is denoted by $L_M$.

In order to give a second-order necessary OC of (MCMOP), we need the following results from Ref. [2].

Let $H \subset R^p$ be the major cone. Then we have

$$\Pi = \{v \in R^n \mid \|v\| \geq \|v\|\},$$

where $K_{i_1,\ldots,i_{\frac{p+1}{2}}} = \{v \in R^n \mid v_i \geq 0, \ldots, v_{i_{\frac{p+1}{2}}} \geq 0\}$ for any $i_1,\ldots,i_{\frac{p+1}{2}}$ belong to $\{1,\ldots,p\}$.

Denote

$$X_{i_1,\ldots,i_{\frac{p+1}{2}}} = \{x \in R^n \mid -g(x) \in K_{i_1,\ldots,i_{\frac{p+1}{2}}}\}.$$

Then the structure of the MC set $X_M$ is represented by

$$X_M = \bigcup X_{i_1,\ldots,i_{\frac{p+1}{2}}},$$

for all $i_1,\ldots,i_{\frac{p+1}{2}}$ belong to $\{1,\ldots,p\}$.

**Lemma 1** Let $h$ from $R^n$ to $R$ and $g$ from $R^n$ to $R^p$ be continuously differentiable at $x^* \in X_M$ and let $x^* \in S_M^L$. For any $i_1,\ldots,i_{\frac{p+1}{2}}$ belong to $\{1,\ldots,p\}$, if $x^* \in X_{i_1,\ldots,i_{\frac{p+1}{2}}}$ and $g_i(x),\ldots,g_{i_{\frac{p+1}{2}}}(x)$ have K-T constraint qualification at $x^*$, then there exists $(\tilde{\lambda}_i,\ldots,\tilde{\lambda}_{i_{\frac{p+1}{2}}}) \in R^{\frac{p+1}{2}}$ satisfying

$$\nabla h(x^*) + \sum_{m=1}^{\frac{p+1}{2}} \tilde{\lambda}_m \nabla g_m(x^*) = 0,$$

$$\tilde{\lambda}_m g_m(x^*) = 0, m = 1,2,\ldots,\lfloor \frac{p+1}{2} \rfloor.$$

**Definition 2** Let $x^* \in S_M^L$. For any $i_1,\ldots,i_{\frac{p+1}{2}}$ belong to $\{1,\ldots,p\}$, if $x^* \in X_{i_1,\ldots,i_{\frac{p+1}{2}}}$, Denoting $F_{i_1,\ldots,i_{\frac{p+1}{2}}} = f(x_{i_1,\ldots,i_{\frac{p+1}{2}}})$, and there exist a convex set $H_{i_1,\ldots,i_{\frac{p+1}{2}}}$ and a neighborhood $N$ of $f(x^*)$ meeting

1) $F_{i_1,\ldots,i_{\frac{p+1}{2}}} \cap N \subset H_{i_1,\ldots,i_{\frac{p+1}{2}}}$,

2) $H_{i_1,\ldots,i_{\frac{p+1}{2}}} \cap (f(x^*) - R^m) = \emptyset$,

then we call that the objective vector function $f(x)$ satisfies the weakly convex inclusion condition at $x^*$.

**Lemma 2** Let $x^* \in S_M^L$ be a MC local Pareto WES of MCMOP and let $f(x)$ satisfy the weakly convex inclusion condition at $x^*$. For any $i_1,\ldots,i_{\frac{p+1}{2}}$ belong to $\{1,\ldots,p\}$, if $x^* \in X_{i_1,\ldots,i_{\frac{p+1}{2}}}$, then there exists $\tilde{\mu}_{i_1,\ldots,i_{\frac{p+1}{2}}} \in R^m$ such that $x^*$ is a local optimal solution (LOS) of the following linear weighting problem (LMP)

$$\begin{align*}
\min & \quad \tilde{\mu}_{i_1,\ldots,i_{\frac{p+1}{2}}}^T f(x), \\
\text{s.t.} & \quad x \in X_{i_1,\ldots,i_{\frac{p+1}{2}}}. 
\end{align*}$$

**Proof** By convex sets separation theorem, this Lemma is easy to know.

Now we give the second order OC for the MC local Pareto WES of MCMOP.
Definition 3.3 Let $S$ be a nonempty subset of $\mathbb{R}^n$ and let $x^* \in c\mathbb{H}S$. The tangent set to $S$ at $x^*$ is defined by
\[ T(x^*, S) = \{ d \mid \text{there exist } x^{(k)} \in S, x^{(k)} \to x^* \text{ and } t_k > 0 \text{ such that } d = \lim_{k \to \infty} (x^{(k)} - x^*)/t_k \}. \]

According to Definition 3, we easily see that if a sequence \( \{x^{(k)}\} \subset S \) converges to $x^*$ satisfying
\[ \lim_{k \to \infty} \frac{x^{(k)} - x^*}{\|x^{(k)} - x^*\|} = d, \]
then $d \in T(x^*, S)$ . For any $i_1, \cdots, i_{\left\lfloor \frac{p+1}{2}\right\rfloor}$ belong to $\{1, \cdots, p\}$ and $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_{\frac{p+1}{2}}) \in R^{\left\lfloor \frac{p+1}{2}\right\rfloor}$ satisfying (1), denote
\[ I_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}} = \{ i_m \in \{i_1, \cdots, i_{\left\lfloor \frac{p+1}{2}\right\rfloor} \} \mid g_{i_m}(x^*) = 0 \} \]
\[ S_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}} = \{ x \in R^n \mid g_{i_m}(x) = 0, i_m \in I_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}, \tilde{\lambda}_m > 0; \tilde{\lambda}_m g_{i_m}(x) \leq 0, i_m \in I_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}, \tilde{\lambda}_m = 0 \}. \]

Let the tangent cone of $S_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}$ at $x^*$ be $T_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}^{\left\lfloor \frac{p+1}{2}\right\rfloor}$.

Denote
\[ G_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}} = \{ d \in R^n \mid \nabla g_{i_m}(x^*)^t d = 0, i_m \in I_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}, \tilde{\lambda}_m > 0; \nabla g_{i_m}(x^*)^t d \leq 0, i_m \in I_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}, \tilde{\lambda}_m = 0 \}. \]

We see that $G_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}} \supseteq T_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}^{\left\lfloor \frac{p+1}{2}\right\rfloor}$, but the reverse is not true, that is, the set $G_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}$ is not necessarily contained in the tangent cone $T_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}^{\left\lfloor \frac{p+1}{2}\right\rfloor}$.

If the condition $G_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}} \subset T_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}^{\left\lfloor \frac{p+1}{2}\right\rfloor}$ is also true, the second order OC for the local Pareto weakly efficient solution of MC programming is given below.

Theorem 1 Let $f$ from $R^n$ to $R^n$ and $g$ from $R^n$ to $R^n$ be second order continuously differentiable at $x^* \in X_M$ and let $x^* \in S_M^L$. For any $\left\lfloor \frac{p+1}{2}\right\rfloor$ numbers $i_1, \cdots, i_{\left\lfloor \frac{p+1}{2}\right\rfloor}$ belong to $\{1, \cdots, p\}$, if $x^* \in X_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}$, $f(x)$ satisfies the weakly convex inclusion condition at $x^*$, $g_{i_1}(x), \cdots, g_{i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}(x)$ have K-T constraint qualification at $x^*$, and $G_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}^{\left\lfloor \frac{p+1}{2}\right\rfloor} = T_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}^{\left\lfloor \frac{p+1}{2}\right\rfloor}$ holds, then there exists
\[ \tilde{\lambda}_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_{\left\lfloor \frac{p+1}{2}\right\rfloor}) \in R^{\left\lfloor \frac{p+1}{2}\right\rfloor} \] such that
\[ \tilde{\lambda}_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_{\left\lfloor \frac{p+1}{2}\right\rfloor}) \in R^{\left\lfloor \frac{p+1}{2}\right\rfloor} \]
\[ \sum_{m=1}^{\left\lfloor \frac{p+1}{2}\right\rfloor} \tilde{\lambda}_m g_{i_m}(x^*) = 0, \]
\[ \tilde{\lambda}_m g_{i_m}(x^*) = 0, \]
\[ m = 1, 2, \cdots \left\lfloor \frac{p+1}{2}\right\rfloor \]
is true, and for each $d \in G_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}$, we have
\[ d^T \nabla^2 L(x^*, \tilde{\lambda}_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}) d \geq 0, \]
where
\[ L(x, \tilde{\lambda}_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}) = \tilde{\lambda}_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}^T f(x) + \sum_{m=1}^{\left\lfloor \frac{p+1}{2}\right\rfloor} \tilde{\lambda}_m g_{i_m}(x), \]
\[ \nabla^2 L(x^*, \tilde{\lambda}_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}) \] is Hesse matrix of function $L(x, \tilde{\lambda}_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}})$ at $x^*$.

Proof let $x^* \in S_M^L$ and let $x^* \in X_{h_{i_1} \cdots i_{\left\lfloor \frac{p+1}{2}\right\rfloor}}$ for some $\left\lfloor \frac{p+1}{2}\right\rfloor$ numbers $i_1, \cdots, i_{\left\lfloor \frac{p+1}{2}\right\rfloor}$ belong to $\{1, \cdots, p\}$. Since $f(x)$ satisfies the weakly convex inclusion condition at $x^*$, by Lemma 2, there exists
\[ \tilde{\mu}_{h_i \cdots h_i}^T = (\tilde{\mu}_1, \cdots, \tilde{\mu}_m)^T \in R^m \]

such that \( x^\ast \) is a LOS of LWP.

Since \( g_i (x), \cdots, g_m (x) \) have K-T constraint qualification at \( x^\ast \), by Lemmas, there exists
\[ \tilde{\lambda}_{h_i \cdots h_i} \in (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_m) \in R^{\frac{m}{2}} \]  

such that (4) holds.

Now for each \( d \neq 0 \), \( d \in G_{h_i \cdots h_i} \), since
\[ G_{h_i \cdots h_i} = T_{h_i \cdots h_i} \]  

it follows that \( d \in T_{h_i \cdots h_i} \). Then there exist \( x^{(k)} \in S_{h_i \cdots h_i} \), \( x^{(k)} \rightarrow x^\ast \) and \( t_k > 0 \) such that
\[ \lim_{k \rightarrow \infty} (x^{(k)} - x^\ast)/t_k = d . \]

Expanding \( L(x, \tilde{\lambda}_{h_i \cdots h_i}) \) at \( x^\ast \) by Taylor’s formula and taking \( x = x^{(k)} \), we have
\[
L(x^{(k)}, \tilde{\lambda}_{h_i \cdots h_i}) = L(x^\ast, \tilde{\lambda}_{h_i \cdots h_i}) + \nabla_x L(x^\ast, \tilde{\lambda}_{h_i \cdots h_i})^T (x^{(k)} - x^\ast) \\
+ \frac{1}{2} (x^{(k)} - x^\ast)^T \nabla_x^2 L(x^\ast, \tilde{\lambda}_{h_i \cdots h_i}) (x^{(k)} - x^\ast) \\
+ o(\|x^{(k)} - x^\ast\|^2) .
\]  

(6)

Since \( x^{(k)} \in S_{h_i \cdots h_i} \), \( x^{(k)} \rightarrow x^\ast \), it follows that
\[ \tilde{\lambda}_{h_i \cdots h_i} \in \int \frac{D + 1}{2} \int \frac{D + 1}{2} . \]  

From (5) we obtain
\[
L(x^{(k)}, \tilde{\lambda}_{h_i \cdots h_i}) = \tilde{\mu}_{h_i \cdots h_i}^T f(x^{(k)}) \\
L(x^\ast, \tilde{\lambda}_{h_i \cdots h_i}) = \tilde{\mu}_{h_i \cdots h_i}^T f(x^\ast) .
\]  

(7)

(8)

Substituting (7) (8) into (6), we have by (4)
\[ \tilde{\mu}_{h_i \cdots h_i}^T f(x^{(k)}) = \tilde{\mu}_{h_i \cdots h_i}^T f(x^\ast) + \frac{1}{2} (x^{(k)} - x^\ast)^T \nabla_x^2 L(x^\ast, \tilde{\lambda}_{h_i \cdots h_i}) (x^{(k)} - x^\ast) \\
+ o(\|x^{(k)} - x^\ast\|^2) .
\]  

(9)

Since \( x^\ast \) is a LOS of LWP, there must be
\[ \tilde{\mu}_{h_i \cdots h_i}^T f(x^{(k)}) \geq \tilde{\mu}_{h_i \cdots h_i}^T f(x^\ast) \] as \( k \) sufficiently large. 

So from (9) for sufficiently large \( k \),
\[ \frac{1}{2} (x^{(k)} - x^\ast)^T \nabla_x^2 L(x^\ast, \tilde{\lambda}_{h_i \cdots h_i}) (x^{(k)} - x^\ast) \\
+ o(\|x^{(k)} - x^\ast\|^2) \geq 0 .
\]

Dividing both the sides of the above equation by \( t_k^2 \) and as \( k \rightarrow \infty \), then
\[ d^T \nabla_x^2 L(x^\ast, \tilde{\lambda}_{h_i \cdots h_i}) d \geq 0 .
\]

III. CONCLUSION

In this paper, the inequality constraints system of the MOP is incompatible, but its major inequality constraints system is compatible. So this paper studies a new class of MOP problem. The new type of MOP is called MC multi-objective programming.

The establishment of second order OCs is very important to a MOP problem. This paper gives the concept of MC local Pareto weakly efficient solution then using the of MC set and its structure representation and some constraint qualification conditions, the second order necessary OC for MC local Pareto weakly efficient solution is obtained.

Then, we will further study the theory such as stability analysis and sensitivity analysis, and we will also study how to design methods for its numerical solutions and so on.

REFERENCES


