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2-Dimension Linguistic Bonferroni Mean Aggregation Operators and Their Application to Multiple Attribute Group Decision Making

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ABSTRACT

The aim of this paper is to provide a multiple attribute group decision making (MAGDM) method based on the 2-dimension linguistic weight Bonferroni mean aggregation (2DLWBMA) operator. Firstly, the new operations of 2-dimension linguistic variables are defined. Then, the 2-dimension linguistic Bonferroni mean aggregation operator is proposed to describe the correlations of input arguments. Subsequently, the 2DLWBMA operator is investigated to consider the importance of attributes. Furthermore, a novel MAGDM method is introduced and two illustrative examples are given.

1. INTRODUCTION

Multiple attribute group decision making (MAGDM) methods refer to ranking the limited alternatives or selecting the best one according to the evaluation results provided by different decision makers. MAGDM methods have been widely applied in many areas, such as economic, society, science and management and so on. Among some MAGDM problems, the decision makers prefer to evaluate the alternatives on the qualitative attributes by linguistic variables (LVs) rather than crisp numbers. Since Zadeh [1–3] introduced the notion of LV, researchers have developed many kinds of methods to rank the alternatives according to the values of LVs [4–10]. However, in some real MAGDM problems, besides the linguistic evaluation values of the attributes, the decision maker still uses LVs to represent his or her self-assessment on the given evaluation results. For describing such phenomena, Zhu *et al.* [11] proposed the concept of 2-dimension linguistic variable (2DLV). A 2DLV includes two classes of LVs, where the *I* class LV is used to represent the linguistic evaluation value of the attribute, and the *II* class LV is used to represent the decision maker's self-assessment.

The main advantage of 2DLVs is that it can distinguish indetermination between decision making problems and subjective understanding. By use of 2DLVs, the decision makers can better express their opinions. Up to now, the research achievements of 2-dimension linguistic information can be classified into three categories: 2DLVs [12–17], 2-dimension uncertain linguistic variables (2DULVs) [18–28], and hesitant fuzzy 2-dimension linguistic variables (HF2DLVs) [29,30]. Since there exist two classes of LVs in a 2DLV, the operations of 2DLVs become more difficult and complex than those of LVs. For simplicity of calculation, many existing 2DLV operations only take the minimum values of the *II* class LVs [12,19–21,23,25,26,31,32]. Although it is easy to operate by taking minimum values, many useful linguistic information may be lost or distorted. Because the *I* class LV of 2DLVs is used to describe the linguistic evaluation value of the attribute, the operation of the *I* class LV is consistent with that of the classical LV proposed by Zadeh [1]. On the other hand, the *II* class LV is used to represent the decision maker's self-assessment. At the same time, fuzzy number is a useful tool to describe the decision maker's subjective attitude. Hence we intend to convert the *II* class LV of 2DLVs into fuzzy number, which can more reasonably describe the decision maker's self-assessment. Concretely, the new operations of 2DLVs take the *I* class LV as the classical LV, and convert the *II* class LV into fuzzy number.

Bonferroni mean (BM) operator, as a useful aggregation operator, has the ability to capture the interrelationships between input arguments. So far, BM operator has been widely extended to fuzzy linguistic environment [6,33–38], intuitionistic fuzzy linguistic environment [39–44], hesitant fuzzy environment [45] and

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pythagorean fuzzy circumstance [46] for handling the increasingly complex fuzzy systems and decision systems. Wei *et al.* [6] developed the uncertain linguistic BM operator and the uncertain linguistic geometric BM operator for aggregating uncertain linguistic information. Liu *et al.* [41] extended BM operator to the intuitionistic linguistic environment. Dutta *et al.* [34] applied the extended Bonferroni Mean (EBM) in linguistic 2-tuple environment to capture heterogeneous interrelationship among the attributes. Tian *et al.* [35,36] combined BM operator with simplified neutrosophic linguistic numbers and gray linguistic numbers to deal with the corresponding MADM problems. Combing the generalized partitioned Bonferroni mean (PBM) operator with linguistic neutrosophic numbers, Wang *et al.* [37] proposed the linguistic neutrosophic generalized weighted partitioned Bonferroni mean (LNGWPBM) aggregation operator to solve MAGDM problems. However, in some 2-dimension linguistic MAGDM problems, because the evaluation values of attributes are extended to two classes of LVs, the relationships between input arguments may be more complicated. In order to capture the interrelationships between input 2DLVs, it is necessary to extend BM operator into 2-dimension linguistic decision making environment.

Further, we list some extended BM methods in different application environments to convenient the readers by using the following tabular (see Table 1).

In 2-dimension linguistic MAGDM problems, Yin *et al.* [32] have combined trapezoidal fuzzy 2-dimensional linguistic information with a PBM operator to address the 2-dimension linguistic decision making problems. Liu *et al.* [25] have proposed the 2-dimensional uncertain linguistic weighted Bonferroni mean (2DULWBM) operator and the 2-dimensional uncertain linguistic improved weighted Bonferroni Harmonic mean (2DULIWBHM) operator. However, this paper combine 2DLVs with BM operators for different aspects. Concretely, the operations of 2DLVs are improved as stated in the above, where the *II* class LV of 2DLVs is converted into a fuzzy number. Further, this paper considers BM operators rather than PBM operators. BM operators focus on the aggregated attributes to capture the interrelationships among them, which assumes that each attribute is related to the rest of the attributes. While PBM operators assume that attributes are partitioned into several unrelated classes and each attribute only has the interrelationship with rest of the attributes in the same class. Moreover, the comparison method of 2DLVs used in this paper [47] is improved more reasonably than many existing methods which can only deal with the total order of alternatives [15,25,32].

Motivated by the above ideas, we intend to

1. Define some new operations of 2DLVs.
2. Establish the 2-dimensional linguistic Boneferroni mean aggregation (2DLBMA) operator and the 2DLWBMA operator.
3. Investigate the properties and special cases of the 2DLBMA operator.
4. Propose a novel decision making method based on the 2DLWBMA operator.
5. Demonstrate the feasibility and practicality of the proposed method.

The remainder of this paper is established as follows: Section 2 reviews 2DLV and BM operator. In Section 3, some new operations of 2DLVs are developed and their properties are discussed. In Section 4, the 2DLBMA operator and the 2DLWBMA operator are proposed for 2DLVs. Section 5 provides a novel decision making approach based on the 2DLWBMA operator. In Section 6, two illustrative examples are given. Section 7 concludes this paper.

2. PRELIMINARIES

This section reviews some basic notions of 2DLV and BM operator.

2.1. 2DLV

Definition 1. [11] Let $S = \{s_0, s_1, \cdots, s_g\}$ and $H = \{h_0, h_1, \cdots, h_t\}$ be two linguistic term sets (LTSs), where $g+1$ is the cardinality of S and $t+1$ is the cardinality of H. $r_i = \left(s_{a_i}, h_{b_i}\right)$ is called a 2DLV, in which $h_{b_i} \in H$ is *I* class LV, which represents the assessment information about the alternative given by the decision maker, while $s_{a_i} \in S$ is *II* class LV, which represents the self-assessment of the decision maker.

For describing the complicated relationships between 2DLVs, Zhu *et al.* [47] proposed the concept of a 2-dimension linguistic lattice implication algebra (2DL-LIA). The method for comparing two 2DLVs is provided in a 2DL-LIA as follows.

Definition 2. [47] Let $(S \times H, \vee, \wedge, \rightarrow, ')$ be a 2DL-LIA, where $S = \{s_0, s_1, \cdots, s_g\}$ and $H = \{h_0, h_1, \cdots, h_t\}$ are two LTSs. $r_i = \left(s_{a_i}, h_{b_i}\right)$ and $r_j = \left(s_{a_j}, h_{b_j}\right)$ are any two 2DLVs of $S \times H$, $a_i, a_j \in \{0, 1, \cdots, g\}$ and $b_i, b_j \in \{0, 1, \cdots, t\}$, δ is a positive number in the real set R.

Table 1 Some extended BM methods in different application environments.

Applications	Some Extended BM Methods
Fuzzy linguistic environment	Beliakov's method [33], Dutta's method [34], Tian's methods [35, 36], Wang's method [37], Wei's method [6], Yager's method [38]
Intuitionistic fuzzy linguistic environment	Das's method [39], Heyd's method [40], Liu's methods [41, 42], Zhang's method [43], Zhou's method [44]
Hesitant fuzzy linguistic environment	Zhu's method [45]
Pythagorean fuzzy linguistic environment	Nierx's method [46]
2-dimension fuzzy linguistic environment	Liu's method [25], Yin's method [32], the proposed method

BM = Bonferroni Mean.

1. If $a_i < a_j$ and $b_i \leqslant b_j$ or $a_i \leqslant a_j$ and $b_i < b_j$, then r_i is less than r_j, denoted by $r_i < r_j$;
2. If $a_i = a_j$ and $b_i = b_j$, then r_i is equivalent to r_j, denoted by $r_i = r_j$;
3. If $b_i \leqslant b_j$ and $0 < a_i - a_j < \delta$, then r_i is weakly less than r_j, denoted by $r_i \leqslant_W r_j$;
4. If $b_i \leqslant b_j$ and $a_i - a_j \geqslant \delta$, then r_i is incomparable to r_j, denoted by $r_i \parallel r_j$.

Usually, the parameter δ takes value less than 1, which can be preseted by the decision maker's preferences.

2.2. BM Operator

Since BM operator can capture the interrelationship between input arguments, it has been widely applied in MAGDM problems.

Definition 3. [48] Let $p, q \geqslant 0$ and $\{a_1, a_2, \cdots, a_n\}$ be a set of nonnegative real numbers. Then the mapping $BM^{p,q} : R^n \to R$ such that

$$BM^{p,q}(a_1, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1,\\ i \neq j}}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}},$$

is called the BM operator.

BM operator satisfies the following properties:

1. If $a_i = a$ for all i, then $BM^{p,q}(a, a, \cdots, a) = a$;
2. If $a_i \leqslant a_i'$ for all i, then $BM^{p,q}(a_1, a_2, \cdots, a_n) \leqslant BM^{p,q}(a_1', a_2', \cdots, a_n')$;
3. If $\underline{a} \leqslant a_i \leqslant \overline{a}$ for all i, then $\underline{a} \leqslant BM^{p,q}(a_1, a_2, \cdots, a_n) \leqslant \overline{a}$.

Especially,

1. $BM^{1,0}(a_1, \cdots, a_n) = \frac{1}{n} \sum_{i=1}^{n} a_i$ is the average mean;
2. $\lim_{p \to 0} BM^{p,0}(a_1, \cdots, a_n) = \left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}}$ is the geometric mean.

3. NEW OPERATIONS OF 2DLVs

According to the Definition 1, in a 2DLV $r = (s_a, h_b)$, h_b represents the linguistic evaluation value of an attribute given by the decision maker, which is consistent with Zadeh's LV. So this paper assumes that the operation of h_b is consistent with that of the classical LV [7]. While s_a represents the decision maker's self-assessment, therefore it is reasonable to convert s_a into fuzzy number $\frac{a}{g}$, where g is the cardinality of S. By use of fuzzy number $\frac{a}{g}$, the subjective attitude of the decision maker can be precisely expressed. Based on the above discussions, some new operational rules of 2DLVs are defined in this section. In the new operations of 2DLVs, the operation of s_a is based on that of fuzzy number, while the operation of h_b is unchanged.

Definition 4. Let $r_i = \left(s_{a_i}, h_{b_i}\right)$, $r_j = \left(s_{a_j}, h_{b_j}\right)$ be two 2DLVs of a 2DL-LIA $S \times H$, where $S = \left\{s_0, s_1, \cdots, s_g\right\}$ and $H = \{h_0, h_1, \cdots, h_t\}$, then the operations are defined as follows:

1. $r_i \oplus r_j = \left(s_{a_i + a_j - \frac{a_i a_j}{g}}, h_{b_i + b_j} \right)$;
2. $r_i \otimes r_j = \left(s_{\frac{a_i a_j}{g}}, h_{b_i b_j} \right)$;
3. $\lambda r_i = \left(s_{g\left[1 - \left(1 - \frac{a_i}{g}\right)^{\lambda}\right]}, h_{\lambda b_i} \right), \quad \lambda > 0$;
4. $r_i^{\lambda} = \left(s_{g\left(\frac{a_i}{g}\right)^{\lambda}}, h_{b_i^{\lambda}} \right), \quad \lambda > 0$.

Remark. In Definition 4, there exist the mutual transformations between the *II* class LVs and fuzzy numbers. Concretely, we transform the *II* class LVs into fuzzy numbers, then implement the operational laws of fuzzy numbers, finally convert the operational results of fuzzy numbers into the *II* class LVs.

Example 1. Let $\left(S \times H, \vee, \wedge, \rightarrow, '\right)$ be a 2DL-LIA, where $S = \{s_0, s_1, s_2, s_3, s_4\}$, $H = \{h_0, h_1, h_2, h_3, h_4, h_5, h_6\}$. Let $r_1 = (s_3, h_2)$, $r_2 = (s_2, h_3)$ and $\lambda = 2$.

By Definition 4, we can compute that

$r_1 \oplus r_2 = (s_{3.5}, h_5)$, $\lambda r_1 = (s_{3.75}, h_4)$,

$r_1 \otimes r_2 = (s_{1.5}, h_6)$, $r_1^{\lambda} = (s_{2.25}, h_4)$.

In the following, some propositions of the new operational rules are discussed.

Proposition 1. *Let $r_i = \left(s_{a_i}, h_{b_i}\right)$, $r_j = \left(s_{a_j}, h_{b_j}\right)$ and $r_k = \left(s_{a_k}, h_{b_k}\right)$ be 2DLVs of a 2DL-LIA $S \times H$, $\lambda, \lambda_1, \lambda_2 > 0$. Then the followings hold:*

1. $r_i \oplus r_j = r_j \oplus r_i$;
2. $\lambda\left(r_i \oplus r_j\right) = \lambda r_i \oplus \lambda r_j$;
3. $\lambda_1 r_i \oplus \lambda_2 r_i = (\lambda_1 + \lambda_2) r_i$;
4. $\left(r_i \oplus r_j\right) \oplus r_k = r_i \oplus \left(r_j \oplus r_k\right)$;
5. $r_i \otimes r_j = r_j \otimes r_i$;
6. $\left(r_i \otimes r_j\right)^{\lambda} = r_i^{\lambda} \otimes r_j^{\lambda}$;
7. $r_i^{\lambda_1} \otimes r_i^{\lambda_2} = r_i^{\lambda_1 + \lambda_2}$;
8. $\left(r_i \otimes r_j\right) \otimes r_k = r_i \otimes \left(r_j \otimes r_k\right)$.

Proof. The proof of (1) can be obtained according to Definition 4.

Firstly, we give the proof of (2). On one hand,

$$\lambda\left(r_i \oplus r_j\right)$$

$$= \lambda\left(s_{a_i+a_j-\frac{a_ia_j}{g}}, h_{b_i+b_j}\right)$$

$$= \left(s_{g\left(1-\left(1-\frac{a_i+a_j}{g}+\frac{a_ia_j}{g^2}\right)^{\lambda}\right)}, h_{\lambda\left(b_i+b_j\right)}\right)$$

$$= \left(s_{g\left(1-\left(1-\frac{a_i}{g}\right)^{\lambda}\left(1-\frac{a_j}{g}\right)^{\lambda}\right)}, h_{\lambda\left(b_i+b_j\right)}\right);$$

On the other hand,

$$\lambda r_i \oplus \lambda r_j$$

$$= \left(s_{g\left(1-\left(1-\frac{a_i}{g}\right)^{\lambda}\right)}, h_{\lambda b_i}\right) \oplus \left(s_{g\left(1-\left(1-\frac{a_j}{g}\right)^{\lambda}\right)}, h_{\lambda b_j}\right)$$

$$= \left(s_{g-g\left(1-\frac{a_i}{g}\right)^{\lambda}\left(1-\frac{a_j}{g}\right)^{\lambda}}, h_{\lambda\left(b_i+b_j\right)}\right)$$

$$= \left(s_{g\left(1-\left(1-\frac{a_i}{g}\right)^{\lambda}\left(1-\frac{a_j}{g}\right)^{\lambda}\right)}, h_{\lambda\left(b_i+b_j\right)}\right);$$

Hence we have $\lambda\left(r_i \oplus r_j\right) = \lambda r_i \oplus \lambda r_j$.

Then, the proof of (4) is given as follows. Since

$$\left(r_i \oplus r_j\right) \oplus r_k$$

$$= \left(s_{a_i+a_j-\frac{a_ia_j}{g}}, h_{b_i+b_j}\right) \oplus \left(s_{a_k}, h_{b_k}\right)$$

$$= \left(s_{a_i+a_j+a_k-\frac{a_ia_j}{g}-\frac{a_ia_k}{g}-\frac{a_ja_k}{g}+\frac{a_ia_ja_k}{g^2}}, h_{b_i+b_j+b_k}\right);$$

On the other hand,

$$r_i \oplus \left(r_j \oplus r_k\right)$$

$$= \left(s_{a_i}, h_{b_i}\right) \oplus \left(s_{a_j+a_k-\frac{a_ja_k}{g}}, h_{b_j+b_k}\right)$$

$$= \left(s_{a_i+a_j+a_k-\frac{a_ia_j}{g}-\frac{a_ia_k}{g}-\frac{a_ja_k}{g}+\frac{a_ia_ja_k}{g^2}}, h_{b_i+b_j+b_k}\right);$$

Hence $\left(r_i \oplus r_j\right) \oplus r_k = r_i \oplus \left(r_j \oplus r_k\right)$.

Similarly, (3) and (5)–(8) can be proved, which are omitted here.

Proposition 2. *Let $r_i = \left(s_{a_i}, h_{b_i}\right)$, $r_i' = \left(s_{a_i'}, h_{b_i'}\right)$, $r_j = \left(s_{a_j}, h_{b_j}\right)$, $r_j' = \left(s_{a_j'}, h_{b_j'}\right)$ be 2DLVs of a 2DL-LIA $S \times H$. If $r_i \leqslant r_i'$, $r_j \leqslant r_j'$, then the followings hold:*

1. $\lambda r_i \leqslant \lambda r_i'$;
2. $r_i^{\lambda} \leqslant r_i'^{\lambda}$;
3. $r_i \oplus r_j \leqslant r_i' \oplus r_j'$;
4. $r_i \otimes r_j \leqslant r_i' \otimes r_j'$;
5. $\bigoplus_{i=1}^{n} r_i \leqslant \bigoplus_{i=1}^{n} r_i'$;
6. $\bigotimes_{i=1}^{n} r_i \leqslant \bigotimes_{i=1}^{n} r_i'$.

Proof. Suppose $r_i \leqslant r_i'$, then according to Definition 2, we have $a_i \leqslant a_i'$ and $b_i \leqslant b_i'$.

(1) Because $\lambda > 0$, we have

$\left(1-\frac{a_i}{g}\right)^{\lambda} \geqslant \left(1-\frac{a_i'}{g}\right)^{\lambda}$, which follows that

$$g\left(1-\left(1-\frac{a_i}{g}\right)^{\lambda}\right) \leqslant g\left(1-\left(1-\frac{a_i'}{g}\right)^{\lambda}\right).$$

And we have $\lambda b_i \leqslant \lambda b_i'$.

Since $\lambda r_i = \left(s_{g\left(1-\left(1-\frac{a_i}{g}\right)^{\lambda}\right)}, h_{\lambda b_i}\right)$ and

$$\lambda r_i' = \left(s_{g\left(1-\left(1-\frac{a_i'}{g}\right)^{\lambda}\right)}, h_{\lambda b_i'}\right),$$

then according to Definition 2, $\lambda r_i \leqslant \lambda r_i'$.

(2) Because $\lambda > 0$, we have

$g\left(\frac{a_i}{g}\right)^{\lambda} \leqslant g\left(\frac{a_i'}{g}\right)^{\lambda}$. And we have $b_i^{\lambda} \leqslant b_i'^{\lambda}$.

Since $r_i^{\lambda} = \left(s_{g\left(\frac{a_i}{g}\right)^{\lambda}}, h_{b_i^{\lambda}}\right)$ and

$r_i'^{\lambda} = \left(s_{g\left(\frac{a_i'}{g}\right)^{\lambda}}, h_{b_i'^{\lambda}}\right)$, then according to Definition 2, $r_i^{\lambda} \leqslant r_i'^{\lambda}$.

(3) Suppose $r_i \leqslant r_i'$ and $r_j \leqslant r_j'$, then by Definition 2, we have $a_i \leqslant a_i'$, $b_i \leqslant b_i'$ and $a_j \leqslant a_j'$, $b_j \leqslant b_j'$.

Then we have

$a_i + a_j - \frac{a_ia_j}{g} = g - g\left(1-\frac{a_i}{g}\right)\left(1-\frac{a_j}{g}\right)$ and

$$a_i' + a_j' - \frac{a_i' a_j'}{g} = g - g\left(1 - \frac{a_i'}{g}\right)\left(1 - \frac{a_j'}{g}\right).$$

Since $a_i \leqslant a_i'$ and $a_j \leqslant a_j'$, we have

$$0 \leqslant 1 - \frac{a_i'}{g} \leqslant 1 - \frac{a_i}{g} \text{ and } 0 \leqslant 1 - \frac{a_j'}{g} \leqslant 1 - \frac{a_j}{g}.$$

It follows that

$$0 \leqslant \left(1 - \frac{a_i'}{g}\right)\left(1 - \frac{a_j'}{g}\right) \leqslant \left(1 - \frac{a_i}{g}\right)\left(1 - \frac{a_j}{g}\right).$$

Therefore

$$g - g\left(1 - \frac{a_i}{g}\right)\left(1 - \frac{a_j}{g}\right) \leqslant g - g\left(1 - \frac{a_i'}{g}\right)\left(1 - \frac{a_j'}{g}\right),$$

that is $a_i + a_j - \frac{a_i a_j}{g} \leqslant a_i' + a_j' - \frac{a_i' a_j'}{g}$.

And $b_i + b_j \leqslant b_i' + b_j'$ holds obviously.

Since $r_i \oplus r_j = \left(s_{a_i + a_j - \frac{a_i a_j}{g}}, h_{b_i + b_j}\right)$ and

$r_i' \oplus r_j' = \left(s_{a_i' + a_j' - \frac{a_i' a_j'}{g}}, h_{b_i' + b_j'}\right)$, then according to Definition 2, we have $r_i \oplus r_j \leqslant r_i' \oplus r_j'$.

Similarly, (4) can be proved, which is omitted here.

(5) We can prove $\bigoplus_{i=1}^{n} r_i \leqslant \bigoplus_{i=1}^{n} r_i'$ by the mathematical induction on n.

When $n = 2$, we have $r_1 \oplus r_2 \leqslant r_1' \oplus r_2'$ by (3).

Assume $n = k$, the equation holds, that is

$$\bigoplus_{i=1}^{k} r_i \leqslant \bigoplus_{i=1}^{k} r_i'.$$

Then, when $n = k + 1$, we have

$$\bigoplus_{i=1}^{k+1} r_i = \left(\bigoplus_{i=1}^{k} r_i\right) \oplus r_{k+1}.$$

Because $\bigoplus_{i=1}^{k} r_i \leqslant \bigoplus_{i=1}^{k} r_i'$ and $r_{k+1} \leqslant r_{k+1}'$,

then by (3), we have

$$\left(\bigoplus_{i=1}^{k} r_i\right) \oplus r_{k+1} \leqslant \left(\bigoplus_{i=1}^{k} r_i'\right) \oplus r_{k+1}' = \bigoplus_{i=1}^{k+1} r_i',$$

that is $\bigoplus_{i=1}^{k+1} r_i \leqslant \bigoplus_{i=1}^{k+1} r_i'$.

Now, we have $\bigoplus_{i=1}^{n} r_i \leqslant \bigoplus_{i=1}^{n} r_i'$ for all n.

(6) can be proved similarly by the mathematical induction.

Proposition 3. *Let $r_i = \left(s_{a_i}, h_{b_i}\right)$ $(i = 1, 2, \cdots, n)$ be 2DLVs of a 2DL-LIA $S \times H$, $w_i \in [0, 1]$ and $\lambda > 0$. Then the followings hold:*

1. $\bigoplus_{i=1}^{n} r_i = \left(s_{g - g\prod_{i=1}^{n}\left(1 - \frac{a_i}{g}\right)}, h_{\sum_{i=1}^{n} b_i}\right)$;
2. $\bigoplus_{i=1}^{n} w_i r_i = \left(s_{g - g\prod_{i=1}^{n}\left(1 - \frac{a_i}{g}\right)^{w_i}}, h_{\sum_{i=1}^{n} w_i b_i}\right)$;
3. $\left(\bigoplus_{i=1}^{n} r_i\right)^{\lambda} = \left(s_{g\left(1 - \prod_{i=1}^{n}\left(1 - \frac{a_i}{g}\right)\right)^{\lambda}}, h_{\left(\sum_{i=1}^{n} b_i\right)^{\lambda}}\right)$;
4. $\bigotimes_{i=1}^{n} r_i = \left(s_{g\prod_{i=1}^{n}\left(\frac{a_i}{g}\right)}, h_{\prod_{i=1}^{n} b_i}\right)$;
5. $\bigotimes_{i=1}^{n} r_i^{w_i} = \left(s_{g\prod_{i=1}^{n}\left(\frac{a_i}{g}\right)^{w_i}}, h_{\prod_{i=1}^{n} b_i^{w_i}}\right)$;
6. $\left(\bigotimes_{i=1}^{n} r_i\right)^{\lambda} = \left(s_{g\left(\prod_{i=1}^{n} \frac{a_i}{g}\right)^{\lambda}}, h_{\left(\prod_{i=1}^{n} b_i\right)^{\lambda}}\right)$.

Proof. (1) We can prove

$$\bigoplus_{i=1}^{n} r_i = \left(s_{g - g\prod_{i=1}^{n}\left(1 - \frac{a_i}{g}\right)}, h_{\sum_{i=1}^{n} b_i}\right),$$

by the mathematical induction on n.

When $n = 2$, we have

$$r_1 \oplus r_2$$
$$= \left(s_{a_1 + a_2 - \frac{a_1 a_2}{g}}, h_{b_1 + b_2}\right)$$
$$= \left(s_{g - g\left(1 - \frac{a_1}{g} - \frac{a_2}{g} + \frac{a_1 a_2}{g^2}\right)}, h_{b_1 + b_2}\right)$$
$$= \left(s_{g - g\prod_{i=1}^{2}\left(1 - \frac{a_i}{g}\right)}, h_{\sum_{i=1}^{2} b_i}\right).$$

Assume $n = k$, the conclusion holds, that is

$$\bigoplus_{i=1}^{k} r_i = \left(s_{g - g\prod_{i=1}^{k}\left(1 - \frac{a_i}{g}\right)}, h_{\sum_{i=1}^{k} b_i}\right).$$

Then, when $n = k + 1$,we have

$$\bigoplus_{i=1}^{k+1} r_i = \left(\bigoplus_{i=1}^{k} r_i\right) \oplus r_{k+1}$$
$$= \left(s_{g - g\prod_{i=1}^{k}\left(1 - \frac{a_i}{g}\right)}, h_{\sum_{i=1}^{k} b_i}\right) \oplus \left(s_{a_{k+1}}, h_{b_{k+1}}\right)$$

$$= \left(s_{g-g\prod_{i=1}^{k}\left(1-\frac{a_i}{g}\right)+\prod_{i=1}^{k}\left(1-\frac{a_i}{g}\right)a_{k+1}}, h_{k+1\sum_{i=1}^{k} b_i} \right)$$

$$= \left(s_{g-g\prod_{i=1}^{k+1}\left(1-\frac{a_i}{g}\right)}, h_{k+1\sum_{i=1}^{k+1} b_i} \right).$$

Now, we have $\bigoplus_{i=1}^{n} r_i = \left(s_{g-g\prod_{i=1}^{n}\left(1-\frac{a_i}{g}\right)}, h_{n\sum_{i=1}^{n} b_i} \right)$ for all n.

Similarly, (2)–(6) can be proved by the mathematical induction, which are omitted.

4. 2DLBMA OPERATOR AND 2DLWBMA OPERATOR

In this section, based on BM operator and the new operational rules of 2DLVs, we develop the 2DLBMA operator to deal with MAGDM problems.

4.1. 2DLBMA Operator

Definition 5. Let $r_i = \left(s_{a_i}, h_{b_i}\right)$ $(i = 1, 2, \cdots, n)$ be 2DLVs of a 2DL-LIA $S \times H$, where $S = \left\{s_0, s_1, \cdots, s_g\right\}$ and $H = \{h_0, h_1, \cdots, h_t\}$, and $p, q \geqslant 0$. Then the mapping $2DLBMA^{p,q} : \Omega^n \to \Omega$ such that

$$2DLBMA^{p,q}(r_1, \cdots, r_n) = \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \left(r_i^p \otimes r_j^q\right) \right)^{\frac{1}{p+q}},$$

is called the 2DLBMA operator.

The 2DLBMA operator is a useful tool to capture the interrelationship between the attributes in 2-dimension linguistic MAGDM environment.

Example 2. Let $\left(S \times H, \vee, \wedge, \rightarrow, '\right)$ be a 2DL-LIA, where $S = \{s_0, s_1, s_2, s_3, s_4\}$, $H = \{h_0, h_1, h_2, h_3, h_4, h_5, h_6\}$. Let $r_1 = (s_3, h_5)$, $r_2 = (s_3, h_3)$, $r_3 = (s_1, h_5)$, $r_4 = (s_2, h_4)$ and $p = 1, q = 1$.

According to Definition 5, we can compute that

$2DLBMA^{1,1}(r_1, r_2, r_3, r_4) = (s_{2.26}, h_{4.22})$.

The following theorem shows that the aggregation value of 2DLVs, which is aggregated by the $2DLBMA$ operator, is still a 2DLV.

Theorem 4. *Let $r_i = \left(s_{a_i}, h_{b_i}\right)$ $(i = 1, 2, \cdots, n)$ be 2DLVs of a 2DL-LIA $S \times H$, where $S = \left\{s_0, s_1, \cdots, s_g\right\}$ and $H = \{h_0, h_1, \cdots, h_t\}$, and $p, q \geqslant 0$. Then $2DLBMA^{p,q}(r_1, \cdots, r_n)$*

$$= \left(s_{g\left(1-\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-\frac{a_i^p a_j^q}{g^{p+q}}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}, h_{\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1,\\ i\neq j}}^{n} b_i^p b_j^q\right)^{\frac{1}{p+q}}} \right).$$

Proof. The proof of Theorem 4 includes three steps, which are given in the following.

Step 1 By Definition 4,

$$r_i^p \otimes r_j^q$$

$$= \left(s_{g\left(\frac{a_i}{g}\right)^p}, h_{b_i^p} \right) \otimes \left(s_{g\left(\frac{a_j}{g}\right)^q}, h_{b_j^q} \right)$$

$$= \left(s_{g\left(\frac{a_i}{g}\right)^p\left(\frac{a_j}{g}\right)^q}, h_{b_i^p b_j^q} \right).$$

then by Proposition 3, we have

$$\bigoplus_{\substack{j=1,\\ j\neq i}}^{n} \left(r_i^p \otimes r_j^q\right)$$

$$= \left(s_{g\left(1-\prod_{\substack{j=1,\\ j\neq i}}^{n}\left(1-\frac{a_i^p a_j^q}{g^{p+q}}\right)\right)}, h_{\sum_{\substack{j=1,\\ j\neq i}}^{n} b_i^p b_j^q} \right).$$

It follows that

$$\bigoplus_{\substack{i,j=1,\\ j\neq i}}^{n} \left(r_i^p \otimes r_j^q\right)$$

$$= \bigoplus_{i=1}^{n} \left(\bigoplus_{\substack{j=1,\\ j\neq i}}^{n} \left(r_i^p \otimes r_j^q\right) \right)$$

$$= \bigoplus_{i=1}^{n} \left(s_{g\left(1-\prod_{\substack{j=1,\\ j\neq i}}^{n}\left(1-\frac{a_i^p a_j^q}{g^{p+q}}\right)\right)}, h_{\sum_{\substack{j=1,\\ j\neq i}}^{n} b_i^p b_j^q} \right)$$

$$= \left(s_{g\left(1-\prod_{\substack{i,j=1,\\ j\neq i}}^{n}\left(1-\frac{a_i^p a_j^q}{g^{p+q}}\right)\right)}, h_{\sum_{\substack{i,j=1,\\ j\neq i}}^{n} b_i^p b_j^q} \right).$$

Step 2 By Definition 4, we can obtain that

$$\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \left(r_i^p \otimes r_j^q\right)$$

$$= \frac{1}{n(n-1)} \left(s_{g\left(1-\prod_{\substack{i,j=1,\\ j\neq i}}^{n}\left(1-\frac{a_i^p a_j^q}{g^{p+q}}\right)\right)}, h_{\sum_{\substack{i,j=1,\\ j\neq i}}^{n} b_i^p b_j^q} \right)$$

$$= \left(s_{g\left(1-\prod_{\substack{i,j=1,\\ j\neq i}}^{n}\left(1-\frac{a_i^p a_j^q}{g^{p+q}}\right)\right)^{\frac{1}{n(n-1)}}}, h_{\frac{1}{n(n-1)}\sum_{\substack{i,j=1,\\ j\neq i}}^{n} b_i^p b_j^q} \right).$$

Step 3 Based on Step 2 and Definition 4, we have

$$2DLBMA^{p,q}(r_1,\cdots,r_n)$$

$$= \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \left(r_i^p \otimes r_j^q\right) \right)^{\frac{1}{p+q}}$$

$$= \left(s_{g\left(1-\left(\prod_{\substack{i,j=1,\\ j\neq i}}^{n}\left(1-\frac{a_i^p a_j^q}{g^{p+q}}\right)\right)^{\frac{1}{n(n-1)}}\right)}, h_{\frac{1}{n(n-1)}\sum_{\substack{i,j=1,\\ j\neq i}}^{n} b_i^p b_j^q} \right)^{\frac{1}{p+q}}$$

$$= \left(s_{g\left(1-\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-\frac{a_i^p a_j^q}{g^{p+q}}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}, h_{\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1,\\ i\neq j}}^{n} b_i^p b_j^q\right)^{\frac{1}{p+q}}} \right).$$

Next, we can prove that the 2*DLBMA* operator satisfies the properties including idempotency, monotonicity, commutativity and boundedness. These properties are important to investigate the applications of 2DLVs.

Theorem 5. *(Idempotency) Let* $r_i = \left(s_{a_i}, h_{b_i}\right)$ $(i = 1, 2, \cdots, n)$ *be 2DLVs of a 2DL-LIA* $S \times H$*, where* $S = \left\{s_0, s_1, \cdots, s_g\right\}$ *and* $H = \{h_0, h_1, \cdots, h_t\}$*, and* $p, q \geqslant 0$*. If* $r_i = r = (s_a, h_b)$ *for all i, then*

$$2DLBMA^{p,q}(r_1,\cdots,r_n) = r.$$

Proof. Because $r_i = r = (s_a, h_b)$ for all i, from Definition 5 and Proposition 1, we have

$$2DLBMA^{p,q}(r_1,\cdots,r_n)$$

$$= \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \left(r_i^p \otimes r_j^q\right) \right)^{\frac{1}{p+q}}$$

$$= \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} r^{p+q} \right)^{\frac{1}{p+q}}$$

$= r.$

Theorem 6. *(Monotonicity) Let* $r_i = \left(s_{a_i}, h_{b_i}\right)$ *and* $r'_i = \left(s_{a'_i}, h_{b'_i}\right)$ $(i = 1, 2, \cdots, n)$ *be 2DLVs of a 2DL-LIA* $S \times H$*, where* $S = \left\{s_0, s_1, \cdots, s_g\right\}$ *and* $H = \{h_0, h_1, \cdots, h_t\}$*, and* $p, q \geqslant 0$*. If* $r_i \leqslant r'_i$ *for all* i*, then*

$$2DLBMA^{p,q}(r_1, \cdots, r_n) \leqslant 2DLBMA^{p,q}\left(r'_1, \cdots, r'_n\right).$$

Proof. Because $r_i \leqslant r'_i$ for all i, from Definition 5 and Proposition 2, we have

$$2DLBMA^{p,q}(r_1, \cdots, r_n) \leqslant 2DLBMA^{p,q}\left(r'_1, \cdots, r'_n\right).$$

Theorem 7. *(Commutativity) Let* $r_i = \left(s_{a_i}, h_{b_i}\right)$ $(i = 1, 2, \cdots, n)$ *be 2DLVs of a 2DL-LIA* $S \times H$*, where* $S = \left\{s_0, s_1, \cdots, s_g\right\}$ *and* $H = \{h_0, h_1, \cdots, h_t\}$*, and* $p, q \geqslant 0$*. If* $(\tilde{r}_1, \cdots, \tilde{r}_n)$ *is a permutation of* $(r_1, \cdots, r_n)$*, then*

$$2DLBMA^{p,q}(r_1, \cdots, r_n) = 2DLBMA^{p,q}(\tilde{r}_1, \cdots, \tilde{r}_n).$$

Proof. According to Proposition 1 (4), the conclusion can be obtained.

Theorem 8. *(Boundedness) Let* $r_i = \left(s_{a_i}, h_{b_i}\right)$ *and* $r'_i = \left(s_{a'_i}, h_{b'_i}\right)$ $(i = 1, 2, \cdots, n)$ *be 2DLVs of a 2DL-LIA* $S \times H$*, where* $S = \left\{s_0, s_1, \cdots, s_g\right\}$ *and* $H = \{h_0, h_1, \cdots, h_t\}$*, and* $p, q \geqslant 0$*. Then*

$$(s_0, h_0) = r_0 \leqslant 2DLBMA^{p,q}(r_1, \cdots, r_n) \leqslant r^* = \left(s_g, h_t\right).$$

Proof. Since for all i, $r_0 \leqslant r_i \leqslant r^*$, then by Theorem 5 and Theorem 6, we have

$$(s_0, h_0) = r_0 = 2DLBMA^{p,q}(r_0, \cdots, r_0)$$

$$\leqslant 2DLBMA^{p,q}(r_1, \cdots, r_n)$$

$$\leqslant 2DLBMA^{p,q}(r^*, \cdots, r^*) = r^* = \left(s_g, h_t\right).$$

Subsequently, some special cases of the $2DLBMA$ operator are discussed as follows:

Case 1 If $q = 0$, then from Definition 5, we have

$$2DLBMA^{p,0}(r_1, \cdots, r_n)$$

$$= \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \left(r_i^p \otimes r_j^0\right) \right)^{\frac{1}{p+q}}$$

$$= \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \left(r_i^p \otimes \left(s_g, h_1\right)\right) \right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{n} \bigoplus_{i=1}^{n} r_i^p \right)^{\frac{1}{p}},$$

which is called the 2-dimension linguistic generalized mean aggregation operator.

Case 2 If $p = 1, \quad q = 0$, then from Definition 5 and Case 1, we have

$$2DLBMA^{1,0}(r_1, \cdots, r_n) = \frac{1}{n} \bigoplus_{i=1}^{n} r_i,$$

which is called the 2-dimension linguistic mean aggregation (2DLMA) operator.

Case 3 If $p \to 0, \quad q = 0$, then from Definition 5, we have

$$\lim_{p\to 0} 2DLBMA^{p,0}(r_1, \cdots, r_n) = \left(\prod_{i=1}^{n} r_i \right)^{\frac{1}{n}},$$

which is called the 2-dimension linguistic geometric mean aggregation (2DLGMA) operator.

4.2. 2DLWBMA Operator

In real MAGDM environment, the weights of attributes are usually different, hence the $2DLWBMA$ operator will be discussed in the following.

Definition 6. Let $r_i = \left(s_{a_i}, h_{b_i}\right)$ $(i = 1, 2, \cdots, n)$ be 2DLVs of a 2DL-LIA $S \times H$, where $S = \left\{s_0, s_1, \cdots, s_g\right\}$ and $H = \{h_0, h_1, \cdots, h_t\}$, and $p, q \geqslant 0$. Then the mapping $2DLWBMA^{p,q} : \Omega^n \to \Omega$ such that

$$2DLWBMA^{p,q}(r_1, \cdots, r_n)$$

$$= \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \left(\beta_i^p \otimes \beta_j^q\right) \right)^{\frac{1}{p+q}},$$

is called the 2-dimension linguistic weight Bonferroni mean aggregation (2DLWBMA) operator, where $\beta_i = nw_i r_i$, $w = (w_1, \cdots, w_n)^T$ is the weight vector of r_i satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Obviously, if the weight vector w is equal to $\left(\frac{1}{n}, \cdots, \frac{1}{n}\right)$, then the 2DLWBMA operator is degenerated to the 2DLBMA operator.

Example 3. Let $\left(S \times H, \vee, \wedge, \rightarrow, '\right)$ be a 2DL-LIA, where $S = \{s_0, s_1, s_2, s_3, s_4\}$, $H = \{h_0, h_1, h_2, h_3, h_4, h_5, h_6\}$. Let $r_1 = (s_3, h_5), r_2 = (s_3, h_3), r_3 = (s_1, h_5), r_4 = (s_2, h_4)$, $w = (0.2, 0.27, 0.3, 0.23)$ and $p = 1, q = 1$.

According to Definition 6, we can compute that

$$2DLWBMA^{1,1}(r_1, r_2, r_3, r_4) = (s_{2.21}, h_{4.19}).$$

The following theorem shows that the aggregated result of 2DLVs by the $2DLWBMA$ operator is still a 2DLV.

Theorem 9. *Let* $r_i = \left(s_{a_i}, h_{b_i}\right)$ $(i = 1, 2, \cdots, n)$ *be 2DLVs of a 2DL-LIA* $S \times H$*, where* $S = \left\{s_0, s_1, \cdots, s_g\right\}$ *and* $H = \{h_0, h_1, \cdots, h_t\}$*,* $w = (w_1, \cdots, w_n)^T$ *be the weight vector of* r_i *satisfying* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1$*,* $p, q \geqslant 0$*. Then*

$$2DLWBMA^{p,q}(r_1,\cdots,r_n)$$

$$=\left(s_{g\left(1-\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)^{p}\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}},\ h_{\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1,\\ i\neq j}}^{n}(w_ib_i)^p(w_jb_j)^q\right)^{\frac{1}{p+q}}}\right).$$

Proof. This proof includes three steps.

Step 1 By Definition 4,

$$(nw_ir_i)^p\otimes\left(nw_jr_j\right)^q$$

$$=\left(s_{g\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)},h_{w_ib_i}\right)^p\otimes\left(s_{g\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)},h_{nw_jb_j}\right)^q$$

$$=\left(s_{g\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)^p},h_{(nw_ib_i)^p}\right)\otimes\left(s_{g\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)^q},h_{\left(nw_jb_j\right)^q}\right)$$

$$=\left(s_{g\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)^p\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)^q},h_{(nw_ib_i)^p\left(nw_jb_j\right)^q}\right).$$

Then by Proposition 3 (1), we have

$$\bigoplus_{\substack{j=1,\\ j\neq i}}^{n}\left((nw_ir_i)^p\otimes\left(nw_jr_j\right)^q\right)$$

$$=\bigoplus_{\substack{j=1,\\ j\neq i}}^{n}\left(s_{g\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)^p\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)^q},h_{(nw_ib_i)^p\left(nw_jb_j\right)^q}\right)$$

$$=\left(s_{g\left(1-\prod_{\substack{j=1,\\ j\neq i}}^{n}\left(1-\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)^p\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)^q\right)\right)},h_{\sum_{\substack{j=1,\\ j\neq i}}^{n}(nw_ib_i)^p\left(nw_jb_j\right)^q}\right)$$

Further, according to Proposition 3 (1), we have

$$\bigoplus_{\substack{i,j=1,\\ j\neq i}}^{n}\left((nw_ir_i)^p\otimes\left(nw_jr_j\right)^q\right)$$

$$=\bigoplus_{i=1}^{n}\left(\bigoplus_{\substack{j=1,\\ j\neq i}}^{n}\left((nw_ir_i)^p\otimes\left(nw_jr_j\right)^q\right)\right)$$

$$=\bigoplus_{i=1}^{n}\left(s_{g\left(1-\prod_{\substack{j=1,\\ j\neq i}}^{n}\left(1-\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)^p\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)^q\right)\right)},h_{\sum_{\substack{j=1,\\ j\neq i}}^{n}(nw_ib_i)^p(nw_jb_j)^q}\right)$$

$$= \left(s_{g\left(1-\prod_{\substack{i,j=1,\\ j\neq i}}^{n}\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)^{p}\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)^{q}\right)}, h_{\sum_{\substack{i,j=1,\\ j\neq i}}^{n}(nw_ib_i)^p(nw_jb_j)^q} \right).$$

Step 2 By Definition 4, we can obtain that

$$\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \left((nw_ir_i)^p \otimes \left(nw_jr_j\right)^q\right)$$

$$= \frac{1}{n(n-1)} \left(s_{g\left(1-\prod_{\substack{i,j=1,\\ j\neq i}}^{n}\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)^{p}\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)^{q}\right)}, h_{\sum_{\substack{i,j=1,\\ j\neq i}}^{n}(nw_ib_i)^p(nw_jb_j)^q} \right)$$

$$= \left(s_{g\left(1-\left(\prod_{\substack{i,j=1,\\ j\neq i}}^{n}\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)^{p}\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)}, h_{\frac{1}{n(n-1)}\sum_{\substack{i,j=1,\\ j\neq i}}^{n}(nw_ib_i)^p(nw_jb_j)^q} \right).$$

Step 3 Based on Step 2 and Definition 4, we have

$2DLWBMA^{p,q}(r_1, \cdots, r_n)$

$$= \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \left((nw_ir_i)^p \otimes \left(nw_ir_j\right)^q\right) \right)^{\frac{1}{p+q}}$$

$$= \left(s_{g\left(1-\left(\prod_{\substack{i,j=1,\\ j\neq i}}^{n}\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)^{p}\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)}, h_{\frac{1}{n(n-1)}\sum_{\substack{i,j=1,\\ j\neq i}}^{n}(nw_ib_i)^p(nw_jb_j)^q} \right)^{\frac{1}{p+q}}$$

$$= \left(s_{g\left(1-\left(\prod_{\substack{i,j=1,\\ j\neq i}}^{n}\left(1-\left(1-\frac{a_i}{g}\right)^{nw_i}\right)^{p}\left(1-\left(1-\frac{a_j}{g}\right)^{nw_j}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}}, \; h_{\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1,\\ i\neq j}}^{n}(w_i b_i)^p (w_j b_j)^q\right)^{\frac{1}{p+q}}} \right).$$

5. A MAGDM METHOD BASED ON THE 2DLWBMA OPERATOR

In a MAGDM problem, let $A = \{A_1, \cdots, A_n\}$ be a set of alternatives and $C = \{C_1, \cdots, C_m\}$ be a set of attributes, where A_i denotes the ith alternative and C_j denotes the jth attribute. Let $D = \{D_1, \cdots, D_l\}$ be a set of experts, where D_k denotes the kth expert. Let $w = (w_1, \cdots, w_m)^T$ be the weight vector of the attributes and $\gamma = (\gamma_1, \cdots, \gamma_l)^T$ be the weight vector of the experts, where $0 \leqslant w_j \leqslant 1$, $\sum_{j=1}^{m} w_j = 1$ and $0 \leqslant \gamma_k \leqslant 1$, $\sum_{k=1}^{l} \gamma_k = 1$. Suppose that there is a given 2DL-LIA $S \times H$, where $S = \left\{s_0, s_1, \cdots, s_g\right\}$ and $H = \{h_0, h_1, \cdots, h_t\}$ be two LTSs.

The expert D_k expresses his (or her) evaluation on the alternative A_i with respect to the attribute C_j by a 2DLV $r_{ij}^k = \left(s_{a_{ij}}^k, h_{b_{ij}}^k\right)$, which is an element of $S \times H$. Suppose $R^k = \left(r_{ij}^k\right)_{n\times m}$ is the 2DLV decision matrix, where $r_{ij}^k = \left(s_{a_{ij}}^k, h_{b_{ij}}^k\right) \in S \times H$ means that the expert D_k gives the evaluation value for the alternative A_i with respect to the attribute C_j. Then, the ranking order of alternatives is required.

The proposed method involves the following steps:

Step 1 Normalize the 2DLV decision matrix.

Since the set of attributes $C = \{C_1, \cdots, C_m\}$ should be classified into the set of benefit attributes $C_{benefit}$ and the set of cost attributes C_{cost}. In order to achieve normalization, we generally transform the cost attribute values into benefit attribute values. Suppose $R^k = \left(r_{ij}^k\right)_{n\times m}$ is the normalized matrix of $Z^k = \left(z_{ij}^k\right)_{n\times m}$, where $z_{ij}^k = \left(s_{a_{ij}}^k, h_{b_{ij}}^k\right)$. Then, the standardizing method is described as follows [16]:

$$r_{ij}^k = \begin{cases} \left(s_{a_{ij}}^k, h_{b_{ij}}^k\right), & z_{ij}^k \in C_{benefit} \\ \left(s_{a_{ij}}^k, h_{t-b_{ij}}^k\right), & z_{ij}^k \in C_{cost} \end{cases},$$

where $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant l$.

Step 2 Calculate the collective decision information by the $2DLWBMA^{p,q}$ operator in the light of Definition 6.

Based on the normalized 2DLV decision matrixes $R^k, (k = 1, \cdots, l)$, utilize the $2DLWBMA^{p,q}$ operator to obtain the aggregated matrix $R = \left(r_{ij}\right)_{n\times m}$, where

$$r_{ij} = \left(s_{a_{ij}}, h_{b_{ij}}\right) = 2DLWBMA^{p,q}\left(r_{ij}^1, \cdots, r_{ij}^l\right) =$$

$$\left(s_{g\left(1-\left(\prod_{\substack{k_1,k_2=1,\\ k_1\neq k_2}}^{l}\left(1-\left(1-\frac{a_{ij}^{k_1}}{g}\right)^{l\gamma_{k_1}}\right)^{p}\left(1-\left(1-\frac{a_{ij}^{k_2}}{g}\right)^{l\gamma_{k_2}}\right)^{q}\right)^{\frac{1}{l(l-1)}}\right)^{\frac{1}{p+q}}}, \; h_{\left(\frac{1}{l(l-1)}\sum_{\substack{k_1,k_2=1,\\ k_1\neq k_2}}^{l}(\gamma_{k_1} b_{ij}^{k_1})^p (\gamma_{k_2} b_{ij}^{k_2})^q\right)^{\frac{1}{p+q}}} \right),$$

where $p, q \geqslant 0, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$.

Step 3 Calculate the overall evaluation value of each alternative base on the $2DLWBMA^{p,q}$ operator.

Based on the aggregated 2DLV decision matrix $R = \left(r_{ij}\right)_{n\times m}$, utilize the $2DLWBMA^{p,q}$ operator to compute the overall evaluation values $r_i = \left(s_{a_i}, h_{b_i}\right)$ as follows:

$$r_i = \left(s_{a_i}, h_{b_i}\right) = 2DLWBMA^{p,q}\left(r_{i1}, \cdots, r_{im}\right) = \left(s_{g\left(1-\left(\prod_{\substack{j_1,j_2=1,\\ j_1\neq j_2}}^{m}\left(1-\left(1-\left(1-\frac{a_{ij_1}}{g}\right)^{mw_{j_1}}\right)^{p}\left(1-\left(1-\frac{a_{ij_2}}{g}\right)^{mw_{j_2}}\right)^{q}\right)\right)^{\frac{1}{m(m-1)}}\right)^{\frac{1}{p+q}}}, h_{\left(\frac{1}{m(m-1)}\sum_{\substack{j_1,j_2=1,\\ j_1\neq j_2}}^{m}(w_{j_1}b_{ij_1})^{p}(w_{j_2}b_{ij_2})^{q}\right)^{\frac{1}{p+q}}}\right).$$

where $p, q \geqslant 0, 1 \leqslant i \leqslant n$.

Step 4 Compare the obtained overall evaluation values $r_i = \left(s_{a_i}, h_{b_i}\right)$ according to Definition 2.

Step 5 Rank all the alternatives or select the best one(s) in accordance with the ranking of r_i. The prior the r_i is, the best the alternative A_i is.

Step 6 Ends.

6. EXAMPLES ILLUSTRATIONS

In this section, we use the proposed MAGDM method based on the $2DLWBMA$ operator to solve the technological innovation ability evaluation problem [23] and the landfill site selection problem [49]. It shows that the effectiveness and advantage of the proposed method.

6.1. Application to the Technological Innovation Ability Evaluation Problem

For the technological innovation ability evaluation problem cited from literature [23], the proposed MAGDM method based on the $2DLWBMA$ operator is applied to rank the candidate enterprises.

Example 4. [23] A practical use of the proposed approach involves the technological innovation ability evaluation of the four enterprises $\{A_1, A_2, A_3, A_4\}$. The set of four attributes is $C = \{C_1, C_2, C_3, C_4\}$, which means that the ability of innovative resources input C_1, the ability of innovation management C_2, the ability of innovation tendency C_3, the ability of research and development C_4. The weight vector of the attributes is $\omega = \{0.25, 0.27, 0.25, 0.23\}^T$.

Let $\left(S \times H, \vee, \wedge, \rightarrow, '\right)$ be a 2DL-LIA, where $S = \{s_0, s_1, s_2, s_3, s_4\}$, $H = \{h_0, h_1, h_2, h_3, h_4, h_5, h_6$. Three experts D_k $(k = 1, 2, 3)$ are respectively required to evaluate the alternatives A_i on attributes C_j with 2DLVs $r_{ij}^k = \left(s_{a_{ij}}^k, h_{b_{ij}}^k\right)$, where $s_{a_{ij}}^k \in S, h_{b_{ij}}^k \in H$. The weight vector of the experts is $\gamma = \{0.4, 0.28, 0.32\}$.

The 2DLV decision matrixes $R^k = \left(r_{ij}^k\right)_{4\times4}$ given by three experts are listed as follows:

$$R^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} (s_3,h_5) & (s_3,h_3) & (s_4,h_5) & (s_2,h_4) \\ (s_3,h_4) & (s_3,h_5) & (s_4,h_3) & (s_2,h_4) \\ (s_3,h_3) & (s_3,h_4) & (s_4,h_4) & (s_2,h_5) \\ (s_3,h_6) & (s_3,h_2) & (s_4,h_3) & (s_2,h_4) \end{pmatrix} \end{array}$$

$$R^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} (s_4,h_4) & (s_3,h_4) & (s_3,h_4) & (s_4,h_6) \\ (s_4,h_5) & (s_3,h_3) & (s_3,h_5) & (s_4,h_3) \\ (s_4,h_4) & (s_3,h_4) & (s_3,h_3) & (s_4,h_4) \\ (s_4,h_5) & (s_3,h_5) & (s_3,h_2) & (s_4,h_4) \end{pmatrix} \end{array}$$

$$R^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} (s_3,h_5) & (s_2,h_3) & (s_4,h_4) & (s_1,h_5) \\ (s_3,h_4) & (s_2,h_5) & (s_4,h_2) & (s_1,h_3) \\ (s_3,h_4) & (s_2,h_5) & (s_4,h_1) & (s_1,h_4) \\ (s_3,h_3) & (s_2,h_3) & (s_4,h_5) & (s_1,h_5) \end{pmatrix} \end{array}.$$

6.1.1. Illustration of the proposed method

The proposed method based on the $2DLWBMA^{p,q}$ operator is applied to rank the alternatives A_i $(i = 1, 2, 3, 4)$ in Example 4. The main process is shown as follows:

Step 1 Normalize the 2DLV decision matrixes.

Since all the attributes are the benefit attributes, the evaluation values of 2DLVs do not need normalizing.

Step 2 Aggregate the evaluation values of each expert by the $2DLWBMA^{p,q}$ operator.

Based on the 2DLV decision matrixes $R^k = \left(r_{ij}\right)_{4\times4}$, $(k = 1, 2, 3)$, we utilize the $2DLWBMA^{1,0}$ operator to obtain the 2DLV aggregated matrix R as follows:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} (s_4,h_{4.72}) & (s_{2.75},h_{3.28}) & (s_4,h_{4.4}) & (s_4,h_{4.88}) \\ (s_4,h_{4.28}) & (s_{2.75},h_{4.44}) & (s_4,h_{3.24}) & (s_4,h_{3.4}) \\ (s_4,h_{3.6}) & (s_{2.75},h_{4.32}) & (s_4,h_{2.76}) & (s_4,h_{4.4}) \\ (s_4,h_{4.76}) & (s_{2.75},h_{3.16}) & (s_4,h_{3.36}) & (s_4,h_{4.32}) \end{pmatrix} \end{array}.$$

Step 3 Calculate the overall evaluation value of each alternative.

Based on the 2DLV aggregated decision matrix $R = \left(r_{ij}\right)_{4\times4}$, we utilize the $2DLWBMA^{p,q}$ operator to obtain the overall 2-dimension linguistic evaluation value r_i of the alternative A_i.

Here let $p = 3, q = 3$, then we can compute that $r_1 = (s_4, h_{4.31})$, $r_2 = (s_4, h_{3.92})$, $r_3 = (s_4, h_{3.83})$, $r_4 = (s_4, h_{3.91})$.

Step 4 Rank r_i according to Definition 2.

According to Definition 2, we can obtain the ranking order of r_i as $r_3 < r_4 < r_2 < r_1$.

Step 5 Rank all the alternatives in accordance with the ranking of r_i. The prior the r_i is, the best the alternative A_i is.

Thus the ranking order of the alternatives is $A_3 < A_4 < A_2 < A_1$, shown as Figure 1.

6.1.2. Exploration of the parameters' influence

Here in this example, for illustrating the influences of the parameters p and q on the ranking order of alternatives, we take different values p and q in the $2DLWBMA^{p,q}$ operator. The ranking order of alternatives solved by the the proposed method with different p and q, are shown in Table 2 and Figure 2.

From Table 2, we can see that the best alternative is always A_1 and the worst alternative is A_3, and the relationships between A_4 and A_2 may be different with various values p and q.

When $p = 1$ and $q = 0$, we obtain that A_2 is less than A_4, that is $A_2 < A_4$. In such case, the $2DLWBMA^{p,q}$ operator is degenerated to the $2DLWMA$ operator.

When $p \to 0$ and $q = 0$, we obtain that A_2 is less than A_4, that is $A_2 < A_4$. In such case, the $2DLWBMA^{p,q}$ operator is degenerated to the $2DLWGMA$ operator.

When $p = 1$ and $q = 1$, we obtain that $A_2 < A_4$. While in other cases, (including $p = 3, q = 3, p = 5, q = 5$ and $p = 10, q = 10$), we obtain that A_4 is less than A_2, that is $A_4 < A_2$.

According to the outcomes of Table 2, it shows that the $2DLWBMA^{p,q}$ operator plays a key role for considering the interrelationships between the attributes. Obviously, when $p \to 0, q = 0$ and $p = 1, q = 0$, we assume that the attributes are independent from each other. However, it can not guarantee that the attributes are always independent from each others in all decision making environment. In this example, the four attributes C_j may be pairwise correlated, and the degree of correlation can be regulated by the parameters p and q. If $p = 1, q = 1$, then the ranking order is still $A_3 < A_2 < A_4 < A_1$. While when $p = 3, q = 3, p = 5, q = 5$, or $p = 10, q = 10$, the ranking order is $A_3 < A_4 < A_2 < A_1$. It shows that the ranking order of alternatives changes with the values of the parameters p and q. In fact, the decision makers can select the values of the parameters p and q according to the real decision making conditions or their experience.

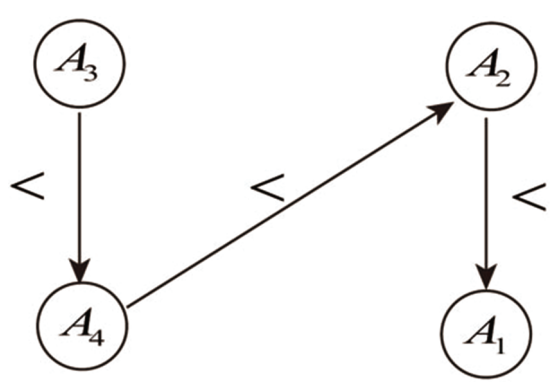

Figure 1 The ranking order of alternatives.

6.1.3. Sensitivity analysis of the weight vector

Sensitivity analysis of the weight vector, as an important part of MAGDM, has been developed to assess the stability of the ranking [16,50,51].

When we conduct the weight sensitivity analysis, only one criterion is focused at a time. That is, if a criterion changes, the other criteria are assumed to be changed uniformly in order to remain normalized.

Memariani *et al.* [51] provided a new method for the weight sensitivity analysis in MADM. Let $w = (w_1, w_2, \cdots, w_m)^T$ be the original weight vector of attributes. If the weight of the pth attribute changes from w_p to w'_p, then the new vector of weights transforms into $w' = \left(w'_1, w'_2, \cdots, w'_m\right)^T$. It can prove that $w'_j = \dfrac{1 - w'_p}{1 - w_p} w_j$, where $j = 1, 2, \cdots, m$ and $j \neq p$.

By use of Memariani's method [51], we only conduct the weight sensitivity analysis of attributes in Example 4. In addition, we only analyse the $2DLWBMA^{p,q}$ operator with $p = 3, q = 3$ in Step 3. The outcomes of the weight sensitivity analysis are shown in Table 3.

In Table 3, it indicates that the ranking order solved by the proposed method is resistant to change in Example 4 for the different stability intervals of weights. The weight stability interval of the attribute C_1 is in $[0.23, 0.26]$, C_2 in $[0.27, 0.29]$, C_3 in $[0.19, 0.28]$ and C_4 in $[0, 0.23]$.

6.1.4. Comparison analysis

In this subsection, different decision making approaches are applied to solve Example 4. These decision making approaches include respectively the decision making approach based on the generalized triangle fuzzy number (TFN) [15], the decision making approach based on the extended Interactive and Multiple Attribute Decision Making (TODIM) method [23], the decision making approach based on the power operator [20] and the decision making approach based on the 2DLV with two 2-tuples [17].

Since the decision making approach based on the generalized TFN [15] can only handle MADM problems in 2-dimension linguistic environment, it may as well select the second decision maker's decision matrix R^2 to rank the alternatives.

Moreover, the decision making approach based on the extended TODIM method [23] mainly considers 2DULVs. Since a 2DLV is the special case of the corresponding 2DULV, we can replace 2DULVs with 2DLVs in this 2-dimension linguistic computational model.

Similarly, the decision making approach based on the power operator [20] considers 2DULVs. we also replace 2DULVs with 2DLVs in this 2-dimension linguistic computational model.

Table 2 Ranking orders of the alternatives under different values of p and q.

The Values of p and q	The Evaluation Values r_i of Alternatives	Ranking Results
$p \to 0, q = 0$	$r_1 = (s_4, h_{4.26})$, $r_2 = (s_4, h_{3.80})$, $r_3 = (s_4, h_{3.70})$, $r_4 = (s_4, h_{3.84})$.	$A_3 < A_2 < A_4 < A_1$
$p = 1, q = 0$	$r_1 = (s_4, h_{4.29})$, $r_2 = (s_4, h_{3.86})$, $r_3 = (s_4, h_{3.77})$, $r_4 = (s_4, h_{3.88})$.	$A_3 < A_2 < A_4 < A_1$
$p = 1, q = 1$	$r_1 = (s_4, h_{4.28})$, $r_2 = (s_4, h_{3.84})$, $r_3 = (s_4, h_{3.75})$, $r_4 = (s_4, h_{3.86})$.	$A_3 < A_2 < A_4 < A_1$
$p = 3, q = 3$	$r_1 = (s_4, h_{4.31})$, $r_2 = (s_4, h_{3.92})$, $r_3 = (s_4, h_{3.83})$, $r_4 = (s_4, h_{3.91})$.	$A_3 < A_4 < A_2 < A_1$
$p = 5, q = 5$	$r_1 = (s_4, h_{4.34})$, $r_2 = (s_4, h_4)$, $r_3 = (s_4, h_{3.91})$, $r_4 = (s_4, h_{3.96})$.	$A_3 < A_4 < A_2 < A_1$
$p = 10, q = 10$	$r_1 = (s_4, h_{4.41})$, $r_2 = (s_4, h_{4.17})$, $r_3 = (s_4, h_{4.04})$, $r_4 = (s_4, h_{4.06})$.	$A_3 < A_4 < A_2 < A_1$

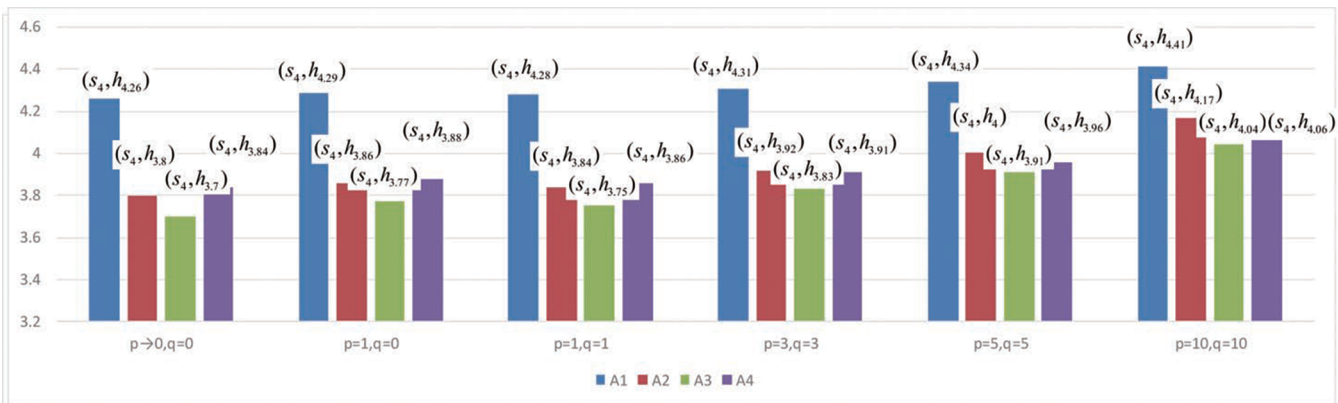

Figure 2 Ranking orders of the alternatives under different values of p and q.

Table 3 Sensitivity analysis for the weight vector of attributes.

Attribute	Original Weights	Stability Intervalsof Weight	
		Min	Max
C_1	0.25	0.23	0.26
C_2	0.27	0.27	0.29
C_3	0.25	0.19	0.28
C_4	0.23	0	0.23

Then we use the decision making approach based on the 2DLV with two 2-tuples [17] to solve Example 4, it should transform the 2DLV decision matrix into the 2DLV decision matrix with two 2-tuples.

The ranking results by different approaches are shown in Table 4.

From Table 4, the best alternative is A_1 and the worst alternative is A_3. While the ranking order of alternatives A_2 and A_4 may be different from various methods. Obviously, the ranking order of alternatives is the same by Yu's method [15], Zhu's method [17] and the proposed method with $p = 1, q = 0$. On the other hand, the ranking order of alternatives is the same by Liu's method [23], Liu's method [20] and the proposed method with $p = 3, q = 3$. The reasons are given in the following.

1. Because Yu's method [15], Zhu's method [17] and the proposed method with $p = 1, q = 0$ do not consider the correlations of attributes, it assume that the attributes are independent from each other. Hence the results solved by these three methods are the same.

Table 4 Ranking results by different methods.

Methods	Ranking Results
Yu's method [15] (based on the 2DLWA operator)	$A_3 < A_2 < A_4 < A_1$
Zhu's method [17] (based on the 2DLWAA operator)	$A_3 < A_2 < A_4 < A_1$
The proposed method with $p = 1, q = 0$	$A_3 < A_2 < A_4 < A_1$
Liu's method [23] (based on the extended TODIM method)	$A_3 < A_4 < A_2 < A_1$
Liu's method [20] (based on the power operator)	$A_3 < A_4 < A_2 < A_1$
The proposed method with $p = 3, q = 3$	$A_3 < A_4 < A_2 < A_1$

2DLWA = 2-dimension linguistic weighted averaging; 2DLWAA = 2-dimension linguistic weighted arithmetic aggregation.

2. Liu's method [20] based on the power operator considers the correlations of attributes. Moreover, Liu's method [23] based on the extended TODIM method considers the bounded rationality of decision makers. Furthermore, the proposed method can not only capture the correlations of attributes, but also describe the rationality of decision makers. Therefore, the results solved by Liu's method [20], Liu's method [23] and the proposed method with $p = 3, q = 3$ are the same.

In a word, the proposed method with 2DLVs for MAGDM problems has the following advantages. Firstly, the new operations of

2DLVs may be more reasonable than the existing operations. This new operations, that is, the *II* class LV of 2DLV is regarded as fuzzy number, can precisely describe the decision maker's self-assessment. Secondly, the proposed 2DLWBMA operator can not only capture the interrelationships between attributes, but also describe the rationality of decision makers in MAGDM problems. Moreover, the proposed 2DLWBMA operator can also deal with the attributes which are independent from each other. When taking special parameter values $p = 1$ and $q = 0$ (or $p = 0$ and $q = 1$), the 2DLWBMA operator can be degenerated to the 2-dimension linguistic weighted averaging (2DLWA) operator.

In fact, the decision makers can select suitable parameter values in the proposed method according to the actual decision situations and their experience or knowledge. In other words, taking different parameter values of p and q in the proposed method, the decision makers can flexibly express the degree of correlation between attributes.

6.2. Application to the Landfill Site Selection Problem

By use of the proposed MAGDM method, we can solve the landfill site selection problem which is cited from literature [49]. It concludes that the proposed method can not only handle MAGDM problems within 2-dimension linguistic environment, but also deal with MAGDM problems within fuzzy linguistic environment.

Example 5. [49] The practical example involves a landfill siting problem in KS City. Considering the dense population of KS City, there are four candidate locations A_1, A_2, A_3, A_4. The seven evaluation attributes for landfill site selection are transportation convenience (C_1), terrain suitability (C_2), community equity (C_3), environmental impact (C_4), ecological impact (C_5), construction cost (C_6) and historic impact (C_7) respectively, where the attributes C_1, C_2, C_3 are maximizing benefit attributes, and the attributes C_4, C_5, C_6, C_7 are minimizing cost attributes. The set of seven evaluation attributes is denoted by $C = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7\}$, with $C_{benefit} = \{C_1, C_2, C_3\}$ and $C_{cost} = \{C_4, C_5, C_6, C_7\}$.

The LTS $H = \{$Absolutely low (AL), Very low (VL),

Low (L), Medium low (ML), Medium(M),

Medium high (MH), High (H), Very High (VH),

Absolutely high (AH) $= \{h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8\}$ is applied to evaluate the four candidate locations A_1, A_2, A_3, A_4 with respective to the seven attributes $C_1, C_2, C_3, C_4, C_5, C_6, C_7$ by decision makers. The weight of the seven attributes, also expressed by linguistic terms in H, is $w = \{H, MH, H, MH, MH, M, ML\} = \{h_6, h_5, h_6, h_5, h_5, h_4, h_3\}$. The overall linguistic evaluation decision matrix $R = \left(r_{ij}\right)_{4\times7}$ is listed as follows:

$$R = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \end{array} \\ \begin{pmatrix} L & AH & AL & L & L & AH & AH \\ AL & VH & H & M & ML & ML & L \\ AH & AL & MH & AH & AH & M & VH \\ H & L & H & VH & MH & M & H \end{pmatrix} \end{array}.$$

6.2.1. Illustration of the proposed method

Example 5 is solved by the proposed method based on the $2DLWBMA^{p,q}$ operator to rank the candidate locations. The main process is shown as follows:

Step 1 Normalize the 2DLV decision matrix.

In order to apply the proposed MAGDM method to rank the four candidate locations, we assume that the decision maker selects the same linguistic term s_2 from the LTS $S = \{Notfamiliar, Familiar, Veryfamilar\} = \{s_0, s_1, s_2\}$ to give his (or her) self-assessment. Here we denote the original 2DLV decision making matrix as $Z = \left(z_{ij}\right)_{4\times7}$, where $z_{ij} = \left(s_2, h_j\right)$.

$$Z = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \end{array} \\ \begin{pmatrix} (s_2,h_2) & (s_2,h_8) & (s_2,h_0) & (s_2,h_2) & (s_2,h_2) & (s_2,h_8) & (s_2,h_8) \\ (s_2,h_0) & (s_2,h_7) & (s_2,h_6) & (s_2,h_4) & (s_2,h_3) & (s_2,h_3) & (s_2,h_2) \\ (s_2,h_8) & (s_2,h_0) & (s_2,h_5) & (s_2,h_8) & (s_2,h_8) & (s_2,h_4) & (s_2,h_7) \\ (s_2,h_6) & (s_2,h_2) & (s_2,h_6) & (s_2,h_7) & (s_2,h_5) & (s_2,h_4) & (s_2,h_6) \end{pmatrix} \end{array}$$

$$R = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \end{array} \\ \begin{pmatrix} (s_2,h_2) & (s_2,h_8) & (s_2,h_0) & (s_2,h_6) & (s_2,h_6) & (s_2,h_0) & (s_2,h_0) \\ (s_2,h_0) & (s_2,h_7) & (s_2,h_6) & (s_2,h_4) & (s_2,h_5) & (s_2,h_5) & (s_2,h_6) \\ (s_2,h_8) & (s_2,h_0) & (s_2,h_5) & (s_2,h_0) & (s_2,h_0) & (s_2,h_4) & (s_2,h_1) \\ (s_2,h_6) & (s_2,h_2) & (s_2,h_6) & (s_2,h_1) & (s_2,h_3) & (s_2,h_4) & (s_2,h_2) \end{pmatrix} \end{array}.$$

Since the attributes C_4, C_5, C_6, C_7 are the minimizing cost attributes, we need to convert the 2DLV decision making matrix $Z = \left(z_{ij}\right)_{4\times7}$ into $R = \left(r_{ij}\right)_{4\times7}$, where the elements $\left(s_2, h_j\right)$ in the 4th, 5th, 6th and 7th columns of Z are changed into $\left(s_2, h_{8-j}\right)$.

The original 2DL decision matrix Z and the normalized 2DL decision matrix R are shown as follows:

Step 2 Calculate the overall evaluation value of each alternative.

Based on the normalized 2DLV decision matrix $R = \left(r_{ij}\right)_{4\times7}$, we utilize the $2DLWBMA^{p,q}$ operator to obtain the overall 2-dimension linguistic evaluation values r_i of the alternative A_i.

Here let $p = 3, q = 3$, then we can compute that $r_1 = (s_2, h_{5.02})$, $r_2 = (s_2, h_{5.26})$, $r_3 = (s_2, h_{4.84})$, $r_4 = (s_2, h_{4.71})$.

Step 3 Rank r_i according to Definition 2.

According to Definition 2, we can obtain the ranking order of r_i as $r_4 < r_3 < r_1 < r_2$.

Step 4 Rank all the alternatives and select the best one(s) in accordance with the ranking of r_i. The prior the r_i is, the best the alternative A_i is.

Thus the ranking order of the alternatives is $A_4 < A_3 < A_1 < A_2$, shown as Figure 3.

6.2.2. Exploration of the parameters influence

For illustrating the influences of the parameters p and q, we take different values p and q in the $2DLWBMA^{p,q}$ operator in this example. The ranking order of alternatives solved by the the proposed method with different p and q, is shown in Table 5.

From Table 5, we can see that the relationships among alternatives A_i $(i = 1, 2, 3, 4)$ change with various values p and q.

When the $2DLWBMA^{p,q}$ operator is degenerated to the $2DLWGMA$ operator, that is $p \to 0$ and $q = 0$, the ranking order is $A_4 < A_3 < A_2 < A_1$.

When the $2DLWBMA^{p,q}$ operator is degenerated to the $2DLWMA$ operator, that is $p = 1$ and $q = 0$, the ranking order of alternatives is $A_3 < A_1 < A_4 < A_2$, which is the same with when $p = 1$ and $q = 1$.

When $p = 3$ and $q = 3$, we obtain that $A_4 < A_3 < A_2 < A_1$. Similarly, when $p = 5$ and $q = 5$, the ranking order is $A_4 < A_2 < A_1 < A_3$. Finally, when $p = 10$ and $q = 10$, the ranking order is $A_2 < A_1 < A_4 < A_3$.

According to the outcomes of Table 5, it shows that the $2DLWBMA^{p,q}$ operator plays a key role for considering the interrelationship between the attributes. It shows that the ranking order of alternatives changes with the values of the parameters p and q.

6.2.3. Sensitivity analysis of the weight vector

In order to illustrate the influences of linguistic weight vector on the ranking results, a sensitivity analysis is conducted in Example 5.

In Subsection 6.2.3, we have discussed Memariani's method [51]. Applying Memariani's method, we can also normalize the linguistic weight vector of attributes.

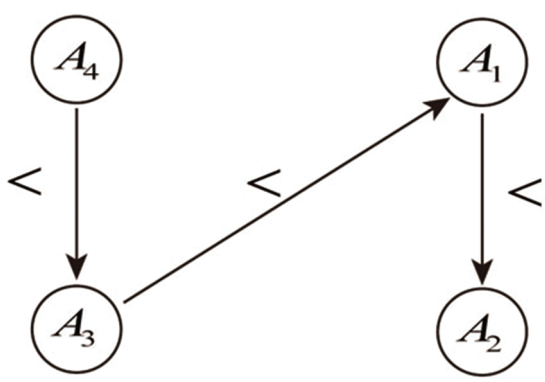

Figure 3 The ranking order of alternatives.

We only discuss the $2DLWBMA^{p,q}$ operator with $p = 1, q = 1$. Table 6 lists the outcomes of the sensitivity analysis for the linguistic weight vector.

From Table 6, we find that the ranking order solved by the proposed method is fully resistant to change in the importance of the attribute C_7. Moreover, we obtain that the linguistic weight stability intervals of attributes C_2, C_4, C_5 are the same interval $[h_3, h_8]$. While the linguistic weight stability intervals of attributes C_1, C_3, C_6 are different from each other: C_1 is in $[h_2, h_8]$, C_3 in $[h_4, h_8]$, C_6 in $[h_1, h_7]$.

7. CONCLUSIONS

Since a 2DLV adds a class of LV to express the decision maker's self-assessment, it can better express fuzzy information. In order to embody the innate character of the decision maker's self-assessment, this paper defined the new operations of 2DLVs. In the new operations of 2DLVs, the operations of the *II* class LV which represents the decision maker's self-assessment are based on fuzzy numbers. Moreover, BM operator has a prominent characteristic which can consider the interrelationships of decision arguments. Through adjusting the parameters p and q, the relationships of decision arguments can be flexibly described by BM operator. Based on BM operator and the new operational rules of 2DLVs, this paper developed the 2DLBMA operator. Further, the 2DLWBMA operator was introduced to consider the importance of attribute weights. Subsequently, a novel MAGDM method based on the 2DLWBMA operator was proposed to deal with the 2-dimension linguistic MAGDM problems evaluated by 2DLVs. Finally, two practical examples were given to illustrate the steps of the proposed method. Then the sensitivity analysis of attributes were conducted to obtain the stability intervals of weights in two examples. Moreover, the proposed method was compared with the other relevant methods to show the effectiveness and flexibility of the proposed method.

In future, based on this new operational rules of 2DLVs, other kinds of aggregation operators should be proposed to deal with the corresponding situations in real 2-dimension linguistic decision making. Moreover, the proposed 2DLWBMA operator should be considered to extend into other fuzzy environments for solving MAGDM problems.

Table 5 Ranking orders of the four candidate locations under different values of p and q.

The Values of p and q	The Evaluation Values r_i of Alternatives	Ranking Results
$p \to 0, q = 0$	$r_1 = (s_2, h_{4.84}), r_2 = (s_2, h_{5.75}),$ $r_3 = (s_2, h_{3.06}), r_4 = (s_2, h_{2.83}).$	$A_4 < A_3 < A_1 < A_2$
$p = 1, q = 0$	$r_1 = (s_2, h_{3.29}), r_2 = (s_2, h_{4.53}),$ $r_3 = (s_2, h_{2.85}), r_4 = (s_2, h_{3.65}).$	$A_3 < A_1 < A_4 < A_2$
$p = 1, q = 1$	$r_1 = (s_2, h_{3.01}), r_2 = (s_2, h_{4.43}),$ $r_3 = (s_2, h_{2.44}), r_4 = (s_2, h_{3.5}).$	$A_3 < A_1 < A_4 < A_2$
$p = 3, q = 3$	$r_1 = (s_2, h_{5.02}), r_2 = (s_2, h_{5.26}),$ $r_3 = (s_2, h_{4.84}), r_4 = (s_2, h_{4.71}).$	$A_4 < A_3 < A_1 < A_2$
$p = 5, q = 5$	$r_1 = (s_2, h_{5.7}), r_2 = (s_2, h_{5.68}),$ $r_3 = (s_2, h_{5.79}), r_4 = (s_2, h_{5.5}).$	$A_4 < A_2 < A_1 < A_3$
$p = 10, q = 10$	$r_1 = (s_2, h_{6.35}), r_2 = (s_2, h_{6.3}),$ $r_3 = (s_2, h_{6.71}), r_4 = (s_2, h_{6.37}).$	$A_2 < A_1 < A_4 < A_3$

Table 6 Sensitivity analysis for the weight vector of attributes.

Attribute	The OriginalLinguistic Weight	Stability Intervalsof Weight	
		Min	Max
C_1	H (h_6)	L (h_2)	AH (h_8)
C_2	MH (h_5)	ML (h_3)	AH (h_8)
C_3	H (h_6)	M (h_4)	AH (h_8)
C_4	MH (h_5)	ML (h_3)	AH (h_8)
C_5	MH (h_5)	ML (h_3)	AH (h_8)
C_6	M (h_4)	VL (h_1)	VH (h_7)
C_7	ML (h_3)	AL (h_0)	AH (h_8)

CONFLICT OF INTEREST

We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

AUTHORS' CONTRIBUTIONS

Jianbin Zhao did the formal analysis, editing, and Software. Hua Zhu was involved with the resources and writing original draft. Both the authors were involved with the conceptualization, methodology, validation, investigation and writing-review.

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