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Interval-Valued Probabilistic Dual Hesitant Fuzzy Sets for Multi-Criteria Group Decision-Making

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ABSTRACT

As a powerful extension to hesitant fuzzy sets (HFSs), dual hesitant fuzzy sets (DHFSs) have been closely watched by many scholars. The DHFSs can reflect the disagreement and hesitancy of decision-makers (DMs) flexibly and conveniently. However, all the evaluation values under the same membership degree are endowed with similar importance. And DHFSs are not able to express DMs' preference degrees on different variables. To overcome this drawback, in this paper, we propose the concept of interval-valued probabilistic dual hesitant fuzzy sets (IVPDHFSs) by providing each element with an interval-valued probability value, which can describe DMs' preferences, hesitancy and disapproval simultaneously. Then we define the basic operation laws, score function and deviation function for interval-valued probabilistic dual hesitant fuzzy elements (IVPDHFEs). Besides, the ordered distance and similarity measures are proposed to calculate the deviation of any two IVPDHFSs and to derive the weight vector for DMs objectively, respectively. To aggregate decision-making information, we present interval-valued probabilistic dual hesitant fuzzy ordered weighted averaging (IVPDHFOWA) operator. Moreover, the water-filling theory is first introduced into IVPDHFSs environment and utilized to obtain unified criteria weights mathematically. Furthermore, a three-phased multi-criteria group decision-making (MCGDM) framework is constructed to address IVPDHFSs information. Finally, a case study concerning Arctic risk evaluation is provided to verify the effectiveness and superiority of the proposed three-phased framework.

1. INTRODUCTION

The process of multi-criteria decision-making (MCDM) [1–5] is usually uncertain and complex in human activities. But how to deal with these vague issues is a challenging question. It is intractable to employ traditional techniques to cope with imprecise information. As a result, Zadeh [6] proposes the fuzzy sets (FSs) theory, which is generally recognized as a convenient model to describe imperfect and uncertain information [7]. FSs have only single membership function to describe the degree to which the given alternative satisfies the DMs. However, to express DMs' hesitancy on the performance of the given alternative, the corresponding models of FSs are limited. To alleviate this issue, Torra [8] introduces the concept of hesitant fuzzy sets (HFSs). Because HFSs allow several values to reflect the membership degree to which an element belongs to the given set, it is very suitable to use HFSs to express the hesitancy of DMs in the decision-making process.

Since HFSs have unique advantages, many scholars pay attention to the study of HFSs theory and obtain many achievements. On the theories of HFSs, Xu and Xia [9] investigate the aforementioned information measures and further define the ordered weighted distance and similarity measures for HFSs; Zhu *et al.* [10] apply proposed hesitant fuzzy geometric Bonferroni mean (HFGBM), hesitant fuzzy Choquet geometric Bonferroni mean (HFCGBM), weighted hesitant fuzzy geometric Bonferroni mean (WHFGBM) and weighted hesitant fuzzy Choquet geometric Bonferroni mean (WHFCGBM) to make MCDM; Zhang [11] develop a wide range of hesitant fuzzy power aggregation operators to aggregate input arguments that take the form of HFSs; Rodríguez *et al.* [12] make a position and perspective analysis of HFSs and a discussion about current proposals; Wei [13] proposes some prioritized aggregation operators to address MCDM problems in which the criteria are in different priority level; Xu and Zhang [14] develop a new approach based on TOPSIS and maximizing deviation method to handle hesitant fuzzy information. Moreover, there are some successful extensions of HFSs, such as hesitant fuzzy linguistic term sets [15,16], interval-valued HFSs [17,18], interval-valued hesitant fuzzy linguistic sets [19], interval-valued intuitionistic HFSs [20,21], DHFSs [22], generalized HFSs [23], triangular HFSs [24], probabilistic HFSs (PHFSs) [25] and interval-valued PHFSs (IVPHFSs) [26,27].

Among all the extensions of HFSs, DHFSs have gained wide popularity since it can depict the membership hesitancy degree and non-membership hesitancy degree in the domain simultaneously. Ye [28] proposes a correlation coefficient of DHFSs and apply it to solve MCDM problems. Su *et al.* [29] introduce the distance and similarity measures for DHFSs. Based on Archimedean t-conorm and t-norm, Wang *et al.* [30] present some dual hesitant fuzzy power aggregation operators for multiple criteria group decision-making

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(MCGDM). Singh [31] presents a new similarity measure and further proposes two algorithms to find the optional solution under DHFSs environments. Wang *et al.* [32] introduce generalized dual hesitant fuzzy Choquet ordered aggregation (GDHFCOA) operator for MCDM. Ren *et al.* [33] propose a comparison method to distinguish DHFSs efficiently and extend the VIKOR method into DHFSs environment for MCGDM. Yu and Li [34] apply the generalized dual hesitant fuzzy weighted averaging (GDHFWA) operator, the generalized dual hesitant fuzzy ordered weighted averaging (GDHFOWA) operator and the generalized dual hesitant fuzzy hybrid averaging (GDHFHA) operator to aggregate dual hesitant fuzzy information. Yu *et al.* [35] propose dual hesitant fuzzy Heronian mean operator and dual hesitant fuzzy geometric Heronian mean operator and utilize them to make group decision-making (GDM) for supplier selection. Zhao *et al.* [36] introduce dual hesitant fuzzy preference relation (DHFPR), which provides a powerful solution to describe the hesitant cognitions of DMs over some feasible alternatives. Ren and Wei [37] employ the proposed correctional score function and the dice similarity measure for DHFSs to address MCDM problems in which the attributes are in different priority levels.

Though these operators and methods can handle dual hesitant fuzzy information effectively, still the issue of DMs' preference on evaluation values has not been addressed. In the membership degree part or the non-membership degree part of DHFSs, all elements have the same importance or weight. Apparently, it is not in conformity in real life. DMs may prefer one element to another one due to the epistemic uncertainty. Up to now, probabilistic approaches are prevalent to model the aleatory uncertainty in terms of the statistical uncertainty but are unable to solve complex and fuzzy MCGDM problems. Thus, it is a hot academic issue that how to combine the randomness in mathematics and vagueness in complex MCDM problems efficiently, which motivates many scholars to make a large number of investigations.

On the whole, the work to incorporate probability theory into fuzzy sets theory can be roughly summarized into the following process: (i) introducing the probability theory and make it available in fuzzy sets theory; (ii) integrating the probability theory into fuzzy operation, measure and aggregation process and (iii) combined methods producing the probabilistic fuzzy values. Followed by this idea, the immediate probability was introduced into the fuzzy decision-making process [38,39]. The immediate probability information can reflect the attitudinal characteristics of DMs precisely and be properly considered as the weight information of the corresponding element in the aggregation process. To transform this incorporated theory into practical applications where the evaluation information is denoted by DHFSs, Hao *et al.* [40] propose the concept of probabilistic dual hesitant fuzzy sets (PDHFSs) and define the operational laws and some aggregation operators for PDHFSs. In the probabilistic dual hesitant fuzzy element (PDHFE), each evaluation value is endowed with an occurring probability to express the confidence and preference of DMs. Meanwhile, Zeng *et al.* [41] present the concept of weighted dual hesitant fuzzy sets (WDHFSs) and weighted dual hesitant fuzzy element (WDHFE) and provide a GDM method under DHFSs environment. For instance, an expert is invited to assess the performance of a central processing unit (CPU) manufacturing company over the technical ability attribute, he/she provide a dual hesitant fuzzy element (DHFE) $\langle (0.4, 0.6, 0.8), (0.2, 0.3) \rangle$ due to the hesitancy character. Nevertheless, he/she has more confidence in the value 0.6 in the satisfaction degree function and the value 0.2 in the dissatisfaction degree function, which is beyond the scope of DHFSs. But PDHFSs and WDHFSs can describe this preference information perfectly. Supposing the weights for $0.4, 0.6, 0.8$ in the membership degree are $0.2, 0.6, 0.2$ respectively and the weights for $0.2, 0.3$ in the non-membership degree are $0.7, 0.3$ respectively, his/ her comment on this statement can be denoted as $\langle (0.4\,(0.2), 0.6\,(0.6), 0.8\,(0.2)), (0.2\,(0.7), 0.3\,(0.3)) \rangle$.

However, we may ignore the fact that DMs are unable to give precise probability preference information for their comments. In some real scenarios, DMs may estimate the preference degree of a certain membership value using linguistic form or interval format. It is unreasonable and irrational to utilize PDHFSs and WDHFSs to express linguistic or interval preference information. As a result, we propose the concept of interval-valued probabilistic dual hesitant fuzzy sets (IVPDHFSs), in which the occurring probability of each element in satisfaction and dissatisfaction degrees is extended to a range covering lower and upper limit values. By contrast, IVPDHFSs are quite suitable to reflect the uncertain preference degree in the decision-making process. Some motivations for this research are summarized as follows:

1. PDHFSs cannot express DMs' hesitant probabilistic preference. Motivated by this weakness, we are devoted to presenting a new concept called IVPDHFSs, which allocates each element with an interval-valued probability value.
2. Motivated by the rationality and consistency of aggregation operator, efforts are made to utilize ordered weight averaging operator to fuse IVPDHFSs information.
3. Motivated by the effectiveness and practicability of score function and deviation function of HFSs, contributions are made to extend score and deviation functions in IVPDHFSs environment.
4. Motivated by the risk preference character of DMs, we propose ordered distance and similarity measures to calculate the difference of any two IVPDHFSs.
5. A large number of models for deriving the criteria weights rarely take the criteria dimensions into account. Motivated by the power of water-filling theory, the achievement is made to remove the influence of criteria dimensions and magnitude the criteria into a consistent scale.
6. Motivated by the efficiency and flexibility of interval-valued probabilistic preference, we regard IVPDHFSs as a novel basic theory and further put forward a three-phased MCGDM framework under IVPDHFSs environment.

Some main contributions of this paper are presented below:

1. We define a new concept of IVPDHFSs so as to describe the hesitant probabilistic preference of DMs.
2. We propose the operational laws for IVPDHFEs and further present the IVPDHFOWA operator to make information fusion.
3. The score function and deviation function is defined to make a simple comparison of any two IVPDHFEs.

4. We present the ordered distance measure to compute the difference of any IVPDHFEs and introduce the ordered similarity measure to derive the weight vector of DMs.
5. The water-filling theory is first introduced into IVPDHFSs environment, and based on this theory, we construct a mathematical model to derive the criteria weights, which eliminates the impact of criteria dimensions.
6. A three-phased MCGDM framework is conceived to handle IVPDHFSs information.

The organization of this paper is constructed as follows. In Section 2, we review some definitions of HFSs, DHFSs and PDHFSs. In Section 3, we give a series of concepts of IVPDHFSs and propose the operational laws and comparison method for IVPDHFEs. Besides, the ordered distance and similarity measures and IVPDHFOWA operator are also presented. In Section 4, we propose a three-phased MCGDM framework within IVPDHFSs. In Section 5, we make a case study to verify the validity of the proposed three-phased framework. Finally, a conclusion is provided in Section 6.

2. PRELIMINARIES

In this section, we review some conceptions related to HFSs, DHFSs and PDHFSs.

2.1. Hesitant Fuzzy Sets

Definition 1. [8] An HFS H on the reference set X is defined in terms of a membership function $h(x)$ that returns a subset of $[0,1]$ when applied to X.

Xia and Xu [42] provide the mathematical symbol of HFS as follows:

$$H = \{\langle x, h(x)\rangle | x \in X\}, \tag{1}$$

where $h(x)$ is a set of several possible values in interval $[0,1]$. For convenience, $h(x)$ is called a hesitant fuzzy element (HFE).

2.2. Dual Hesitant Fuzzy Sets

Definition 2. [22] A DHFS D on the reference set X is defined in terms of the membership hesitancy function $h(x)$ and non-membership hesitancy function $g(x)$ that both return a set of $[0,1]$ when applied to X. Mathematically, it can be expressed by the following symbol:

$$D = \{\langle x, h(x), g(x)\rangle | x \in X\}, \tag{2}$$

where $h(x)$ and $g(x)$ satisfy the following conditions: $\forall x \in X, \gamma \geq 0$, $\eta \leq 1, 0 \leq \gamma^+ + \eta^+ \leq 1, \gamma \in h(x), \eta \in g(x), \gamma^+ = \bigcup_{\gamma \in h(x)} \max\{\gamma\}$ and $\eta^+ = \bigcup_{\eta \in g(x)} \max\{\eta\}$. For simplicity, the pair $\langle h(x), g(x)\rangle$ is called a dual hesitant fuzzy element (DHFE).

Zhu *et al.* [22] also give basic operations for DHFSs.

Definition 3. [22] Let $d = \{h, g\}$, $d_1 = \{h_1, g_1\}$ and $d_2 = \{h_2, g_2\}$ be any three DHFEs, then the operational laws are as follows:

$$\oplus - union: d_1 \oplus d_2 = \{h_1 \oplus h_2, g_1 \otimes g_2\} = \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2 \\ \eta_1 \in g_1, \eta_2 \in g_2}} \{(\gamma_1 + \gamma_2 - \gamma_1\gamma_2), (\eta_1\eta_2)\}; \tag{3}$$

$$\otimes - intersection: d_1 \otimes d_2 = \{h_1 \otimes h_2, g_1 \oplus g_2\} = \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2 \\ \eta_1 \in g_1, \eta_2 \in g_2}} \{(\gamma_1\gamma_2), (\eta_1 + \eta_2 - \eta_1\eta_2)\}; \tag{4}$$

$$\lambda d = \bigcup_{\gamma \in h, \eta \in g} \left\{1 - (1-\gamma)^\lambda, \eta^\lambda\right\}, \lambda \geq 0; \tag{5}$$

$$d^\lambda = \bigcup_{\gamma \in h, \eta \in g} \left\{\gamma^\lambda, 1 - (1-\eta)^\lambda\right\}, \lambda \geq 0. \tag{6}$$

2.3. Probabilistic Dual Hesitant Fuzzy Sets

Definition 4. [40] A PDHFS P on the reference set X is defined as the following symbol:

$$P = \left\{\langle x, h(x) | p(x), g(x) | q(x)\rangle | x \in X\right\}, \tag{7}$$

where $h(x)$ and $g(x)$ represent the membership hesitancy function and non-membership hesitancy function, respectively. $p(x)$ and $q(x)$ denote the corresponding single probability value for the elements in these two possible degrees. Besides, the two parts $h(x) | p(x)$ and $g(x) | q(x)$ satisfy the following conditions: $\forall x \in X$, $\gamma \geq 0, \eta \leq 1, 0 \leq \gamma^+ + \eta^+ \leq 1, \gamma \in h(x), \eta \in g(x), \gamma^+ = \bigcup_{\gamma \in h(x)} \max\{\gamma\}, \eta^+ = \bigcup_{\eta \in g(x)} \max\{\eta\}, p(x) \in [0,1], q(x) \in [0,1]$, $\sum_{p_i \in p(x)} p_i = 1$ and $\sum_{q_j \in q(x)} q_j = 1$. For the sake of convenience, the pair $\langle h(x) | p(x), g(x) | q(x)\rangle$ is called as a PDHFE.

Hao *et al.* [40] also give the basic operational laws for PDHFSs as follows:

Definition 5. [40] Let $P = \{h | p, g | q\}$, $P_1 = \{h_1 | p_1, g_1 | q_1\}$, $P_2 = \{h_2 | p_2, g_2 | q_2\}$ be three PDHFEs, then

$$P_1 \oplus P_2 = \{h_1 \oplus h_2, g_1 \otimes g_2\} = \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, p_{\gamma_1} \in p_1, p_{\gamma_2} \in p_2 \\ \eta_1 \in g_1, \eta_2 \in g_2, q_{\eta_1} \in q_1, q_{\eta_2} \in q_2}} \left\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2 | p_{\gamma_1}p_{\gamma_2}, \eta_1\eta_2 | q_{\eta_1}q_{\eta_2}\right\}; \tag{8}$$

$$P_1 \otimes P_2 = \{h_1 \otimes h_2, g_1 \oplus g_2\} = \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, p_{\gamma_1} \in p_1, p_{\gamma_2} \in p_2 \\ \eta_1 \in g_1, \eta_2 \in g_2, q_{\eta_1} \in q_1, q_{\eta_2} \in q_2}} \left\{\gamma_1\gamma_2 | p_{\gamma_1}p_{\gamma_2}, \eta_1 + \eta_2 - \eta_1\eta_2 | q_{\eta_1}q_{\eta_2}\right\}; \tag{9}$$

$$\lambda P = \bigcup_{\substack{\gamma \in h, p_r \in p \\ \eta \in g, q_\eta \in q}} \left\{1-(1-\gamma)^\lambda \,|p_r, \eta^\lambda|q_\eta\right\}, \lambda \geq 0; \quad (10)$$

$$P^\lambda = \bigcup_{\substack{\gamma \in h, p_r \in p \\ \eta \in g, q_\eta \in q}} \left\{\gamma^\lambda|p_r, 1-(1-\eta)^\lambda \,|q_\eta\right\}, \lambda \geq 0. \quad (11)$$

3. INTERVAL-VALUED PROBABILISTIC DUAL HESITANT FUZZY SETS

In this section, to describe the probabilistic hesitant information flexibly and reasonably, we propose the concept of IVPDHFSs and investigate basic operational laws and its comparison method. Besides, we define the ordered distance and similarity measures of generalized IVPDHFSs. To fuse information, the interval-valued probabilistic dual hesitant fuzzy ordered weighted averaging (IVPDHFOWA) operator is presented.

3.1. The Concept of IVPDHFSs

As has been discussed above, it is difficult for the DMs to give precise probabilistic preference degrees on their evaluation values. Sometimes, they prefer to use interval-valued probability to express their opinions instead of single-valued probability. Thus, we present the concept of IVPDHFSs as follows:

Definition 6. Let X be the reference set, an IVPDHFS on X is defined by the following expression:

$$D_{IVP} = \left\{\left\langle x, h(x) \,|\, \left[p^l(x), p^u(x)\right], g(x) \,|\, \left[q^l(x), q^u(x)\right]\right\rangle |x \in X\right\}. \quad (12)$$

The components $h(x) \,|\, \left[p^l(x), p^u(x)\right]$ and $g(x) \,|\, \left[q^l(x), q^u(x)\right]$ are the two distinct membership functions, in which $h(x)$ and $g(x)$ are the satisfaction function and dissatisfaction function of x to the set of X, respectively, $\left[p^l(x), p^u(x)\right]$ and $\left[q^l(x), q^u(x)\right]$ are the interval probabilistic preference values corresponding to membership functions $h(x)$ and $g(x)$, respectively, satisfying $\gamma \geq 0$, $\eta \leq 1$, $0 \leq \gamma^+ + \eta^+ \leq 1$, $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ = \bigcup_{\gamma \in h(x)} \max\{\gamma\}$, $\eta^+ = \bigcup_{\eta \in g(x)} \max\{\eta\}$, $0 \leq p^l_\gamma \leq 1$, $0 \leq p^u_\gamma \leq 1$, $p^l_\gamma \leq p^u_\gamma$, $0 \leq q^l_\eta \leq 1$, $0 \leq q^u_\eta \leq 1$, $q^l_\eta \leq q^u_\eta$, $\left[p^l_\gamma, p^u_\gamma\right] \in \left[p^l(x), p^u(x)\right]$, $\left[q^l_\eta, q^u_\eta\right] \in \left[q^l(x), q^u(x)\right]$, $\sum_{p^u_\gamma \in p^u(x)} p^u_\gamma \leq 1$ and $\sum_{q^u_\eta \in q^u(x)} q^u_\eta \leq 1$ for any $x \in X$.

For the sake of simplicity, the pair $\left\langle x, h(x) \,|\, \left[p^l(x), p^u(x)\right], g(x) \,|\, \left[q^l(x), q^u(x)\right]\right\rangle$ is called as interval-valued probabilistic dual hesitant fuzzy element (IVPDHFE), and simply denoted by $d_{IVP} = \left\{h| \left[p^l_h, p^u_h\right], g| \left[q^l_g, q^u_g\right]\right\}$.

In real problems, some evaluation values may be lost due to complex decision-making environment and unexpected ignorance behaviors. Thus, we provide the definition of generalized IVPDHFSs and discuss some special forms of IVPDHFSs.

Definition 7. Let $D_{IVP} = \left\{\left\langle x, h(x)|\left[p^l(x), p^u(x)\right], g(x)|\left[q^l(x), q^u(x)\right]\right\rangle |x \in X\right\}$ be an IVPDHFS, if $\#h(x) \geq 2$, $\#g(x) \geq 2$, $p^l(x) < p^u(x)$ and $q^l(x) < q^u(x)$, where $\#h(x)$ and $\#g(x)$ denote the total numbers of the elements in membership function $h(x)$ and non-membership function $g(x)$, respectively, then we call D_{IVP} generalized IVPDHFS. For short, generalized IVPDHFS is also denoted as D_{IVP}.

Remark 1

Let $D_{IVP} = \left\{\left\langle x, h(x) \,|\, \left[p^l(x), p^u(x)\right], g(x) \,|\, \left[q^l(x), q^u(x)\right]\right\rangle |x \in X\right\}$ be an IVPDHFS, if $\#h(x) \geq 2$, $\#g(x) \geq 2$, $p^l(x) = p^u(x)$ and $q^l(x) = q^u(x)$, where $\#h(x)$ and $\#g(x)$ denote the total numbers of the elements in membership function $h(x)$ and non-membership function $g(x)$, respectively, then IVPDHFS is reduced to PDHFS; if $\#h(x) \geq 2$, $g(x) = \varnothing$ and $p^l(x) < p^u(x)$, where $\#h(x)$ denotes the total number of the elements in membership function $h(x)$, then IVPDHFS is reduced to IVPHFS; if $\#h(x) \geq 2$, $g(x) = \varnothing$ and $p^l(x) = p^u(x)$, where $\#h(x)$ denotes the total number of the elements in membership function $h(x)$, then IVPDHFS reduces to PHFS.

Very often in the decision-making process, the elements in given IVPDHFEs are disordered, which may lead to difficulties in operation. Thus, we propose the concept of ordered IVPDHFE as follows:

Definition 8. Let $d_{IVP} = \left\{h| \left[p^l_h, p^u_h\right], g| \left[q^l_g, q^u_g\right]\right\}$ be a generalized IVPDHFE, if the elements $\bigcup_{\gamma \in h}\{\gamma\}$ in membership function h are sorted based on the values of $\gamma\left(\left(p^l_\gamma\right)^2 + \left(p^u_\gamma\right)^2\right)$ in descending order and the elements $\bigcup_{\eta \in g}\{\eta\}$ in membership function g are sorted based on the values of $\eta\left(\left(q^l_\eta\right)^2 + \left(q^u_\eta\right)^2\right)$ in descending order, then this IVPDHFE is called an ordered IVPDHFE.

Remark 2

Specially, if the values of $\gamma\left(\left(p^l_\gamma\right)^2 + \left(p^u_\gamma\right)^2\right)$ are equal, then the elements $\bigcup_{\gamma \in h}\{\gamma\}$ in membership function h are sorted based on the values of γ in descending order. And if the values of $\eta\left(\left(q^l_\eta\right)^2 + \left(q^u_\eta\right)^2\right)$ are equal, then the elements $\bigcup_{\eta \in g}\{\eta\}$ in membership function g are sorted based on the values of η in descending order.

Example 1. Given an IVPDHFE $d_{IVP} = \langle\{0.8| [0.2, 0.3], 0.6| [0.5, 0.7]\}, \{0.6| [0.3, 0.4], 0.4| [0.3, 0.6]\}\rangle$, if we calculate and line up the values of $\gamma\left(\left(p^l_\gamma\right)^2 + \left(p^u_\gamma\right)^2\right)$ and $\eta\left(\left(q^l_\eta\right)^2 + \left(q^u_\eta\right)^2\right)$, respectively, then a new IVPDHFE is acquired. Since $0.8 \times \left(0.2^2 + 0.3^2\right) = 0.104$, $0.6 \times \left(0.5^2 + 0.7^2\right) = 0.444$, $0.6 \times \left(0.3^2 + 0.4^2\right) = 0.15$ and $0.4 \times \left(0.3^2 + 0.6^2\right) = 0.18$, the new IVPDHFE $\dot{d}_{IVP} = \langle\{0.6| [0.5, 0.7], 0.8| [0.2, 0.3]\}, \{0.4| [0.3, 0.6], 0.6| [0.3, 0.4]\}\rangle$ is an ordered IVPDHFE.

For an IVPDHFE $d_{IVP} = \{h|[p_h^l, p_h^u], g|[q_g^l, q_g^u]\}$, if $\sum_{p_\gamma^u \in p_h^u} p_\gamma^u = 1$ and $\sum_{q_\eta^u \in q_g^u} q_\eta^u = 1$, then the probability distribution of all elements is complete; if $\sum_{p_\gamma^u \in p_h^u} p_\gamma^u < 1$ or $\sum_{q_\eta^u \in q_g^u} q_\eta^u < 1$, then the probability distribution of all elements is incomplete, namely, some probabilistic information is lost and ignored. To overcome this cognitive limitation, we give the concept of normalized IVPDHFSs.

Definition 9. Let $D_{IVP} = \{\langle x, h(x)|[p^l(x), p^u(x)], g(x)|[q^l(x), q^u(x)]\rangle | x \in X\}$ be an IVPDHFS, then its normalized form is denoted by

$$\overline{D}_{IVP} = \left\{\left\langle x, h(x)\left|\left[\overline{p}^l(x), \overline{p}^u(x)\right], g(x)\right|\left[\overline{q}^l(x), \overline{q}^u(x)\right]\right\rangle | x \in X\right\}, \quad (13)$$

where the elements $[p_\gamma^l, p_\gamma^u] \in [p^l(x), p^u(x)]$ and $[q_\eta^l, q_\eta^u] \in [q^l(x), q^u(x)]$ are normalized as

$$\overline{p}_\gamma^l = \frac{p_\gamma^l}{\sqrt{\sum_{p_\gamma^l \in p^l(x), p_\gamma^u \in p^u(x)} \frac{(p_\gamma^l)^2 + (p_\gamma^u)^2}{2}}}, \quad (14)$$

$$\overline{p}_\gamma^u = \frac{p_\gamma^u}{\sqrt{\sum_{p_\gamma^l \in p^l(x), p_\gamma^u \in p^u(x)} \frac{(p_\gamma^l)^2 + (p_\gamma^u)^2}{2}}}, \quad (15)$$

$$\overline{q}_\eta^l = \frac{q_\eta^l}{\sqrt{\sum_{q_\eta^l \in q^l(x), q_\eta^u \in q^u(x)} \frac{(q_\eta^l)^2 + (q_\eta^u)^2}{2}}}, \quad (16)$$

$$\overline{q}_\eta^u = \frac{q_\eta^u}{\sqrt{\sum_{q_\eta^l \in q^l(x), q_\eta^u \in q^u(x)} \frac{(q_\eta^l)^2 + (q_\eta^u)^2}{2}}}. \quad (17)$$

3.2. The Basic Operations for IVPDHFSs

Definition 10. Let $d_{IVP} = \{h|[p_h^l, p_h^u], g|[q_g^l, q_g^u]\}$, $d_{IVP1} = \{h_1|[p_{h_1}^l, p_{h_1}^u], g_1|[q_{g_1}^l, q_{g_1}^u]\}$, $d_{IVP2} = \{h_2|[p_{h_2}^l, p_{h_2}^u], g_2|[q_{g_2}^l, q_{g_2}^u]\}$ be any three generalized IVPDHFEs, then we have

$$d_{IVP1} \oplus d_{IVP2} = \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2 \\ p_{\gamma_1}^l \in p_{h_1}^l, p_{\gamma_1}^u \in p_{h_1}^u, p_{\gamma_2}^l \in p_{h_2}^l, p_{\gamma_2}^u \in p_{h_2}^u \\ q_{\eta_1}^l \in q_{g_1}^l, q_{\eta_1}^u \in q_{g_1}^u, q_{\eta_2}^l \in q_{g_2}^l, q_{\eta_2}^u \in q_{g_2}^u}} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2|[p_{\gamma_1}^l p_{\gamma_2}^l, p_{\gamma_1}^u p_{\gamma_2}^u], \eta_1\eta_2|[q_{\eta_1}^l q_{\eta_2}^l, q_{\eta_1}^u q_{\eta_2}^u]\}; \quad (18)$$

$$d_{IVP1} \otimes d_{IVP2} = \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2 \\ p_{\gamma_1}^l \in p_{h_1}^l, p_{\gamma_1}^u \in p_{h_1}^u, p_{\gamma_2}^l \in p_{h_2}^l, p_{\gamma_2}^u \in p_{h_2}^u \\ q_{\eta_1}^l \in q_{g_1}^l, q_{\eta_1}^u \in q_{g_1}^u, q_{\eta_2}^l \in q_{g_2}^l, q_{\eta_2}^u \in q_{g_2}^u}} \{\gamma_1\gamma_2|[p_{\gamma_1}^l p_{\gamma_2}^l, p_{\gamma_1}^u p_{\gamma_2}^u], \eta_1 + \eta_2 - \eta_1\eta_2|[q_{\eta_1}^l q_{\eta_2}^l, q_{\eta_1}^u q_{\eta_2}^u]\}; \quad (19)$$

$$\lambda d_{IVP} = \bigcup_{\substack{\gamma \in h, \eta \in g \\ p_\gamma^l \in p_h^l, p_\gamma^u \in p_h^u \\ q_\eta^l \in q_g^l, q_\eta^u \in q_g^u}} \{1 - (1-\gamma)^\lambda |[p_\gamma^l, p_\gamma^u], \eta^\lambda|[q_\eta^l, q_\eta^u]\}, \lambda \geq 0; \quad (20)$$

$$d_{IVP}^\lambda = \bigcup_{\substack{\gamma \in h, \eta \in g \\ p_\gamma^l \in p_h^l, p_\gamma^u \in p_h^u \\ q_\eta^l \in q_g^l, q_\eta^u \in q_g^u}} \{\gamma^\lambda|[p_\gamma^l, p_\gamma^u], 1 - (1-\eta)^\lambda|[q_\eta^l, q_\eta^u]\}, \lambda \geq 0; \quad (21)$$

$$d_{IVP}^c = \bigcup_{\substack{\gamma \in h, \eta \in g \\ p_\gamma^l \in p_h^l, p_\gamma^u \in p_h^u \\ q_\eta^l \in q_g^l, q_\eta^u \in q_g^u}} \{\eta|[q_\eta^l, q_\eta^u], \gamma|[p_\gamma^l, p_\gamma^u]\}. \quad (22)$$

Remark 3

In IVPDHFSs, the probability distribution of all elements is denoted by interval values with lower limits and upper limits. For each evaluation value, if the lower limit is equal to the upper limit, namely, interval-valued probability reduces to single probability, then Eqs. (18–21) reduce to Eqs. (8–11); if all the elements in membership set and non-membership set have equal importance and weight, namely, $p_h^l = p_h^u = \frac{1}{\#h}$, $q_g^l = q_g^u = \frac{1}{\#g}$, where $\#h$ and $\#g$ represent the total numbers of all elements in membership function and non-membership function, respectively, then balanced probability information cannot reflect preference character of DMs, and Eqs. (18–21) reduce to Eqs. (3–6).

Property 1. The complement set of generalized IVPDHFS is involutive.

***Proof*.** According to the Eq. (22), we can conclude that

$$\left(\{h|[p_h^l, p_h^u], g|[q_g^l, q_g^u]\}^c\right)^c = \left(\bigcup_{\substack{\gamma \in h, \eta \in g \\ p_\gamma^l \in p_h^l, p_\gamma^u \in p_h^u \\ q_\eta^l \in q_g^l, q_\eta^u \in q_g^u}} \{\eta|[q_\eta^l, q_\eta^u], \gamma|[p_\gamma^l, p_\gamma^u]\}\right)^c = \{h|[p_h^l, p_h^u], g|[q_g^l, q_g^u]\}.$$

Property 2. Commutative

$$d_{IVP1} \oplus d_{IVP2} = d_{IVP2} \oplus d_{IVP1}; \quad (23)$$
$$d_{IVP1} \otimes d_{IVP2} = d_{IVP2} \otimes d_{IVP1}; \quad (24)$$

Property 3. Associative

$$d_{IVP1} \oplus (d_{IVP2} \oplus d_{IVP3}) = (d_{IVP1} \oplus d_{IVP2}) \oplus d_{IVP3}; \tag{25}$$

$$d_{IVP1} \otimes (d_{IVP2} \otimes d_{IVP3}) = (d_{IVP1} \otimes d_{IVP2}) \otimes d_{IVP3}; \tag{26}$$

Proof. The proofs for Properties 2 and 3 are straightforward and simple. Here, we concrete on these Properties alone.

Property 4. Distributive

$$\lambda\,(d_{IVP1} + d_{IVP2}) = \lambda d_{IVP1} + \lambda d_{IVP2}, \lambda \geq 0; \tag{27}$$

$$(\lambda_1 + \lambda_2)\, d_{IVP} = \lambda_1 d_{IVP} + \lambda_2 d_{IVP}, \lambda_1, \lambda_2 \geq 0; \tag{28}$$

Proof. For the left part of Eq. (27), we have

$$\begin{aligned}\lambda\,(d_{IVP1} + d_{IVP2}) &= \lambda\left(\bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2\\ \eta_1 \in g_1, \eta_2 \in g_2\\ p^l_{\gamma_1} \in p^l_{h_1}, p^u_{\gamma_1} \in p^u_{h_1}\\ p^l_{\gamma_2} \in p^l_{h_2}, p^u_{\gamma_2} \in p^u_{h_2}\\ q^l_{\eta_1} \in q^l_{g_1}, q^u_{\eta_1} \in q^u_{g_1}\\ q^l_{\eta_2} \in q^l_{g_2}, q^u_{\eta_2} \in q^u_{g_2}}} \left\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2 \middle| \left[p^l_{\gamma_1} p^l_{\gamma_2}, p^u_{\gamma_1} p^u_{\gamma_2}\right], \eta_1\eta_2 \middle| \left[q^l_{\eta_1} q^l_{\eta_2}, q^u_{\eta_1} q^u_{\eta_2}\right]\right\}\right)\\
&= \lambda\left(\bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2\\ \eta_1 \in g_1, \eta_2 \in g_2\\ p^l_{\gamma_1} \in p^l_{h_1}, p^u_{\gamma_1} \in p^u_{h_1}\\ p^l_{\gamma_2} \in p^l_{h_2}, p^u_{\gamma_2} \in p^u_{h_2}\\ q^l_{\eta_1} \in q^l_{g_1}, q^u_{\eta_1} \in q^u_{g_1}\\ q^l_{\eta_2} \in q^l_{g_2}, q^u_{\eta_2} \in q^u_{g_2}}} \left\{1 - (1 - \gamma_1)(1 - \gamma_2) \middle| \left[p^l_{\gamma_1} p^l_{\gamma_2}, p^u_{\gamma_1} p^u_{\gamma_2}\right], \eta_1\eta_2 \middle| \left[q^l_{\eta_1} q^l_{\eta_2}, q^u_{\eta_1} q^u_{\eta_2}\right]\right\}\right)\\
&= \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2\\ \eta_1 \in g_1, \eta_2 \in g_2\\ p^l_{\gamma_1} \in p^l_{h_1}, p^u_{\gamma_1} \in p^u_{h_1}\\ p^l_{\gamma_2} \in p^l_{h_2}, p^u_{\gamma_2} \in p^u_{h_2}\\ q^l_{\eta_1} \in q^l_{g_1}, q^u_{\eta_1} \in q^u_{g_1}\\ q^l_{\eta_2} \in q^l_{g_2}, q^u_{\eta_2} \in q^u_{g_2}}} \left\{1 - (1 - \gamma_1)^{\lambda}(1 - \gamma_2)^{\lambda} \middle| \left[p^l_{\gamma_1} p^l_{\gamma_2}, p^u_{\gamma_1} p^u_{\gamma_2}\right], \eta_1^{\lambda}\eta_2^{\lambda} \middle| \left[q^l_{\eta_1} q^l_{\eta_2}, q^u_{\eta_1} q^u_{\eta_2}\right]\right\}\end{aligned}$$

For the right part of Eq. (27), we have

$$\begin{aligned}
\lambda d_{IVP1} + \lambda d_{IVP2} &= \bigcup_{\substack{\gamma_1 \in h_1, \eta_1 \in g_1 \\ p^l_{\gamma_1} \in p^l_{h_1}, p^u_{\gamma_1} \in p^u_{h_1} \\ q^l_{\eta_1} \in q^l_{g_1}, q^u_{\eta_1} \in q^u_{g_1}}} \left\{1-(1-\gamma_1)^\lambda \middle| \left[p^l_{\gamma_1}, p^u_{\gamma_1}\right], \eta_1^\lambda \middle| \left[q^l_{\eta_1}, q^u_{\eta_1}\right]\right\} \\
&\quad + \bigcup_{\substack{\gamma_2 \in h_2, \eta_2 \in g_2 \\ p^l_{\gamma_2} \in p^l_{h_2}, p^u_{\gamma_2} \in p^u_{h_2} \\ q^l_{\eta_2} \in q^l_{g_2}, q^u_{\eta_2} \in q^u_{g_2}}} \left\{1-(1-\gamma_2)^\lambda \middle| \left[p^l_{\gamma_2}, p^u_{\gamma_2}\right], \eta_2^\lambda \middle| \left[q^l_{\eta_2}, q^u_{\eta_2}\right]\right\} \\
&= \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2 \\ \eta_1 \in g_1, \eta_2 \in g_2 \\ p^l_{\gamma_1} \in p^l_{h_1}, p^u_{\gamma_1} \in p^u_{h_1} \\ p^l_{\gamma_2} \in p^l_{h_2}, p^u_{\gamma_2} \in p^u_{h_2} \\ q^l_{\eta_1} \in q^l_{g_1}, q^u_{\eta_1} \in q^u_{g_1} \\ q^l_{\eta_2} \in q^l_{g_2}, q^u_{\eta_2} \in q^u_{g_2}}} \left\{\begin{matrix} 1-(1-\gamma_1)^\lambda + 1-(1-\gamma_2)^\lambda - \\ \left(1-(1-\gamma_1)^\lambda\right)\left(1-(1-\gamma_2)^\lambda\right) \end{matrix} \middle| \left[p^l_{\gamma_1}p^l_{\gamma_2}, p^u_{\gamma_1}p^u_{\gamma_2}\right], \eta_1^\lambda\eta_2^\lambda \middle| \left[q^l_{\eta_1}q^l_{\eta_2}, q^u_{\eta_1}q^u_{\eta_2}\right]\right\} \\
&= \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2 \\ \eta_1 \in g_1, \eta_2 \in g_2 \\ p^l_{\gamma_1} \in p^l_{h_1}, p^u_{\gamma_1} \in p^u_{h_1} \\ p^l_{\gamma_2} \in p^l_{h_2}, p^u_{\gamma_2} \in p^u_{h_2} \\ q^l_{\eta_1} \in q^l_{g_1}, q^u_{\eta_1} \in q^u_{g_1} \\ q^l_{\eta_2} \in q^l_{g_2}, q^u_{\eta_2} \in q^u_{g_2}}} \left\{1-(1-\gamma_1)^\lambda(1-\gamma_2)^\lambda \middle| \left[p^l_{\gamma_1}p^l_{\gamma_2}, p^u_{\gamma_1}p^u_{\gamma_2}\right], \eta_1^\lambda\eta_2^\lambda \middle| \left[q^l_{\eta_1}q^l_{\eta_2}, q^u_{\eta_1}q^u_{\eta_2}\right]\right\}.
\end{aligned}$$

Thus, Eq. (27) is kept.

Similarly, Eq. (28) can be proved.

Theorem 1. *All the results acquired from Eqs. (18–28) are also IVPDHFSs.*

Proof. *The proof is easy and straightforward.*

Theorem 2. *Let* $d_{IVP} = \left\{h \middle| \left[p^l_h, p^u_h\right], g \middle| \left[q^l_g, q^u_g\right]\right\}$, $d_{IVP1} = \left\{h_1 \middle| \left[p^l_{h_1}, p^u_{h_1}\right], g_1 \middle| \left[q^l_{g_1}, q^u_{g_1}\right]\right\}$, $d_{IVP2} = \left\{h_2 \middle| \left[p^l_{h_2}, p^u_{h_2}\right], g_2 \middle| \left[q^l_{g_2}, q^u_{g_2}\right]\right\}$ *be any three generalized IVPDHFEs, then we have the following properties:*

$$d^\lambda_{IVP1} \times d^\lambda_{IVP2} = \left(d_{IVP1} \times d_{IVP2}\right)^\lambda, \lambda \geq 0; \tag{29}$$

$$d^{\lambda_1}_{IVP} \times d^{\lambda_2}_{IVP} = d^{\lambda_1+\lambda_2}_{IVP}, \lambda_1, \lambda_2 \geq 0. \tag{30}$$

Proof. *For the left part of Eq. (29), we have*

$$\begin{aligned}
d^\lambda_{IVP1} \times d^\lambda_{IVP2} &= \bigcup_{\substack{\gamma_1 \in h_1, \eta_1 \in g_1 \\ p^l_{\gamma_1} \in p^l_{h_1}, p^u_{\gamma_1} \in p^u_{h_1} \\ q^l_{\eta_1} \in q^l_{g_1}, q^u_{\eta_1} \in q^u_{g_1}}} \left\{\gamma_1^\lambda \middle| \left[p^l_{\gamma_1}, p^u_{\gamma_1}\right], 1-(1-\eta_1)^\lambda \middle| \left[q^l_{\eta_1}, q^u_{\eta_1}\right]\right\} \\
&\quad \times \bigcup_{\substack{\gamma_2 \in h_2, \eta_2 \in g_2 \\ p^l_{\gamma_2} \in p^l_{h_2}, p^u_{\gamma_2} \in p^u_{h_2} \\ q^l_{\eta_2} \in q^l_{g_2}, q^u_{\eta_2} \in q^u_{g_2}}} \left\{\gamma_2^\lambda \middle| \left[p^l_{\gamma_2}, p^u_{\gamma_2}\right], 1-(1-\eta_2)^\lambda \middle| \left[q^l_{\eta_2}, q^u_{\eta_2}\right]\right\} \\
&= \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2 \\ p^l_{\gamma_1} \in p^l_{h_1}, p^u_{\gamma_1} \in p^u_{h_1}, p^l_{\gamma_2} \in p^l_{h_2}, p^u_{\gamma_2} \in p^u_{h_2} \\ q^l_{\eta_1} \in q^l_{g_1}, q^u_{\eta_1} \in q^u_{g_1}, q^l_{\eta_2} \in q^l_{g_2}, q^u_{\eta_2} \in q^u_{g_2}}} \left\{\gamma_1^\lambda\gamma_2^\lambda \middle| \left[p^l_{\gamma_1}p^l_{\gamma_2}, p^u_{\gamma_1}p^u_{\gamma_2}\right], \begin{matrix} 1-(1-\eta_1)^\lambda + 1-(1-\eta_2)^\lambda - \\ \left(1-(1-\eta_1)^\lambda\right)\left(1-(1-\eta_2)^\lambda\right) \end{matrix} \middle| \left[q^l_{\eta_1}q^l_{\eta_2}, q^u_{\eta_1}q^u_{\eta_2}\right]\right\} \\
&= \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2 \\ p^l_{\gamma_1} \in p^l_{h_1}, p^u_{\gamma_1} \in p^u_{h_1}, p^l_{\gamma_2} \in p^l_{h_2}, p^u_{\gamma_2} \in p^u_{h_2} \\ q^l_{\eta_1} \in q^l_{g_1}, q^u_{\eta_1} \in q^u_{g_1}, q^l_{\eta_2} \in q^l_{g_2}, q^u_{\eta_2} \in q^u_{g_2}}} \left\{\gamma_1^\lambda\gamma_2^\lambda \middle| \left[p^l_{\gamma_1}p^l_{\gamma_2}, p^u_{\gamma_1}p^u_{\gamma_2}\right], 1-(1-\eta_1)^\lambda(1-\eta_2)^\lambda \middle| \left[q^l_{\eta_1}q^l_{\eta_2}, q^u_{\eta_1}q^u_{\eta_2}\right]\right\}
\end{aligned}$$

For the right part of Eq. (29), we have

$$\begin{aligned}
(d_{IVP1}\times d_{IVP2})^{\lambda} &= \left(\bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2\\ p^l_{\gamma_1}\in p^l_{h_1},p^u_{\gamma_1}\in p^u_{h_1},p^l_{\gamma_2}\in p^l_{h_2},p^u_{\gamma_2}\in p^u_{h_2}\\ q^l_{\eta_1}\in q^l_{g_1},q^u_{\eta_1}\in q^u_{g_1},q^l_{\eta_2}\in q^l_{g_2},q^u_{\eta_2}\in q^u_{g_2}}}\left\{\gamma_1\gamma_2|\left[p^l_{\gamma_1}p^l_{\gamma_2},p^u_{\gamma_1}p^u_{\gamma_2}\right],\eta_1+\eta_2-\eta_1\eta_2|\left[q^l_{\eta_1}q^l_{\eta_2},q^u_{\eta_1}q^u_{\eta_2}\right]\right\}\right)^{\lambda}\\
&= \bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2\\ p^l_{\gamma_1}\in p^l_{h_1},p^u_{\gamma_1}\in p^u_{h_1},p^l_{\gamma_2}\in p^l_{h_2},p^u_{\gamma_2}\in p^u_{h_2}\\ q^l_{\eta_1}\in q^l_{g_1},q^u_{\eta_1}\in q^u_{g_1},q^l_{\eta_2}\in q^l_{g_2},q^u_{\eta_2}\in q^u_{g_2}}}\left\{\gamma_1^{\lambda}\gamma_2^{\lambda}|\left[p^l_{\gamma_1}p^l_{\gamma_2},p^u_{\gamma_1}p^u_{\gamma_2}\right],1-(1-(1-(1-\eta_1)(1-\eta_2)))^{\lambda}\,|\left[q^l_{\eta_1}q^l_{\eta_2},q^u_{\eta_1}q^u_{\eta_2}\right]\right\}\\
&= \bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2\\ p^l_{\gamma_1}\in p^l_{h_1},p^u_{\gamma_1}\in p^u_{h_1},p^l_{\gamma_2}\in p^l_{h_2},p^u_{\gamma_2}\in p^u_{h_2}\\ q^l_{\eta_1}\in q^l_{g_1},q^u_{\eta_1}\in q^u_{g_1},q^l_{\eta_2}\in q^l_{g_2},q^u_{\eta_2}\in q^u_{g_2}}}\left\{\gamma_1^{\lambda}\gamma_2^{\lambda}|\left[p^l_{\gamma_1}p^l_{\gamma_2},p^u_{\gamma_1}p^u_{\gamma_2}\right],1-(1-\eta_1)^{\lambda}(1-\eta_2)^{\lambda}\,|\left[q^l_{\eta_1}q^l_{\eta_2},q^u_{\eta_1}q^u_{\eta_2}\right]\right\}
\end{aligned}$$

Thus, Eq. (29) is kept.

Similarly, Eq. (30) can be proved.

Theorem 3. *Let* $d_{IVP}=\left\{h|\left[p^l_h,p^u_h\right],g|\left[q^l_g,q^u_g\right]\right\}$, $d_{IVP1}=\left\{h_1|\left[p^l_{h_1},p^u_{h_1}\right],g_1|\left[q^l_{g_1},q^u_{g_1}\right]\right\}$, $d_{IVP2}=\left\{h_2|\left[p^l_{h_2},p^u_{h_2}\right],g_2|\left[q^l_{g_2},q^u_{g_2}\right]\right\}$ *be any three generalized IVPDHFEs, then based on Eqs. (18–22), we can obtain*

$$d^c_{IVP1}\oplus d^c_{IVP2}=(d_{IVP1}\otimes d_{IVP2})^c; \tag{31}$$

$$d^c_{IVP1}\otimes d^c_{IVP2}=(d_{IVP1}\oplus d_{IVP2})^c; \tag{32}$$

$$\left(d^c_{IVP}\right)^{\lambda}=(\lambda d_{IVP})^c,\lambda\geq 0; \tag{33}$$

$$\lambda d^c_{IVP}=\left(d^{\lambda}_{IVP}\right)^c,\lambda\geq 0. \tag{34}$$

Proof. *For the left part of Eq. (31), we have*

$$\begin{aligned}
d^c_{IVP1}\oplus d^c_{IVP2} &= \bigcup_{\substack{\gamma_1\in h_1,\eta_1\in g_1\\ p^l_{\gamma_1}\in p^l_{h_1},p^u_{\gamma_1}\in p^u_{h_1}\\ q^l_{\eta_1}\in q^l_{g_1},q^u_{\eta_1}\in q^u_{g_1}}}\left\{\eta_1|\left[q^l_{\eta_1},q^u_{\eta_1}\right],\gamma_1|\left[p^l_{\gamma_1},p^u_{\gamma_1}\right]\right\}\oplus\bigcup_{\substack{\gamma_2\in h_2,\eta_2\in g_2\\ p^l_{\gamma_2}\in p^l_{h_2},p^u_{\gamma_2}\in p^u_{h_2}\\ q^l_{\eta_2}\in q^l_{g_2},q^u_{\eta_2}\in q^u_{g_2}}}\left\{\eta_2|\left[q^l_{\eta_2},q^u_{\eta_2}\right],\gamma_2|\left[p^l_{\gamma_2},p^u_{\gamma_2}\right]\right\}\\
&= \bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2\\ \eta_1\in g_1,\eta_2\in g_2\\ p^l_{\gamma_1}\in p^l_{h_1},p^u_{\gamma_1}\in p^u_{h_1}\\ p^l_{\gamma_2}\in p^l_{h_2},p^u_{\gamma_2}\in p^u_{h_2}\\ q^l_{\eta_1}\in q^l_{g_1},q^u_{\eta_1}\in q^u_{g_1}\\ q^l_{\eta_2}\in q^l_{g_2},q^u_{\eta_2}\in q^u_{g_2}}}\left\{\eta_1+\eta_2-\eta_1\eta_2|\left[q^l_{\eta_1}q^l_{\eta_2},q^u_{\eta_1}q^u_{\eta_2}\right],\gamma_1\gamma_2|\left[p^l_{\gamma_1}p^l_{\gamma_2},p^u_{\gamma_1}p^u_{\gamma_2}\right]\right\},
\end{aligned}$$

and for the right part of Eq. (31), we have

$$\begin{aligned}
(d_{IVP1}\otimes d_{IVP2})^c &= \left(\bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2\\ p^l_{\gamma_1}\in p^l_{h_1},p^u_{\gamma_1}\in p^u_{h_1},p^l_{\gamma_2}\in p^l_{h_2},p^u_{\gamma_2}\in p^u_{h_2}\\ q^l_{\eta_1}\in q^l_{g_1},q^u_{\eta_1}\in q^u_{g_1},q^l_{\eta_2}\in q^l_{g_2},q^u_{\eta_2}\in q^u_{g_2}}}\left\{\gamma_1\gamma_2|\left[p^l_{\gamma_1}p^l_{\gamma_2},p^u_{\gamma_1}p^u_{\gamma_2}\right],\eta_1+\eta_2-\eta_1\eta_2|\left[q^l_{\eta_1}q^l_{\eta_2},q^u_{\eta_1}q^u_{\eta_2}\right]\right\}\right)^c\\
&= \bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2\\ p^l_{\gamma_1}\in p^l_{h_1},p^u_{\gamma_1}\in p^u_{h_1},p^l_{\gamma_2}\in p^l_{h_2},p^u_{\gamma_2}\in p^u_{h_2}\\ q^l_{\eta_1}\in q^l_{g_1},q^u_{\eta_1}\in q^u_{g_1},q^l_{\eta_2}\in q^l_{g_2},q^u_{\eta_2}\in q^u_{g_2}}}\left\{\eta_1+\eta_2-\eta_1\eta_2|\left[q^l_{\eta_1}q^l_{\eta_2},q^u_{\eta_1}q^u_{\eta_2}\right],\gamma_1\gamma_2|\left[p^l_{\gamma_1}p^l_{\gamma_2},p^u_{\gamma_1}p^u_{\gamma_2}\right]\right\}.
\end{aligned}$$

Thus, Eq. (31) is hold.

Similarly, Eqs. (32–34) can be proved.

3.3. The Comparison Method for IVPDHFEs

For an IVPDHFS, the elements in satisfaction function and dissatisfaction function are almost in the partial order. However, it is useful and necessary to rank a set of IVPFDHFEs in decision-making problems. Hence, we propose the score function and deviation function for IVPDHFEs, providing reliable access to the comparison of IVPDHFEs.

Definition 11. Let $d_{IVP} = \left\{h \middle| \left[p_h^l, p_h^u\right], g \middle| \left[q_g^l, q_g^u\right]\right\}$ be the generalized IVPDHFE, then the score function for IVPDHFE is expressed by the following mathematical symbol:

$$s\left(d_{IVP}\right) = \sum_{\gamma \in h}^{\#h} \left(\gamma p_\gamma^l + \gamma p_\gamma^u\right) - \sum_{\eta \in g}^{\#g} \left(\eta q_\eta^l + \eta q_\eta^u\right). \tag{35}$$

For any two generalized IVPDHFEs d_{IVP1} and d_{IVP2}, if $s\left(d_{IVP1}\right) > s\left(d_{IVP2}\right)$, then the IVPDHFE d_{IVP1} is superior to the IVPDHFE d_{IVP2}, mathematically denoted as $d_{IVP1} \succ d_{IVP2}$; if $s\left(d_{IVP1}\right) < s\left(d_{IVP2}\right)$, then the IVPDHFE d_{IVP1} is inferior to the IVPDHFE d_{IVP2}, mathematically denoted as $d_{IVP1} \prec d_{IVP2}$; if $s\left(d_{IVP1}\right) = s\left(d_{IVP2}\right)$, then it is inappropriate to draw the conclusion that these two IVPDHFEs are equal and identical. In this situation, another indicator is needed, thus, we define the deviation function for IVPDHEs to make a further comparison.

Definition 12. Let $d_{IVP} = \left\{h \middle| \left[p_h^l, p_h^u\right], g \middle| \left[q_g^l, q_g^u\right]\right\}$ be the generalized IVPDHFE, then the deviation function for IVPDHFE is denoted by the following expression:

$$\sigma\left(d_{IVP}\right) = \left(\sum_{\gamma \in h}^{\#h} \left(\left(\gamma - s(d_{IVP})\right)^2 (p_\gamma^l + p_\gamma^u)\right) + \sum_{\eta \in g}^{\#g} \left(\left(\eta - s(d_{IVP})\right)^2 (q_\eta^l + q_\eta^u)\right)\right)^{\frac{1}{2}}. \tag{36}$$

As expressed in Eq. (36), the deviation function $\sigma\left(d_{IVP}\right)$ reflects the overall fluctuation degree of an IVPDHFE from the average level. The smaller value of $\sigma\left(d_{IVP}\right)$ implies higher coherence while the bigger value of $\sigma\left(d_{IVP}\right)$ indicates lower coherence.

Considering the score function and deviation function for IVPDHFEs, we further present the comparison method for IVPDHFEs.

Definition 13. Let $d_{IVP1} = \left\{h_1 \middle| \left[p_{h_1}^l, p_{h_1}^u\right], g_1 \middle| \left[q_{g_1}^l, q_{g_1}^u\right]\right\}$, $d_{IVP2} = \left\{h_2 \middle| \left[p_{h_2}^l, p_{h_2}^u\right], g_2 \middle| \left[q_{g_2}^l, q_{g_2}^u\right]\right\}$ be any two generalized IVPDHFEs, then

- If $s\left(d_{IVP1}\right) > s\left(d_{IVP2}\right)$, then the IVPDHFE d_{IVP1} is superior to the IVPDHFE d_{IVP2}, mathematically denoted as $d_{IVP1} \succ d_{IVP2}$.
- If $s\left(d_{IVP1}\right) < s\left(d_{IVP2}\right)$, then the IVPDHFE d_{IVP1} is inferior to the IVPDHFE d_{IVP2}, mathematically denoted as $d_{IVP1} \prec d_{IVP2}$.
- If $s\left(d_{IVP1}\right) = s\left(d_{IVP2}\right)$, then

i. if $\sigma\left(d_{IVP1}\right) < \sigma\left(d_{IVP2}\right)$, then the IVPDHFE d_{IVP1} is superior to the IVPDHFE d_{IVP2}, mathematically denoted as $d_{IVP1} \succ d_{IVP2}$.

ii. if $\sigma\left(d_{IVP1}\right) > \sigma\left(d_{IVP2}\right)$, then the IVPDHFE d_{IVP1} is inferior to the IVPDHFE d_{IVP2}, mathematically denoted as $d_{IVP1} \prec d_{IVP2}$.

iii. if $\sigma\left(d_{IVP1}\right) = \sigma\left(d_{IVP2}\right)$, then the IVPDHFE d_{IVP1} is equivalent to the IVPDHFE d_{IVP2}, mathematically denoted as $d_{IVP1} \sim d_{IVP2}$.

3.4. The Ordered Distance and Similarity Measures for Generalized IVPDHFSs

The distance and similarity measures are classical tools to address decision-making problems in which the evaluation on alternative projects is denoted by FSs, HFSs, DHFSs and PHFSs. These measures can also reflect the closeness degree of any two items easily in practical applications. Hence, it is essential to study the distance and similarity measures under IVPDHFSs environment. To this end, we introduce the axioms for distance and similarity measures for IVPDHFSs.

Definition 14. Let M and N be two generalized IVPDHFSs on the reference set X, then the distance measure between M and N is expressed as $d\left(M, N\right)$, satisfying the following properties:

1. $0 \le d\left(M, N\right) \le 1$;
2. $d\left(M, N\right) = 0$ if and only if $M = N$;
3. $d\left(M, N\right) = d\left(N, M\right) = d\left(M^c, N^c\right) = d\left(N^c, M^c\right)$.

Definition 15. Let M and N be two generalized IVPDHFSs on the reference set X, then the similarity measure between M and N is expressed as $s\left(M, N\right)$, satisfying the following properties:

1. $0 \le s\left(M, N\right) \le 1$;
2. $s\left(M, N\right) = 1$ if and only if $M = N$;
3. $s\left(M, N\right) = s\left(N, M\right) = s\left(M^c, N^c\right) = s\left(N^c, M^c\right)$.

Since there are complement relationships between the distance measure and the similarity measure, we mainly concrete on the distance measure of IVPDHFSs, and the similarity measure for IVPDHFSs can be obtained easily.

In most circumstance, for any $x \in X$, $\#h_M\left(x\right) \neq \#h_N\left(x\right)$ or $\#g_M\left(x\right) \neq \#g_N\left(x\right)$, namely, the membership sets or non-membership sets in two IVPDHFSs have unequal length. Inspired by the idea of Xu and Xia [9] and Liu *et al.* [43], we propose the ordered distance measure for IVPDHFSs,

$$
\begin{aligned}
d(M,N) = \frac{1}{4n}\sum_{x\in X}\Bigg(& \frac{1}{\#h_M(x)}\sum_{i=1}^{\#h_M(x)} \big|h_M^{\sigma(i)}(x)\big(p_M^l(x)+p_M^u(x)\big) \\
& - h_N^{\sigma(i)}(x)\big(p_N^l(x)+p_N^u(x)\big)\big| \\
& + \frac{1}{\#h_N(x)}\sum_{j=1}^{\#h_N(x)} \big|h_N^{\sigma(j)}(x)\big(p_N^l(x)+p_N^u(x)\big) \\
& - h_M^{\sigma(j)}(x)\big(p_M^l(x)+p_M^u(x)\big)\big| \\
& + \frac{1}{\#g_M(x)}\sum_{s=1}^{\#g_M(x)} \big|g_M^{\sigma(s)}(x)\big(q_M^l(x)+q_M^u(x)\big) \\
& - g_N^{\sigma(s)}(x)\big(q_N^l(x)+q_N^u(x)\big)\big| \\
& + \frac{1}{\#g_N(x)}\sum_{t=1}^{\#g_N(x)} \big|g_N^{\sigma(t)}(x)\big(q_N^l(x)+q_N^u(x)\big) \\
& - g_M^{\sigma(t)}(x)\big(q_M^l(x)+q_M^u(x)\big)\big|\Bigg)
\end{aligned}
\tag{37}
$$

where $\sigma(i)$, $\sigma(j)$, $\sigma(s)$ and $\sigma(t)$ are the i-th, j-th, s-th and t-th largest values in corresponding sets, respectively.

Based on Eq. (37), we can obtain the distance measure for IVPDHFEs easily:

$$
\begin{aligned}
d(m,n) = \frac{1}{4}\Bigg(& \frac{1}{\#h_m}\sum_{i=1}^{\#h_m}\left|h_m^{\sigma(i)}(p_m^l+p_m^u) - h_n^{\sigma(i)}(p_n^l+p_n^u)\right| \\
& + \frac{1}{\#h_n}\sum_{j=1}^{\#h_n}\left|h_n^{\sigma(j)}(p_n^l+p_n^u) - h_m^{\sigma(j)}(p_m^l+p_m^u)\right| \\
& + \frac{1}{\#g_m}\sum_{s=1}^{\#g_m}\left|g_m^{\sigma(s)}(q_m^l+q_m^u) - g_n^{\sigma(s)}(q_n^l+q_n^u)\right| \\
& + \frac{1}{\#g_n}\sum_{t=1}^{\#g_n}\left|g_n^{\sigma(t)}(q_n^l+q_n^u) - g_m^{\sigma(t)}(q_m^l+q_m^u)\right|\Bigg)
\end{aligned}
\tag{38}
$$

Remark 4

Before measuring the distance between any two IVPDHFEs by utilizing Eq. (38), we need to make the following two-step preparation.

Firstly, we need to line up all the elements in IVPDHFEs in descending order by referring to the Definition 8.

Secondly, for the membership set, two IVPDHFEs may have unequal length. Hence, we need to extend the shorter membership set until both of them have equal length. There are many methods to extend the shorter one, such as adding some elements in it. In most cases, we almost add some same values at the end or beginning of the shorter one, which depends on the DMs' attitude towards possible risks. Optimistic DMs are willing to add the element $\gamma\left|\left[p_\gamma^l, p_\gamma^u\right]\right.$ with maximum value of $\gamma\left(p_\gamma^l\right)^2 + \left(p_\gamma^u\right)^2$ at the beginning of the shorter membership set until its length is equal to that of the longer membership set, while pessimistic DMs usually add the element $\gamma\left|\left[p_\gamma^l, p_\gamma^u\right]\right.$ with minimum value of $\gamma\left(p_\gamma^l\right)^2 + \left(p_\gamma^u\right)^2$ at the end of the shorter membership set until its length is equal to that of the longer membership set. It is also the same for the non-membership set.

According to the ordered distance measure for IVPDHFSs, the ordered similarity measure is denoted by the following expression:

$$
\begin{aligned}
s(M,N) &= 1 - d(M,N) \\
&= 1 - \frac{1}{4n}\sum_{x\in X}\Bigg(\frac{1}{\#h_M(x)}\sum_{i=1}^{\#h_M(x)} \big|h_M^{\sigma(i)}(x)\big(p_M^l(x)+p_M^u(x)\big) \\
&\quad - h_N^{\sigma(i)}(x)\big(p_N^l(x)+p_N^u(x)\big)\big| \\
&\quad + \frac{1}{\#h_N(x)}\sum_{j=1}^{\#h_N(x)} \big|h_N^{\sigma(j)}(x)\big(p_N^l(x)+p_N^u(x)\big) \\
&\quad - h_M^{\sigma(j)}(x)\big(p_M^l(x)+p_M^u(x)\big)\big| \\
&\quad + \frac{1}{\#g_M(x)}\sum_{s=1}^{\#g_M(x)} \big|g_M^{\sigma(s)}(x)\big(q_M^l(x)+q_M^u(x)\big) \\
&\quad - g_N^{\sigma(s)}(x)\big(q_N^l(x)+q_N^u(x)\big)\big| \\
&\quad + \frac{1}{\#g_N(x)}\sum_{t=1}^{\#g_N(x)} \big|g_N^{\sigma(t)}(x)\big(q_N^l(x)+q_N^u(x)\big) \\
&\quad - g_M^{\sigma(t)}(x)\big(q_M^l(x)+q_M^u(x)\big)\big|\Bigg)
\end{aligned}
\tag{39}
$$

3.5. Interval-Valued Probabilistic Dual Hesitant Fuzzy Ordered Weighted Averaging Operator

In fuzzy MCGDM problems, it is essential to fuse DMs' assessments into comprehensive information. Based on Definitions 8 and 10, we propose a basic aggregation operator under IVPDHFSs environment.

Definition 16. Let $d_{IVPi} = \left\{h_i\left|\left[p_{h_i}^l, p_{h_i}^u\right], g_i\right|\left[q_{g_i}^l, q_{g_i}^u\right]\right\}$ $(i = 1, 2, \cdots, n)$ be a collection of IVPDHFEs, $d_{IVP\sigma(i)}$ be the i-th largest element of them, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the associated weight vector such that $0 \le \omega_i \le 1$ and $\sum_{i=1}^{n}\omega_i = 1$, then the IVPDHFOWA operator is a mapping $IVPDHFOWA: D^n \to D$, where

$$
\begin{aligned}
& IVPDHFOWA(d_{IVP1}, d_{IVP2}, \cdots, d_{IVPn}) = \bigoplus_{i=1}^{n}\left(\omega_i d_{IVP\sigma(i)}\right) = \\
& \bigcup_{\substack{\gamma_1\in h_{\sigma(1)}, \gamma_2\in h_{\sigma(2)}, \cdots, \gamma_n\in h_{\sigma(n)} \\ \eta_1\in g_{\sigma(1)}, \eta_2\in g_{\sigma(2)}, \cdots, \eta_n\in g_{\sigma(n)} \\ p_{\gamma_1}^l\in p_{h_{\sigma(1)}}^l, p_{\gamma_1}^u\in p_{h_{\sigma(1)}}^u, p_{\gamma_2}^l\in p_{h_{\sigma(2)}}^l, p_{\gamma_2}^u\in p_{h_{\sigma(2)}}^u, \cdots, p_{\gamma_n}^l\in p_{h_{\sigma(n)}}^l, p_{\gamma_n}^u\in p_{h_{\sigma(n)}}^u \\ q_{\eta_1}^l\in q_{g_{\sigma(1)}}^l, q_{\eta_1}^u\in q_{g_{\sigma(1)}}^u, q_{\eta_2}^l\in q_{g_{\sigma(2)}}^l, q_{\eta_2}^u\in q_{g_{\sigma(2)}}^u, \cdots, q_{\eta_n}^l\in q_{g_{\sigma(n)}}^l, q_{\eta_n}^u\in q_{g_{\sigma(n)}}^u}} \\
& \left\{1 - \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i} \left| \left[\prod_{i=1}^{n} p_{\gamma_i}^l, \prod_{i=1}^{n} p_{\gamma_i}^u\right], \prod_{i=1}^{n}\eta_i^{\omega_i} \right|\left| \left[\prod_{i=1}^{n} q_{\eta_i}^l, \prod_{i=1}^{n} q_{\eta_i}^u\right]\right.\right\}
\end{aligned}
\tag{40}
$$

Theorem 4. *(Idempotency). If all* $d_{IVPi} = \left\{h_i\left|\left[p_{h_i}^l, p_{h_i}^u\right], g_i\right|\left[q_{g_i}^l, q_{g_i}^u\right]\right\}$ $(i = 1, 2, \cdots, n)$ *are totally equal, i.e., for any i,* $d_{IVPi} = d_{IVP}$*, then*

$$
IVPDHFOWA(d_{IVP1}, d_{IVP2}, \cdots, d_{IVPn}) = d_{IVP} \tag{41}
$$

Theorem 5. *(Boundedness). Let* $d_{IVPi} = \left\{h_i | \left[p^l_{h_i}, p^u_{h_i}\right], g_i | \left[q^l_{g_i}, q^u_{g_i}\right]\right\}$ $(i = 1, 2, \cdots, n)$ *be a collection of IVPDHFEs,* $d^-_{IVP} = \left\langle \min h_i | [\min p^l_{h_i}, \min p^u_{h_i}], \max g_i | [\max q^l_{g_i}, \max q^u_{g_i}] \right\rangle$ *and* $d^+_{IVP} = \left\langle \max h_i | [\max p^l_{h_i}, \max p^u_{h_i}], \min g_i | [\min q^l_{g_i}, \min q^u_{g_i}] \right\rangle$*, then*

$$\begin{aligned} IVPDHFOWAd^-_{IVP} &\leq IVPDHFOWA\left(d_{IVP1}, d_{IVP2}, \cdots, d_{IVPn}\right) \\ &\leq IVPDHFOWAd^+_{IVP} \end{aligned} \tag{42}$$

Theorem 6. *(Monotonicity). Let* $d_{IVPi} = \left\{h_i | \left[p^l_{h_i}, p^u_{h_i}\right], g_i | \left[q^l_{g_i}, q^u_{g_i}\right]\right\}$ *and* $d^*_{IVPi} = \left\{h^*_i | \left[p^l_{h^*_i}, p^u_{h^*_i}\right], g^*_i | \left[q^l_{g^*_i}, q^u_{g^*_i}\right]\right\}(i = 1, 2, \cdots, n)$ *be two sets of IVPDHFEs,* $d_{IVP\sigma(i)}$ *and* $d^*_{IVP\sigma(i)}$ *be the i-th largest element in the corresponding set. For the elements in* $d_{IVP\sigma(i)}$ *and* $d^*_{IVP\sigma(i)}$*, if* $\gamma_{h_{\sigma(i)}} \leq \gamma_{h^*_{\sigma(i)}}, \eta_{g_{\sigma(i)}} \geq \eta_{g^*_{\sigma(i)}}, p^l_{\gamma_{h_{\sigma(i)}}} = p^u_{\gamma_{h^*_{\sigma(i)}}}$ *and* $q^i_{\eta_{g_{\sigma(i)}}} = q^u_{\eta_{g^*_{\sigma(i)}}}$*, then*

$$\begin{aligned} &IVPDHFOWA\left(d_{IVP1}, d_{IVP2}, \cdots, d_{IVPn}\right) \leq \\ &IVPDHFOWA\left(d^*_{IVP1}, d^*_{IVP2}, \cdots, d^*_{IVPn}\right) \end{aligned} \tag{43}$$

Theorem 7. *(Commutativity). Let* $d_{IVPi} = \left\{h_i | \left[p^l_{h_i}, p^u_{h_i}\right], g_i | \left[q^l_{g_i}, q^u_{g_i}\right]\right\}$ *and* $d'_{IVPi} = \left\{h'_i | \left[p^l_{h'_i}, p^u_{h'_i}\right], g'_i | \left[q^l_{g'_i}, q^u_{g'_i}\right]\right\}$ *be two sets of IVPDHFEs and* d'_{IVPi} *is any permutation of* d_{IVPi} $(i = 1, 2, \cdots, n)$*, then*

$$\begin{aligned} &IVPDHFOWA\left(d_{IVP1}, d_{IVP2}, \cdots, d_{IVPn}\right) = \\ &IVPDHFOWA\left(d'_{IVP1}, d'_{IVP2}, \cdots, d'_{IVPn}\right) \end{aligned} \tag{44}$$

Proof. *The proofs for Theorems 4–7 are simple and straightforward, and hence, we leave them to the readers for exploration.*

4. A THREE-PHASED MCGDM FRAMEWORK WITH IVPDHFSs

In this section, an MCGDM problem under IVPDHFSs environment is described. Then, a similarity-based measure is developed to derive the weight information of DMs. In addition, inspired by the water-filling theory, we construct a mathematical modeling to obtain the weight vector of criteria under IVPDHFSs circumstance. Further, an extended fuzzy TODIM MCGDM method is constructed to generate the ranking order of alternatives. Finally, a three-phase MCGDM framework is summarized to cope with IVPDHFSs information.

4.1. Description for Interval-Valued Probabilistic Dual Hesitant Fuzzy MCGDM

For an interval-valued probabilistic dual hesitant fuzzy (IVPDHF) MCGDM problem, let $A = \{a_1, a_2, \ldots, a_m\}$ be the set of m candidate alternatives and $C = \{c_1, c_2, \ldots, c_n\}$ be the set of n criteria whose weight vector is $w = \{w_1, w_2, \ldots, w_n\}$, satisfying $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Let $E = \{e_1, e_2, \ldots, e_q\}$ be the set of DMs and $\eta = \{\eta_1, \eta_2, \ldots, \eta_q\}$ be the corresponding weight vector, which satisfies $\eta_k > 0$ and $\sum_{k=1}^{q} \eta_k = 1$. Then, for each DM, his/her assessment on the performance of alternative a_i $(i = 1, 2, \cdots, m)$ over criterion c_j $(j = 1, 2, \cdots, n)$ is depicted in the form of IVPDHFE $d^k_{ij} = \left\{h^k_{ij} | \left[p^l_{h^k_{ij}}, p^u_{h^k_{ij}}\right], g^k_{ij} | \left[q^l_{g^k_{ij}}, q^u_{g^k_{ij}}\right]\right\}$ $(k = 1, 2, \cdots, q)$. Therefore, this MCGDM problem is converted into IVPDHF decision-making matrices $D^k = \left(d^k_{ij}\right)_{m \times n}$ accurately.

In this IVPDHF MCGDM problem, there are two types of criteria, namely, benefit and cost criteria. Besides, the probabilistic information provided by DMs may be incomplete and lost due to the decision-making background and DMs' expertise. Thus, it is essential to normalize the decision-making matrix (DMM). Combing the ideas in Eqs. (13) and (22), we normalize individual DMM $D^k = \left(d^k_{ij}\right)_{m \times n}$ into $\overline{D}^k = \left(\overline{d}^k_{ij}\right)_{m \times n}$ where

$$\overline{d}^k_{ij} = \begin{cases} \left\{h^k_{ij} | \left[\overline{p}^l_{h^k_{ij}}, \overline{p}^u_{h^k_{ij}}\right], g^k_{ij} | \left[\overline{q}^l_{g^k_{ij}}, \overline{q}^u_{g^k_{ij}}\right]\right\} \text{for benefit criterion} \\ \left\{g^k_{ij} | \left[\overline{q}^l_{g^k_{ij}}, \overline{q}^u_{g^k_{ij}}\right], h^k_{ij} | \left[\overline{p}^l_{h^k_{ij}}, \overline{p}^u_{h^k_{ij}}\right]\right\} \text{for cost criterion} \end{cases} \tag{45}$$

4.2. A Similarity-Based Method to Derive the Weight Vector of DMs

In MCGDM problems, it is impossible to have a batch of DMs whose attitude, experience and knowledge level are the same. Thus, it is essential to derive the weights of DMs to describe their contribution to the issue. The DM who has richer experience, more positive attitude and higher knowledge level should be endowed with higher weight. Currently, the similarity-based measure is one of the most preferential approaches to derive the weight vector for DMs. The core idea of similarity-based weight method is to make a pairwise comparison of decision-making matrices. Based on the Eq. (39), the pairwise comparison matrix can be established as

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ c_{q1} & c_{q2} & \cdots & c_{qq} \end{bmatrix}_{q \times q}, \tag{46}$$

where $c_{lv} = S\left(\overline{D}^l, \overline{D}^v\right)(l, v = 1, 2, \cdots, q)$.

Then the weight information $\eta = \{\eta_1, \eta_2, \ldots, \eta_q\}$ for DMs can be acquired by the following equation:

$$\eta_l = \frac{1 + \sum_{v=1}^{q} c_{lv}}{\sum_{l=1}^{q} \left(1 + \sum_{v=1}^{q} c_{lv}\right)} \tag{47}$$

From Eq. (47), it is clear that a higher weight should be assigned to the DM who has slighter divergence with other DMs.

After obtaining the weight vector for DMs, we can aggregate normalized individual decision-making matrices $\overline{D}^k = \left(\overline{d}^k_{ij}\right)_{m \times n}$ $(k = 1, 2, \cdots q)$ into group DMM $D = \left(d_{ij}\right)_{m \times n}$ in terms of DMs' preference character.

4.3. Water-Filling Theory-Based Modeling for Determining the Criteria Weights

Water-filling theory was initially applied to address power optimization selection problem in wireless communication field [44]. It is argued that the sub-channel with a lower signal to noise ratio (SNR) should be allocated with smaller transmit power. The aim of this theory is to maximize the channel capacity.

And this theory can be expressed by the following symbol:

$$T = \sum_{i=1}^{n} \log_2 \left(1 + p_i \frac{\alpha_i^2}{\sigma_i^2}\right), \tag{48}$$

where T denotes the channel capacity, p_i, α_i and σ_i represent the assigned transmit power, gain and noise variance of the i-th sub-channel, respectively.

The main advantage of applying the water-filling theory into MCGDM is that it can unify the criteria dimensions into a compatible scale. If we compare decision criteria to channel, then the weight for each criterion can be regarded as the assigned power of each sub-channel [45].

Inspired by the total capacity of attributes proposed by Liu *et al.* [46], we propose a mathematic modeling to derive the criteria weights under IVPDHFSs environment as follows:

$$\max T = \sum_{j=1}^{n} \log_2 \left[1 + w_j \left(\frac{1}{m} \sum_{i=1}^{m} \frac{s\left(d_{ij}\right)}{\sigma\left(d_{ij}\right)}\right)^2\right]$$

$$s.t. \begin{cases} \sum_{j=1}^{n} w_j = 1 \\ 0 \le w_j \le 1, j = 1, 2, \cdots, n \end{cases}, \tag{M-1}$$

where $s\left(d_{ij}\right)$ and $\sigma\left(d_{ij}\right)$ represent the score function and deviation function of alternative a_i under criterion c_j, respectively.

To solve this model, we introduce the following Lagrange function:

$$L\left(w_j, \lambda\right) = \sum_{j=1}^{n} \log_2 \left[1 + w_j \left(\frac{1}{m} \sum_{i=1}^{m} \frac{s(d_{ij})}{\sigma(d_{ij})}\right)^2\right] + \lambda \left(\sum_{j=1}^{n} w_j - 1\right), \tag{49}$$

where λ is the Lagrange multiplier.

Computing the partial derivative and set

$$\begin{cases} \frac{\partial L}{\partial w_j} = \frac{1}{\ln 2 \cdot \left(1 + w_j \left(\frac{1}{m} \sum_{i=1}^{m} \frac{s\left(d_{ij}\right)}{\sigma\left(d_{ij}\right)}\right)^2\right)} \left(\frac{1}{m} \sum_{i=1}^{m} \frac{s\left(d_{ij}\right)}{\sigma\left(d_{ij}\right)}\right)^2 + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = \sum_{j=1}^{n} w_j - 1 = 0 \end{cases} \tag{50}$$

Solving Eq. (50), we can get

$$w_j = \frac{1}{n} \left(1 + \sum_{j=1}^{n} \left(\frac{1}{m} \sum_{i=1}^{m} \frac{s\left(d_{ij}\right)}{\sigma\left(d_{ij}\right)}\right)^2\right) - \left(\frac{1}{m} \sum_{i=1}^{m} \frac{s\left(d_{ij}\right)}{\sigma\left(d_{ij}\right)}\right)^2 \tag{51}$$

4.4. Acquisition of the Ranking Order by Fuzzy TODIM Method

Based on the weight vector of criteria obtained in Section 4.3, we can calculate the relative weight vector for criteria $\varpi = (\varpi_1, \varpi_2, \cdots, \varpi_n)$, where

$$\varpi_j = \frac{w_j}{\max_j \left\{w_j\right\}} \left(j = 1, 2, ..., n\right) \tag{52}$$

Then the dominance of alternative a_i over each alternative a_s under criterion c_j can be computed using the expression

$$\Phi_j\left(a_i, a_s\right) = \begin{cases} \sqrt{\frac{\varpi_j}{\sum_{j=1}^{n} \varpi_j}} \cdot d\left(a_{ij}, a_{sj}\right), & \text{if } a_{ij} \succ a_{sj} \\ 0, & \text{if } a_{ij} \sim a_{sj} \\ -\frac{1}{t}\sqrt{\frac{\sum_{j=1}^{n} \varpi_j}{\varpi_j}} \cdot d\left(a_{ij}, a_{sj}\right), & \text{if } a_{ij} \prec a_{sj} \end{cases}, \tag{53}$$

where the parameter t represents the attenuation factor of the loss such that $t > 0$. In Eq. (53), three cases can occur: (i) if $a_{ij} \succ a_{sj}$, then it represents a gain; (ii) if $a_{ij} \sim a_{sj}$, then it is nil; (iii) if $a_{ij} \prec a_{sj}$, it represents a loss.

Further, the overall dominance degree of alternative a_i over each alternative a_s can be computed by

$$\Phi\left(a_i, a_s\right) = \sum_{j=1}^{n} \Phi_j\left(a_i, a_s\right) \tag{54}$$

Finally, the comprehensive value of alternative a_i can be obtained via the following equation:

$$\zeta_i = \frac{\sum_{s=1}^{m} \Phi\left(a_i, a_s\right) - \min_i \left\{\sum_{s=1}^{m} \Phi\left(a_i, a_s\right)\right\}}{\max_i \left\{\sum_{s=1}^{m} \Phi\left(a_i, a_s\right)\right\} - \min_i \left\{\sum_{s=1}^{m} \Phi\left(a_i, a_s\right)\right\}} \tag{55}$$

Therefore, the ranking order of alternatives can be acquired by descending the values of $\zeta_i \left(i = 1, 2, \cdots, m\right)$. The higher the comprehensive value ζ_i is, the better the alternative a_i performs.

4.5. A Three-Phased MCGDM Framework with IVPDHFSs

According to the above analysis, a three-phased MCGDM framework with IVPDHFSs is constructed, of which the procedure is summarized as follows:

Phase I: Determination of the weight vector for DMs.

Step 1: Collect the evaluation information from each DMs, and construct individual DMM $D^k = \left(d_{ij}^k\right)_{m\times n}$ $(k = 1, 2, \cdots q)$.

Step 2: Normalize the individual DMM $D^k = \left(d_{ij}^k\right)_{m\times n}$ into $\overline{D}^k = \left(\overline{d}_{ij}^k\right)_{m\times n}$ $(k = 1, 2, \cdots q)$.

Step 3: Calculate the relative similarity degrees of any two decision-making matrices by using Eq. (39).

Step 4: Construct the pairwise similarity comparison matrix based on Eq. (46).

Step 5: Derive the weight vector $\eta = \{\eta_1, \eta_2, \dots, \eta_q\}$ for DMs based on Eq. (47).

Phase II: Determination of criteria weight.

Step 6: Based on the weight information for DMs in Step 5 and IVPDHFOWA operator in Eq. (40), the normalized individual DMM $\overline{D}^k = \left(\overline{d}_{ij}^k\right)_{m\times n}$ $(k = 1, 2, \cdots q)$ can be aggregated into a comprehensive DMM $D = \left(d_{ij}\right)_{m\times n}$.

Step 7: Calculate the score function s and deviation function σ of the performance of alternative a_i over criterion c_j by using Eqs. (35) and (36), respectively.

Step 8: Obtain the weight vector $w = \{w_1, w_2, \dots, w_n\}$ of criteria by solving Model (M-1) mathematically.

Phase III: Acquisition of the ranking order.

Step 9: Calculate the relative criteria weight vector $\varpi = (\varpi_1, \varpi_2, \cdots, \varpi_n)$ by Eq. (52).

Step 10: Calculate the dominance degree of alternative a_i over each alternative a_s under criterion c_j by Eq. (53).

Step 11: Compute the overall dominance degree of alternative a_i over each alternative a_s under all criteria by Eq. (54).

Step 12: Obtain the comprehensive value ζ_i of alternative a_i by using Eq. (55).

Step 13: Rank the alternatives by descending the values of $\zeta_i\,(i = 1, 2, \cdots, m)$ and select the optimal solution.

5. A CASE STUDY CONCERNING ARCTIC GEOPOLITICS RISK EVALUATION

In this section, an example of Arctic geopolitics risk evaluation is cited to demonstrate the applicability and effectiveness of our proposed three-phased MCGDM framework. In addition, comparative analysis and sensitivity analysis are provided to verify the superiority and efficiency of our proposed framework.

5.1. Description of Arctic Geopolitics Risk Evaluation Problems

With the warming of the climate, the Arctic has developed into a hot spot in world politics. Especially since the beginning of the 21st century, the international competition in the Arctic has become more and more fierce. Relevant countries have been competing for the initiative of Arctic affairs from different fields to seize the commanding heights of geopolitics. At the same time, international coordination and cooperation in the Arctic region are also under way, making international relations in the Arctic region complex and confused. Thus, how to grasp the investment opportunity and manage the risk in the Arctic is the prominent decision-making problem.

To solve the above issues, Hao *et al.* [40] conduct a geopolitical risk evaluation for the Arctic region by considering peripheral countries' exploitation actions. Five countries adjacent to the Arctic area are taken into consideration first, such as Russia, Canada, the USA, Denmark and Norway. Besides, China that intends to gain a seat in the Arctic investment and exploitation is also taken into account. Next, a committee of domain experts is organized to make positive risk evaluation on these countries' Arctic resource exploitation actions. After discussion at the meeting, it is agreed that four evaluation criteria are selected, such as c_1 potential military conflicts (MCs), c_2 diplomatic disputes (DDs), c_3 dependence on energy imports (EIs) and c_4 control over marine routes (MRs). The meeting also reaches a consensus that all experts should utilize IVPDHFSs or PDHFSs to express their cognitive preferences on these criteria. All the appraisal data from domain experts is collected and presented in corresponding DMM, as is listed in Tables 1–3.

Remark 5

For an IVPDHFE $d_{IVP} = \left\{h\left|\left[p_h^l, p_h^u\right], g\right|\left[q_g^l, q_g^u\right]\right\}$, if $p_h^l = p_h^u$ and $q_g^l = q_g^u$, then the IVPDHFE reduces to the PDHFE $P = \{h|p, g|q\}$. Thus, the PDHFSs is a special version of IVPDHFSs, and the IVPDHFSs is the generalized form of PDHFSs. From the data in Tables 1–3, we can see that all the expressions are denoted both in the form of IVPDHFE and PDHFE. In conclusion, it is proper to cite Hao *et al.*'s [40] example to make feasible and applicable analysis.

5.2. The Implementation Procedure of Proposed Three-Phased Framework

The procedure of the proposed three-phased framework is summarized as follows:

Step 1: Normalize individual DMM.

The probabilistic information in Tables 1–3 is already normalized.

Step 2: Compute the relative similarity degrees of any two decision-making matrices.

Based on Eq. (39), we can calculate the relative similarity degrees of pairwise matrices as follows:

Step 3: Construct the relative similarity degree matrix.

According to the results in Table 4 and the structure of Eq. (46), the relative similarity degree matrix is constructed as

$$C = \begin{bmatrix} 1 & 0.6133 & 0.6660 \\ 0.6133 & 1 & 0.5815 \\ 0.6660 & 0.5815 & 1 \end{bmatrix} \tag{56}$$

Table 1 The decision-making matrix provided by the domain expert e_1.

	c_1(MC)	c_2(DD)	c_3(EI)	c_4(MR)
The USA	$\langle\{0.7\|0.2, 0.6\|0.2, 0.5\|0.6\}, \{0.2\|1\}\rangle$	$\langle\{0.7\|1\}, \{0.25\|1\}\rangle$	$\langle\{0.2\|1\}, \{0.2\|1\}\rangle$	$\langle\{0.7\|0.5, 0.6\|0.5\}, \{0.3\|1\}\rangle$
Canada	$\langle\{0.1\|1\}, \{0.4\|1\}\rangle$	$\langle\{0.3\|1\}, \{0.7\|1\}\rangle$	$\langle\{0.7\|1\}, \{0.3\|0.5, 0.2\|0.5\}\rangle$	$\langle\{0.3\|1\}, \{0.3\|1\}\rangle$
Russia	$\langle\{0.6\|1\}, \{0.35\|1\}\rangle$	$\langle\{0.56\|1\}, \{0.2\|1\}\rangle$	$\langle\{0.1\|1\}, \{0.7\|1\}\rangle$	$\langle\{0.2\|0.6, 0.4\|0.4\}, \{0.4\|1\}\rangle$
Denmark	$\langle\{0.05\|0.7, 0.2\|0.3\}, \{0.5\|1\}\rangle$	$\langle\{0.3\|0.5, 0.2\|0.5\}, \{0.6\|0.5, 0.5\|0.5\}\rangle$	$\langle\{0.8\|1\}, \{0.15\|1\}\rangle$	$\langle\{0.2\|1\}, \{0.6\|1\}\rangle$
China	$\langle\{0.15\|1\}, \{0.8\|1\}\rangle$	$\langle\{0.5\|1\}, \{0.5\|1\}\rangle$	$\langle\{0.8\|0.6, 0.6\|0.4\}, \{0.15\|1\}\rangle$	$\langle\{0.2\|1\}, \{0.7\|0.9, 0.6\|0.1\}\rangle$
Norway	$\langle\{0.08\|1\}, \{0.6\|1\}\rangle$	$\langle\{0.1\|0.6, 0.3\|0.4\}, \{0.7\|1\}\rangle$	$\langle\{0.3\|1\}, \{0.65\|1\}\rangle$	$\langle\{0.5\|1\}, \{0.2\|0.3, 0.4\|0.7\}\rangle$

MC = military conflict; DD = diplomatic dispute; EI = energy import; MR = marine route.

Table 2 The decision-making matrix provided by the domain expert e_2.

	c_1(MC)	c_2(DD)	c_3(EI)	c_4(MR)
The USA	$\langle\{0.5\|1\}, \{0.5\|1\}\rangle$	$\langle\{0.2\|1\}, \{0.4\|0.8, 0.6\|0.2\}\rangle$	$\langle\{0.7\|0.4, 0.4\|0.6\}, \{0.3\|0.7, 0.2\|0.3\}\rangle$	$\langle\{0.6\|0.7, 0.7\|0.3\}, \{0.25\|1\}\rangle$
Canada	$\langle\{0.3\|0.5, 0.5\|0.5\}, \{0.4\|1\}\rangle$	$\langle\{0.1\|1\}\{0.6\|0.6, 0.8\|0.4\}\rangle$	$\langle\{0.4\|0.8, 0.3\|0.2\}, \{0.5\|0.3, 0.4\|0.7\}\rangle$	$\langle\{0.2\|0.3, 0.3\|0.7\}, \{0.6\|1\}\rangle$
Russia	$\langle\{0.1\|0.1, 0.2\|0.9\}, \{0.5\|1\}\rangle$	$\langle\{0.3\|0.5, 0.2\|0.5\}, \{0.3\|0.5, 0.2\|0.5\}\rangle$	$\langle\{0.2\|1\}, \{0.7\|0.6, 0.5\|0.4\}\rangle$	$\langle\{0.5\|1\}, \{0.4\|1\}\rangle$
Denmark	$\langle\{0.2\|1\}, \{0.7\|0.1, 0.6\|0.9\}\rangle$	$\langle\{0.1\|1\}, \{0.7\|1\}\rangle$	$\langle\{0.2\|1\}, \{0.6\|1\}\rangle$	$\langle\{0.2\|0.8, 0.1\|0.2\}, \{0.3\|0.4, 0.2\|0.6\}\rangle$
China	$\langle\{0.2\|1\}, \{0.7\|1\}\rangle$	$\langle\{0.45\|1\}, \{0.5\|1\}\rangle$	$\langle\{0.8\|0.9, 0.6\|0.1\}, \{0.11\|1\}\rangle$	$\langle\{0.3\|1\}, \{0.2\|1\}\rangle$
Norway	$\langle\{0.4\|0.4, 0.5\|0.6\}, \{0.5\|1\}\rangle$	$\langle\{0.3\|0.4, 0.4\|0.6\}, \{0.7\|1\}\rangle$	$\langle\{0.3\|1\}, \{0.6\|1\}\rangle$	$\langle\{0.2\|1\}, \{0.6\|1\}\rangle$

MC = military conflict; DD = diplomatic dispute; EI = energy import; MR = marine route.

Table 3 The decision-making matrix provided by the domain expert e_3.

	c_1(MC)	c_2(DD)	c_3(EI)	c_4(MR)
The USA	$\langle\{0.4\|1\}, \{0.5\|1\}\rangle$	$\langle\{0.9\|1\}, \{0.1\|1\}\rangle$	$\langle\{0.3\|1\}, \{0.5\|0.4, 0.6\|0.6\}\rangle$	$\langle\{0.6\|1\}, \{0.3\|1\}\rangle$
Canada	$\langle\{0.75\|1\}, \{0.2\|1\}\rangle$	$\langle\{0.4\|1\}, \{0.6\|1\}\rangle$	$\langle\{0.2\|0.7, 0.4\|0.3\}, \{0.2\|1\}\rangle$	$\langle\{0.3\|1\}, \{0.6\|1\}\rangle$
Russia	$\langle\{0.6\|0.6, 0.8\|0.4\}, \{0.1\|1\}\rangle$	$\langle\{0.5\|1\}, \{0.2\|1\}\rangle$	$\langle\{0.1\|1\}, \{0.8\|1\}\rangle$	$\langle\{0.2\|0.7, 0.4\|0.3\}, \{0.6\|1\}\rangle$
Denmark	$\langle\{0.2\|1\}, \{0.7\|1\}\rangle$	$\langle\{0.5\|0.6, 0.7\|0.4\}, \{0.1\|1\}\rangle$	$\langle\{0.3\|0.3, 0.5\|0.7\}, \{0.2\|0.5, 0.5\|0.5\}\rangle$	$\langle\{0.1\|0.6, 0.3\|0.4\}, \{0.6\|1\}\rangle$
China	$\langle\{0.3\|0.7, 0.4\|0.3\}, \{0.4\|0.6, 0.5\|0.4\}\rangle$	$\langle\{0.6\|1\}, \{0.2\|0.5, 0.1\|0.5\}\rangle$	$\langle\{0.7\|1\}, \{0.2\|1\}\rangle$	$\langle\{0.1\|0.45, 0.3\|0.55\}, \{0.5\|0.5, 0.65\|0.5\}\rangle$
Norway	$\langle\{0.2\|0.2, 0.1\|0.8\}, \{0.7\|1\}\rangle$	$\langle\{0.2\|1\}, \{0.8\|1\}\rangle$	$\langle\{0.2\|0.8, 0.3\|0.2\}, \{0.6\|1\}\rangle$	$\langle\{0.35\|1\}, \{0.5\|0.5, 0.6\|0.5\}\rangle$

MC = military conflict; DD = diplomatic dispute; EI = energy import; MR = marine route.

Table 4 The relative similarity degrees of any two matrices.

	Tables 1 and 2	Tables 1 and 3	Tables 2 and 3
Relative similarity degree	0.6133	0.6660	0.5815

Step 4: Determine the weight vector for DMs.

Referring to Eq. (47), we can derive the weight vector for DMs, as is shown in Table 5.

Step 5: Aggregate all individual decision-making matrices into a comprehensive group DMM.

Based on proposed IVFDHFOWA operator and derived weight vector for DMs, we can aggregate all the individual decision-making matrices into a comprehensive group DMM, as is displayed in Table 6.

Step 6: Compute the score function and deviation function of each alternative.

To make comparison effective and obtain criteria weights efficiently, we calculate the score function and deviation function of the performance of each alternative over criteria by utilizing Eqs. (35) and (36), as is outlined in Table 7.

Table 5 The weight information of DMs.

DM	e_1	e_2	e_3
Weight	0.3373	0.3286	0.3340

DM = decision-maker.

Step 7: Determine the criteria weights.

By putting the score function and deviation function obtained in Step 6 into the Eq. (51), we can derive the weight vector for criteria, which is shown in Table 8.

Step 8: Compute the relative criteria weights.

From the results in Table 8, it is clear that criterion c_4 is endowed with the highest weight. Thus, the relative criteria weights can be calculated by utilizing the Eq. (52), as is shown in Table 9.

Step 9: Compute the dominance degree of each alternative under each criterion.

We use the score function and deviation function obtained in Step 6 to make a simple comparison, then based on the relative criteria weights and Eqs. (38) and (53), we can obtain the dominance degree of alternative a_i over each alternative a_s under criterion

Table 6 The comprehensive group decision-making matrix.

	c_1(MC)
The USA	$\langle\{0.4686\|0.6, 0.5527\|0.2, 0.5071\|0.2\}, \{0.3671\|1\}\rangle$
Canada	$\langle\{0.5164\|0.5, 0.4598\|0.5\}, \{0.3173\|1\}\rangle$
Russia	$\langle\{0.4977\|0.54, 0.6015\|0.36, 0.4778\|0.06, 0.5858\|0.04\}, \{0.2590\|1\}\rangle$
Denmark	$\langle\{0.1523\|0.7, 0.0878\|0.3\}, \{0.5940\|0.9, 0.6249\|0.1\}\rangle$
China	$\langle\{0.2191\|0.7, 0.2583\|0.3\}, \{0.6074\|0.6, 0.6544\|0.4\}\rangle$
Norway	$\langle\{0.2526\|0.48, 0.2064\|0.32, 0.2814\|0.12, 0.2370\|0.08\}, \{0.5950\|1\}\rangle$
	c_2(DD)
The USA	$\langle\{0.7131\|1\}, \{0.2148\|0.8, 0.2454\|0.2\}\rangle$
Canada	$\langle\{0.2779\|1\}, \{0.6320\|0.6, 0.6947\|0.4\}\rangle$
Russia	$\{0.4651\|0.5, 0.4411\|0.5\}, \{0.2285\|0.5, 0.2\|0.5\}$
Denmark	$\langle\{0.3206\|0.3, 0.2893\|0.3, 0.4271\|0.2, 0.4008\|0.2\}, \{0.3469\|0.5, 0.3262\|0.5\}\rangle$
China	$\langle\{0.5211\|1\}, \{0.3682\|0.5, 0.2921\|0.5\}\rangle$
Norway	$\langle\{0.2427\|0.36, 0.3042\|0.24, 0.2033\|0.24, 0.2681\|0.16\}, \{0.7319\|1\}\rangle$
	c_3(EI)
The USA	$\langle\{0.3039\|0.6, 0.4457\|0.4\}, \{0.3298\|0.42, 0.3103\|0.28, 0.2887\|0.18, 0.2716\|0.12\}\rangle$
Canada	$\langle\{0.4772\|0.56, 0.5251\|0.24, 0.4500\|0.14, 0.5004\|0.06,$ $\{0.2880\|0.35, 0.2512\|0.35, 0.3099\|0.15, 0.2703\|0.15\}\rangle$
Russia	$\langle\{0.1342\|1\}, \{0.7319\|0.6, 0.6553\|0.4\}\rangle$
Denmark	$\langle\{0.5716\|0.7, 0.5207\|0.3\}, \{0.3537\|0.5, 0.2604\|0.5\}\rangle$
China	$\langle\{0.7710\|0.54, 0.7107\|0.36, 0.7124\|0.06, 0.6367\|0.04\}, \{0.1491\|1\}\rangle$
Norway	$\langle\{0.2681\|0.8, 0.3\|0.2\}, \{0.6164\|1\}\rangle$
	c_4(MR)
The USA	$\langle\{0.6370\|0.35, 0.6\|0.35, 0.6697\|0.15, 0.6361\|0.15\}, \{0.2826\|1\}\rangle$
Canada	$\langle\{0.3\|0.7, 0.2686\|0.3\}, \{0.4749\|1\}\rangle$
Russia	$\langle\{0.3145\|0.42, 0.3779\|0.28, 0.3773\|0.18, 0.4349\|0.12\}, \{0.4580\|1\}\rangle$
Denmark	$\langle\{0.1679\|0.48, 0.2349\|0.32, 0.1351\|0.12, 0.2047\|0.08\}, \{0.4182\|0.6, 0.4778\|0.4\}\rangle$
China	$\langle\{0.2438\|0.55, 0.1776\|0.45\}, \{0.4524\|0.45, 0.4145\|0.45, 0.4295\|0.05, 0.3935\|0.05\}\rangle$
Norway	$\langle\{0.3630\|1\}, \{0.5233\|0.35, 0.4924\|0.35, 0.4142\|0.15, 0.3897\|0.15\}\rangle$

MC = military conflict; DD = diplomatic dispute; EI = energy import; MR = marine route.

Table 7 The score function and deviation function of alternatives.

	c_1(MC)		c_2(DD)	
	Score Function	**Deviation Function**	**Score Function**	**Deviation Function**
The USA	0.2521	0.3805	0.9843	1.1458
Canada	0.3415	0.2138	−0.7584	2.4813
Russia	0.5569	0.4284	0.4777	0.3751
Denmark	−0.9284	2.6285	0.0239	0.6423
China	−0.7907	2.4706	0.3821	0.2167
Norway	−0.7099	2.2828	−0.9597	2.9433
	c_3(EI)		**c_4(MR)**	
	Score Function	**Deviation Function**	**Score Function**	**Deviation Function**
The USA	0.1013	0.4817	0.6925	0.5877
Canada	0.4211	0.2299	−0.3687	1.5142
Russia	−1.1342	3.1557	−0.2000	1.2216
Denmark	0.4986	0.2924	−0.5073	1.6657
China	1.1825	1.5904	−0.4344	1.5306
Norway	−0.6839	2.2845	−0.2261	1.2982

MC = military conflict; DD = diplomatic dispute; EI = energy import; MR = marine route.

Table 8 Criteria weights.

Criteria	c_1(MC)	c_2(DD)	c_3(EI)	c_4(MR)
Weight	0.2895	0.1711	0.0658	0.4737

MC = military conflict; DD = diplomatic dispute; EI = energy import; MR = marine route.

Table 9 Relative criteria weights.

Criteria	c_1(MC)	c_2(DD)	c_3(EI)	c_4(MR)
Relative weight	0.6111	0.3611	0.1389	1

MC = military conflict; DD = diplomatic dispute; EI = energy import; MR = marine route.

$c_j\left(i, s = 1, 2, \cdots, m; j = 1, 2, \cdots, n\right)$. Here, we set the value of the attenuation factor t to 1.

Step 10: Compute the overall dominance degrees of pairwise alternatives.

We can calculate the overall dominance degree of alternative a_i over each alternative a_s under all criteria by utilizing the Eq. (54), as is displayed in Table 10.

Step 11: Compute the comprehensive value of alternatives.

Based on the data in Table 10, we can calculate the overall dominance degree of alternative a_i and obtain the comprehensive value ζ_i of alternative a_i by using Eq. (55), as is represented in Table 11.

Step 12: Rank the alternatives.

By descending the comprehensive values of six countries in Table 11, we can obtain the ranking order which is the USA ≻ China ≻ Canada ≻ Russia ≻ Denmark ≻ Norway.

5.3. Comparative Analysis

In this subsection, a comparative analysis is conducted to verify the validity and rationality of our three-phased MCGDM framework.

5.3.1. Comparison with Hao *et al.*'s visualization method based on the entropy of PDHFSs [40]

We cite the example from Hao *et al.* [40], so we can make comparison and discussion directly. The result acquired from Hao *et al.*'s visualization method based on the entropy of PDHFSs [40] is the USA ≻ Denmark ≻ China ≻ Canada ≻ Russia ≻ Norway. Compared with the ranking obtained by the proposed three-phased framework, although the best and the worst countries are the same, three are some differences. And we make further discussion from the following points:

First, Hao *et al.* [40] present the concept of PDHFSs, which is a special case of IVPDHFSs. Thus, IVPDHFSs has more advantages than PDHFSs, especially in the representation of probabilistic hesitant preference.

Second, in Hao *et al.*'s visualization method based on the entropy of PDHFSs [40], the weight information of DMs and criteria is subjectively assumed, which may not reflect the relative professional level of DMs and the objective importance degree of criteria. In this paper, we employ the relative similarity degree of decision-making matrices to derive the weight vector of DMs objectively. Besides, the water-filling theory is first introduced to IVPDHFSs environment to obtain criteria weights mathematically.

Third, Hao *et al.* [40] propose the entropy of PDHFSs, which is denoted by the following symbol:

$$E\left(p\right)=\frac{1}{l}\sum_{i=1}^{l}e\left(\xi^{\delta(i)},\zeta^{\delta(i)}\right), \tag{57}$$

where $e\left(x,y\right)=1-\frac{x^{\alpha}+(1-y)^{\beta}}{2}$, $\xi^{\delta(i)}=h_{\delta(i)}\cdot p_{h_{\delta(i)}}-g_{\delta(i)}\cdot q_{g_{\delta(i)}}$ and $\zeta^{\delta(i)}=1-\left(h_{\delta(i)}\cdot p_{h_{\delta(i)}}+g_{\delta(i)}\cdot q_{g_{\delta(i)}}\right)$. Then Hao *et al.* [40] give the limited condition that $E\left(p\right)=1$ if and only if $\xi=0$ and $\zeta=1$. But on second thought, if $\xi=0$ and $\zeta=1$, then $h\cdot p-g\cdot q=0$ and $h\cdot p+g\cdot q=0$, thus we can get $h\cdot p=g\cdot q=0$, i.e., the elements in membership function and non-membership function are all zero, and further, this PDHFE is empty. Consequently, the entropy measure proposed by Hao *et al.* [40] is defective and unreliable, leading to an imperfect and inadequate ranking order. By contrast, we apply the classical fuzzy TODIM method to make risk evaluation in IVPDHFSs and obtain a reasonable and sufficient result.

5.3.2. Comparison with Wang *et al.*'s dual hesitant fuzzy weighted average operator [47]

It is acknowledged that IVPDHFSs and PDHFSs are the extensions of DHFSs. Hence, we can transform the PDHFE into the DHFE by multiplying the element value and its corresponding probability value. Let $P=\left\langle\underset{\gamma\in h,p_{\gamma}\in p_h}{\cup}\left\{\gamma|p_{\gamma}\right\},\underset{\eta\in g,q_{\eta}\in q_g}{\cup}\left\{\eta|q_{\eta}\right\}\right\rangle$ be a PDHFE, then $D=\left\langle\underset{\gamma\in h,p_{\gamma}\in p_h}{\cup}\left\{\gamma\cdot p_{\gamma}\right\},\underset{\eta\in g,q_{\eta}\in q_g}{\cup}\left\{\eta\cdot q_{\eta}\right\}\right\rangle$ can be the translated DHFE. Based on the example described in Section 5.1, the ranking order acquired from Wang *et al.*'s dual hesitant fuzzy weighted average operator [47] is USA ≻ China ≻ Norway ≻ Denmark ≻ Canada ≻ Russia. Compared with the result obtained from

Table 10 The overall dominance degrees of pairwise alternatives.

$\Phi(a_i,a_s)$	Dominance Degree	$\Phi(a_i,a_s)$	Dominance Degree	$\Phi(a_i,a_s)$	Dominance Degree
$\Phi(a_1,a_2)$	−0.0306	$\Phi(a_3,a_1)$	−11.5888	$\Phi(a_5,a_1)$	−3.5836
$\Phi(a_1,a_3)$	1.5773	$\Phi(a_3,a_2)$	−5.2394	$\Phi(a_5,a_2)$	−2.5831
$\Phi(a_1,a_4)$	−1.1509	$\Phi(a_3,a_4)$	−4.3763	$\Phi(a_5,a_3)$	−6.1922
$\Phi(a_1,a_5)$	−1.9894	$\Phi(a_3,a_5)$	−5.3326	$\Phi(a_5,a_4)$	1.7288
$\Phi(a_1,a_6)$	3.6525	$\Phi(a_3,a_6)$	−0.9679	$\Phi(a_5,a_6)$	−1.5383
$\Phi(a_2,a_1)$	−7.5690	$\Phi(a_4,a_1)$	−7.8652	$\Phi(a_6,a_1)$	−23.3851
$\Phi(a_2,a_3)$	−3.6376	$\Phi(a_4,a_2)$	−4.1633	$\Phi(a_6,a_2)$	−16.9297
$\Phi(a_2,a_4)$	−5.0951	$\Phi(a_4,a_3)$	−5.9432	$\Phi(a_6,a_3)$	−13.5330
$\Phi(a_2,a_5)$	−4.7533	$\Phi(a_4,a_5)$	−8.8851	$\Phi(a_6,a_4)$	−13.4002
$\Phi(a_2,a_6)$	−0.4164	$\Phi(a_4,a_6)$	−2.1818	$\Phi(a_6,a_5)$	−18.0163

Table 11 The overall dominance degrees and comprehensive values of alternatives.

	The USA	Canada	Russia	Denmark	China	Norway
Dominance degree	2.0588	−21.4714	−27.5050	−29.0386	−12.1683	−85.2643
Comprehensive value	1	0.7305	0.6614	0.6439	0.8371	0
Ranking	1	3	4	5	2	6

our three-phased framework, the optimal and suboptimal countries are the same, but three are still some differences in theory and method.

In theory, DHFSs does not have the ability to reflect DMs' probabilistic preference, even interval probabilistic preference. Thus, while applying Wang *et al.*'s dual hesitant fuzzy weighted average operator [47] to tackle the problem described in Section 5.1, the probabilistic preference information is lost and ignored, which is unreliable and unrealistic.

In terms of method, Wang *et al.* [47] utilize the weighted average operator to aggregate information, while we apply the proposed ordered weighted averaging operator to make information fusion. In this paper, all the elements with corresponding probabilistic information are reordered by descending before information fusion, which can consider the risk preference characteristics of DMs and enable the final result more acceptable and reasonable. In addition, Wang *et al.* [47] use the score function to measure the overall performance of each alternative, which is regarded as a simple and single operation. In contrary, we employ the fuzzy TODIM method which takes DMs' attitude toward loss into account and acquire the final result by pairwise comparison instead of simple calculation.

As discussed above, we can conclude that our three-phased MCGDM framework within IVPDHFSs is more effective and efficient.

5.4. Sensitive Analysis of Parameter *t*

In the fuzzy TODIM method, the parameter t in Eq. (53) represents the attenuation coefficient of loss, which can influence the partial dominance degree when there is loss and the final result to some extent.

When there is a loss, the partial dominance degree changes according to the value of t. Different values of t also influence the shape of prospect function. Figure 1 depicts the prospect value function with two different values of the parameter t, i.e., $t = 1$ and $t = 2.5$. From Figure 1, it is clear that in the first quadrant, the prospect functions with $t = 1$ and $t = 2.5$ have the same shape, which indicates that there is a gain in the first quadrant. Thus, the parameter t has no effect on the partial dominance degree when there is a gain. But in the third quadrant, the prospect function with $t = 1$ has distinct shape from that with $t = 2.5$, moreover, the shape of the prospect function with $t = 1$ is deeper. Thus, we can conclude when there is a loss, the larger the value of t is, the greater the partial dominance degree will be, and the flatter the shape of prospect function will become.

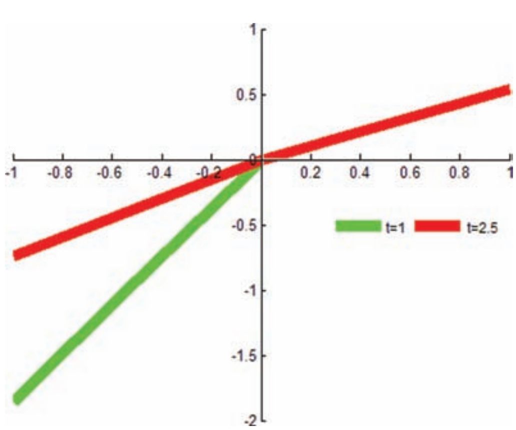

Figure 1 The prospect function with $t = 1$ and $t = 2.5$.

Furthermore, the ranking result is also varied with different values of t. To this end, we make further investigation on the final ranking order with the value of t varying from 0.025 to 25, as is outlined in Table 12.

From the results in Table 12, we can observe that the ranking order has slight changes as the value of t varies from 0.025 to 25. The best candidate is always the USA and the worst one is always Norway.

In theory, when $t < 1$, the partial dominance degree has a reverse relationship with the value of t, namely, the loss is gradually strengthened with the value of t changing from 1 to 0, but when $t \geq 1$, the partial dominance degree has a positive relation with the value of t, that is, the loss is increasingly receded as the value of t increases. In this paper, this phenomenon is proved.

According to previous research results, the results are more reliable and convincing when the value of t is between 2 and 2.5. In this paper, the ranking order of six countries remains unaltered when the value of t varies from 2 to 2.5, indicating the robustness of the fuzzy TODIM method.

6. CONCLUSIONS

To overcome the drawbacks of PDHFSs, we extend single-valued occurring probability into interval-valued probability and propose the concept of IVPDHFSs. First, a series of IVPDHFSs are defined, such as generalized IVPDHFSs, ordered IVPDHFSs and normalized IVPDHFSs. We also give some operations and comparison method for IVPDHFEs. In addition, the ordered distance measure is defined to calculate the deviation of any two IVPDHFSs, the ordered similarity measure is also defined to derive the weight information of DMs. Moreover, to fuse the information provided by multiple DMs, the IVPDHFOWA operator is proposed. Based on the criteria weights derived from the water-filling theory- based model, a three-phased MCGDM framework is designed under IVPDHFSs circumstance. Finally, an example regarding Arctic risk evaluation is introduced to demonstrate the feasibility and acceptance of our three-phased MCGDM framework. The results of comparative analysis and sensitive analysis also show that the

Table 12 The ranking order with different values of t.

t	Ranking Order
$t = 0.025$	The USA ≻ China ≻ Canada ≻ Denmark ≻ Russia ≻ Norway
$t = 0.25$	The USA ≻ China ≻ Canada ≻ Denmark ≻ Russia ≻ Norway
$t = 0.5$	The USA ≻ China ≻ Canada ≻ Denmark ≻ Russia ≻ Norway
$t = 1$	The USA ≻ China ≻ Canada ≻ Russia ≻ Denmark ≻ Norway
$t = 2$	The USA ≻ China ≻ Canada ≻ Russia ≻ Denmark ≻ Norway
$t = 2.25$	The USA ≻ China ≻ Canada ≻ Russia ≻ Denmark ≻ Norway
$t = 2.5$	The USA ≻ China ≻ Canada ≻ Russia ≻ Denmark ≻ Norway
$t = 5$	The USA ≻ China ≻ Russia ≻ Canada ≻ Denmark ≻ Norway
$t = 10$	The USA ≻ Russia ≻ China ≻ Canada ≻ Denmark ≻ Norway
$t = 25$	The USA ≻ Russia ≻ Canada ≻ China ≻ Denmark ≻ Norway

proposed three-phased MCGDM framework is reasonable and efficient.

In future research, we intend to investigate IVPDHFSs theory and apply more MCDM techniques to cope with practical problems within IVPDHFSs environment.

CONFLICT OF INTEREST

The authors declare no conflicts of interest.

AUTHORS' CONTRIBUTIONS

Peide Liu: Conceptualization, methodology, formal analysis, writing–writing–review and editing, supervision, funding acquisition.

Shufeng Cheng: Conceptualization, methodology, validation, investigation, writing–original draft preparation, visualization.

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