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Search for the optimal branching structure from paired copulas when forming an investment portfolio

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***Abstract* — The article is devoted to the problem of modeling the stock returns of companies in various industry and country affiliations. The main attention is paid to comparison of two methods of structuring a constructure from paired copulas: the maximum spanning tree method and the author's method, which involves the use of industry and country stock indices. The use of stock indices in structuring a constructure from paired copulas allows one to identify the systemic risk component and reduce the impact of temporary false correlations between assets on investment decisions made. Comparison of the effective boundaries of portfolios built using different profitability models showed the advantage of double copula designs compared to simple Archimedean copulas. The results of the Kupiec test testify that the author's method of structuring the yield model allows one to obtain more conservative risk assessments of the investment portfolio in comparison with the maximum spanning tree method. As part of testing two strategies for optimizing an investment portfolio over a sufficiently long period of time, it was shown that the author's method of structuring a constructure from paired copulas allows an investor to receive a greater accumulated income. The results will be useful to portfolio managers, as well as private investors using quantitative methods in managing their investment portfolio.**

Keywords — copula, pair-copula construction, international diversification, stock portfolio management

I. INTRODUCTION

The current stage of development of the global economy is characterized by an increase in the relationship between national financial markets, which leads to the spread of effects such as volatility flows and "financial infection". The varying degrees of exposure of national financial markets to these effects creates an asymmetry in the relationship between the returns of financial instruments. This asymmetry lies in the fact that individual pairs of assets with varying degrees are prone to a joint decrease and growth. In today's highly competitive environment, an investor needs to take this asymmetry into account when evaluating the return and risk of an investment. The problem of accounting for asymmetries in the relationship between the returns of securities for the international investment portfolio is of particular importance.

The portfolio theory is based on statistical analysis, the purpose of which is to choose the most optimal ratio between profitability and investment risk. The most common method of modeling stock returns used by professional portfolio managers is CAPM (Capital Assets Pricing Model) [1]. The main advantage of the classic CAPM is its computational simplicity, however, it has a number of disadvantages. One of these drawbacks is the assumption of a normal distribution of securities returns. The current level of computer technology development allows the use of more complex models that take into account the characteristic features of the joint distribution of returns, such as asymmetry in the relationship between various assets and markets [2, 3, 4].

Constructions from paired copulas (CPCs) are a relatively new way of modeling the joint distribution of random variables, allowing the asymmetry of the relationship between them to be taken into account. As an application to modeling in financial markets, CPKs are a way of defining a joint distribution of asset returns in the form of a combination of pair copula-functions of conditional and unconditional private distributions of returns of each of them. A pair copula is a joint distribution function of two random variables, in which two partial distribution functions (conditional or unconditional) are considered as arguments. The use of conditional paired copulas in modeling multidimensional distributions was first considered in the work by G. Joe [5]. In the works of Bedford and Cook [6, 7], a graphical model of regular vines was proposed, which is a convenient way of representing the decomposition of the distribution density function into the product of the densities of paired copulas and densities of particular distributions. In [2], two particular cases of regular branching were considered: canonical vine, abbr. C-vine) and drawable vine, abbr. D-vine), an algorithm for estimating the parameters of C- and D-branches, as well as the generation of observations, is proposed. In [8], algorithms for estimating the parameters of regular branches and generating

observations are presented, a method for choosing the optimal branch, based on the search for the minimum spanning tree, is proposed. This method involves maximizing the sum of the Kendall correlation coefficients on each branch. There are other ways of forming the structure of the CPC, for example, maximizing the function of the tail dependence, tested in the work of A.I. Travkin [9].

A different way of constructing a branch is described in the work of Hayk Christian Brehman and Claudia Kzado [10]. This method involves the allocation of the systemic risk component by introducing into the model the distribution of returns of country stock indices. At the same time, the authors significantly limit the structure of the CPC, the relationship between national financial markets is modeled using a pair of independent copulas, and at the higher levels of branching, a Gaussian copula is used to model the relationships between assets. These restrictions were established in order to reduce the time for evaluating the parameters of the CPC, as well as to bring the model to CAPM as close as possible. In our opinion, these restrictions can significantly reduce the accuracy of modeling the relationship between financial markets and individual assets, given that at the level of the first branch of the CPC, the most significant relationships between assets are reflected. In this paper, we consider the author's method of modeling the joint distribution of asset returns of various industry and country affiliations. This method involves the formation of the first branch of the CPC by industry and country principle using stock indices. The remaining branches are proposed to be structured according to the maximum spanning tree method. The considered method of structuring CPC, on the one hand, allows us to identify the systemic component of risk and reduce the effect of temporary false correlations on the quality of the model, and on the other hand, takes into account the asymmetry in the relationships between the stock markets of individual countries. The author’s method of forming the CPC structure (R-Vine-INDX) is compared with the classical maximum spanning tree method (R-Vine-MST), which is based on maximizing the sum of the Kendall correlation coefficients at each branch level.

II. Methods

In order to evaluate the effectiveness of using various methods of structuring CPC on the correct branches when managing a stock portfolio, three problems were solved in this paper. First, effective VaR (Value at Risk) boundaries are constructed for various ways of modeling the joint distribution of securities returns. Secondly, various copula models were tested for accuracy of VaR prediction. Thirdly, two problems of portfolio optimization were tested over a sufficiently long time horizon, while various optimality criteria and various methods for constructing CPC branching (with the author’s restriction on the structure of the first tree and without restriction) were examined.

A. Initial data

The source data of the study are the daily and weekly ruble returns of liquid stocks of Russian and American companies from various sectors of the economy for the period from January 1, 2007 to December 23, 2016, 10 shares for each of the countries, as well as data on the rate MIBOR for a period of 31 to 90 days. As part of the study of effective boundaries, and testing various portfolio optimization strategies, daily periodicity data were used to evaluate the copula parameters, and data on weekly stock and index returns were used to test the adequacy of risk assessment by various CPC models. A complete list of stocks and indices used in testing various copula models is given in Table 1.

B. Method for estimating the parameters of copula functions

Evaluation of Archimedean and elliptical copulas is carried out by the classical method IFM (Inference Function for Margins) - pseudo-observations are first calculated using empirical functions of private distributions of asset returns, then the parameters of the copula function are estimated using the maximum likelihood method. For the purpose of estimating CCP, pseudo-observations are also first calculated using empirical particular distribution functions, after which the CPC structure is formed and the parameters of paired copulas in the branch edges are evaluated simultaneously. The structure of each tree is formed by the maximum spanning tree method (taking into account the restrictions imposed by the type of branching, as well as the restrictions on the first tree established by the author). The generation of observations on the obtained models is carried out according to the procedure described in [8].

For the purpose of evaluating the parameters of copulas and PDA models, function libraries for the R language were used: "copula", "CDVine" and "VineCopula". The following copulas were used in the formation of the CPC: Student, Gauss, Clayton, Gumbel, Frank, Town, Joe, BB1, BB6, BB7, BB8 and their rotation.

TABLE I. List of Stocks and Stock Indices

Ticker	Company/index
Shares of Russian companies	
AFLT	Aeroflot - Russian Airlines, PJSC
GAZP	Gazprom, PJSC
GMKN	MMC Norilsk Nickel, PJSC
LKOH	Lukoil, PJSC
MGNT	Magnit, PJSC
NLMK	Novolipetsk Metallurgical Plant, PJSC
RASP	Raspadskaya, PJSC
ROSN	Rosneft Oil Company, PJSC
SBER	Sberbank, PJSC
TATN	Tatneft, PJSC
US stocks	
AAPL	Apple Inc.
AMZN	Amazon.com, Inc.
GOOGL	Alphabet Inc.
KO	The coca-cola company
MCD	Mcdonald's corporation
MSFT	Microsoft Corporation
NKE	NIKE Inc.
PEP	PepsiCo, Inc.
PG	The Procter & Gamble Company
WMT	Walmart Inc.
Sectoral and Country Stock Indices	
W1DOW	Dow Jones Global Index
MOEX	Moscow Exchanges Index
MICEX O&G	MICEX Index of Oil and Gas Companies

MICEX M&M	MICEX Index of the Mining and Metallurgical Sector
MSCI US	MSCI USA Index
MSCI US CS	MSCI USA Consumer Staples Index
MSCI US CD	MSCI USA Consumer Discretionary Index
MSCI US IT	MSCI USA Information Technology Index

C. Modeling the effective border of the investment portfolio using various copula models

In [11], the effective boundaries obtained for various models of copulas were already investigated, however, these boundaries were constructed based on the monthly returns of industry indices of stocks of American companies, and portfolios consisting of 3-12 indices were considered. Such a set of "assets" does not allow to properly assess the advantages of the CPC due to their small number and insufficient variety. This study examines a much wider range of assets with different industry and country affiliations.

To build effective boundaries, the following copula models are used:

- Gaussian copula;
- Student's t copula;
- Clayton copula;
- Survival Gumbel copula;
- CPC on the correct branch, the structure is formed by the method of maximum spanning tree (R-Vine-MST);
- CPC on the correct branching, the structure of the first tree is formed according to the industry and country principles, the structure of the remaining trees is formed by the maximum spanning tree method (R-Vine-INDX);
- CPC on canonical branching, the structure is formed by the method of maximum spanning tree;
- CPC on elongated branching, the structure of the first tree is formed by the maximum spanning tree method (the structure of the remaining trees is determined by the structure of the first tree).

Multivariate Clayton copulas and Survival Gumbel copulas are considered as an alternative to the CPC because they allow such a stylized fact as the left-side asymmetry of the relationship between stock returns.

In paper [11], two parametric models of partial distributions are considered: the normal and the skew-t distribution. In this study, comparison of copula models is emphasized; empirical distribution functions are used as marginal distributions.

The procedure for forming an effective border consists of three stages:

1) distribution models are estimated using data on daily returns on securities and indices for the period from January 1, 2007 to December 23, 2016;
2) according to the estimates of distribution models, 100,000 daily yield observations for each security are generated;
3) the construction of an effective boundary based on the generated observations is being carried out.

D. Assessing the quality of risk prediction by various copul models

A number of studies [12, 13, 10, 14] have already demonstrated the benefits of CPC for the purpose of assessing the risk of an investment portfolio. In [13], the authors compare the quality of risk prediction (VaR and CVaR) with different PDAs, a model with a dynamic DCC-GARCH correlation matrix, and also a historical modeling method. To evaluate the quality of forecasting VaR, the authors used the Kupiec test [15], and to assess the quality of forecasting CVaR, they used the McNeill test [16]. The initial data were the daily returns on 10 country wide stock indices. The results of the study showed that CPCs provide a more accurate estimate of the VaR of an equilibrium portfolio compared to other methods, while CPCs on elongated branches give the most accurate assessment of CVaR.

In order to evaluate the effectiveness of using various copulas to evaluate VaR, the following sequence of actions is performed:

1) During the time interval from December 24, 2014 to December 21, 2016, in the middle of the week, six models of the joint distribution of stock returns and indices are evaluated based on data on weekly returns for the previous 200 weeks. We consider Archimedean copulas as distribution models, CPCs on canonical, elongated, and arbitrary regular branches. For PDAs on arbitrary regular branches, two ways of forming the first tree are considered - the method of maximum spanning tree (R-Vine-MST) and structuring according to the industry and stand-by principle using stock indices (R-Vine-INDX).
2) According to the results of evaluating the copula models, for each of them 100,000 observations are generated and the predicted VaR is calculated with a significance level of 0.05 (VaR is calculated as a loss in % and is taken with a negative sign) for a portfolio with equal weights.
3) The actual return on a balanced portfolio over the next week is compared with the forecasted level of VaR.
4) Likelihood statistics are calculated from the predicted VaR values and the actual return on equal-weighted portfolio.

In order to calculate the likelihood ratio statistics, a vector is formed from ξ the data obtained using the above procedure:

$$\xi_t = \begin{cases} 1, \text{если } r_t < VaR_t \\ 0, \text{иначе} \end{cases} \quad (1)$$

where:

r_t - historical portfolio return over a period of time t;
VaR_t - forecast value of VaR for a portfolio with equal weights at the beginning of the period t.

Assuming that the vector ξ has a Bernoulli distribution, the likelihood function can be written as follows:

$$L(\pi) = \prod_{t=1}^{T} (1-\pi)^{1-\xi_t} \pi^{\xi_t} = (1-\pi)^{K_0} \pi^{K_1} \quad , \qquad (2)$$

where: T - number of observations;

π - the proportion of observations for which the yield was below the forecast level VaR_t;

K_0 - the number of observations in which the actual portfolio return exceeded the forecast level VaR_t;

K_1 - the number of observations in which the actual portfolio return was below the forecast level VaR_t.

Zero and alternative hypotheses for the test are formulated as follows:

H_0 : $\alpha = \pi$ (the model provides a fairly accurate estimate of the portfolio VaR);

H_1 : $\alpha < \pi$ (the model does not provide an accurate estimate of the portfolio VaR),

where: α - a given level of significance for VaR_t.

The estimate of the likelihood function (1) from actual observations of portfolio returns is calculated using the following formula:

$$L(\hat{\pi}) = (1-\hat{\pi})^{K_0} \hat{\pi}^{K_1} \qquad (3)$$

For a known α maximum likelihood function value can be calculated by the formula:

$$L(a) = (1-\alpha)^{K_0} \alpha^{K_1} \qquad (4)$$

The likelihood ratio coefficient in this case is determined as follows:

$$LR = -2ln\left\{\frac{L(\alpha)}{L(\hat{\pi})}\right\} \qquad (5)$$

Statistics LR have χ_1^2 a (Chi-squared) distribution with one degree of freedom. For a confidence level of 0.95, the critical value of the statistics LR will be approximately 3.84, if $LR > 3.84$, then it is concluded that the joint distribution model does not allow to obtain a sufficiently accurate estimate of the portfolio VaR.

E. Multiple Periodic Testing of Portfolio Optimization Models

The profitability and risk of the investment portfolio obtained as a result of the search procedure for optimal asset weights depend on the quality of approximation by the model of the true joint distribution of asset returns. In order to assess the suitability of using various methods of structuring PDA branches, we consider the results of multi-period portfolio optimization for various methods of forming a PDA structure (R-Vine-MST and R-Vine-INDX). The minimum CVaR (Conditional Value at Risk), as well as the maximum Sharpe ratio are used as criteria for optimizing the portfolio.

To test portfolio optimization models, the sliding window method was used, characterized by the following conditions:

- as the horizon for testing optimization models, the period from January 1, 2009 to December 12, 2016 was selected;
- the portfolio was reviewed at the beginning of each quarter;
- CPC parameters were estimated using data on daily, simple stock returns and stock indices for two years preceding the moment of portfolio optimization;
- as a risk-free return for calculating the Sharpe ratio, the MIBOR rate was used with a period of 31 to 90 days;
- empirical functions were considered as private distributions of returns.

Additional restrictions were established on the structure of assets in the portfolio - the weight of one security in the portfolio is not negative, and should not exceed 0.15, this helped to avoid excessive concentration of capital in individual securities as a result of a numerical solution of the optimization problem.

The algorithm for determining the weight of the optimal portfolio at the beginning of each quarter is as follows:

1) The joint distribution of daily returns on stocks of Russian and American companies, as well as stock indices, is estimated using various PDA models (R-Vine-MST and R-Vine-INDX). To assess the distribution models, data on the daily ruble returns of stocks and indices for two years preceding the date of portfolio formation are used;

2) N = 10,000 independent observations of daily stock returns and indices are generated using estimates of co-distribution models;

3) The method of Lagrange multipliers solves the problems of optimizing an investment portfolio.

III. RESULTS

The author's method of specifying the CPC structure used in this work involves grouping the vertices of the first branch tree first according to the country and then branch principles, the remaining branch trees are structured using the maximum spanning tree method. A distinctive feature of the author's method is the use of industry and country stock indices MSCI and Moscow Exchange to structure branching. Fig. 1 and 2 show the first branch tree obtained for two methods for specifying the structure of the CPC. When constructing the branching, daily stock and index returns for the period from January 1, 2007 to December 23, 2016 were used. Next to each edge of the tree is the type of paired copula, its parameters in parentheses and the value of the Kendall rank correlation coefficient in square brackets.

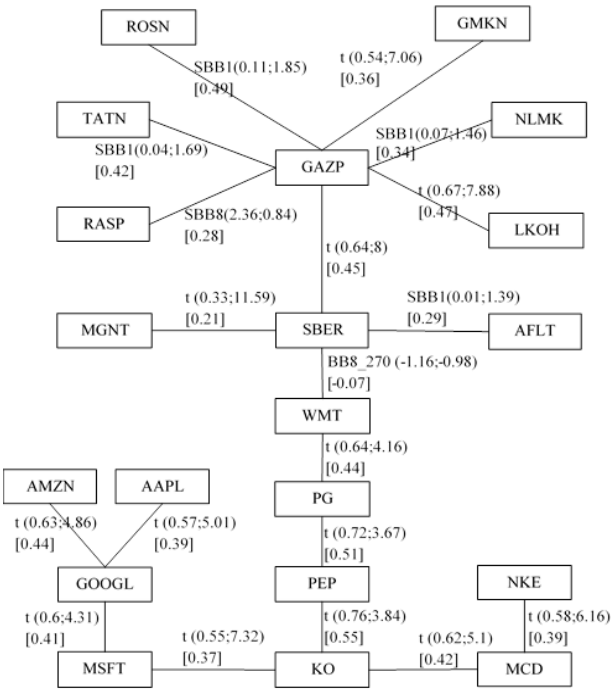

Fig. 1. Method of maximum spanning tree.

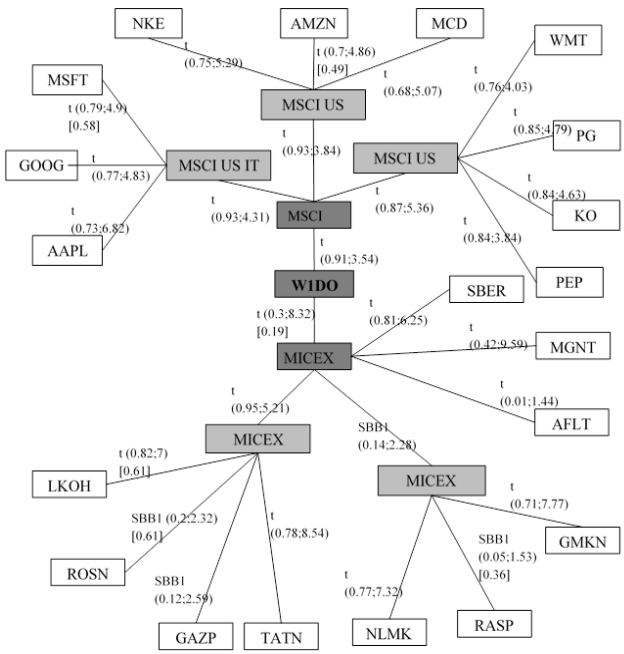

Fig. 2. Author's method of structuring CPC

It can be seen from the figures that Student's paired copulas predominate in the structure, which is associated with a rather long (10 years) period of parameter estimation, while the asymmetry in the relationship between stock returns manifests itself at shorter intervals (up to three years). Figure 1 (a) also shows that securities are prone to clustering according to industry and country principles using the maximum spanning tree method.

A. Modeling the effective border of the investment portfolio using various copula models

Figure 3 shows the effective boundaries obtained using various copula models.

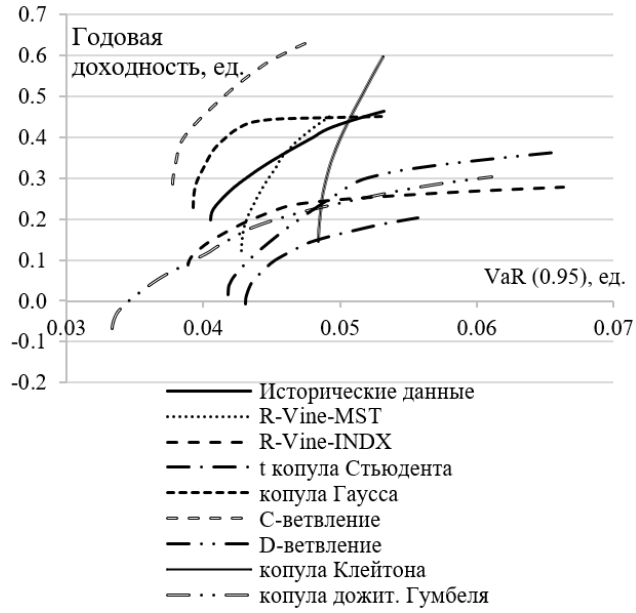

Fig. 3. The boundaries of effective portfolios.

It can be seen from the figure that the effective boundaries obtained using CPC on regular branches without restrictions on the Gaussian structure and copula are closer than the others to the effective boundary obtained from historical data. The R-Vine-INDX model provides a more conservative assessment of the risk of an effective portfolio compared to a normal copula and historical boundary.

It is noteworthy that estimates of the effective boundary obtained using CPC on canonical branches and CPC on elongated branches diverge quite strongly. The effective boundary for C-branching lies significantly higher than the historical effective boundary, while the effective boundary obtained for D-branching is much lower. The copula of the survival of Gumbel and the copula of Clayton showed a bad result - the first underestimates profitability too much, and the second overestimates risk (the effective border takes too steep form). Student's t-copula showed the most conservative result (the effective boundary for this model is located in Figure 1 below the others).

Based on a graphical analysis of the obtained effective boundaries, we can conclude that for the purposes of portfolio optimization, CPC models on the correct branches are more suitable.

B. Modeling the effective border of the investment portfolio using various copula models

Table 2 presents the results of the Kupiec test for various copula models (the total number of observations is 113).

TABLE II. KUPIEC TEST RESULTS

Model	K_1	K_0	$\hat{\pi}$	LR	$F_{\chi_1^2}(LR)$
R-vine-mst	6	107	0.05310	0.022389	0.118943
R-vine-indx	4	109	0.03540	0.562342	0.546681
t-Student copula	6	107	0.05310	0.022389	0.118943
Gaussian copula	5	108	0.04425	0.081751	0.225062
C-vine	5	108	0.04425	0.081751	0.225062
D-vine	7	106	06.06195	0.316613	0.4263497

All considered copula models pass the Kupiec test with a significance level of 0.05, the smallest number of periods during which the loss of the equally weighted portfolio exceeded the predicted levelVaR is observed for the R-Vine-INDX model. The largest number of "failures" in the level prediction $VaR$is observed in the CPC model on the extended branch.

C. *Multiple Periodic Testing of Portfolio Optimization Models*

The accumulated income obtained using various copula models as part of a multi-period solution of optimization problems is presented in Fig. 4.

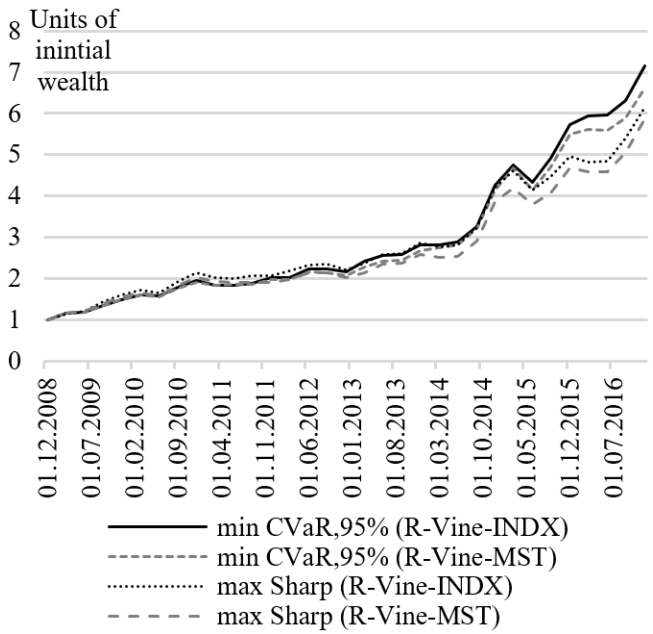

Fig. 4. The dynamics of accumulated income.

For both of the portfolio optimality criteria considered, the R-Vine-INDX model showed the best results in terms of accumulated income.

The main results of portfolio optimization as described above are presented in Table 3.

TABLE III. PORTFOLIO OPTIMIZATION RESULTS (ALL, EXCEPT SHARPE RATIO IN UNITS OF INITIAL WEALTH)

Criteria	CVaR (0.95) minimization		Sharpe's ratio maximization	
	MST	INDX	MST	INDX
Expected quarter return	0.0635	0.0661	0.0601	0.0615
Standard deviation of quarter return	0.0768	0.0773	0.0842	0.0824
Minimum quarterly return	-0.0756	-0.0613	09.0936	-0.1021
Maximum quarterly return	0.2961	0.3108	0.3228	0.3082
Sharpe ratio	0.8268	0.8553	0.7129	0.7470
Terminal wealth	6.6064	7.1463	5.8578	6.1487

Table 3 shows that the standard deviation of the yield is approximately the same for various methods of forming the CPC. It is noteworthy that the highest Sharpe ratio is observed for portfolios that minimize CVaR. In the event that CVaR minimum is considered as a criterion of portfolio optimality, the largest quarterly drawdown is observed for a CPC structured by the maximum spanning tree method (-7.556%), for a CPC structured using stock indices, the drawdown was -6.132%. In the event that the maximum of the Sharpe coefficient is considered as an optimality criterion, the largest quarterly loss is observed for the R-Vine-INDX model (10.206%). For the R-Vine-MST model, the maximum loss for the quarter was 9.359%.

IV. DISCUSSION

The obtained research results are consistent with the results obtained earlier in foreign and domestic studies [11, 13, 10, 14]. The effective boundary for CPCs on regular branches without restrictions on the structure of the first tree allows one to obtain a more accurate effective boundary in comparison with other multidimensional copula models, which is consistent with the results obtained in [11]. It is noteworthy that CPC on the correct branches with the author's restriction on the structure of the first tree allows you to get the most conservative and reliable risk estimates. In our opinion, the R-Vine-INDX model provides more conservative risk assessments compared to the R-Vine-MST due to the greater number of “assets” included in the structure of the CPC. Adding stock indices to the model allows you to use more information when constructing a branch.

The structure of the CPC has a significant impact on the quality of the model. The right choice of structure allows one to reflect the most significant information about the relationship of asset returns at the first branching levels, and get a more accurate assessment of the return and risk of investments. Taking into account the asymmetry in the relationship between industries and national financial markets allows us to obtain a better model for the joint distribution of asset returns. The proposed CPC structuring method takes an intermediate place between the structuring methods described in the works of A.I. Travkin [14] and A. Brehman and K. Kzado [10].

Further improvement of the process of improving the methods of forming the structure of the CPC may be associated with the use of neural networks for its formation, as well as the use of models with switchable structure regimes or methods of its formation.

V. Conclusion

Despite the fact that designs from paired copulas appeared relatively recently, their popularity is growing, especially in the field of financial modeling. The reason is that it is possible to take into account the asymmetry of the relationship between assets in the tails of private distributions more accurately. The most flexible model are PDAs on arbitrary regular branches.

The tests conducted in this study showed that CPCs on the correct branches repeat the effective portfolio boundary, constructed using historical data more precisely than Archimedean copulas and the simplest *C*- and *D*-branches, and at the same time, provide no less accurate estimates of the investment portfolio risk. The author's method of structuring the CPC as part of a multi-period solution to the portfolio optimization problem showed a higher accumulated income, in comparison with the maximum spanning tree method. Correct branches, the first tree of which is structured using stock indices, are more reminiscent of traditional CAPM, where the expected return on the asset is modeled relative to the market portfolio. In our opinion, the introduction of stock indices into the model allows us to highlight the systematic risk of securities and reduce the impact of temporary false correlations on investment decisions. The method of structuring CPC models proposed in the article may be of particular value in the implementation of international portfolio diversification.

References

[1] W.F. Sharpe, "Capital asset prices: a theory of market equilibrium under conditions of risk," in *Journal of Finance, vol. 45,* 1964, pp. 425-442.

[2] K. Aas, C. Czado, A. Frigessi and H. Bakken, "Pair-copula constructions of multiple dependence," in *Insurance: Mathematics and Economics,* vol. 44, issue 2, 2009, pp. 182-198.

[3] P. Christoffersen, V. Errunza, K. Jacobs and H. Langlois, "Is the potential for international diversification disappearing? A dynamic copula approach," in *Review of financial studies, vol. 25, issue 12, 2012, pp. 3711-3751.*

[4] A. Patton, "Modelling asymmetric exchange rate dependence," in *International Economic Review,* vol. 47, Issue 2, 2006, pp. 527-556.

[5] H. Joe, "Families of m-variate distributions with given margins and m(m-1)/2 bivariate dependence parameters," in *Institute of Mathematical Statistics Lecture Notes, Distributions with fixed marginals and related topics, Monograph Series,* Institute of Mathematical Statistics, Hayward, CA., 1996, pp. 120-141.

[6] T. Bedford and R.M. Cooke, "Probability density decomposition for conditionally dependent random variables modeled by vines," in *Annals of Mathematics and Artificial Intelligence,* vol. 32, 2001, pp. 245-268.

[7] T. Bedford and R.M. Cooke, "Vines - a new graphical model for dependent random variables," in *Annals of Statistics,* vol. 30, 2002, pp. 1031-1068.

[8] J. Dissman, E.C. Brechmann, C. Czado and D. Kurowicka, "Selecting and estimating regular vine copulae and application to financial returns," in *Computational Statistics and Data Analysis, vol.* 59, 2013, pp. 52-69.

[9] A.V. Travkin, "Postroenie konstruktsiy iz parnykh kopul na osnove empiricheskikh funktsiy khvostovoy zavisimosti v prilozhenii k rossiyskomu rynku aktsiy," v Zhurnal Novoy ekonomicheskoy assotsiatsii, № 1, tom 25, 2015, s. 39-55.

[10] E.C. Brechmann and C. Czado, "Risk management with high-dimensional vine copulas: An analysis of the Euro Stoxx 50," in *Statistics and Risk Modeling,* vol. 30, issue 4, 2013, pp. 307-342.

[11] R.K.W. Low, J. Alcock, R. Faff and T. Brailsford, "Canonical vine copulas in the context of modern portfolio management:Are they worth it?," in Journal of Banking & Finance, vol. 37, issue 8, 2013, pp. 3085-3099.

[12] L. Chollete, A. Heinen and A. Valdesogo, "Modeling international financial returns with a multivariate regime-switching copula," in Journal of Financial Econometrics, vol. 7, issue 4, 2009, pp. 437-480.

[13] B. Zhang, Y. Wei, J. Yu, X. Lai and Z. Peng, "Forecasting VaR and ES of stock index portfolio: A Vine copula method," in Physica A: Statistical Mechanics and its Applications, vol. 416, 2014, pp. 112-124.

[14] A.V. Travkin, "Konstruktsii iz parnykh kopul v zadache formirovaniya portfelya aktsiy," v Prikladnaya ekonometrika, № 32, tom 4, 2013, s. 110-133.

[15] P.H. Kupiec, "Techniques for Verifying the Accuracy of Risk Measurement Models," in The Journal of Derivatives, vol. 3, issue 2, 1995, pp. 73-84.

[16] A. McNeil, and R. Frey, "Estimation of tail-related risk measures for heteroscedastic financial time series: An extreme value approach," in Journal of Empirical Finance, vol. 7, issue 3-4, 2000, pp. 271-300.