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Odd Hyperbolic Cosine-FG Family of Lifetime Distributions

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ABSTRACT

In the present paper, a new family of lifetime distributions is introduced via odd ratio function, the well-known concept in survival analysis and reliability engineering. Some important properties of the proposed model including survival function, quantile function, hazard function, order statistic are obtained in a general setting. A special case of this new family is taken up by considering exponential and Lindley models as the parent distributions. In addition, estimation of the unknown parameters of the special model will be examined from the perspective of the classic statistics. A simulation study is presented to investigate the bias and mean square error of the maximum likelihood estimators. Moreover, an example of real data set is studied; point and interval estimations of all parameters are obtained by maximum likelihood and bootstrap (parametric and non-parametric) procedures. Finally, the superiority of the proposed model in terms of the parent exponential distribution over other fundamental statistical distributions is shown via the example of real observations.

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1. INTRODUCTION

The statistical distribution theory has been widely explored by researchers in recent years. Given the fact that the data from our surrounding environment follow various statistical models, it is necessary to extract and develop appropriate high-quality models.

Recently, Alzaatreh *et al.* [2] have introduced a new model of lifetime distributions, which the researchers refer to its especial case as $odd-G$ distribution. It is based on the combination of an arbitrary cumulative distribution function *CDF* with odd ratio of the baseline *CDF* as G. The integration form of the new *CDF* is

$$H(x) = \int_{-\infty}^{\frac{G(x)}{1-G(x)}} f(t)\,dt, \tag{1}$$

where f is the probability density function *PDF* of arbitrary *CDF*. This interesting method attracted the attention of some researchers. We refer the reader to [4,1,8,9,14]. Generating new model based on this method resulted in creating very flexible statistical model.

In the present paper, we introduce a new family of lifetime distributions based on the odd ratio of a parent distribution G for general hyperbolic cosine-F (*HCF*) family of lifetime distributions that recently proposed by Kharazmi and Saadatinik [7]. This new model will be denoted by $odd-HCF-G\,(or\ OHCFG)$ distribution. One of our main motivation to introduce this new category of distributions is to provide more flexibility for fitting real datasets compare to other well-known classic statistical distributions.

In summary, we first obtain the fundamental and statistical properties of the *OHCFG* in general setting and then we propose a special case of this model by considering exponential distributions in place of the parent distribution F and Lindley distribution instead of the parent distribution G. It is referred to as Odd Hyperbolic Cosine Exponential Lindley (*OHCEL*) distribution. We provide a comprehensive discussion about statistical and reliability properties of new *OHCEL* model. Furthermore, we consider maximum likelihood and bootstrap estimation procedures in order to estimate the unknown parameters of the new model for a complete data set. In addition, parametric and non-parametric bootstrap CIs are calculated.

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The rest of the paper organized as follows. In the Section 2, a new category of lifetime distributions is introduced based on the fundamental odd quantity and then the main statistical and reliability properties are obtained in general setting. In Section 3, by considering the two exponential and Lindley distributions as the base distributions, a new model is presented according to the general model discussed in Section 1 and its prominent characteristics are studied. This new model refer to as *OHCEL* distribution. In Section 4, we examine the inferential procedures for estimation unknown parameters of the *OHCEL* model. In this section, we provide discussions about the maximum likelihood and bootstrap procedures. Applications and numerical analysis of three real data sets are presented in Section 5. Finally, in Section 6 the paper is concluded.

2. NEW GENERAL MODEL AND ITS PROPERTIES

In this section, we provide the structure of our new model and some main properties of the proposed model in general setting. Motivated by idea of Alzaatreh *et al.* [2], a new class of statistical distributions is proposed. The new model is constructed by applying Alzaattreh idea to the *HCF* family of lifetime distributions that recently have seen proposed by Kharazmi and Saadatinik [7]. According to Kharazmi and Saadatinik [7] a random variable X has a HCF distribution if its *CDF* is given by

$$M(x) = \frac{2e^a}{e^{2a}-1}\sinh\left(aF(x)\right), \tag{2}$$

where $x > 0,\ a > 0$.

The CDF of the new model is defined as

$$\begin{aligned} H(x) &= \int_0^{\frac{G(x)}{1-G(x)}} \frac{2a\ e^a}{e^{2a}-1} f(t)\cosh\left(aF(t)\right)\mathrm{d}t \\ &= \frac{2e^a}{e^{2a}-1}\sinh\left(aF\left(\frac{G(x)}{1-G(x)}\right)\right). \end{aligned} \tag{3}$$

The probability density function (*PDF*) is

$$h(x) = \frac{2a\ e^a}{e^{2a}-1}\frac{g(x)}{(1-G(x))^2} f\left(\frac{G(x)}{1-G(x)}\right)\cosh\left(aF\left(\frac{G(x)}{1-G(x)}\right)\right), \tag{4}$$

where $a > 0$ is a parameter.

Using (3) and (4), the survival $\overline{H}(x)$ and the hazard rate $r(x)$ functions for OHCFG distribution are given, respectively, by:

$$\overline{H}(x) = 1 - \frac{2e^a}{e^{2a}-1}\sinh\left(aF\left(\frac{G(x)}{1-G(x)}\right)\right),$$

and

$$r(x) = \frac{\frac{2a\ e^a}{e^{2a}-1}\frac{g(x)}{(1-G(x))^2} f\left(\frac{G(x)}{1-G(x)}\right)\cosh\left(aF\left(\frac{G(x)}{1-G(x)}\right)\right)}{1-\frac{2e^a}{e^{2a}-1}\sinh\left(aF\left(\frac{G(x)}{1-G(x)}\right)\right)}.$$

The *pth* quantile x_p of the OHCFG distribution can be obtained as

$$x_p = G^{-1}\left(\frac{F^{-1}\left(\frac{1}{a}arcsinh\left(\frac{e^{2a}-1}{2e^a}p\right)\right)}{1+F^{-1}\left(\frac{1}{a}arcsinh\left(\frac{e^{2a}-1}{2e^a}p\right)\right)}\right), \quad 0 \le p \le 1. \tag{5}$$

In view of $arcsinh(x) = \ln\left(x+\sqrt{x^2+1}\right)$, we can simplify (5) as

$$x_p = G^{-1}\left(\frac{F^{-1}\left(\frac{1}{a}\log\left(\frac{e^{2a}-1}{2e^a}p+\sqrt{\left(\frac{e^{2a}-1}{2e^a}p\right)^2+1}\right)\right)}{1+F^{-1}\left(\frac{1}{a}\log\left(\frac{e^{2a}-1}{2e^a}p+\sqrt{\left(\frac{e^{2a}-1}{2e^a}p\right)^2+1}\right)\right)}\right).$$

Hence, if the baseline F and G distributions are invertible then we can easily generate random samples from the *OHCFG* distribution.

3. *OHCEL* DISTRIBUTION AS A SPECIAL CASE OF *OHCFG* MODEL

In this section, we specialize previous mentioned general model by choosing special cases for baseline distributions F and G. We apply the *OHCFG* method to a specific case of baseline distributions, namely to an exponential distribution and a Lindley distribution and call this proposed model, three-parameter *OHCEL* distribution.

Definition. A random variable X has *OHCEL* (a, λ, β) distribution, if its *PDF* is given by

$$h(x) = \frac{2a\, e^{a}}{e^{2a}-1} \frac{\beta^2 (1+\beta)(1+x)\, e^{\beta x}}{(1+\beta+\beta x)^2} \lambda\, e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x} e^{\beta x}-1\right)} \cosh\left(a\left(1-e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x} e^{\beta x}-1\right)}\right)\right), \tag{6}$$

where $x > 0,\ a, \lambda, \beta > 0$.

The CDF corresponding to (6) is

$$H(x) = \frac{2\, e^{a}}{e^{2a}-1} \sinh\left(a\left(1-e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x} e^{\beta x}-1\right)}\right)\right), \quad x \geq 0.$$

The survival and hazard rate functions are

$$\overline{H}(x) = 1 - \frac{2\, e^{a}}{e^{2a}-1} \sinh\left(a\left(1-e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x} e^{\beta x}-1\right)}\right)\right),$$

and

$$r(x) = \frac{h(x)}{\overline{H}(x)} = \frac{\frac{2e^{a}\, \beta^2(1+\beta)(1+x)e^{\beta x}}{(1+\beta+\beta x)^2} \lambda\, e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x} e^{\beta x}-1\right)} \cosh\left(a\left(1-e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x} e^{\beta x}-1\right)}\right)\right)}{e^{2a}-1-2e^{a}\, \sinh\left(a\left(1-e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x} e^{\beta x}-1\right)}\right)\right)},$$

respectively. Some shapes of *PDF* and hazard function for the selected values of parameters are given in Figures 1 and 2.

3.1. Some Properties of the *OHCEL* Distribution

In this section, we obtain some properties of the *OHCEL* distribution, involving quantiles, moments, moment generating function and order statistics distribution. The characterizations of *OHCEL* distribution are presented in Subsection 3.5.

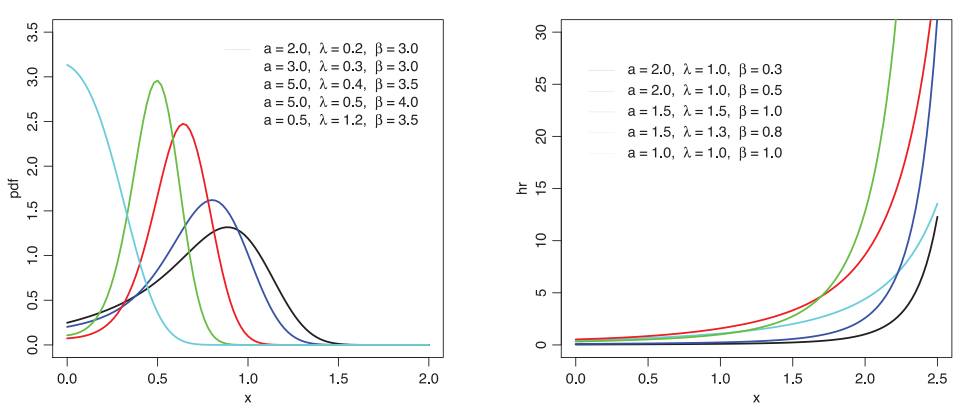

Figure 1 | Plots of the *OHCEL* (α, λ, β) (left) and failure rate function (right) for selected values of α, λ, β.

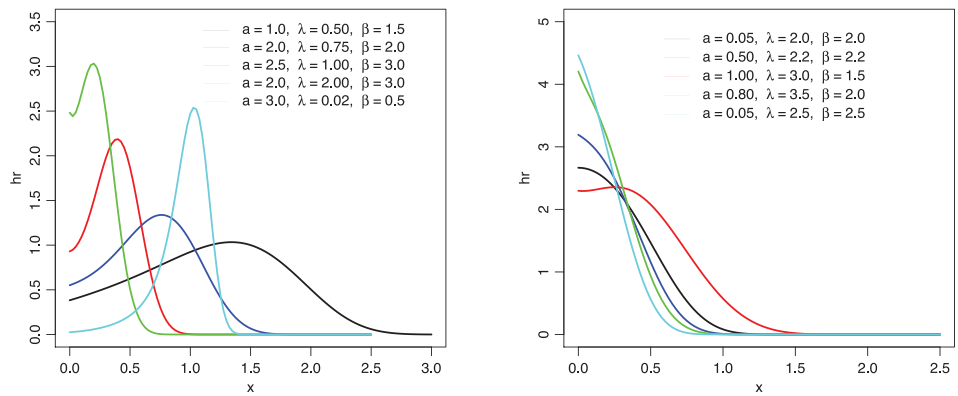

Figure 2 Failure rate function shapes for selected values of the parameters.

3.2. Quantiles

For the *OHCEL* distribution, the *pth* quantile x_p is the solution of $H\left(x_p\right) = p$, hence

$$x_p = -1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(\frac{-(1+\beta)\lambda\, e^{-1-\beta}}{\lambda - \log\left(1 - \frac{1}{a}\sinh^{-1}\left(\frac{p\left(e^{2a}-1\right)}{2e^{a}}\right)\right)}\right)$$

$$= -1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(\frac{-(1+\beta)\lambda\, e^{-1-\beta}}{\lambda - \log\left(1 - \frac{1}{a}\log\left[\frac{e^{2a}-1}{2e^{a}}p + \sqrt{\left(\frac{e^{2a}-1}{2e^{a}}p\right)^{2} + 1}\right]\right)}\right); \qquad 0 \le p \le 1,$$

which is the base of generating *OHCEL* random variates, where W_{-1} denotes the negative branch of the Lambert function.

3.3. Moments and Moment Generating Function

In this subsection, moments and related measures including coefficients of variation, skewness and kurtosis are presented. Tables of values for the first six moments, standard deviation (*SD*), coefficient of variation (*CV*), coefficient of skewness (*CS*) and coefficient of kurtosis (*CK*) are also presented. The *rth* moment of the *OHCEL* distribution, denoted by μ'_r, is

$$\mu_r' = E\left(X^r\right) = \sum_{n,j,t,l=0}^{\infty}\sum_{k=0}^{2n} \frac{a^{2n+1}(-1)^k \binom{2n}{k}\binom{-2-j}{t} e^{\lambda(k+1)}\lambda^{j+1}(k+1)^j(j+1)^l\,\beta^t}{(2n)!\;j!\;l!\;(1+\beta)^t}\, E_{X_L}\left[X^{r+t+l}\right],$$

where $X_L \sim Lindley\,(\beta)$ and $E_{X_L}[X^{r+t+l}] = \frac{(r+t+l)!(\beta+r+t+l+1)}{\beta^{r+l+t}(\beta+1)}$.

The variance, *CV*, *CS*, and *CK* are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1}, \tag{7}$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E\left(X-\mu\right)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{\left(\mu_2' - \mu^2\right)^{3/2}}, \tag{8}$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E\left(X-\mu\right)^2\right]^{2}} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{\left(\mu_2' - \mu^2\right)^{2}}, \tag{9}$$

respectively.

In the next, we show that the *rth* moment can be calculated numerically in terms of the selection of different values of the parameters. Table 1 lists the first six moments of the *OHCEL* distribution for selected values of the parameters, when $a = 2$. Table 2 lists the first six moments of the *OHCEL* distribution for selected values of the parameters, when $\beta = 0.5$. These values can be determined numerically using R.

The moment generating function of the *OHCEL* distribution is given by

$$E\left(e^{tX}\right) = \sum_{n,j,q,l=0}^{\infty} \sum_{k=0}^{2n} \frac{a^{2n+1}(-1)^k \binom{2n}{k} \binom{-2-j}{q} e^{\lambda(k+1)} \beta^q \lambda^{j+1} (k+1)^j}{(2n)!\ l!\ j!} E_{X_L}\left[X^{q+l}\right],$$

where $E_{X_L}\left[X^{q+l}\right] = \frac{(q+l)!(\beta+q+l+1)}{\beta^{q+l}(\beta+1)}$.

In order to investigate and analyze amount of skewness and kurtosis of the new model under the three parameters a, β and λ, 3D diagrams are presented in Figures 3–5. Analysis of theses graphs shows that all three parameter are effective in variation of skewness and kurtosis.

3.4. Order Statistics

Order statistics play an important role in probability and statistics. In this subsection, we present the distribution of the ith order statistic from the *OHCEL* distribution. The *PDF* of the *ith* order statistic from the *OHCEL PDF*, $f_{OHCEL}(x)$, is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!\,(n-i)!} f_{OHCEL}(x) \left[F_{OHCEL}(x)\right]^{i-1} \left[1 - F_{OHCEL}(x)\right]^{n-i} \\ &= \frac{n!}{(i-1)!\,(n-i)!} f_{OHCEL}(x) \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m \left[F_{OHCEL}(x)\right]^{m+i-1}. \end{aligned}$$

Using the binomial expansion $\left[1 - F_{OHCEL}(x)\right]^{n-i} = \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m \left[F_{OHCEL}(x)\right]^m$, we have

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m \left[F_{OHCEL}(x)\right]^{m+i-1} f_{OHCEL}(x).$$

Table 1 Moments of the *OHCEL* distribution for selected parameter values when $a = 2$.

μ'_r	$\lambda = 0.5, \beta = 0.5$	$\lambda = 0.5, \beta = 1$	$\lambda = 1.5, \beta = 0.5$	$\lambda = 1.5, \beta = 1$
μ'_1	3.941234	1.776483	2.252314	0.9715466
μ'_2	18.52086	3.837182	6.5022	1.248397
μ'_3	95.44743	9.157249	21.37342	1.850604
μ'_4	523.8631	23.37855	76.7875	3.022521
μ'_5	3017.057	62.83307	295.3096	5.314822
μ'_6	18070.29	176.066	1200.363	9.921312
SD	1.728447	0.8254029	1.195526	0.5518099
CV	0.4385548	0.4646275	0.530799	0.5679706
CS	−0.2124006	−0.1423441	0.1697464	0.2741677
CK	2.401186	2.342314	2.416298	2.459799

CK, coefficient of kurtosis; CS, coefficient of skewness; CV, coefficient of variation; SD, standard deviation.

Table 2 Moments of the *OHCEL* distribution for selected parameter values when $\beta = 0.5$.

μ'_r	$a = 0.3, \lambda = 0.5$	$a = 0.5, \lambda = 1$	$a = 0.8, \lambda = 1.2$	$a = 1, \lambda = 1.8$
μ'_1	3.247616	2.291018	2.13056	1.718264
μ'_2	13.50444	7.099715	6.173539	4.117847
μ'_3	63.89187	25.72169	20.95563	11.70906
μ'_4	329.3444	103.3036	78.96134	37.28532
μ'_5	1808.908	447.6086	321.3765	129.1265
μ'_6	10444.89	2058.578	1389.934	477.896
SD	1.719718	1.360497	1.278379	1.079544
CV	0.5295325	0.5938394	0.6000202	0.6282761
CS	0.1622998	0.3871795	0.4014882	0.49958
CK	2.312932	2.489167	2.507828	2.653294

CK, coefficient of kurtosis; CS, coefficient of skewness; CV, coefficient of variation; SD, standard deviation.

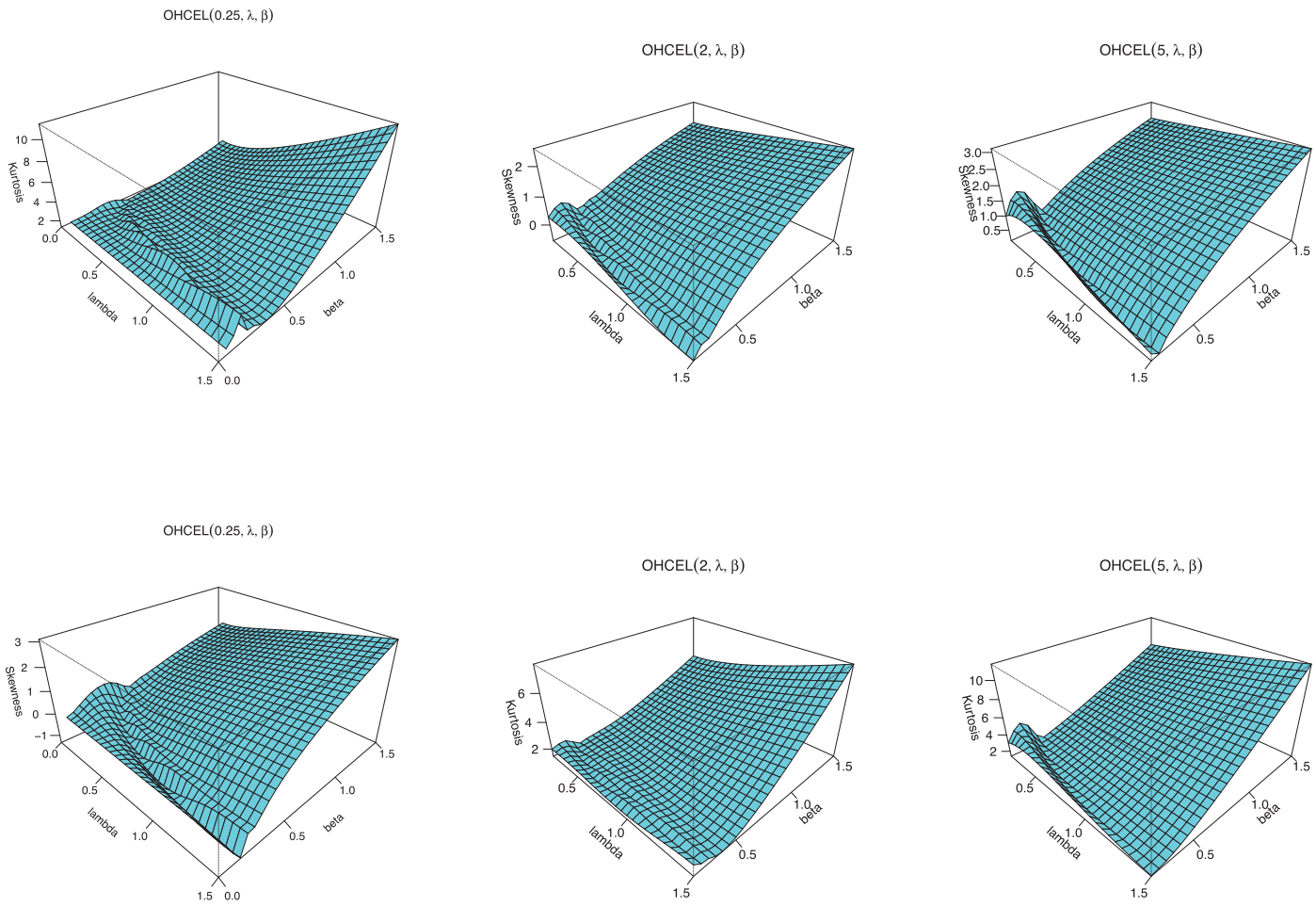

Figure 3 3D plots of skewness and kurtosis of *OHCEL* distribution for some fixed values of parameter α.

3.5. Characterization Results

This section is devoted to the characterizations of the OHCEL distribution based on the ratio of two truncated moments. Note that our characterizations can be employed also when the CDF does not have a closed form. We would also like to mention that due to the nature of OHCEL distribution, our characterizations may be the only possible ones. Our first characterization employs a theorem due to Glänzel [6]. The result, however, holds also when the interval H is not closed, since the condition of the Theorem is on the interior of H.

Proposition 3.5.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \left[\cosh\left(a\left(1 - e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}\right)\right)\right]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}$ for $x > 0$. The random variable X has PDF (6) if and only if the function η defined in Theorem 1 Glänzel [6] is of the form

$$\eta(x) = \frac{1}{2} e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}, \; x > 0.$$

Proof. Suppose the random variable X has PDF (6), then

$$(1 - F(x))\, E\left[q_1(X) \,|X \geq x\right] = \frac{2ae^a}{(e^{2a} - 1)} e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}, \; x > 0,$$

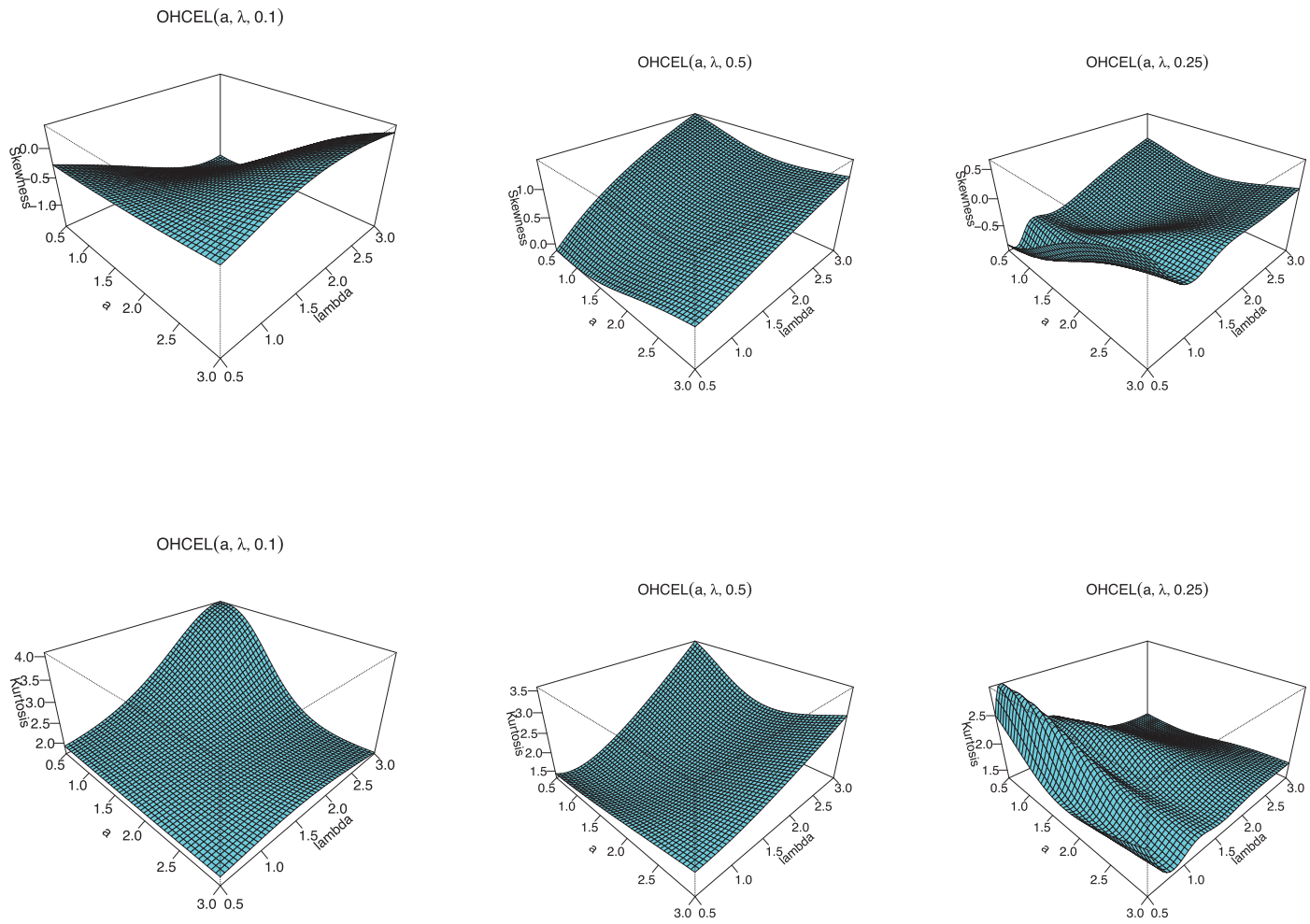

Figure 4 3D plots of skewness and kurtosis of *OHCEL* distribution for some fixed values of parameter β.

and

$$(1-F(x))\,E\left[q_2(X)\,|X\geq x\right]=\frac{2ae^{a}}{2\left(e^{2a}-1\right)}e^{-2\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)},\ x>0.$$

Further,

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\frac{2ae^{a}}{\left(e^{2a}-1\right)}e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}<0,\ for\, x>0.$$

Conversely, if η is of the above form, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda\beta^2\left(1+\beta\right)(1+x)\,e^{\beta x}}{\left(1+\beta+\beta x\right)^2},\quad x>0,$$

and consequently

$$s(x)=\frac{\lambda\left(1+\beta\right)e^{\beta x}}{1+\beta+\beta x},\ x>0.$$

Now, according to Theorem 1, X has density (6). □

Corollary 3.5.1. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 3.5.1. The random variable X has PDF (6) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the following differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda\beta^2(1+\beta)(1+x)\,e^{\beta x}}{(1+\beta+\beta x)^2},\quad x>0.$$

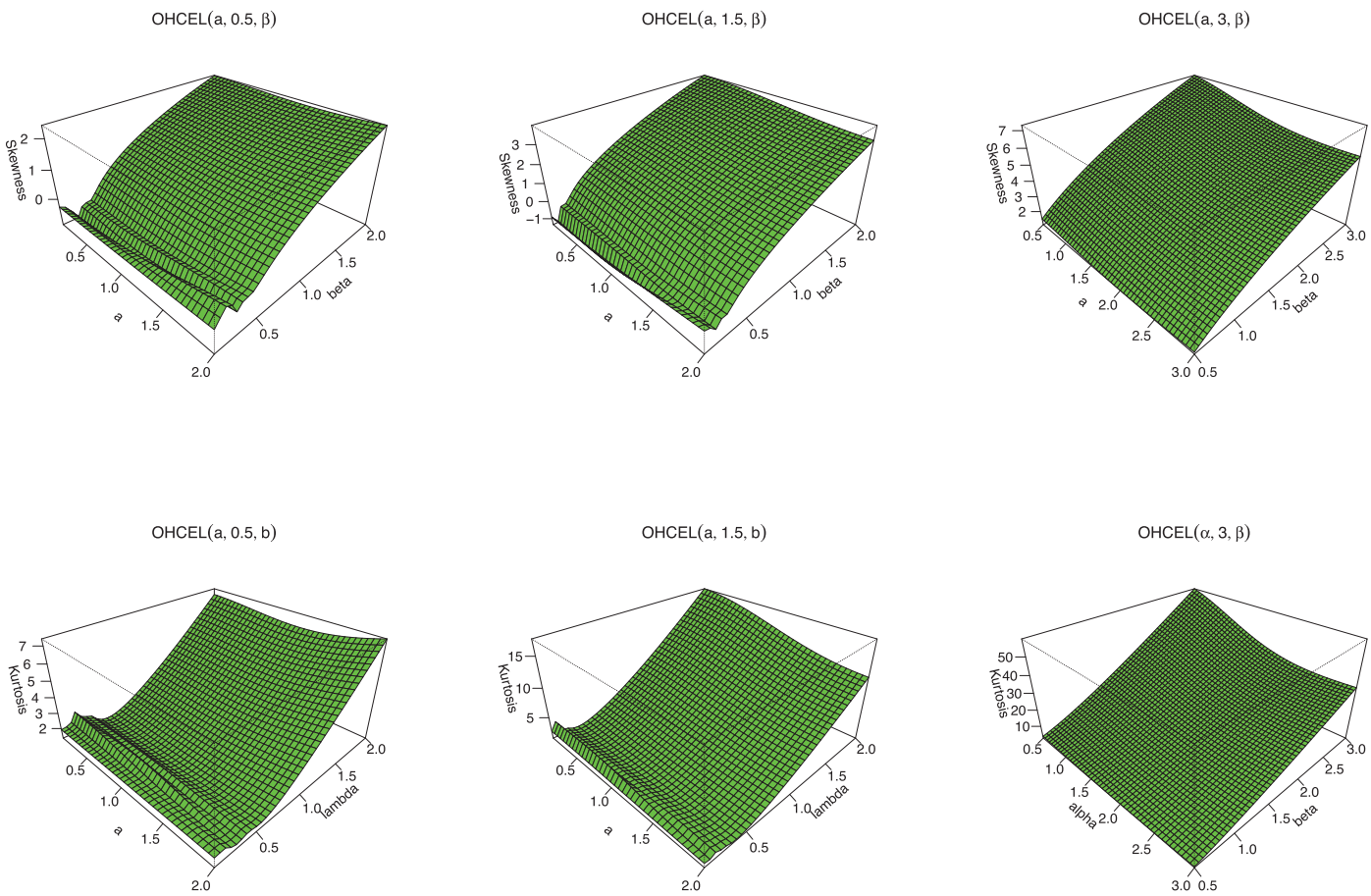

Figure 5 | 3D plots of skewness and kurtosis of *OHCEL* distribution for some fixed values of parameter λ.

Proof. Let $q_1(x)$ and $q_2(x)$ be as in Proposition 3.5.1, then

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\eta'(x)}{\eta(x)-e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}}=\frac{\lambda\beta^2(1+\beta)(1+x)\,e^{\beta x}}{(1+\beta+\beta x)^2},\quad x>0,$$

or

$$\begin{aligned}&\eta'(x)-\left[\frac{\lambda\beta^2(1+\beta)(1+x)\,e^{\beta x}}{(1+\beta+\beta x)^2}\right]\eta(x)\\&=-\left[\frac{\lambda\beta^2(1+\beta)(1+x)\,e^{\beta x}}{(1+\beta+\beta x)^2}\right]e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)},\end{aligned}$$

or

$$\begin{aligned}&\eta'(x)\,e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}-\left[\frac{\lambda\beta^2(1+\beta)(1+x)\,e^{\beta x}}{(1+\beta+\beta x)^2}\right]e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}\eta(x)\\&=-\left[\frac{\lambda\beta^2(1+\beta)(1+x)\,e^{\beta x}}{(1+\beta+\beta x)^2}\right]e^{-2\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)},\end{aligned}$$

from which we have

$$\frac{d}{dx}\left\{\eta(x)\,e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}\right\}=-\left[\frac{\lambda\beta^2(1+\beta)(1+x)\,e^{\beta x}}{(1+\beta+\beta x)^2}\right]e^{-2\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}.$$

From the last equation, we have

$$\begin{aligned}\eta(x)\, e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)} &= \left[-\int\left[\frac{\lambda\beta^2(1+\beta)(1+x)\,e^{\beta x}}{(1+\beta+\beta x)^2}\right]e^{-2\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}dx\right]\\ &= \frac{1}{2}e^{-2\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)},\end{aligned}$$

or

$$\eta(x) = \frac{1}{2}e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}, x > 0.$$

□

Corollary 3.5.2. The general solution of the differential equation in Corollary 3.5.1 is

$$\eta(x) = e^{\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}\left[-\int\left(\frac{\lambda\beta^2(1+\beta)(1+x)\,e^{\beta x}}{(1+\beta+\beta x)^2}\right)e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x}e^{\beta x}-1\right)}\left(q_1(x)\right)^{-1}q_2(x)\,dx + D\right],$$

where D is a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 3.5.1 with $D = 0$. Clearly, there are other triplets $\left(q_1, q_2, \eta\right)$ which satisfy conditions of Theorem1.

4. INFERNCE PROCEDURE

In this section, we consider estimation of the unknown parameters of the $OHCEL(a, \lambda, \beta)$ distribution via maximum likelihood method and bootstrap estimation.

4.1. Maximum Likelihood Estimation

Let $x_1, \ldots, x_n$ be a random sample from the $OHCEL$ distribution and $\Delta = (a, \lambda, \beta)$ be the vector of parameters. The log-likelihood function is given by

$$\begin{aligned}L = L(\Delta) = {} & n\log\left(\frac{2a\,e^a}{e^{2a}-1}\right) + 2n\log(\beta) + n\log(1+\beta) + \sum_{i=1}^{n}\log(1+x_i) + \beta\sum_{i=1}^{n}x_i\\ & -2\sum_{i=1}^{n}\log(1+\beta+\beta x_i) + n\log(\lambda) - \lambda\sum_{i=1}^{n}\left(\frac{1+\beta}{1+\beta+\beta x_i}e^{\beta x_i}-1\right)\\ & +\sum_{i=1}^{n}\log\left(\cosh\left(a\left(1-e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x_i}e^{\beta x_i}-1\right)}\right)\right)\right).\end{aligned} \tag{10}$$

The elements of the score vector are given by

$$\frac{dL}{da} = n\frac{2e^{3a}(1-a) - 2e^a(1+a)}{2ae^a(e^{2a}-1)} + \sum_{i=1}^{n}\left(1-e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x_i}e^{\beta x_i}-1\right)}\right)\tanh\left(a\left(1-e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x_i}e^{\beta x_i}-1\right)}\right)\right) = 0,$$

$$\begin{aligned}\frac{dL}{d\lambda} = {} & \frac{n}{\lambda} - \sum_{i=1}^{n}\left(\frac{1+\beta}{1+\beta+\beta x_i}e^{\beta x_i}-1\right)\\ & + a\sum_{i=1}^{n}\left(\frac{1+\beta}{1+\beta+\beta x_i}e^{\beta x_i}-1\right)e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x_i}e^{\beta x_i}-1\right)}\tanh\left(a\left(1-e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x_i}e^{\beta x_i}-1\right)}\right)\right) = 0,\end{aligned}$$

and

$$\begin{aligned}\frac{dL}{d\beta} &= \frac{2n}{\beta} + \frac{n}{1+\beta} - \sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} \frac{1+x_i}{1+\beta+\beta x_i} - \lambda \sum_{i=1}^{n} \frac{\left[(1+\beta+\beta x_i) - (1+x_i)(1+\beta)\right] e^{\beta x_i}}{(1+\beta+\beta x_i)^2} \\ &\quad - \lambda\beta \sum_{i=1}^{n} \left(\frac{1+\beta}{1+\beta+\beta x_i} e^{\beta x_i} \right) + a\lambda \sum_{i=1}^{n} \left(\frac{1+\beta}{1+\beta+\beta x_i} e^{\beta x_i} \right) \tanh\left(a \left(1 - e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x_i} e^{\beta x_i} - 1 \right)} \right) \right) \\ &\quad + a\lambda \sum_{i=1}^{n} \frac{\left[(1+\beta+\beta x_i) - (1+x_i)(1+\beta)\right] e^{\beta x_i}}{(1+\beta+\beta x_i)^2} \tanh\left(a \left(1 - e^{-\lambda\left(\frac{1+\beta}{1+\beta+\beta x_i} e^{\beta x_i} - 1 \right)} \right) \right) = 0,\end{aligned}$$

respectively.

The maximum likelihood estimate, $\hat{\Delta}$ of $\Delta = (a, \lambda, \beta)$ is obtained by solving the nonlinear equations $\frac{dL}{da} = 0$, $\frac{dL}{d\lambda} = 0$, $\frac{dL}{d\beta} = 0$. These equations are not in closed form and the values of the parameters a, λ and β must be found using iterative methods. Therefore, the maximum likelihood estimate, $\hat{\Delta}$ of $\Delta = (a, \lambda, \beta)$ can be determined using an iterative method such as the Newton–Raphson procedure.

4.2. Bootstrap Estimation

The parameters of the fitted distribution can be estimated by parametric (resampling from the fitted distribution) or non-parametric (resampling with replacement from the original data set) bootstraps resampling (see [5]). These two parametric and non-parametric bootstrap procedures are described as below. **Parametric bootstrap procedure:**

1. Estimate θ (vector of unknown parameters), say $\hat{\theta}$, by using the maximum likelihood estimation (*MLE*) procedure based on a random sample.
2. Generate a bootstrap sample $\{X_1^*, \dots, X_m^*\}$ using $\hat{\theta}$ and obtain the bootstrap estimate of θ, say $\widehat{\theta^*}$, from the bootstrap sample based on the *MLE* procedure.
3. Repeat Step 2 *NBOOT* times.
4. Order $\widehat{\theta^*}_1, \dots, \widehat{\theta^*}_{NBOOT}$ as $\widehat{\theta^*}_{(1)}, \dots, \widehat{\theta^*}_{(NBOOT)}$. Then obtain γ-quantiles and $100\,(1-\alpha)\,\%$ CIs for the parameters.

In case of the *OHCEL* distribution, the parametric bootstrap estimators (PBs) of a, λ and β, are $\hat{a}_{PB}, \hat{\lambda}_{PB}$ and $\hat{\beta}_{PB}$, respectively.

Non-parametric bootstrap procedure

1. Generate a bootstrap sample $\{X_1^*, \dots, X_m^*\}$, with replacement from the original data set.
2. Obtain the bootstrap estimate of θ with MLE procedure, say $\widehat{\theta^*}$, by using the bootstrap sample.
3. Repeat Step 2 *NBOOT* times.
4. Order $\widehat{\theta^*}_1, \dots, \widehat{\theta^*}_{NBOOT}$ as $\widehat{\theta^*}_{(1)}, \dots, \widehat{\theta^*}_{(NBOOT)}$. Then obtain γ-quantiles and $100\,(1-\alpha)\,\%$ CIs for the parameters.

In case of the *OHCEL* distribution, the non-parametric bootstrap estimators (NPBs) of a, λ and β, are $\hat{a}_{NPB}, \hat{\lambda}_{NPB}$ and $\hat{\beta}_{NPB}$, respectively.

5. ALGORITHM AND A SIMULATION STUDY

In this section, we give two different algorithms for generating the random data $x_1, \dots, x_n$ from the *OHCEL* distribution and hence a simulation study is obtained to evaluate the performance of MLEs.

5.1. Algorithms

Here, we obtain two algorithms for generating the random data $x_1, \dots, x_n$ from the *OHCEL* distribution as follows.

i. The first algorithm is based on generating random data from the Lindley distribution by using the exponential gamma mixture distribution.

ii. The second algorithm is based on generating random data from the inverse cdf of the *OHCEL* distribution.

Algotithm 1.

- Generete $U_i \sim U(0,1)\,, \quad i = 1, \dots, n$
- Generete $V_i \sim Exponential\,(\lambda)$
- Generete $W_i \sim Gamma\,(2, \lambda)$
- If $\dfrac{-\log\left[1-\frac{1}{a}\sinh^{-1}\left(\frac{U(e^{2a}-1)}{2e^a}\right)\right]}{\lambda-\log\left[1-\frac{1}{a}\sinh^{-1}\left(\frac{U(e^{2a}-1)}{2e^a}\right)\right]} \leq \frac{\beta}{1+\beta}$ set $X_i = V_i$, otherwise set $X_i = W_i,\ i = 1, \dots, n$.

Algotithm 2.

- Generate $U_i \sim Uniform\,(0,1)\,;\ i = 1, \dots, n,$
- set

$$X_i = -1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(\frac{-(1+\beta)\,\lambda\, e^{-1-\beta}}{\lambda - \log\left[1 - \frac{1}{a}\sinh^{-1}\left(\frac{U_i(e^{2a}-1)}{2e^a}\right)\right]}\right).$$

where W_{-1} denote the negative branch of lambert function.

5.2. Monte Carlo Simulation Study

In this subsection, we assess the performance of the *MLEs* of the parameters with respect to the sample size n for the $OHCEL\,(a, \lambda, \beta)$ distribution. We used the above Algorithms to generate data from the *OHCEL* distribution. The assessment of the performance is based on a simulation study using the Monte Carlo method. Let $\hat{a}, \hat{\lambda}$ and $\hat{\beta}$ be the *MLEs* of the parameters a, λ and β, respectively. We compute the mean square error (*MSE*) and bias of the *MLEs* of the parameters a, λ and β, based on the simulation results of $N = 2000$ independente replications. results are summarized in Table 3 for selected values of n, a, λ and β. From Table 3 the results verify that *MSE* and bias of the *MLEs* of the parameters decrease as sample size n increases. Hence, we can see the *MLEs* of a, λ and β, are consistent estimators.

Table 3 MSEs and Average biases (values in parentheses) of the simulated estimates.

		$a = 0.2$	$\lambda = 0.5$	$\beta = 0.5$
n	30	0.5836 (0.2563)	8.1080 (0.3897)	0.0222 (0.0242)
	50	0.5580 (0.2396)	3.3838 (0.2796)	0.0127 (0.0016)
	100	0.5255 (0.2381)	1.2838 (0.2227)	0.0078 (-0.0123)
	200	0.4734 (0.2604)	0.1464 (0.1494)	0.0045 (-0.0170)
		$a = 0.8$	$\lambda = 1.5$	$\beta = 1$
n	30	48.0955 (4.2138)	355.3437 (10.8196)	1.1098 (0.2865)
	50	44.1961 (4.1982)	352.1701 (11.3449)	0.9312 (0.1912)
	100	40.5445 (4.0471)	329.3157 (11.1429)	0.8281 (0.1532)
	200	40.0010 (4.0069)	322.3770 (11.0732)	0.7625 (0.1342)
		$a = 1$	$\lambda = 1$	$\beta = 1$
n	30	1.1752 (−0.0833)	45.0169 (1.7125)	0.1701 (0.0510)
	50	0.9521 (−0.0733)	32.4758 (1.3976)	0.1131 (0.0000)
	100	0.8157 (−0.0811)	13.4573 (0.9355)	0.0737 (−0.0286)
	200	0.6613 (−0.0884)	6.0679 (0.6140)	0.0511 (−0.0284)
		$a = 1.5$	$\lambda = 1.5$	$\beta = 1.5$
n	30	1.5531 (−0.1629)	123.4354 (3.3165)	0.6466 (0.1583)
	50	1.2556 (−0.1908)	91.9078 (2.8051)	0.4630 (0.0964)
	100	1.0043 (−0.2377)	35.0604 (1.4781)	0.2926 (0.0576)
	200	0.7890 (−0.1934)	12.3476 (0.8496)	0.1965 (0.0190)
		$a = 0.5$	$\lambda = 2$	$\beta = 2$
n	30	1.0248 (0.1155)	161.4630 (4.1734)	1.4181 (0.1981)
	50	0.8239 (0.1032)	120.0046 (3.5063)	0.8292 (0.0220)
	100	0.6550 (0.0403)	56.6965 (2.2262)	0.4844 (−0.0672)
	200	0.6057 (0.0399)	41.3022 (1.9812)	0.3273 (−0.1509)

6. PRACTICAL DATA APPLICATION

In this section, we present the application of the *OHCEL* model to a practical data set to illustrate its flexibility among a set of competitive models.

The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the nonstructural outer ply or failure of the heating system. These failures do not result in damage to the aircraft but do result in replacement of the windshield. We consider the data on service times for a particular model windshield given in Table 16.11 of Murthy *et al.* [11]. These data were recently studied by Ramos *et al.* [17]. These data are:

0.046 1.436 2.592 0.140 1.492 2.600 0.150 1.580 2.670 0.248 1.719 2.717 0.280 1.794 2.819 0.313 1.915 2.820 0.389 1.920 2.878 0.487 1.963 2.950 0.622 1.978 3.003 0.900 2.053 3.102 0.952 2.065 3.304 0.996 2.117 3.483 1.003 2.137 3.500 1.010 2.141 3.622 1.085 2.163 3.665 1.092 2.183 3.695 1.152 2.240 4.015 1.183 2.341 4.628 1.244 2.435 4.806 1.249 2.464 4.881 1.262 2.543 5.140.

Graphical measure: The total time test (*TTT*) plot due to Aarset [3] is an important graphical approach to verify whether the data can be applied to a specific distribution or not. According to Aarset [3], the empirical version of the *TTT* plot is given by plotting $T(r/n) = \left[\sum_{i=1}^{r} y_{i:n} + (n-r)\, y_{r:n}\right] / \sum_{i=1}^{n} y_{i:n}$ against r/n, where $r = 1, \ldots, n$ and $y_{i:n}$ $(i = 1, \ldots, n)$ are the order statistics of the sample. Aarset [3] showed that the hazard function is constant if the *TTT* plot is graphically presented as a straight diagonal, the hazard function is increasing (or decreasing) if the *TTT* plot is concave (or convex). The hazard function is U-shaped if the *TTT* plot is convex and then concave, if not, the hazard function is unimodal. The *TTT* plots for data set is presented in Figure 6. These plots indicate that the empirical hazard rate functions of the data set is increasing. Therefore, the *OHCEL* distribution is appropriate to fit this data set.

Here we obtain point and 95% confidence interval (*CI*) estimation of parameters of the *OHCEL* distribution by parametric bootstrap method for the real data set. We provide results of bootstrap estimation based on 1000 bootstrap replicates in Table 4. It is interesting to look at the joint distribution of the bootstrapped values in a scatter plot in order to understand the potential structural correlation between parameters.

In the next, we fit the *OHCEL* distribution to the one data set and compare it with the *HCE*, Lidley, Generalized Lindley (*GL*), Gamma Lindley (*GaL*), Power Lindley (*PL*), Exponential Lindley (*EL*), gamma, generalized exponential (GE) and Weibull densities. Table 5 shows the *MLEs* of parameters, log-likelihood, Akaike information criterion (*AIC*), Cramrvon Mises (W^*), Bayesian information criterion *(BIC)*, Kolmogorov-Smirnov *(K.S)*, AndersonDarling (A^*) and $p-value$ (P) statistics for the data set. The *OHCEL* distribution provides the best fit for the data set as it shows the lowest AIC, A^* and W^* than other considered models. The relative histograms, fitted *OHCEL*, *HCE*, Lindley, *GL*, *GaL*, *PL*, *EL*, gamma, GE and Weibull *PDFs* and the plots of empirical and fitted survival functions for data are plotted in Figure 7. The $P-P$ plots and $Q-Q$ plots for the *OHCEL* and other fitted distributions are displayed in Figure 8. These plots also support the results in Table 5. We compare the *OHCEL* model with a set of competitive models, namely:

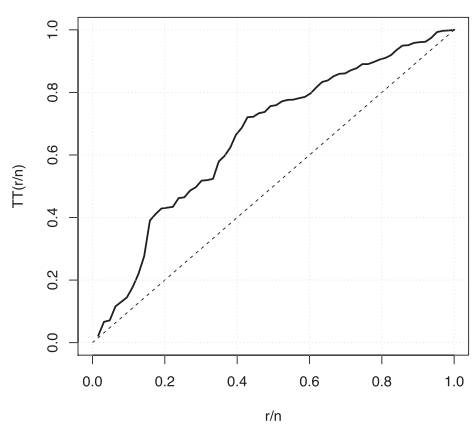

Figure 6 | Scaled-plot of the data set.

Table 4 | Bootstrap point and interval estimation of the parameters a, λ and β.

	Parametric Bootstrap		Non-parametric Bootstrap	
	Point Estimation	***CI***	**Point Estimation**	***CI***
a	2.096	(0.505, 4.949)	2.028	(0.517, 3.362)
λ	5.952	(3.063, 8.836)	5.888	(0.712, 34.743)
β	0.256	(0.054, 1.642)	0.290	(0.108, 0.661)

CI, confidence interval.

Table 5 Parameter estimates (standard errors), log-likelihood values and goodness of fit measures.

Model	*MLEs* of Parameters (s.e)	Log-likelihood	*AIC*	*BIC*	A^*	W^*	*KS*	*P*
OHCEL	$\hat{a} = 2.08\,(0.90)$	−98.02	202.04	208.47	0.22	0.02	0.06	0.95
	$\hat{\lambda} = 5.94\,(14.99)$							
	$\hat{\beta} = 0.26\,(0.33)$							
HCE	$\hat{a} = 3.69\,(0.67)$	−99.81	203.63	207.92	0.45	0.07	0.10	0.51
	$\hat{\lambda} = 0.89\,(0.09)$							
Lindley	$\hat{\beta} = 0.75\,(0.07)$	−104.57	211.15	213.29	2.13	0.41	0.15	0.08
GL	$\hat{\theta} = 1.04\,(0.17)$	−101.60	209.20	215.63	0.92	0.16	0.12	0.22
	$\hat{\alpha} = 1.43\,(0.44)$							
	$\hat{\beta} = 3.20\,(3.85)$							
PL	$\hat{\theta} = 0.54\,(0.08)$	−99.58	203.17	207.45	0.49	0.06	0.09	0.56
	$\hat{\alpha} = 1.38\,(0.12)$							
EL	$\hat{\theta} = 0.92\,(0.10)$	−101.88	207.77	212.06	0.96	0.16	0.12	0.21
	$\hat{\alpha} = 1.55\,(0.28)$							
GaL	$\hat{\theta} = 0.90\,(0.09)$	−102.10	208.20	212.48	1.10	0.21	0.13	0.16
	$\hat{\alpha} = 4.59\,(4.85)$							
Weibull	$\hat{\alpha} = 1.62\,(0.16)$	−100.31	204.63	208.92	0.64	0.09	0.10	0.41
	$\hat{\beta} = 2.30\,(0.18)$							
Gamma	$\hat{\alpha} = 1.9\,(0.31)$	−102.83	209.66	213.95	1.16	0.2	0.58	0.84
	$\hat{\theta} = 0.91\,(0.17)$							
GE	$\hat{\alpha} = 1.89\,(0.34)$	−103.54	211.09	215.37	1.31	0.23	0.14	0.13
	$\hat{\lambda} = 0.69\,(0.09)$							

AIC, Akaike information criterion; EL, Exponential Lindley; GE, generalized exponential; GL, Generalized Lindley; GaL, Gamma Lindley; BIC, Bayesian information criterion; K.S, Kolmogorov-Smirnov.

i. Hyperbolic Cosine-Exponential distribution (*HCE*) [7]. The two-parameter *HCE* density function is given by

$$f(x; a, \lambda) = \frac{2a\ e^{a}}{e^{2a} - 1} \lambda\ e^{-\lambda x} \cosh\left(a\left(1 - e^{-\lambda x}\right)\right); \quad x > 0.$$

where $a > 0$ and $\lambda > 0$.

ii. Lindley distribution [10]). The one-parameter Lindley density function is given by

$$f(x; \beta) = \frac{\beta^2}{1 + \beta}(1 + x)\ e^{-\beta x}; \quad x > 0,$$

where $\beta > 0$.

iii. *GL* distribution [16]. The three-parameter *GL* density function is given by

$$f(x; \theta, \alpha, \beta) = \frac{\theta^{\alpha+1}}{(\theta + \beta)\,\Gamma(\alpha + 1)} x^{\alpha-1}\ (\alpha + \beta x)\ e^{-\theta x}; \quad x > 0,$$

where $\theta > 0, \alpha > 0$ and $\beta > 0$.

iv. Exponentiated *EL* distribution [12]. The two-parameter *EL* density function is given by

$$f(x; \theta, \alpha) = \frac{\alpha\theta^2}{(1 + \theta)}(1 + x)\, e^{-\theta x}\left[1 - \left(1 + \frac{\theta x}{1 + \theta}\right) e^{-\theta x}\right]^{\alpha-1}; \quad x > 0,$$

where $\theta > 0$ and $\alpha > 0$.

v. *PL* distribution [13]. The two-parameter *PL* density function is given by

$$f(x; \theta, \alpha) = \frac{\alpha\ \theta^2}{\theta + 1}(1 + x^{\alpha})\, x^{\alpha-1} e^{-\theta x^{\alpha}}; \quad x > 0.$$

where $\alpha > 0$ and $\theta > 0$.

vi. *GaL* distribution [15]. The two-parameter *GaL* density function is given by

$$f(x; \theta, \alpha) = \frac{\theta^2}{\alpha\,(1 + \theta)}\left[(\alpha + \alpha\theta - \theta)\, x + 1\right] e^{-\theta x}; \quad x > 0,$$

where $\theta > 0$ and $\alpha > 0$.

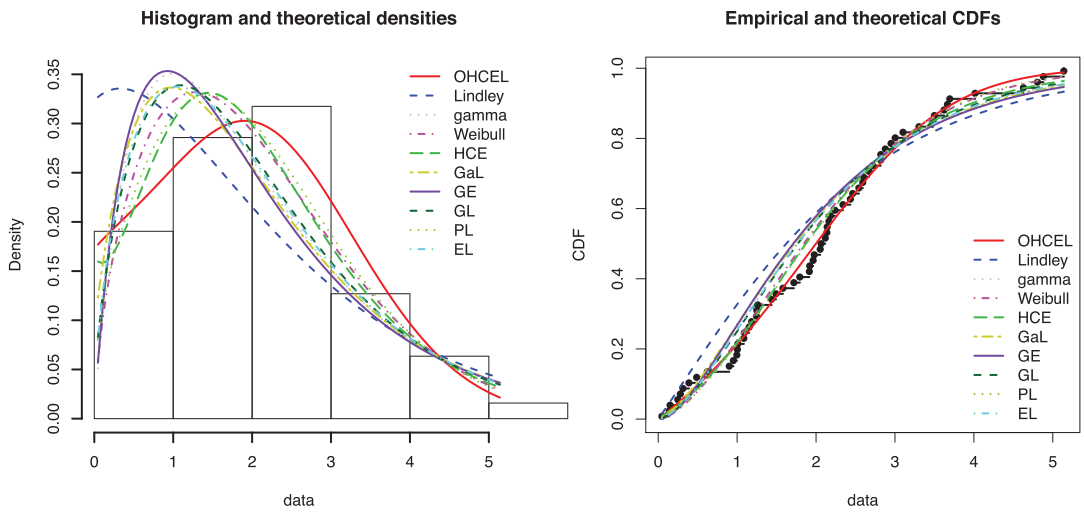

Figure 7 | Estimated densities and Empirical and Estimated cdf for the data set.

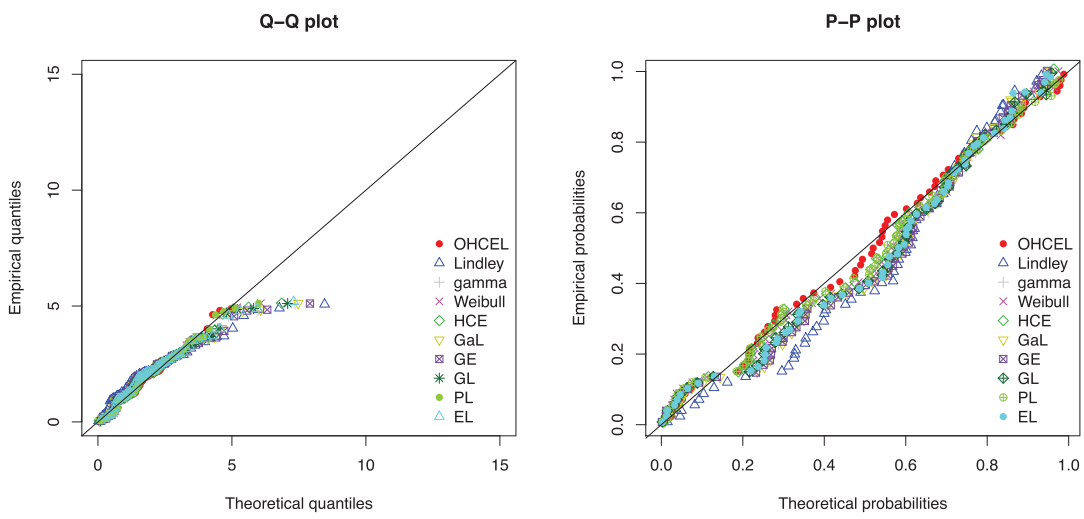

Figure 8 | Q-Q and P-P plots for the data set.

vii. The two-parameter Weibull distribution is given by

$$f(x;\alpha,\beta) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}; \quad x > 0,$$

where $\alpha > 0$ and $\beta > 0$.

viii. The two-parameter Gamma distribution is given by

$$f(x;\alpha,\theta) = \frac{1}{\theta^{\alpha}\,\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-(x/\theta)}; \quad x > 0$$

where $\alpha > 0$ and $\theta > 0$ and $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}\, e^{-t} dt$.

ix. The two-parameter (GE) distribution is given by

$$f(x;\alpha,\lambda) = \alpha\lambda\, e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1}; \quad x > 0,$$

where $\alpha > 0$ and $\lambda > 0$.

As mentioned in inference section, there are not closed expression for MLE estimation of parameters a, λ and β. We use numerical methods to obtain MLE estimation of these parameters. To evaluate the results of MLE estimation, We provide profile-likelihood plots of $OHCEL$ distribution for each parameter in Figure 9.

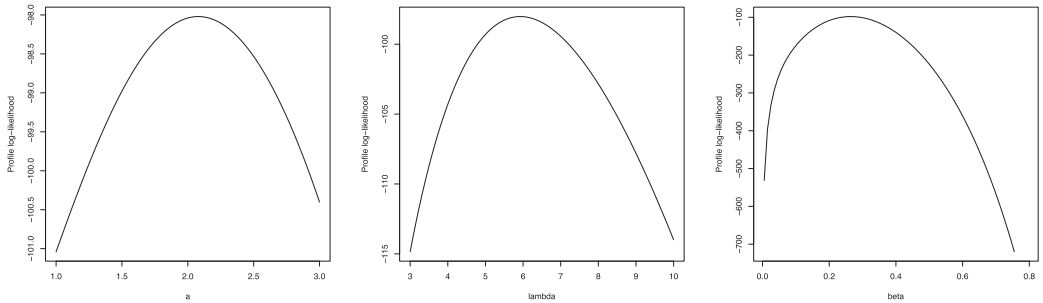

Figure 9 | Profile-likelihood plots of distribution.

7. CONCLUSION

In this article, a new model for the lifetime distributions is introduced and its main properties are discussed. A special submodel of this family is taken up by considering exponential distributions in place of the parent distribution F and Lindley distribution instead of the parent distribution G. We also show that the proposed distribution has variability of hazard rate shapes such as increasing, decreasing and upside-down bathtub shapes. Numerical results of maximum likelihood and bootstrap procedures for a set of real data are presented in separate tables. From a practical point of view, we show that the proposed distribution is more flexible than some common statistical distributions.

CONFLICT OF INTEREST

The authors are declare no conflict of interest.

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