A Decision Making Approach with Linguistic Weight and Unavoidable Incomparable Ranking

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ABSTRACT

In order to deal with the decision making problem including some linguistic values uncertainty information, we propose an approach for decision making with linguistic weighted and unavoidable incomparable ranking based on Linguistic-valued lattice implication algebra (LV-LIA). The properties of binary operations $\otimes$ and $\oplus$ are discussed in LV-LIA, and used to handle importance degree of the attributes expressed by linguistic values. We define a linguistic-real valuation function which is a positive valuation function and a linguistic-real metric distance implied by the linguistic-real valuation function is introduced, to process incomparable linguistic values in the results which need further procedure to make a certain decision. Illustrating examples show the effectiveness of the proposed approach which can rank the incomparable elements elastic.

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1. INTRODUCTION

In terms of common sense, people use linguistic terms in nature language for evaluating, reasoning and making decision, rather than crisp numbers. Hence, collection with qualitative information is always got, for example, object-attribute assessment for a car may be “very cheap” in price, “slightly comfortable” in comfort, “somewhat dangerous” in safety. To handle the qualitative information, many linguistic terms models have been proposed based on computing with words (CWW) [1,2], such as fuzzy set theory [3–5], symbolic approaches [6,7] and linguistic truth-valued mode in lattice order [8,9].

Fuzzy set theory was proposed by Zadeh in 1965 using a membership function to model the uncertain information [5]. But it is difficult to find appropriate membership function when we confronted with the circumstances where various concepts are given in various contexts. Type-2 fuzzy set is the extension of ordinary fuzzy set based on the theory of computing with words (CWW) [2]. Unlike a type-1 fuzzy set where the membership grade is a crisp number in [0, 1], the membership grade of Type-2 fuzzy set was extended to an ordinary fuzzy set [2–4]. Type-2 fuzzy set plays a role in wider applications and processes the uncertain information more flexibly [10–12].

Symbolic approaches use linguistic symbols to represent linguistic information directly without the numerical approximation required by fuzzy set based methods, and aggregate or compute on the indexes of these symbols to obtain the final result [13,14].

Symbolic approach has performed well in decision making problems especially semantic interpretation of linguistic terms in nature language [15,16]. But there exists some loss of information when the results come out from the initial expression domain. Francisco Herrera and Luis Martinez proposed a 2-tuple model which are composed by a linguistic term and a numeric value assessed in [−0.5, 0.5] [17,18]. The 2-tuple model is widely used in many real problems for that it can avoid information losing in linguistic information processing and many aggregation operators of 2-tuple model were provided [19–22].

The conventional symbolic approach for linguistic value uses a linear order structure. However, in the natural language, some linguistic values are incomparable, for example, “Exactly True” and “Somewhat False.” Obviously, it is difficult to describe the incomparable values utilizing a linear order structure. Lattice implication algebra (LIA) structure can imitate the uncertain and both comparable and incomparable characteristic [7]. Many researchers have studied in varied direction of LIA [23,24]. Jun Liu *et al.* have proposed an axiomatizable lattice ordered qualitative linguistic truth-valued logic system, which is a foundation for establishing formal linguistic truth-valued logic [25]. Xingxing He *et al.* have unified method for finding the structure of k -IESF in linguistic truth-valued lattice-valued propositional logic which provided a theoretical foundations and algorithms for α -resolution automated reasoning [26]. Yang Xu *et al.* have extended the binary α -resolution to multiary α -resolution in lattice-valued propositional logic $LP(X)$ and lattice-valued first-order logic $LF(X)$, obtained a result that multiary α -resolution principle in $LF(X)$ can be equivalently transformed into that in $LP(X)$ [27]. Yi Liu and Vassilis

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G. Kaburlasos *et al.* have introduced a LIA with implication values in a complete lattice of intervals on the real number axis which followed a capacity to optimize [28]. Li Zou proposed a linguistic-valued knowledge representation mode and approximated reasoning approach with linguistic-valued credibility factors [29].

All the above researches provided the feasibility of decision making with linguistic values based on linguistic-valued lattice implication algebra (LV-LIA). Besides object-attribute assessment, the importance of the attributes is often expressed using linguistic terms. However, the linguistic weights are always transformed to numbers or values on a linear order structure in a decision making problem [30]. This paper is aiming at decision making problems not only with the linguistic evaluation set but also the linguistic important degree, i.e., linguistic weight. We will propose a decision making model based on LV-LIA which has a flexible linguistic weighted method.

LV-LIA can express both comparable and incomparable linguistic information, which is consistent with nature language characteristic. In decision making there often exist some incomparable linguistic-valued decision making results. The positive valuation function (PVF) which implies a metric distance as well as an inclusion measure function can mapping a lattice to a real set [31]. We define a linguistic-real valuation function and get a rank with incomparable linguistic values through the linguistic-real metric distance (LRMD) elastic according to the individual preference degree.

Based on the aforementioned academic ideas, we will introduce a decision making approach based on 18-element LV-LIA.

The rest of paper is organized as follows: In section 2, we briefly review the concepts of LIA and LV-LIA. In section 3, we discuss the properties of the operations $\otimes$ and $\oplus$ of LV-LIA, and propose a decision making model with linguistic weighted method. In section 4, we introduce a linguistic-real valuation function implied a metric distance to rank the incomparable decision making results. In section 5, some conclusions are summarized.

2. LINGUISTIC-VALUED LATTICE IMPLICATION ALGEBRA

We briefly review some concepts of linguistic truth-valued LIA. We refer to the related Ref. [7].

Definition 1. [7] Let $(L, \vee, \wedge, O, I)$ be a bounded lattice with universal boundaries O (the least element) and I (the greatest element) respectively, and “ $'$ ” be an order-reversing involution. For any $x, y, z \in L$, if mapping $\rightarrow: L \times L \rightarrow L$ satisfies:

$$\begin{aligned} (I1) : & x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z); \\ (I2) : & x \rightarrow x = I; \\ (I3) : & x \rightarrow y = y' \rightarrow x'; \\ (I4) : & x \rightarrow y = y \rightarrow x = I \text{ implies } x = y; \\ (I5) : & (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x; \\ (I6) : & (x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z); \\ (I7) : & (x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z). \end{aligned} \quad (1)$$

Then $(L, \vee, \wedge, ', \rightarrow, O, I)$ is an LIA.

Definition 2. [7] Let $L_{n+1} = \{d_0, d_1, \dots, d_n\}$, $d_0 < d_2 < \dots < d_n$, $L_2 = \{b_1, b_2\}$, $b_1 < b_2$, $(L_{n+1}, \wedge_{(L_{n+1})}, \wedge'_{(L_{n+1})}, \rightarrow_{(L_{n+1})}, d_0, d_n)$ and $(L_2, \vee_{(L_2)}, \wedge_{(L_2)}, \wedge'_{(L_2)}, \rightarrow_{(L_2)}, b_1, b_2)$ be Lukasiewicz implication algebra. For any $(d_i, b_j), (d_k, b_m) \in L_{n+1} \times L_2$, if

$$\begin{aligned} (d_i, b_j) \vee (d_k, b_m) &= (d_i \vee_{(L_{n+1})} d_k, b_j \vee_{(L_2)} b_m), \\ (d_i, b_j) \wedge (d_k, b_m) &= (d_i \wedge_{(L_{n+1})} d_k, b_j \wedge_{(L_2)} b_m), \\ (d_i, b_j)' &= (d_i'^{(L_{n+1})}, b_j'^{(L_2)}), \\ (d_i, b_j) \rightarrow (d_k, b_m) &= (d_i \rightarrow_{(L_{n+1})} d_k, b_j \rightarrow_{(L_2)} b_m), \end{aligned}$$

then $(L_{n+1} \times L_2, \vee, \wedge, ', \rightarrow, (d_1, b_1), (d_n, b_2))$ is a LIA, denote as $\mathcal{L}_n \times \mathcal{L}_2$.

Definition 3. [7] Let $AD_{n+1} = \{h_0, h_1, \dots, h_n\}$ be a set of n hedge operators and $h_0 < h_1 < \dots < h_n$, $MT = \{f, t\}$ be “false(f)” and “true(t)”, denote $f < t$ and $L_{V((n+1) \times 2)} = AD_{n+1} \times MT$. Define the mapping $g: L_{V((n+1) \times 2)} \rightarrow \mathcal{L}_{n+1} \times \mathcal{L}_2$ as follows.

$$g((h_i, mt)) = \begin{cases} (d_i', b_1), & mt = f, \\ (d_i, b_2), & mt = t. \end{cases} \quad (2)$$

Then g is bijection. Its inverse mapping is g^{-1} . For any $x, y \in L_{V((n+1) \times 2)}$, define

$$\begin{aligned} x \vee y &= g^{-1}(g(x) \vee g(y)), \\ x \wedge y &= g^{-1}(g(x) \wedge g(y)), \\ x' &= g^{-1}(g(x)'), \\ x \rightarrow y &= g^{-1}(g(x) \rightarrow g(y)). \end{aligned} \quad (3)$$

Then $\mathcal{L}_{V((n+1) \times 2)} = (L_{V((n+1) \times 2)}, \vee, \wedge, ', \rightarrow, (h_n, f), (h_n, t))$ is called linguistic truth-valued LIA from AD_{n+1} and MT (Figure 1). g is an isomorphic mapping from $(L_{V((n+1) \times 2)}, \vee, \wedge, ', \rightarrow, (h_n, f), (h_n, t))$ to $\mathcal{L}_{n+1} \times \mathcal{L}_2$.

The operation “ $\rightarrow$ ” is

$$\begin{cases} (h_i, c_2) \rightarrow (h_j, c_1) = (h_{\max\{0, i+j-n\}}, c_1), \\ (h_i, c_1) \rightarrow (h_j, c_2) = (h_{\min\{n, i+j\}}, c_2), \\ (h_i, c_2) \rightarrow (h_j, c_2) = (h_{\min\{n, n-i+j\}}, c_2), \\ (h_i, c_1) \rightarrow (h_j, c_1) = (h_{\min\{n, n-j+i\}}, c_1). \end{cases} \quad (4)$$

Figure 1 | The Hasse diagram of $\mathcal{L}_{V((n+1) \times 2)}$.

Note that n should be an even number when we construct the $\mathcal{L}_{V((n+1) \times 2)}$.

3. THE DECISION MAKING APPROACH WITH LINGUISTIC WEIGHTED BASED ON LV-LIA

In a multi-attributes decision making problem, sometimes attribute importance is expressed with linguistic terms. We explore a linguistic weighted method expressing these linguistic weights with a linguistic lattice structure based on LV-LIA and apply it into decision making with linguistic weight in this section.

3.1. The Operations in LV-LIA

In a LIA L , there exist two binary operations $\otimes$ and $\oplus$ as follows [7].

For any $x, y \in L$,

$$x \otimes y = (x \rightarrow y')', \quad (5)$$

$$x \oplus y = x' \rightarrow y. \quad (6)$$

Similarly, we extend two binary operations $\otimes$ and $\oplus$ into LV-LIA.

Definition 4. For any $(h_i, c_k), (h_j, c_l) \in \mathcal{L}_{V((n+1) \times 2)}$, the binary operation $\otimes$ and $\oplus$ are defined as follows,

$$\begin{aligned} (h_i, c_k) \otimes (h_j, c_l) &= ((h_i, c_k) \rightarrow (h_j, c_l)')', \\ (h_i, c_k) \oplus (h_j, c_l) &= (h_i, c_k)' \rightarrow (h_j, c_l). \end{aligned} \quad (7)$$

According to Eqs. (4) and (7), an equal definition is obtained.

For operation $\otimes$, there exist

$$\begin{cases} (h_i, c_2) \otimes (h_j, c_1) = (h_{\min(n, n-i+j)}, c_1), \\ (h_i, c_1) \otimes (h_j, c_2) = (h_{\min(n, n-j+i)}, c_1), \\ (h_i, c_2) \otimes (h_j, c_2) = (h_{\max(0, j+i-n)}, c_2), \\ (h_i, c_1) \otimes (h_j, c_1) = (h_{\min(n, i+j)}, c_1). \end{cases} \quad (8)$$

And for operation $\oplus$, there exist

$$\begin{cases} (h_i, c_2) \oplus (h_j, c_1) = (h_{\min(n, n-j+i)}, c_2), \\ (h_i, c_1) \oplus (h_j, c_2) = (h_{\min(n, n-i+j)}, c_2), \\ (h_i, c_2) \oplus (h_j, c_2) = (h_{\min(n, i+j)}, c_2), \\ (h_i, c_1) \oplus (h_j, c_1) = (h_{\max(0, i+j-n)}, c_1). \end{cases} \quad (9)$$

As $\mathcal{L}_{V((n+1) \times 2)}$ is a LIA, for any $(h_i, c_k), (h_j, c_l), (h_s, c_t) \in \mathcal{L}_{V((n+1) \times 2)}$, we get the following properties,

- i. $(h_i, c_k) \otimes (h_j, c_l) = (h_j, c_l) \otimes (h_i, c_k),$
 $(h_i, c_k) \oplus (h_j, c_l) = (h_j, c_l) \oplus (h_i, c_k).$
- ii. $(h_i, c_k) \otimes (h_n, c_2) = (h_i, c_k),$
 $(h_i, c_k) \otimes (h_n, c_1) = (h_i, c_k),$
 $(h_i, c_k) \oplus (h_n, c_2) = (h_n, c_2),$
 $(h_i, c_k) \oplus (h_n, c_1) = (h_i, c_k).$

- iii. $(h_i, c_k) \otimes (h_n, c_2) \otimes (h_s, c_t) = (h_i, c_k) \otimes ((h_n, c_2) \otimes (h_s, c_t)),$
 $(h_i, c_k) \otimes (h_n, c_2) \otimes (h_s, c_t) = (h_i, c_k) \otimes ((h_n, c_2) \otimes (h_s, c_t)).$

Some other special properties of the operations $\otimes$ and $\oplus$ in LV-LIA are listed as follows.

Theorem 1. For any $(h_i, c_k), (h_j, c_l) \in \mathcal{L}_{V((n+1) \times 2)}$,

$$\begin{aligned} (h_i, c_k) \otimes (h_j, c_l) &\leq (h_i, c_k), \\ (h_i, c_k) \otimes (h_j, c_l) &\leq (h_j, c_l), \\ (h_i, c_k) \oplus (h_j, c_l) &\geq (h_i, c_k), \\ (h_i, c_k) \oplus (h_j, c_l) &\geq (h_j, c_l). \end{aligned}$$

Proof. We prove it from four situations.

- i. $c_k = c_2$ and $c_l = c_1$;
- ii. $c_k = c_1$ and $c_l = c_2$;
- iii. $c_k = c_2$ and $c_l = c_2$;
- iv. $c_k = c_1$ and $c_l = c_1$.

Without loss of generality, we only prove situation at that $c_k = c_2$ and $c_l = c_1$. The other situations can be proved similarly.

From Eq. (8), we know that

$$(h_i, c_2) \otimes (h_j, c_1) = (h_{\min(n, n-i+j)}, c_1).$$

$$(h_{\min(n, n-i+j)}, c_1) \leq (h_{n-i}, c_1) < (h_i, c_2), \text{ for } n-i+j \geq n-i.$$

$$(h_{\min(n, n-i+j)}, c_1) \leq (h_j, c_1), \text{ for } n-i+j \geq j.$$

Hence, $(h_i, c_k) \otimes (h_j, c_l) \leq (h_j, c_l)$.

From Eq. (9), we know that

$$(h_i, c_2) \oplus (h_j, c_1) = (h_{\min(n, n-j+i)}, c_2).$$

$$(h_{\min(n, n-j+i)}, c_2) \geq (h_i, c_2), \text{ as } n-j+i \geq i.$$

$$(h_{\min(n, n-j+i)}, c_2) \geq (h_{n-j}, c_2) > (h_j, c_1), \text{ for } n-j+i \geq n-j.$$

Hence, $(h_i, c_k) \oplus (h_j, c_l) \geq (h_j, c_l)$.

It also can be proved in the other situations that

$$\begin{aligned} (h_i, c_k) \otimes (h_j, c_l) &\leq (h_j, c_l), \\ (h_i, c_k) \oplus (h_j, c_l) &\geq (h_j, c_l). \end{aligned}$$

From the properties of the operations $\otimes$ and $\oplus$ in LV-LIA,

$$\begin{aligned} (h_i, c_k) \otimes (h_j, c_l) &\leq (h_i, c_k), \\ (h_i, c_k) \oplus (h_j, c_l) &\geq (h_i, c_k). \end{aligned}$$

Theorem 2. For any $(h_i, c_k), (h_j, c_l), (h_s, c_t) \in \mathcal{L}_{V((n+1) \times 2)}$, if $(h_i, c_k) \leq (h_j, c_l)$, then

$$\begin{aligned} (h_i, c_k) \otimes (h_s, c_t) &\leq (h_j, c_l) \otimes (h_s, c_t), \\ (h_i, c_k) \oplus (h_s, c_t) &\leq (h_j, c_l) \oplus (h_s, c_t). \end{aligned}$$

It can be proved from the Eqs. (8) and (9) respectively.

According to the properties of the operations $\otimes$ and $\oplus$ in LV-LIA, we obtain

$$(h_s, c_t) \otimes (h_i, c_k) \leq (h_s, c_t) \otimes (h_j, c_l),$$

$$(h_s, c_t) \oplus (h_i, c_k) \leq (h_s, c_t) \oplus (h_j, c_l).$$

Suppose that x_i, x_j are two assessments with respect to the i th and the j th attribute of an alternative expressed by linguistic values on LV-LIA and ω_i, ω_j are two linguistic weights on LV-LIA. Then according to the Theorem 1, we have

$$x_i \otimes \omega_i < x_j.$$

According to the Theorem 2, if $x_i \leq x_j$ or $\omega_i \leq \omega_j$, we have

$$\omega_i \otimes x_i \leq \omega_j \otimes x_i,$$

$$\omega_i \otimes x_i \leq \omega_i \otimes x_j.$$

That shows that when the assessments of alternatives are the same, more important the attribute is, higher evaluation the alternative achieves. Accordingly, when the weights of attributes are the same, larger the assessment of alternative is, higher evaluation the alternative achieves.

3.2. The Decision Making Approach Based on 18LV-LIA

In generally, linguistic terms with linguistic hedges can be seen as linguistic modifiers and prime terms. In the following, we divide the linguistic terms into nine linguistic hedges (modifiers) applying to two prime terms. The linguistic modifiers of LV-LIA are *{Slightly (Sl for short), Somewhat (So), Rather (Ra), Almost (Al), Exactly (Ex), Quite (Qu), Very (Ve), Highly (Hi), Absolutely (Ab)}* with the semantic ordering relationship $Sl < So < Ra < Al < Ex < Qu < Ve < Hi < Ab$, i.e., the hedge set is $AD_9 = \{h_0 = Sl, h_1 = So, h_2 = Ra, h_3 = Al, h_4 = Ex, h_5 = Qu, h_6 = Ve, h_7 = Hi, h_8 = Ab\}$. The prime terms are *{Dissatisfied (Ds for short), Satisfied (Sa)}* representing a pair opposite meta vague concept with $Ds < Sa$, i.e., the prime term set $MT = \{c_1 = Ds, c_2 = Sa\}$. Applying the linguistic modifiers of AD_9 to the prime terms MT , we obtain a partially ordered lattice $\mathcal{L}_{V(9 \times 2)} = (L_{V(9 \times 2)}, \vee, \wedge, ', \rightarrow, (h_8, c_2), (h_8, c_1))$ called 18-element linguistic-valued lattice implication algebra (18LV-LIA) (Figure 2) [32].

A decision making model is constructed based on 18LV-LIA. Weights of the attributes are expressed with linguistic terms on LV-LIA. The operation $\otimes$ can be used to calculate the linguistic weight. The attribute aggregation can be done according to the aggregation rules which can be set flexibly based on the actual cases.

There are some aggregation function about the logical relation “or” and “and”.

For relation “ (h_i, c_k) and (h_j, c_l) ” can be aggregated as,

Figure 2 The Hasse diagram of $\mathcal{L}_{V(9 \times 2)}$.

$$i. F_{and}((h_i, c_k), (h_j, c_l)) = \quad (10)$$

$$\begin{cases} (h_{\lfloor \frac{i+j}{2} \rfloor}, c_l), & \text{if } k = l \\ (h_{\lfloor \frac{8+i \cdot \text{sgn}(k-1.5)+j \cdot \text{sgn}(l-1.5)}{2} \rfloor}, c_2), & \text{if } k \neq l \end{cases},$$

where $\text{sgn}(\cdot)$ is the sign function, the maximum subscript of the hedges in 18LV-LIA is 8, and 1.5 is the median point of the subscript of prime term, in fact, it can be any point in interval (1, 2).

$$ii. F_{and}((h_i, c_k), (h_j, c_l)) = (h_i, c_k) \wedge (h_j, c_l). \quad (11)$$

For relation “ (h_i, c_k) or (h_j, c_l) ” can be aggregated as,

$$iii. F_{or}((h_i, c_k), (h_j, c_l)) = \begin{cases} (h_u, c_2), & \text{if } k \parallel l \\ \max((h_i, c_k), (h_j, c_l)), & \text{others} \end{cases} \quad (12)$$

where $u = \max(n, 8 + i \cdot \text{sgn}(k - 1.5) + j \cdot \text{sgn}(l - 1.5))$, and the parameters means the same with parameters in Eq. (10).

$$iv. F_{or}((h_i, c_k), (h_j, c_l)) = (h_i, c_k) \vee (h_j, c_l). \quad (13)$$

They all satisfy the associative law and commutative law.

A linguistic multi-attribute decision making problem consists of a finite and non-empty alternative set $A = \{A_1, A_2, \dots, A_k\}$, where each alternative is defined by means of a finite set of attribute, $C = \{C_1, C_2, \dots, C_m\}$ which is assessed using linguistic expressions, and the corresponding weight set for attributes is $W = (\omega_1, \omega_2, \dots, \omega_m)$.

The decision making approach based on 18LV-LIA consists mainly of the flowing steps.

Step 1. Linguistic assessment matrix and linguistic weight transforming. The linguistic assessment matrix is

$$E = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & \ddots & & a_{2m} \\ \vdots & & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{Nm} \end{pmatrix}.$$

In the linguistic assessment matrix, $a_{ij} \in \mathcal{L}_{V(9 \times 2)}$ is the assessment about the attribute C_j of alternative A_i .

The attribute weights given by the experts are the linguistic terms in $\mathcal{L}_{V(9 \times 2)}$ as

$$W = \{\omega_1, \omega_2, \dots, \omega_m\}.$$

In this step, the attribute assessment collection of alternatives is transformed to linguistic terms on $\mathcal{L}_{V(9 \times 2)}$.

Step 2. Weighted evaluation matrix Construction. Weighted evaluation matrix is constructed from Eq. (8).

$$M = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & \ddots & & b_{2m} \\ \vdots & & \ddots & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{Nm} \end{pmatrix} = E \otimes W,$$

where $b_{ij} = a_{ij} \otimes \omega_j$.

Step 3. Aggregation function setting and comprehensive evaluations aggregation.

The comprehensive evaluation

$$b_i = F(b_{i1}, b_{i2}, \dots, b_{im}),$$

where $F(\cdot)$ is the aggregation function of the weighted evaluation.

Step 4. Decision making process. The comprehensive evaluations b_i ($i = 1, 2, \dots, N$) are ranked with respect to each alternative according to Fig. 2.

Here are a few notes. The aggregation function is set as

$$F(b_{i1}, b_{i2}, \dots, b_{im}) = F_{m-1}(\dots F_2(F_1(b_{i1}, b_{i2}), b_{i3}) \dots, b_{im}),$$

where $F_k(\cdot)$ is the aggregation operators of b_{ik} and $b_{i,k-1}$ related to their logical relation, b_i is the comprehensive evaluation of the i th alternative. The function can be a logical operator, an arithmetical operator or even a hybrid one.

It is also noted that incomparable linguistic results are allowed.

3.3. Examples Illustration

Example 1. A teaching evaluation system with linguistic information.

To show how the proposed approach works, we give a teaching evaluation system which help decision makers evaluate the teacher's teaching quality. Suppose that the teaching quality depends on four attributes, responsibility, lesson preparation, professional proficiency and teaching efficiency. The attribute set is denoted as $C = \{C_1, C_2, C_3, C_4\}$. The alternative set is $A = \{A_1, A_2, A_3, A_4\}$ representing four teachers respectively.

Step 1. The assessment collection with respect to the attributes of four teachers about teaching quality is shown in Table 1. The linguistic terms are the linguistic values in $\mathcal{L}_{V(9 \times 2)}$. We get the initial assessment matrix.

$$E = \begin{pmatrix} (h_4, c_2) & (h_5, c_2) & (h_4, c_2) & (h_8, c_2) \\ (h_4, c_2) & (h_1, c_1) & (h_4, c_2) & (h_8, c_1) \\ (h_7, c_2) & (h_5, c_2) & (h_7, c_2) & (h_4, c_1) \\ (h_0, c_1) & (h_3, c_1) & (h_0, c_1) & (h_4, c_2) \end{pmatrix}.$$

W_1, W_2 and W_3 are three weight vectors.

$$W_1 = ((h_8, c_2), (h_8, c_2), (h_8, c_2), (h_8, c_2)),$$

$$W_2 = ((h_6, c_1), (h_4, c_2), (h_6, c_2), (h_8, c_1)),$$

$$W_3 = ((h_8, c_1), (h_8, c_1), (h_8, c_1), (h_8, c_1)).$$

W_1 mans that all of the attribute are equally important. W_2 mans that the attributes are with different importance. W_3 means that all of the attributes are not important at all, where the case is that unavailable attributes are collected.

Step 2. The weighted matrix are got utilizing the operation $\otimes$ according to Eq. (8).

$$M_1 = \begin{pmatrix} (h_4, c_2) & (h_5, c_2) & (h_4, c_2) & (h_8, c_2) \\ (h_4, c_2) & (h_1, c_1) & (h_4, c_1) & (h_8, c_1) \\ (h_7, c_2) & (h_5, c_2) & (h_7, c_2) & (h_4, c_1) \\ (h_0, c_1) & (h_3, c_1) & (h_0, c_1) & (h_4, c_2) \end{pmatrix},$$

$$M_2 = \begin{pmatrix} (h_8, c_1) & (h_1, c_2) & (h_2, c_2) & (h_8, c_1) \\ (h_8, c_1) & (h_5, c_1) & (h_2, c_2) & (h_8, c_1) \\ (h_7, c_1) & (h_1, c_2) & (h_5, c_2) & (h_8, c_1) \\ (h_6, c_1) & (h_7, c_1) & (h_2, c_1) & (h_8, c_1) \end{pmatrix},$$

$$M_3 = \begin{pmatrix} (h_8, c_1) & (h_8, c_1) & (h_8, c_1) & (h_8, c_1) \\ (h_8, c_1) & (h_8, c_1) & (h_8, c_1) & (h_8, c_1) \\ (h_8, c_1) & (h_8, c_1) & (h_8, c_1) & (h_8, c_1) \\ (h_8, c_1) & (h_8, c_1) & (h_8, c_1) & (h_8, c_1) \end{pmatrix}.$$

Step 3. Suppose there is facts that a teacher who is responsible or prepare lesson fully, and professional proficiency or teaching efficiency should be with high teaching quality. It is a rule, as a teacher,

IF (responsible or prepare lesson fully) and (professional proficiency or teaching efficiency) THEN (with high teaching quality).

The aggregation function for relation “or” is set as the logical operation Eq. (13). The aggregation function for relation “and” is set as the arithmetic operations Eq. (10). The weighted evaluations under W_2 are listed in the Table 2.

And by the aggregation rule, we get the assessments of four teachers with different attribute weights (Table 3).

Step 4: The results in Table 3 are ranked (Table 4).

Table 1 | The assessment collection about quality of teaching.

	C_1	C_2	C_3	C_4
A_1	(Ex, Sa)	(Qu, Sa)	(Ex, Sa)	(Ab, Sa)
A_2	(Ex, Sa)	(So, Ds)	(Ex, Sa)	(Ab, Ds)
A_3	(Hi, Sa)	(Qu, Sa)	(Hi, Sa)	(Ex, Ds)
A_4	(Sl, Ds)	(Al, Ds)	(Sl, Ds)	(Ex, Sa)

Table 2 | The weighted evaluations values under W_2 .

	A_1	A_2	A_3	A_4
C_1	(h_8, c_1)	(h_1, c_2)	(h_2, c_2)	(h_8, c_1)
C_2	(h_8, c_1)	(h_5, c_1)	(h_2, c_2)	(h_8, c_1)
C_3	(h_7, c_1)	(h_1, c_2)	(h_5, c_2)	(h_8, c_1)
C_4	(h_6, c_1)	(h_7, c_1)	(h_2, c_1)	(h_8, c_1)

Table 3 | Comprehensive evaluations.

	A_1	A_2	A_3	A_4
W_1	(h_6, c_2)	(h_5, c_2)	(h_7, c_2)	(h_8, c_2)
W_2	(h_1, c_2)	(h_3, c_2)	(h_3, c_2)	(h_4, c_1)
W_3	(h_8, c_1)	(h_8, c_1)	(h_8, c_1)	(h_8, c_1)
Without weights	(h_6, c_2)	(h_5, c_2)	(h_7, c_2)	(h_8, c_2)

Table 4 | The final results with different weights.

W_1	W_2	W_3
$A_4 > A_3 > A_1 > A_2$,	$A_2 = A_3 > A_1$, $A_1 \parallel A_4$, $A_2 \parallel A_4$, $A_3 \parallel A_4$.	$A_1 = A_2 = A_3 = A_4$.

From Table 2, we can see that the four teachers get the same assessments in attribute C_1 and C_3 but they get different weighted evaluations because the weights of attribute C_1 and C_3 are different. In fact, the two weights has the same linguistic hedges but the different prime terms, that can't expressed utilizing a linear structure.

From Table 3, we can see that when all attributes are absolutely important, the results equal to the results without any weight. Therefore, the present method not considering weights is a special case of the proposed approach.

When the weights of attribute are different, the results will be changed. For example, Teacher A_2 get higher score under the weight W_2 than it under the weight W_1 .

From the results, we can see that when we consider all attribute absolutely unimportant, i.e., the weight vector is W_3 , the four teachers get the same and lowest evaluation. It means that we collected unavailable data.

From Table 4, there are incomparable values in the results. For example, there exists the relation for W_2 as follows (Figure 3),

It is reasonable for that the incomparable cases exist in reality because of the characteristic of the linguistic value in nature language.

Remark: In step 3, taking the teacher A_1 when the weight set is W_2 as an example, we show the aggregation process.

$$\begin{aligned}
 F &= F_{and}(F_{or}((h_8, c_1), (h_1, c_2)), F_{or}((h_2, c_2), (h_8, c_1))) \\
 &= F_{and}(((h_8, c_1) \vee (h_1, c_2)), ((h_2, c_2) \vee (h_8, c_1))) \\
 &= \left(h_{\left[\frac{1+2}{2} \right]}, c_2 \right) \\
 &= (h_1, c_2).
 \end{aligned}$$

Figure 3 | Relations of teachers on $\mathcal{L}_{V(9 \times 2)}$ under W_2 .

Example 2. To illustrate the effectiveness of the method, an example of car evaluation in Ref. [30] is considered.

Suppose that there are three kinds of cars: BMW (x_1), Hyundai (x_2) and Passat (x_3) are under evaluation according to four attributes: safety (f_1), price (f_2), comfort (f_3) and fuel economy (f_4). The linguistic judgments for evaluating the cars on $\mathcal{L}_{V(9 \times 2)}$ is as Table 5. The weights associated with the four criteria are supposed to be

$$W_1 = [(h_8, c_2), (h_6, c_2), (h_3, c_2), (h_5, c_2)],$$

or

$$W_2 = [(h_8, c_1), (h_6, c_2), (h_3, c_1), (h_5, c_2)].$$

The comprehensive evaluations is as Table 6.

From Table 6, we can see that when the weights are positive prime terms, the two methods get the same results. But when the weights consist of negative prime terms, the method in Ref. [30] is hard to handle it.

In Ref. [30], Shuwei Chen *et al.* used the linguistic multi-criteria decision making approach based on logical reasoning. Our method is more widely in weight setting and more flexible in comprehensive way.

In Ref. [30], weights are from the positive prime terms only. But in this paper, weights can be positive prime terms or negative prime terms. And the aggregation function can be logical operators or arithmetic operators or hybrid operators in this paper. Our method can handle more widely cases. It is more flexible facing the complex problems in reality.

Table 5 | Linguistic assessment about cars on $\mathcal{L}_{V(9 \times 2)}$.

	f_1	f_2	f_3	f_4
x_1	(h_6, c_2)	(h_4, c_1)	(h_7, c_2)	(h_1, c_2)
x_2	(h_1, c_1)	(h_5, c_2)	(h_1, c_2)	(h_4, c_2)
x_3	(h_6, c_2)	(h_1, c_2)	(h_5, c_2)	(h_2, c_1)

Table 6 | The comprehensive evaluations from two methods.

Weight	W_1			W_2		
Method	x_1	x_2	x_3	x_1	x_2	x_3
Our Method	(h_6, c_2)	(h_7, c_2)	(h_6, c_2)	(h_4, c_2)	(h_3, c_2)	(h_3, c_2)
Chen's method	(h_6, c_2)	(h_7, c_2)	(h_6, c_2)	–	–	–

From the results in Table 4, incomparable linguistic values are existed rationally because of the inherent characteristic of linguistic value. However, a certain decision is demanded sometimes. Then some further works should be done in this case.

4. RANKING METHOD FOR INCOMPARABLE LINGUISTIC VALUES

From the Figure 2, we know that that (h_i, c_2) and (h_{n-1-i}, c_1) are incomparable. They can describe the incomparable linguistic terms in the nature language. The characteristic brings incomparable results. We utility a PVF to make a certain ranking when it is necessary. So this section is extension and supplement of the above.

4.1. PVF for Linguistic-Value Lattice

A valuation on a crisp lattice L is a real-valued function $v: L \rightarrow R$ which satisfies $v(a) + v(b) = v(a \vee b) + v(a \wedge b)$, $a, b \in L$. A valuation is called monotone if and only if $a \leq b$ in L implies $v(a) \leq v(b)$ and positive if and only if $a < b$ implies $v(a) < v(b)$ [11].

A PVF $v(\cdot)$ in a lattice L implies a metric distance $d: L \times L \rightarrow R$ given by $d(x_1, x_2) = v(x_1 \vee x_2) - v(x_1 \wedge x_2)$ for L [11].

Utilizing a PVF, a value on a lattice can be mapped to a real number which keeping the order of these comparable elements on a lattice. Aiming to handle the incomparable problem of the linguistic values on LV-LIA, a PVF is defined called linguistic-real positive-valuation function (LR-PVF) on LV-LIA.

Definition 5. Let $\mathcal{L}_{V((n+1) \times 2)}$ be a linguistic-value lattice, $(h_i, c_k) \in \mathcal{L}_{V((n+1) \times 2)}$ ($i = 0, 1, 2, \dots, n$ and $k = 1, 2$). Let $\alpha, \beta \in [0, 1]$ be two preference parameters which represent the preference degree to positive side and negative side respectively. The LR-PVF is defined as

$$v: \mathcal{L}_{V((n+1) \times 2)} \rightarrow R.$$

$$v((h_i, c_k)) = \alpha \times (k - 1) + \beta \times (n \times (2 - k) + s(k) \times i), \quad (14)$$

where $s(\cdot)$ is a function defined as

$$s(k) = \begin{cases} -1, & k = 1, \\ 1, & k = 2, \end{cases}$$

The preference parameters α, β make the LR-PVF be elastic by the individual performances. $(k - 1)$, $(2 - k)$ and $s(k)$ are to distinguish the prime terms. An equal equipment is obtained by substituting k ,

$$v((h_i, c_k)) = \begin{cases} \beta \times (n - i), & k = 1, \\ \alpha + \beta \times i, & k = 2. \end{cases} \quad (15)$$

Theorem 3. The linguistic-real valuation function $v: \mathcal{L}_{V((n+1) \times 2)} \rightarrow R$ is a PVF.

It is proved in Appendix A.

Definition 6. LRMD which implied by PVF $v(\cdot)$ in LV-LIA is defined as

$$dl: \mathcal{L}_{V((n+1) \times 2)} \rightarrow R,$$

$$dl(x_1, x_2) = v(x_1 \vee x_2) - v(x_1 \wedge x_2). \quad (16)$$

Theorem 4. Suppose $(h_i, c_k), (h_j, c_l) \in \mathcal{L}_{V((n+1) \times 2)}$, the LRMD between $(h_i, c_k) \wedge (h_j, c_l)$ and (h_i, c_k) is denoted as Dc_1 , the LRMD between $(h_i, c_k) \wedge (h_j, c_l)$ and (h_j, c_l) is denoted as Dc_2 . Similarly, the LRMD between $(h_i, c_k) \vee (h_j, c_l)$ and (h_i, c_k) is denoted as Dd_1 , the LRMD between $(h_i, c_k) \vee (h_j, c_l)$ and (h_j, c_l) is denoted as Dd_2 .

If $Dc_1 < Dc_2$ or $Dd_1 > Dd_2$, then $(h_i, c_k) < (h_j, c_l)$.

$$\begin{aligned} \text{Proof. } Dc_1 &= dl((h_i, c_k), ((h_i, c_k) \wedge (h_j, c_l))) \\ &= v((h_i, c_k) \vee ((h_i, c_k) \wedge (h_j, c_l))) \\ &\quad - v((h_i, c_k) \wedge ((h_i, c_k) \wedge (h_j, c_l))) \\ &= v((h_i, c_k)) - v((h_i, c_k) \wedge (h_j, c_l)); \end{aligned} \quad (17)$$

$$\begin{aligned} Dc_2 &= dl((h_j, c_l), ((h_i, c_k) \wedge (h_j, c_l))) \\ &= v((h_j, c_l) \vee ((h_i, c_k) \wedge (h_j, c_l))) \\ &\quad - v((h_j, c_l) \wedge ((h_i, c_k) \wedge (h_j, c_l))) \\ &= v((h_j, c_l)) - v((h_i, c_k) \wedge (h_j, c_l)). \end{aligned} \quad (18)$$

If $Dc_1 < Dc_2$, i.e., $v((h_i, c_k)) - v((h_i, c_k) \wedge (h_j, c_l)) < v((h_j, c_l)) - v((h_i, c_k) \wedge (h_j, c_l))$. Then we know $v((h_i, c_k)) < v((h_j, c_l))$. Since the LR-PVF $v(\cdot)$ is positive, we get $(h_i, c_k) < (h_j, c_l)$.

$$\begin{aligned} Dd_1 &= dl((h_i, c_k), ((h_i, c_k) \vee (h_j, c_l))) \\ &= v((h_i, c_k) \vee ((h_i, c_k) \vee (h_j, c_l))) \\ &\quad - v((h_i, c_k) \wedge ((h_i, c_k) \vee (h_j, c_l))) \\ &= v((h_i, c_k) \vee (h_j, c_l)) - v((h_i, c_k)). \end{aligned} \quad (19)$$

$$\begin{aligned} Dd_2 &= dl((h_j, c_l), ((h_i, c_k) \vee (h_j, c_l))) \\ &= v((h_j, c_l) \vee ((h_i, c_k) \vee (h_j, c_l))) \\ &\quad - v((h_j, c_l) \wedge ((h_i, c_k) \vee (h_j, c_l))) \\ &= v((h_i, c_k) \vee (h_j, c_l)) - v((h_j, c_l)). \end{aligned} \quad (20)$$

If $Dd_1 > Dd_2$, i.e., $v((h_i, c_k) \vee (h_j, c_l)) - v((h_i, c_k)) > v((h_i, c_k) \vee (h_j, c_l)) - v((h_j, c_l))$. Then we know $v((h_i, c_k)) < v((h_j, c_l))$. Since the LR-PVF $v(\cdot)$ is positive, we get $(h_i, c_k) < (h_j, c_l)$.

Definition 7. Let $(h_i, c_k), (h_j, c_l) \in \mathcal{L}_{V((n+1) \times 2)}$, the LRMD between $(h_i, c_k) \wedge (h_j, c_l)$ and (h_i, c_k) is denoted as Dc_1 and the LRMD between $(h_i, c_k) \wedge (h_j, c_l)$ and (h_j, c_l) is denoted as Dc_2 . Similarly, the LRMD between $(h_i, c_k) \vee (h_j, c_l)$ and (h_i, c_k) is denoted as Dd_1 and the LRMD between $(h_i, c_k) \vee (h_j, c_l)$ and (h_j, c_l) is denoted as Dd_2 .

If $Dc_1 < Dc_2$ or $Dd_1 > Dd_2$, then we say (h_i, c_k) is ranked after (h_j, c_l) , denoted as $(h_i, c_k) \lessdot (h_j, c_l)$.

It is obvious that if $(h_i, c_k) < (h_j, c_l)$, then $(h_i, c_k) \lessdot (h_j, c_l)$.

4.2. The Ranking Method Base on Linguistic-Real Valuation Function

Individual preference to the positive side and the negative side is different. We adjust the preference parameters in linguistic-real valuation function according to individual's preference for incomparable ranking.

Suppose $(h_i, c_k), (h_j, c_l) \in \mathcal{L}_{V(n+1) \times 2}$ are two incomparable linguistic values in the decision making problems, we rank them utilizing LRMD from Theorem 4.

The flowchart of the developing ranking process by considering the incomparable values ranking is shown in Figure 4.

Step 1. If two values in comprehensive results are comparable, then go to step 2; else go to step 3.

Step 2. The comparable values can be ranked by the lattice structure according to the Figure 1.

Step 3. For the incomparable values are mapped to real number by LR-PVF. For example, suppose $(h_i, c_k), (h_j, c_l)$ are incomparable, then $(h_i, c_k), (h_j, c_l)$, and $(h_i, c_k) \wedge (h_j, c_l)$ or $(h_i, c_k) \vee (h_j, c_l)$ are mapped to real number according to Eq. (15),

$$\begin{aligned} v((h_i, c_k)) &= r_1, \\ v((h_j, c_l)) &= r_2, \end{aligned}$$

$$v((h_i, c_k) \wedge (h_j, c_l)) = p \text{ or } v((h_i, c_k) \vee (h_j, c_l)) = q.$$

Step 4. Linguistic-real distance computing. According to Eqs. (17) and (18), denote the linguistic-real distance between (h_i, c_k) and $(h_i, c_k) \wedge (h_j, c_l)$ as Dc_1 , and denote the linguistic-real distance between (h_j, c_l) and $(h_i, c_k) \wedge (h_j, c_l)$ as Dc_2 respectively. Or according to Eqs. (19) and (20), denote the linguistic-real distance between (h_i, c_k) and $(h_i, c_k) \vee (h_j, c_l)$ as Dd_1 and (h_j, c_l) and $(h_i, c_k) \vee (h_j, c_l)$ as Dd_2 respectively.

$$\begin{aligned} Dc_1 &= v((h_i, c_k)) - v((h_i, c_k) \wedge (h_j, c_l)) = r_1 - p, \\ Dc_2 &= v((h_j, c_l)) - v((h_i, c_k) \wedge (h_j, c_l)) = r_2 - p. \end{aligned}$$

or

$$\begin{aligned} Dd_1 &= v((h_i, c_k) \vee (h_j, c_l)) - v((h_i, c_k)) = q - r_1, \\ Dd_2 &= v((h_i, c_k) \vee (h_j, c_l)) - v((h_j, c_l)) = q - r_2. \end{aligned}$$

Figure 4 The flowchart of ranking with incomparable results.

Step 5. Incomparable ranking process. If $Dc_1 < Dc_2$ or $Dd_1 > Dd_2$, then $(h_i, c_k) \prec (h_j, c_l)$.

Example 3. In the teaching evaluation system, when the weight vector is W_2 , teacher A_2, A_3 and teacher A_4 are incomparable. We show the incomparable ranking process by rank $A_2 (= A_3) = (h_3, c_2)$ and $A_4 = (h_4, c_1)$.

The preference parameters are set as $\alpha = 0.2, \beta = 0.8$ by experience. It means that the preference degree to the positive side is lower than negative side on 18LV-LIA, then

$$v((h_i, c_k)) = \begin{cases} 0.8 \cdot (8 - i), & k = 1, \\ 0.2 + 0.8 \cdot i, & k = 2. \end{cases} \quad (21)$$

Step 1. The comparable values has been ranked in the Section 3.

Step 2. Linguistic valuation function computing:

According to Eq. (21),

$$v((h_3, c_2)) = 2.6,$$

$$v((h_4, c_1)) = 3.2,$$

$$v((h_3, c_2) \wedge (h_4, c_1)) = v((h_5, c_1)) = 2.4 \text{ or}$$

$$v((h_3, c_2) \vee (h_4, c_1)) = v((h_4, c_2)) = 3.4.$$

Step 3. Linguistic-real distance: The linguistic-real distance Dc_1 and Dc_2 are got as follows.

$$\begin{aligned} Dc_1 &= dl((h_3, c_2), ((h_3, c_2) \wedge (h_4, c_1))) \\ &= v((h_3, c_2)) - v((h_3, c_2) \wedge (h_4, c_1)) = 0.2, \\ Dc_2 &= dl((h_4, c_1), ((h_3, c_2) \wedge (h_4, c_1))) \\ &= v((h_4, c_1)) - v((h_3, c_2) \wedge (h_4, c_1)) = 0.8. \end{aligned}$$

Step 4. Ranking process:

Since $Dc_1 < Dc_2$, so $(h_3, c_2) \prec (h_4, c_1)$.

We also can do that in another way. In step 3, linguistic-real distance between (h_3, c_2) and $(h_3, c_2) \vee (h_4, c_1)$ noted as Dd_1 , (h_4, c_1) and $(h_3, c_2) \vee (h_4, c_1)$ noted as Dd_2 .

$$\begin{aligned} Dd_1 &= dl((h_3, c_2), ((h_3, c_2) \vee (h_4, c_1))) \\ &= v((h_3, c_2) \vee (h_4, c_1)) - v((h_3, c_2)) = 0.8, \\ Dd_2 &= dl((h_4, c_1), ((h_3, c_2) \vee (h_4, c_1))) \\ &= v((h_3, c_2) \vee (h_4, c_1)) - v((h_4, c_1)) = 0.2. \end{aligned}$$

It is obvious that $Dd_1 > Dd_2$, so $(h_3, c_2) \prec (h_4, c_1)$.

From above, the best one in teaching quality is teacher A_4 if the system highlight the negative evaluations.

When the preference changes, the ranking results would change. Figure 5 shows Dc_1 and Dc_2 relation under different preference parameters.

From Figure 5, Dc_1 and Dc_2 is changing following preference parameters. When $\alpha = \beta$, $Dc_1 = Dc_2$; When $\alpha > \beta$, $Dc_1 < Dc_2$; When $\beta < \alpha$, $Dc_1 > Dc_2$. It shows that the incomparable ranking results can be adjust elastic related to people's preference to positive side and negative side.

Figure 5 | D_{c1} and D_{c2} with different preference parameters.

5. CONCLUSIONS

LV-LIA can deal with linguistic information by manipulating directly linguistic terms without numerical approximation. We proposed a linguistic approach of decision making with linguistic weight based on LV-LIA to deal with the attribute in different importance degree. The contrast example illustrated the proposed approach is more widely in setting weight and more flexible in linguistic aggregation. The attribute weight can be positive prime terms or negative prime terms which are on a lattice structure rather than a linear construction. The linguistic aggregation function for weighted evaluations comprehensive can be logical operations or arithmetic operations or hybrid ones. It makes the proposed approach more widely applied.

Incomparable linguistic results sometimes exist and are rational. In this paper we defined a PVF named LR-PVF by which the LV-LIA can be mapped to a real set. The distance implied by LV-LIA was used to do further works when these incomparable linguistic results need to be ranked. It is elastic by adjusting the performance parameters setting by individual performance to the positive side and negative side.

The proposed approach can be extended into more fields such as pattern recognition, risk analysis, quality evaluation and so on. The data and the parameters in the example are by experience in this paper. It will lead more convincing conclusion with practical data.

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APPENDIX A

Proof of Theorem 3

The proof of Theorem 3 is as below.

Proof. There are two conditions should be proved to convince that $v : \mathcal{L}_{V((n+1) \times 2)} \rightarrow R$ is a positive valuation function. For any $(h_i, c_k), (h_j, c_l) \in \mathcal{L}_{V((n+1) \times 2)}$,

- i. $v((h_i, c_k) \vee (h_j, c_l)) + v((h_i, c_k) \wedge (h_j, c_l)) = v((h_i, c_k)) + v((h_j, c_l))$,
- ii. If and only if $(h_i, c_k) < (h_j, c_l)$ implies $v((h_i, c_k)) < v((h_j, c_l))$.
- i. The first condition is proved as the following.
To prove $v((h_i, c_k) \vee (h_j, c_l)) + v((h_i, c_k) \wedge (h_j, c_l)) = v((h_i, c_k)) + v((h_j, c_l))$. We formulate the operation “ $\vee$ ” and “ $\wedge$ ” are shown in Figure 1.

$$(h_i, c_k) \vee (h_j, c_l) = \begin{cases} (h_{i \vee j}, c_2), & k = l = 2, \\ (h_{i \wedge j}, c_1), & k = l = 1, \\ (h_{i \vee (n-k)}, c_2), & k = 2, l = 1. \end{cases}$$

$$(h_i, c_k) \wedge (h_j, c_l) = \begin{cases} (h_{i \wedge j}, c_2), & k = l = 2, \\ (h_{i \vee j}, c_1), & k = l = 1, \\ (h_{(n-i) \vee j}, c_1), & k = 2, l = 1. \end{cases}$$

Then we'll prove it in three cases according to the operations “ $\vee$ ” and “ $\wedge$ ”.

- (a) When $j = l = 2$, according to Eq. (15),

$$v((h_i, c_k)) + v((h_j, c_l)) = (\alpha + \beta \times i) + (\alpha + \beta \times j) = 2\alpha + \beta(i + j).$$

And

$$\begin{aligned} v((h_i, c_k) \vee (h_j, c_l)) &= v((h_{i \vee j}, c_2)) = \alpha + \beta \times (i \vee j), \\ v((h_i, c_k) \wedge (h_j, c_l)) &= v((h_{i \wedge j}, c_2)) = \alpha + \beta \times (i \wedge j). \end{aligned}$$

Then

$$\begin{aligned} v((h_i, c_k) \vee (h_j, c_l)) + v((h_i, c_k) \wedge (h_j, c_l)) \\ = 2\alpha + \beta \times ((i \vee j) + (i \wedge j)) = 2\alpha + \beta(i + j). \end{aligned}$$

Therefore, we get

$$\begin{aligned} v((h_i, c_k) \vee (h_j, c_l)) + v((h_i, c_k) \wedge (h_j, c_l)) \\ = v((h_i, c_k)) + v((h_j, c_l)). \end{aligned}$$

- (b) when $j = l = 1$, according to Eq. (15),

$$\begin{aligned} v((h_i, c_k)) + v((h_j, c_l)) &= \beta(n - i) + \beta(n - k) \\ &= \beta(2n - 2 - i - k). \end{aligned}$$

And

$$\begin{aligned} v((h_i, c_j) \vee (h_k, c_l)) &= v((h_{i \wedge k}, c_1)) = \beta(n - (i \wedge k)), \\ v((h_i, c_j) \wedge (h_k, c_l)) &= v((h_{i \vee k}, c_1)) = \beta(n - (i \vee k)), \end{aligned}$$

Then

$$\begin{aligned} v((h_i, c_j) \vee (h_k, c_l)) + v((h_i, c_j) \wedge (h_k, c_l)) \\ = \beta(2n - (i \wedge k) - (i \vee k)) = \beta(2n - i - k). \end{aligned}$$

Therefore,

$$\begin{aligned} v((h_i, c_j) \vee (h_k, c_l)) + v((h_i, c_j) \wedge (h_k, c_l)) \\ = v((h_i, c_k)) + v((h_j, c_l)). \end{aligned}$$

- (c) When $j \neq l$, without loss of generality, let $j = 2, l = 1$,
 $v((h_i, c_k)) + v((h_j, c_l)) = (\alpha + \beta \times i) + \beta(n - j) = \alpha + \beta(n + i - j)$.

And

$$\begin{aligned} v((h_i, c_k) \vee (h_j, c_l)) &= v((h_{i \vee (n-j)}, c_2)) \\ &= (\alpha + \beta \times (i \vee (n - k))), \end{aligned}$$

$$\begin{aligned} v((h_i, c_k) \wedge (h_j, c_l)) &= v((h_{(n-i) \vee j}, c_1)) \\ &= \beta(n - ((n - i) \vee j)). \end{aligned}$$

Then

$$\begin{aligned} v((h_i, c_k) \vee (h_j, c_l)) + v((h_i, c_k) \wedge (h_j, c_l)) \\ = (\alpha + \beta \times (i \vee (n - j))) + \beta(n - ((n - i) \vee j)). \end{aligned}$$

If $i > n - j$, it is easy to know $j > n - i$, so

$$\begin{aligned} v((h_i, c_k) \vee (h_j, c_l)) + v((h_i, c_k) \wedge (h_j, c_l)) \\ = (\alpha + \beta \times i) + \beta \times (n - j) = \alpha + \beta(n + i - j) \end{aligned}$$

If $n + 1 - k > i$, it is easy to know $n + 1 - i > k$, so

$$\begin{aligned} v((h_i, c_k) \vee (h_j, c_l)) + v((h_i, c_k) \wedge (h_j, c_l)) \\ = (\alpha + \beta \times ((n - j))) + \beta \times (n - (n - i)) \\ = \alpha + \beta(n + i - j). \end{aligned}$$

Therefore, we get

$$\begin{aligned} v((h_i, c_k) \vee (h_j, c_l)) + v((h_i, c_k) \wedge (h_j, c_l)) \\ = v((h_i, c_k)) + v((h_j, c_l)). \end{aligned}$$

- ii. The second condition is proved as the following.

According to

- (a) when $k = l = 2$, if $(h_i, c_k) < (h_j, c_l)$, then $i < j$

$$v((h_i, c_k)) = \alpha + \beta \times i < \alpha + \beta \times j = v((h_j, c_l)).$$

If $v((h_i, c_k)) < v((h_j, c_l))$, i.e. $\alpha + \beta \times i < \alpha + \beta \times j$, then $i < j$, hence $(h_i, c_k) < (h_j, c_l)$.

- (b) When $k = l = 1$, if $(h_i, c_k) < (h_j, c_l)$, then $i > j$.

$$v((h_i, c_k)) = \beta(n - i) < \beta(n - j) = v((h_j, c_l)).$$

If $v((h_i, c_k)) < v((h_j, c_l))$, i.e. $\beta(n - i) < \beta(n - j)$, then $i > j$, hence $(h_i, c_k) < (h_j, c_l)$.

- (c) When $k = 1, l = 2$, if $(h_i, c_k) < (h_j, c_l)$, then $i \geq n - j$.

$v((h_i, c_k)) = \beta(n - i) \leq \beta \times j < \alpha + \beta \times j = v((h_j, c_l))$.
If $v((h_i, c_k)) < v((h_j, c_l))$, that is to say, $\beta(n - i) < \alpha + \beta \times j < \beta(j + 1)$ for $\alpha < \beta$, then $n - i < j + 1$. Because n, i, j are positive integers, $n - i \leq j$, i.e., $i \geq n - j$. Then $(h_i, c_k) < (h_j, c_l)$.