A 2D Flutter Equation Transformation Using Chebyshev Expansion Method

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Abstract. The classical two-dimensional airfoil flutter equations can be established by many ways. For the simplicity of solving it, a sinusoidal structure motion hypothesis and some kind of aerodynamic theory must be proposed in advance. However, when a wing flutter occurs, its structural movement is likely to be more complex. Furthermore, the well-known harmonic balance method may not sufficiently accurate due to those higher order terms are ignored, which might lead to larger errors. In this paper, we propose a parametric method such that the original equations are parameterized, namely Chebyshev expansion method. A simple example is used to illustrate our strategy.

Introduction

As is known to all, it’s easy to induce the continuous or divergent vibration form when the elastic structure of the aircraft in uniform flow is impacted coupling by the air force, elastic force and inertial force. This phenomenon is called “flutter”, and it is one of the most important questions in the pneumatic elastic mechanics [1].

In recent years, with the development of computer hardware and software technologies, the coupled flutter computation of the research based on the Computational Fluid Dynamic (CFD) and the Computational Structure Dynamics (CSD) began to prevail. The flutter calculation and research based on the two-dimensional airfoil can be divided into categories, the qualitative and the quantitative, which is similar to the three-dimensional airfoil flutter problem on the mechanism [2]. The former makes research on the stability of the system, and the latter focuses on the flutter amplitude, frequency, phase and so on. For a 2D wing prediction of flutter, the structural motion was scribed in a sine function in the pass, and the aerodynamics model established by the aid of Theodorsen unsteady aerodynamic force theory. Calculation results can be used in the primitive engineering practice [3].

Based on the sine motion hypothesis and the common harmonic balance method, the accuracy of quantitative calculation may not be very high [4], namely large errors may occur [5] because of the standard harmonic balance method ignoring the higher frequencies. Other works of modeling an aeroelasticity system or calculating air forces can be found in references [6,7] behind.

This paper proposes a parametric approach to transform the original flutter equations: using Chebyshev expansion method. The method doesn’t only limit the form of structure movement, but also it can be applied to those nonlinear aeroelastic problems. Finally, one can obtain sufficiently precise result when solving the new equations expressed by Chebyshev series.

Properties of the First Chebyshev Polynomial

Since Chebyshev polynomial is put forward, it is widely used in academic fields such as in system analysis, parameter identification, optimal control, model reduction and so on [8]. In the field of aeronautics and astronautics, for example, the model identification problem, we can effectively improve the accuracy and efficiency by using the good properties of Chebyshev orthogonal basis functions.

The first Chebyshev polynomial $T_n$ which are defined in $[0,1]$ satisfy orthogonal relationship according to weight function $w(t) = [t(1-t)]^{-1/2}$. 
where \( m, n \) are non-negative integers.

As Chebyshev polynomial functions constitute a complete orthogonal basis in the closed interval \([0, 1]\), an arbitrary continuous function in the time domain can be approximated represented by Chebyshev polynomials

\[
f(t) = \sum_{i=0}^{\infty} a_i T_i(t)
\]

where the coefficient \( a_i \) can be determined by

\[
a_0 = \frac{1}{\pi} \int_0^1 w(t) f(t) T_0(t) dt, \quad a_n = \frac{2}{\pi} \int_0^1 w(t) f(t) T_n(t) dt, n > 0
\]

**Establishment of the Equations of Motion of a Two-dimensional Airfoil**

Assuming that a two-dimensional airfoil is moving in the form of sine when flutter occurs. Here it is justified to establish the equations of motion with Theodorsen theory of the unsteady aerodynamic force.

Approximatively, when flutter occurs, the motion of the airfoil can be given by

\[
h = h_0 e^{i\omega t}, \quad \alpha = \alpha_0 e^{i\omega t}
\]

where \( h \) represents the plunge motion and \( \alpha \) represents the pitch motion. The amplitudes of \( h_0, \alpha_0 \) are allowed to be complex form which represents that there is a phase difference between the two structural motions, \( \omega \) the frequency of motion. According to Theodorsen theory, the unsteady aerodynamic force is determined by

\[
L = \pi \rho b^2 (V \dot{\alpha} + \dot{h} - ab \dot{\alpha}) + 2\pi \rho V b C(k) [V \alpha + \dot{h} + (0.5 - a)b \dot{\alpha}]
\]

\[
M = \pi \rho b^2 (ab (V \dot{\alpha} + \dot{h} - ab \dot{\alpha}) - 0.5V b \dot{\alpha} - 0.125b^2 \ddot{\alpha})
\]

\[
+ 2\pi \rho V b^2 (0.5 + a) \cdot C(k) [V \alpha + \dot{h} + (0.5 - a)b \dot{\alpha}]
\]

where \( V \) is the flow velocity, \( b \) semi-chord of the airfoil, \( a \cdot b \) the distance between mid-chord and the elastic axis, \( k \) reduced frequency and \( C(k) = F(k) + iG(k) \) Theodorsen function which is approximate in the form as

\[
C(k) = 1 - \frac{0.165}{1 - 0.045i}, k \leq 0.5 \quad C(k) = 1 - \frac{0.165}{1 - 0.32i}, k > 0.5
\]

As shown in Fig.1, a two-dimensional airfoil has pitch and plunge degrees of freedom. With Lagrange’s equations or the principle of virtual work, the equations of motion of the two-dimensional airfoil can be expressed as
\[ m\ddot{h} + S_\alpha \ddot{\alpha} + c_\beta \dot{h} + K_\beta h = -L \], or
\[ S_\alpha \dot{h} + I_\alpha \ddot{\alpha} + c_\alpha \dot{\alpha} + K_\alpha \alpha = M \]

where \( h, \alpha \) represent the harmonic motions of the two-dimensional airfoil with \( \omega \) as the frequency. \( L \) and \( M \) are the unsteady aerodynamic force and moment based on Theodorsen theory, \( S_\alpha = m\alpha b \) the mass static moment about the elastic axis, \( I_\alpha = m r_\alpha ^2 b^2 \) the mass inertial about the shaft, and \( r_\alpha \) the radius of rotation about the elastic axis and the dimension of which is 1.

### A Parametric Modeling to Replace the Original Flutter Equations by Chebyshev Expansion Method

First, letting \( C_h = 0 \) and \( C_\alpha = 0 \), (7) can be rewritten as

\[ \begin{bmatrix} 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ mx_\alpha b & I_\alpha & 0 & 0 \\ mx_\alpha b & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{h} \\ \dot{\alpha} \\ \dot{h} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 & c_\beta & 0 & 0 \\ 0 & 0 & c_\alpha & 0 \\ k_\beta & 0 & 0 & 0 \\ 0 & k_\alpha & 0 & 0 \end{bmatrix} \begin{bmatrix} h \\ \alpha \\ \dot{h} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} -L \\ 0 \\ M \end{bmatrix} \]  

(8)

Then letting \( \Delta = I_\alpha m - (S_\alpha)^2 \), (9) can be rewritten as

\[ \begin{align*}
\ddot{h} &= \frac{I_\alpha K_\beta h}{\Delta} - \frac{S_\alpha K_\alpha \alpha}{\Delta} - \frac{LL_\alpha - MS_\alpha}{\Delta} \\
\ddot{\alpha} &= \frac{S_\alpha K_\beta h}{\Delta} - \frac{mK_\alpha \alpha}{\Delta} + \frac{LS_\alpha - Mm}{\Delta}
\end{align*} \]

(10)

Letting \( X = (x_1 \ x_2 \ x_3 \ x_4)^T = (h \ \dot{h} \ \dot{\alpha})^T \), the form of (10) in the state space is

\[ \frac{dX}{dt} = BX + F \], or

\[ \begin{pmatrix} h \\ \dot{h} \\ \alpha \\ \dot{\alpha} \end{pmatrix} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{I_\alpha K_\beta}{\Delta} & \frac{S_\alpha K_\alpha}{\Delta} & 0 & 0 \\
\frac{S_\alpha K_\beta}{\Delta} & -\frac{mK_\alpha}{\Delta} & 0 & 0
\end{pmatrix} \begin{pmatrix} h \\ \dot{h} \\ \alpha \\ \dot{\alpha} \end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
\frac{LL_\alpha - MS_\alpha}{\Delta} \\
\frac{LS_\alpha - Mm}{\Delta}
\end{pmatrix} \]

(12)

Rewrite (4) as
where the flow velocity can be eliminated as the reduced frequency

\[ V \]

where the flow velocity can be eliminated as the reduced frequency

\[ \frac{V}{\alpha} \]

when flutter occurs which is to be determined.

Substituting (14) to (13), we can get

\[ h = h_0 + \sum_{i=0}^{m-1} c_{1,i} T_i(s) + j \sum_{i=0}^{m-1} c_{2,i} T_i(s) \]

\[ \alpha = \alpha_0 + \sum_{i=0}^{m-1} j c_{1,i} T_i(s) \]

\[ \quad = h_0 \sum_{i=0}^{m-1} (c_{1,i} + j c_{2,i}) T_i(s) \]

\[ = \alpha_0 \sum_{i=0}^{m-1} (c_{1,i} + j c_{2,i}) T_i(s) \]

Meanwhile, we can get the form of \( h, \dot{h} \) and \( \alpha, \dot{\alpha} \).

Then, the final form of (12) is

\[
\begin{align*}
\dot{h} & = \dot{h}_0 + J + J_{\alpha} T_i(s) + \frac{2 \pi \rho V b C(k)}{\omega} \left( \sum_{i=0}^{m-1} c_{1,i} T_i(s) + j \sum_{i=0}^{m-1} c_{2,i} T_i(s) \right) \\
\dot{\alpha} & = \dot{\alpha}_0 + J_{\alpha} T_i(s)
\end{align*}
\]

where \( L, M \) are the aerodynamic force and moment which can be determined by (5) and (6).

The unsteady aerodynamic force and moment can be simplified as

\[
L = \pi \rho b^2 (V \dot{\alpha} + h^* \alpha - \dot{h} \alpha - \dot{h} \alpha) + 2 \pi \rho V b C(k) (V \dot{\alpha} + h^* \alpha - \dot{h} \alpha)
\]

\[
= \pi \rho b^2 [ab \dot{\alpha} + j \omega \alpha - \dot{h} \alpha + \dot{h} \alpha] + 2 \pi \rho b^2 b C(k) \left( \frac{\dot{h}}{k} (V \dot{\alpha} + h^* \alpha - \dot{h} \alpha) \right) \\
= \pi \rho b^2 \dot{\alpha} [ab \dot{\alpha} + j \omega \alpha - \dot{h} \alpha + \dot{h} \alpha] + 2 \pi \rho b^2 b C(k) \left( \frac{\dot{h}}{k} (V \dot{\alpha} + h^* \alpha - \dot{h} \alpha) \right)
\]

\[
M = \pi \rho b^2 [ab (V \dot{\alpha} + h^* \alpha - \dot{h} \alpha) - \dot{h} \alpha] + 2 \pi \rho b^2 (0.5 + a) C(k) \left[ \frac{\dot{h}}{k} (V \dot{\alpha} + h^* \alpha - \dot{h} \alpha) \right]
\]

\[
= \pi \rho b^2 \dot{\alpha} [ab \dot{\alpha} + j \omega \alpha - \dot{h} \alpha + \dot{h} \alpha] + 2 \pi \rho b^2 b C(k) \left( \frac{\dot{h}}{k} (V \dot{\alpha} + h^* \alpha - \dot{h} \alpha) \right) \\
\]

\[
= \pi \rho b^2 \dot{\alpha} [ab \dot{\alpha} + j \omega \alpha - \dot{h} \alpha + \dot{h} \alpha] + 2 \pi \rho b^2 b C(k) \left( \frac{\dot{h}}{k} (V \dot{\alpha} + h^* \alpha - \dot{h} \alpha) \right)
\]

where the flow velocity can be eliminated as the reduced frequency \( \frac{b \omega}{V} \).
Next, \( \sum_{i=0}^{m-1} (c_{1,i} + jc_{2,i})T_i \) can be further rewritten as

\[
\sum_{i=0}^{m-1} (c_{1,i} + jc_{2,i})T_i = \left[ \begin{array}{c} c_{1,0} \\ c_{1,1} \\ \vdots \\ c_{1,m-1} \end{array} \right] \begin{pmatrix} T_0 \\ T_1 \\ \vdots \\ T_{m-1} \end{pmatrix} + j \left[ \begin{array}{c} c_{2,0} \\ c_{2,1} \\ \vdots \\ c_{2,m-1} \end{array} \right] \begin{pmatrix} T_0 \\ T_1 \\ \vdots \\ T_{m-1} \end{pmatrix}
\]

(19)

According to (16)-(19), we can come to a conclusion that the process to introduce Chebyshev polynomials is no other than the process to introduce more parameters to the system that are to be identified.

**Summary**

In this paper, the traditional airfoil structural movement in sinusoidal form is given up because this is not very reasonable when flutter occurs. We propose a new parametric modeling to replace the original differential model by using Chebyshev expansion. By this technique the periodic solution of dynamic system of interest can be expressed in the form of the high order Chebyshev orthogonal basis. There are many advantages: 1) from the point of quantitative analysis and system identification, it can effectively improve the accuracy when flutter occurs. 2) From the point of qualitative analysis, the Chebyshev expansion method can deal with the flutter problem of even more general structural motion.

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**References**


