T-Normed Fuzzy TM-Subalgebra of TM-Algebras

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ABSTRACT

The concept of T-normed fuzzy TM-subalgebras is introduced by applying the notion of t-norm to fuzzy TM-algebra and its properties are investigated. The ideas based on minimum t-norm are generalized to all widely accepted t-norms in a fuzzy TM-subalgebra. The characteristics of an idempotent T-normed fuzzy TM-subalgebra are studied. The properties of image and the inverse image of a T-normed fuzzy TM-subalgebra under homomorphism is discussed. The T-direct product and T-product of T-normed fuzzy TM-subalgebras are also considered.

1. INTRODUCTION

Triangular norms (abbreviation t-norms) were first appeared in the background of statistical metric spaces, introduced by K. Menger [1] and studied later by Schweizer and Sklar [2,3]. Klement et al. [4–6] conducted a systematic study on the related properties of t-norms. The concept of fuzzy sets were introduced by Zadeh [7]. Rosenfeld [8] applied this concept to group theory and introduced fuzzy subgroups leading to the fuzzification of different algebraic structures. Alsina et al. [9,10] and Prade [11] suggested to use a t-norm for fuzzy intersection and its t-conorm for fuzzy union, following some attempts of Hohle [12] in introducing t-norms into the area of fuzzy logics. This was extended by combining the notions of fuzzy sets and t-norm to different algebraic structures such as group [13–17], BCK-algebra [18], BCC-algebra [19], B-algebra [20], KU-algebra [21,22], BG-algebra [23], and so on, and defined different types of product of fuzzy substructures on them.

TM-algebra is a class of logical algebra based on propositional calculus, introduced by Megalai and Tamilarasi [24]. They have investigated several characterizations of it and relation between TM-algebras and other algebras. They [25] applied the concept of fuzzy set to TM-algebra and studied the properties of the newly obtained algebraic structure called fuzzy TM-algebra. Some operations on fuzzy TM-subalgebra were discussed and fuzzy ideals were also defined. Several fuzzy substructures in TM-algebras were considered by many researchers (see [26–28]).

Speaking in terms of t-norm, fuzzy TM-subalgebra was actually defined using the concept of minimum t-norm. Hence we generalize this concept by taking an arbitrary t-norm. The whole paper is arranged as follows: Relevant definitions and theorems needed in sequel are included in Section 2. In Section 3, we introduced the notion of T-normed fuzzy TM-subalgebra with suitable examples and the characteristics are studied. An idempotent T-normed fuzzy TM-subalgebra is defined depending on whether the image set of the membership function becomes a subset of the subsemigroup of idempotents of the semigroup ([0, 1], T) or not and its properties are studied. The properties of image and the inverse image of a T-normed fuzzy TM-subalgebra under homomorphism are investigated. In Section 4, some properties of the T-product and T-direct product of T-normed fuzzy TM-subalgebras and the relationship between them are also considered. The conclusion and a comparison with the existing results are given in the last section.

2. PRELIMINARIES

We recall some definitions and results that will be required in the sections that follow:

Definition 1. [24] A TM-algebra is a triple $(X, *, \varnothing)$, where $X (\neq \varnothing)$ is a set with a fixed element $\varnothing$ and $*$ is a binary operation such that the conditions

i. $x * \varnothing = x$

ii. $(x * y) * (x * z) = z * y$

hold for all $x, y, z \in X$. 

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A nonempty subset $S$ of a TM-algebra $X$ is called a TM-subalgebra of $X$ if $x \ast y \in S$ for all $x, y \in S$.

**Definition 2.** [29] Let $(X_1, \ast_1, \theta_1)$ and $(X_2, \ast_2, \theta_2)$ be two TM-algebras. The direct product $X = X_1 \times X_2$ is also a TM-algebra with the binary operation $\ast$ defined as $(x_1, x_2) \ast (y_1, y_2) = (x_1 \ast_1 y_1, x_2 \ast_2 y_2)$ for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ and $\theta = (\theta_1, \theta_2)$.

**Definition 3.** [7] A fuzzy set $A$ in a set $X$ is a pair $(X, \mu_A)$, where the function $\mu_A : X \to [0, 1]$ is called the membership function of $A$. For $x \in X$ in $[0, 1]$, the set $U(\mu_A : x) = \{x \in X \mid \mu_A(x) \geq x\}$ is called an upper level set of $A$.

**Definition 4.** [7] Let $A = (X, \mu_A)$ and $B = (Y, \eta_B)$ are fuzzy sets in $X$ and $Y$, respectively, and $f$ is a mapping defined from $X$ into $Y$. Then $f(A)$ is a fuzzy set in $f(X)$, where $\mu_{f(A)}$ is defined by

$$f(\mu_A)(y) = \begin{cases} \sup \{\mu_A(x) \mid x \in f^{-1}(y) \neq \emptyset, x \in X, y \in Y \} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

for all $y \in f(X)$ and is called the image of $A$ under $f$. $A$ is said to have sup property, if for every subset $P \subseteq X$, there exists $p_0 \in P$ such that $\mu_A(p_0) = \sup \{\mu_A(p) \mid p \in P\}$. The inverse image $f^{-1}(B)$ in $X$ is also a fuzzy set of $X$, where $\eta_{f^{-1}(B)}$ is defined by $f^{-1}(\eta_B)(x) = \eta_B(f(x))$ for all $x \in X$ is also a fuzzy set of $X$.

When $X$ is taken as a TM-algebra, then we have the following definition:

**Definition 5.** [25] A fuzzy set $A = (X, \mu_A)$ of a TM-algebra $X$ is called a fuzzy TM-subalgebra of $X$ if $\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}$, for all $x, y \in X$.

**Theorem 1.** [25] Let $f : X \to Y$ be a homomorphism from a TM-algebra $X$ onto a TM-algebra $Y$. If $A = (X, \mu_A)$ is a fuzzy TM-subalgebra of $X$, then the image $f(A) = (Y, f(\mu_A))$ of $A$ under $f$ is a fuzzy TM-subalgebra of $Y$.

Now we recall some preliminary ideas on t-norm.

**Definition 6.** [5] A t-norm is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies

i. $T(x, 1) = x$

ii. $T(x, y) = T(y, x)$

iii. $T(x, T(y, z)) = T(T(x, y), z)$

iv. Drastic t-norm $T_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$ for all $x, y \in [0, 1]$.

Some useful properties of a t-norm $T$ used in the sequel are the following:

i. $T(x, 0) = 0$ for all $x \in [0, 1]$.

ii. $T_D(x, y) \leq T(x, y) \leq T_{\text{min}}(x, y)$ for any t-norm $T$ and all $x, y \in [0, 1]$.

iii. $T(T(x, y), T(z, t)) = T(T(x, z), T(y, t)) = T(T(x, t), T(y, z))$ for all $x, y, z$ and $t \in [0, 1]$.

**Definition 7.** Let $T$ be a t-norm. Denote by $E_T$ the set of all idempotents with respect to $T$, that is, $E_T = \{x \in [0, 1] \mid T(x, x) = x\}$. A fuzzy set $A$ in $X$ is called an idempotent T-normed fuzzy set if $\text{Im}(\mu_A) \subseteq E_T$.

**Definition 8.** [16] A t-norm $T_1$ dominates a t-norm $T_2$, or equivalently, $T_2$ is dominated by $T_1$, and write $T_1 \gg T_2$ if $T_1(T_2(x, y), T_2(a, b)) \geq T_2(T_1(x, a), T_1(y, b))$ for all $x, y, a, b \in [0, 1]$.

We can extend these concepts by generalizing the domain of t-norm to $\prod_{i=1}^{n} [0, 1]$ to define the function $t_n$-norm.

**Definition 9.** [16] The function $T_n : \prod_{i=1}^{n} [0, 1] \to [0, 1]$ is defined by $T_n(x_1, x_2, \cdots, x_n) = T(x_n, T_{n-1}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n))$ for all $1 \leq i \leq n$, where $n \geq 2$, $T_2 = T$ and $T_1 = i_2$ (identity).

For a t-norm $T$ and every $x_i, y_i \in [0, 1]$, where $1 \leq i \leq n$ and $n \geq 2$, we have $T_n(T(x_1, y_1), T(x_2, y_2), \cdots, T(x_n, y_n)) = T(T_n(x_1, x_2, \cdots, x_n), T_n(y_1, y_2, \cdots, y_n))$.

### 3. T-NORMED FUZZY TM-SUBALGEBRA OF A TM-ALGEBRA

We first apply the notion of t-norm to obtain a new fuzzy substructure called T-normed fuzzy TM-subalgebra in a TM-algebra.

**Definition 10.** Let $(X, \ast, \theta)$ be a TM-algebra and $A = (X, \mu_A)$ be a fuzzy set in $X$. Then the set $A$ is a T-normed fuzzy TM-subalgebra over the binary operation $\ast$ if it satisfies $\mu_A(x \ast y) \geq T(\mu_A(x), \mu_A(y))$ for all $x, y \in X$.

**Example 1.** Define a fuzzy set $A$ in the TM-algebra $X$ given in Table 1, by $\mu_A(\theta) = 0.5, \mu_A(a) = 0.3, \mu_A(b) = 0.7$, and $\mu_A(c) = 0.6$. Then $A$ is a $T_{\text{Luk}}$-normed fuzzy TM-subalgebra of $X$.

**Definition 11.** A T-normed fuzzy TM-subalgebra $A$ is called an idempotent T-normed fuzzy TM-subalgebra of $X$ if $\text{Im}(\mu_A) \subseteq E_T$.

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<th>Table 1 Cayley Table</th>
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Example 2. Consider a TM-algebra $X = \{\emptyset, a, b, c\}$ defined in Table 1. Define a fuzzy set $A$ in $X$ by $\mu_A(x) = 0$, if $x \in \{\emptyset, a\}$ and $\mu_A(x) = 0.6$, if $x \in \{b, c\}$. Consider a $t$-norm $T_k$ defined in [30] by

$$T_k(x, y) = \begin{cases} \min \{x, y\} & \text{if } \max \{x, y\} = 1 \\ 0 & \text{if } \max \{x, y\} < 1, x + y \leq 1 + k \\ k & \text{otherwise} \end{cases}$$

for all $x, y \in [0, 1]$. Take $k = 0.6$. It is easy to check that $\mu_A(x * y) \geq T_k(\mu_A(x), \mu_A(y))$ for all $x, y \in X$. Also $Im(\mu_A) \subseteq F_k$. Hence $A$ is an idempotent $T_k$-normed fuzzy TM-subalgebra of $X$ when $k = 0.6$.

Proposition 2. If $A$ is an idempotent $T$-normed fuzzy TM-subalgebra of TM-algebra $X$, then we have the following results for all $x \in X$:

i. $\mu_A(\emptyset) \geq \mu_A(x)$

ii. $\mu_A(x * x) \geq \mu_A(x)$

iii. If there exists a sequence $x_n$ in $X$ such that $\lim_{n \to \infty} \mu_A(x_n) = 1$ then $\mu_A(\emptyset) = 1$

Proof. Let $x \in X$.

i. Then by using the two conditions in Definition 1, we get $\mu_A(\emptyset) = \mu_A(\emptyset * \emptyset) = \mu_A((x * \emptyset) * (x * \emptyset)) = \mu_A(x * x) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x)$.

ii. $\mu_A(x * x) \geq T(\mu_A(x), \mu_A(x)) = T[\mu_A(x), \mu_A(x)] \geq T[\mu_A(x), \mu_A(x), \mu_A(x)] = \mu_A(x)$ since it is idempotent.

iii. By (i), $\mu_A(\emptyset) \geq \mu_A(x)$ for all $x \in X$, therefore $\mu_A(\emptyset) \geq \mu_A(x_n)$ for every positive integer $n$. Consider, $1 \geq \mu_A(\emptyset) \geq \lim_{n \to \infty} \mu_A(x_n) = 1$. Hence, $\mu_A(\emptyset) = 1$.

Theorem 3. Let $A_1$ and $A_2$ be two $T$-normed fuzzy TM-subalgebras of $X$. Then $A_1 \cap A_2$ is a $T$-normed fuzzy TM-subalgebra of $X$.

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and $A_2$. Now,

$$\mu_{A_1 \cap A_2}(x * y) = \min \{\mu_{A_1}(x * y), \mu_{A_2}(x * y)\} \geq \min \{T[\mu_{A_1}(x), \mu_{A_1}(y)], T[\mu_{A_2}(x), \mu_{A_2}(y)]\} \geq T[\min \{\mu_{A_1}(x), \mu_{A_1}(x)\}, \min \{\mu_{A_2}(x), \mu_{A_2}(x)\}]$$

Hence, $A_1 \cap A_2$ is a $T$-normed fuzzy TM-subalgebra of $X$.

This can be generalized to obtain the following theorem:

Theorem 4. Let $\{A_i\}_{i \in I}$ be a family of $T$-normed fuzzy TM subalgebras of a TM-algebra $X$. Then $\bigcap_{i \in I} A_i$ is also a $T$-normed fuzzy TM-subalgebra of $X$, where $\bigcap_{i \in I} A_i = \{x, \inf_{i \in I} \mu_A(x) : x \in X\}$.

Proof. For any $x, y \in X$, we have $\mu_A(x) \geq \inf_{i \in I} \mu_{A_i}(x)$ and $\mu_A(y) \geq \inf_{i \in I} \mu_{A_i}(y)$. Hence for every $i \in I$,

$$T(\inf_{i \in I} \mu_{A_i}(x), \mu_{A_i}(y)) \geq \inf_{i \in I} T(\mu_{A_i}(x), \mu_{A_i}(y)) \geq T(\mu_{A}(x), \mu_{A}(y))$$

It follows that

$$\mu_{\bigcap_{i \in I} A_i}(x * y) = \inf_{i \in I} \mu_{A_i}(x * y) \geq \inf_{i \in I} T(\mu_{A_i}(x), \mu_{A_i}(y)) \geq T(\mu_{A}(x), \mu_{A}(y))$$

This completes the proof.

Theorem 5. Let $T$ be a $t$-norm and let $A$ be a fuzzy set in a TM-algebra $X$ with $Im(\mu_A) = \{a_1, a_2, \ldots, a_n\}$, where $a_i < a_j$ whenever $i > j$. Suppose that there exists an ascending chain of subalgebras $S_0 \subset S_1 \subset \cdots \subset S_n = X$ of $X$ such that $\mu_A(S_k) = a_k$, where $S_k = S_i \cap S_{i+1}$ for $k = 1, \ldots, n$ and $S_0 = S_0$. Then $A$ is a $T$-normed fuzzy TM-subalgebra of $X$.

Proof. Let $x, y \in X$. If $x$ and $y$ belong to the same $S_k$, then $\mu_A(x) = \mu_A(y) = a_k$ and $x * y \in S_k$. Hence $\mu_A(x * y) \geq a_k = min\{\mu_A(x), \mu_A(y)\} \geq T(\mu_A(x), \mu_A(y))$. Assume that $x, y \in \overline{S_i}$ and $y \in S_j$ for some $i < j$. Without loss of generality we may assume that $i > j$. Then $\mu_A(x) = a_i < a_j = \mu_A(y)$ and $x * y \in G_i$. It follows that $\mu_A(x * y) \geq a_i = min\{\mu_A(x), \mu_A(y)\} \geq T(\mu_A(x), \mu_A(y))$. Consequently, $A$ is a $T$-normed fuzzy TM-subalgebra of $X$.

Theorem 6. Let $A$ be an idempotent $T$-normed fuzzy TM-subalgebra of $X$, then the set $I_A = \{x \in X | \mu_A(x) \leq \mu_A(\emptyset)\}$ is a $T$-normed fuzzy TM-subalgebra of $X$.

Proof. Let $x, y \in I_A$. Then $\mu_A(x) = \mu_A(\emptyset) = a_k$ and so, $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)) = T(\mu_A(x), \mu_A(y)) = \mu_A(x)$. By using Proposition 2, we know that $\mu_A(x * y) \leq \mu_A(x)$. Hence $\mu_A(x * y) = \mu_A(\emptyset)$ or equivalently $x * y \in I_A$. Therefore, the set $I_A$ is TM-subalgebra of $X$.

Theorem 7. If $A$ is a $T$-subalgebra of $X$, then the characteristic function $\chi_A$ is a $T$-normed fuzzy TM-subalgebra of $X$.

Proof. Let $x, y \in X$. We consider here three cases:

Case (i). If $x, y \in A$, then $x * y \in A$ since $A$ is a TM-subalgebra of $X$. Then $\chi_A(x * y) = 1 \geq T(\chi_A(x), \chi_A(y))$.

Case (ii). If $x, y \notin A$, then $\chi_A(x) = 0 = \chi_A(y)$. Thus $\chi_A(x * y) \geq 0 = min\{0, 0\} = T[\chi_A(x), \chi_A(y)]$.

Case (iii). If $x \in A$ and $y \notin A$ (or $x \notin A$ and $y \in A$), then $\chi_A(x) = 1, \chi_A(y) = 0$. Thus $\chi_A(x * y) \geq 0 = T[0, 1] = T[\chi_A(x), \chi_A(y)]$.

Therefore, the characteristic function $\chi_A$ is a $T$-normed fuzzy TM-subalgebra of $X$.

Theorem 8. Let $A$ be a non-empty subset of $X$. If $\chi_A$ satisfies $\chi_A(x * y) \geq T(\chi_A(x), \chi_A(y))$, then $A$ is a $T$-subalgebra of $X$.

Proof. Let $x, y \in A$. Then $\chi_A(x * y) \geq T(\chi_A(x), \chi_A(y)) = T[1, 1] = 1$ so that $\chi_A(x * y) = 1$, i.e., $x * y \in A$. Hence, $A$ is a TM-subalgebra of $X$.

Proposition 9. Let $Y$ be a TM-subalgebra of $X$ and $A$ be a fuzzy set in $X$ defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in Y \\ \tau, & \text{otherwise} \end{cases}$$
for all $\lambda, \tau \in [0, 1]$ with $\lambda \geq \tau$. Then $A$ is a $T_{\text{fuzz}}$-normed fuzzy subalgebra of $X$. In particular if $\lambda = 1$ and $\tau = 0$ then $A$ is an idempotent $T_{\text{fuzz}}$-normed fuzzy subalgebra of $X$. Moreover, $I_{\mu_A} = Y$.

**Proof.** Let $x, y \in X$. We consider here three cases:

Case (i). If $x, y \in Y$ then

$$T_{\text{fuzz}}(\mu_A(x), \mu_A(y)) = T_{\text{fuzz}}(\lambda, \lambda) = \max(2\lambda - 1, 0)$$

$$= \begin{cases} 
2\lambda - 1 & \text{if } \lambda \geq \frac{1}{2} \\
0 & \text{otherwise}
\end{cases}$$

$$\leq \lambda = \mu_A(x \ast y)$$

Case (ii). If $x \in Y$ and $y \not\in Y$ (or, $x \not\in Y$ and $y \in Y$), then

$$T_{\text{fuzz}}(\mu_A(x), \mu_A(y)) = T_{\text{fuzz}}(\lambda, \tau) = \max(\lambda + \tau - 1, 0)$$

$$= \begin{cases} 
\lambda + \tau - 1 & \text{if } \lambda + \tau \geq 1 \\
0 & \text{otherwise}
\end{cases}$$

$$\leq \tau = \mu_A(x \ast y)$$

Case (iii). If $x, y \not\in Y$, then

$$T_{\text{fuzz}}(\mu_A(x), \mu_A(y)) = T_{\text{fuzz}}(\lambda, \tau) = \max(2\tau - 1, 0)$$

$$= \begin{cases} 
2\tau - 1 & \text{if } \tau \geq \frac{1}{2} \\
0 & \text{otherwise}
\end{cases}$$

$$\leq \tau = \mu_A(x \ast y)$$

Hence, $A$ is an $T_{\text{fuzz}}$-normed fuzzy TM-subalgebra of $X$.

Assume that $\lambda = 1$ and $\tau = 0$. Then $T_{\text{fuzz}}(\lambda, \lambda) = \max(\lambda + \lambda - 1, 0) = 1 = \lambda$ and $T_{\text{fuzz}}(\tau, \tau) = \max(\tau + \tau - 1, 0) = 0 = \tau$. Thus $\lambda, \tau \in E_{T_{\text{fuzz}}}$, that is, $\text{Im}(\mu_A) \subseteq E_{T_{\text{fuzz}}}$. So, $A$ is an idempotent $T_{\text{fuzz}}$-normed fuzzy TM-subalgebra of $X$.

Also,

$$I_{\mu_A} = \{x \in X | \mu_A(x) = \mu_A(\emptyset)\} = \{x \in X, \mu_A(x) = \lambda\} = Y.$$ 

Therefore, $I_{\mu_A} = Y$.

**Theorem 10.** Let $A$ be a $T$-normed fuzzy $TM$-subalgebra of $X$ and $\alpha \in [0, 1]$. Then if $\alpha = 1$, the upper level set $U(\mu_A; \alpha)$ is either empty or a $TM$-subalgebra of $X$.

**Proof.** Let $\alpha = 1$. Suppose $U(\mu_A; \alpha)$ is not empty and let $x, y \in U(\mu_A; \alpha)$. Then $\mu_A(x) \geq 1$ and $\mu_A(y) \geq 1$. It follows that $\mu_A(x \ast y) \geq T(\mu_A(x), \mu_A(y)) \geq T(1, 1) = 1$ so that $x \ast y \in U(\mu_A; \alpha)$. Hence, $U(\mu_A; \alpha)$ is a $TM$-subalgebra of $X$ when $\alpha = 1$.

**Theorem 11.** If $A$ is an idempotent $T$-normed fuzzy $TM$-subalgebra of $X$, then the upper level set $U(\mu_A; \alpha)$ of $A$ is a $TM$-subalgebra of $X$.

**Proof.** Assume that $x, y \in U(\mu_A; \alpha)$. Then $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$. It follows that $\mu_A(x \ast y) \geq T(\mu_A(x), \mu_A(y)) \geq T(\alpha, \alpha) = \alpha$ so that $x \ast y \in U(\mu_A; \alpha)$. Hence, $U(\mu_A; \alpha)$ is a $TM$-subalgebra of $X$. 

**Theorem 12.** Let $A$ be a fuzzy set in $X$ such that the set $U(\mu_A; \alpha)$ is a $TM$-subalgebra of $X$ for every $\alpha \in [0, 1]$. Then $A$ is a $T$-normed fuzzy $TM$-subalgebra of $X$.

**Proof.** Let for every $\alpha \in [0, 1]$, $U(\mu_A; \alpha)$ is a $TM$-subalgebra of $X$. In contrary, let $x_0, y_0 \in X$ be such that $\mu_A(x_0 \ast y_0) < T(\mu_A(x_0), \mu_A(y_0))$. Let us consider, $\alpha_0 = \frac{1}{2} \{\mu_A(x_0 \ast y_0) + T(\mu_A(x_0), \mu_A(y_0))\}$. Then $\mu_A(x_0 \ast y_0) < \alpha_0 \leq T(\mu_A(x_0), \mu_A(y_0)) \leq \mu_A(x_0) \ast \mu_A(y_0)$ and so $x_0 \ast y_0 \not\in U(\mu_A; \alpha_0)$ but $x_0, y_0 \in U(\mu_A; \alpha_0)$. This is a contradiction and hence $\alpha$ satisfies the inequality $\mu_A(x \ast y) \geq T(\mu_A(x), \mu_A(y))$ for all $x, y \in X$.

**Theorem 13.** Let $f : X \rightarrow Y$ be a homomorphism of $TM$-algebras $(X, \ast, \emptyset)$ onto $(Y, \ast_1, \emptyset_1)$. If $B = (Y, \mu_B)$ is a $T$-normed fuzzy TM-subalgebra of $Y$, then the pre-image $f^{-1}(B) = (X, f^{-1}(\mu_B))$ of $B$ under $f$ is a $T$-normed fuzzy TM-subalgebra of $X$.

**Proof.** Assume that $B$ is a $T$-normed fuzzy TM-subalgebra of $Y$ and let $x, y \in X$. Then

$$f^{-1}(\mu_B)(x \ast y) = \mu_B(f(x \ast y)) = \mu_B(f(x) \ast y) \geq T(\mu_B(f(x)), \mu_B(f(y))) = T(f^{-1}(\mu_B(x)), f^{-1}(\mu_B(y)))$$

Therefore, $f^{-1}(B)$ is a $T$-normed fuzzy TM-subalgebra of $X$.

**Theorem 14.** Let $T$ be a continuous t-norm and let $f$ be an epimorphism of $TM$-algebras $(X, \ast, \emptyset)$ onto $(Y, \ast_1, \emptyset_1)$. If $A$ is a $T$-normed fuzzy TM-subalgebra of $X$, then $f(A)$ is a $T$-normed fuzzy TM-subalgebra of $Y$.

**Proof.** Let $y_1, y_2 \in Y$. Take $A_1 = f^{-1}(y_1), A_2 = f^{-1}(y_2)$ and $A_3 = f^{-1}(y_1 \ast y_2)$. Consider the set

$$A_1 \ast A_2 = \{x \in X | x = a_1 \ast a_2 \text{ for some } a_1 \in A_1 \text{ and } a_2 \in A_2\}.$$ 

If $x \in A_1 \ast A_2$, then $x = x_1 \ast x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ and so $f(x) = f(x_1 \ast x_2) = f(x_1) \ast f(x_2) = y_1 \ast y_2$, that is, $x \in f^{-1}(y_1 \ast y_2) = A_3$. Thus $A_1 \ast A_2 \subseteq A_3$. It follows that

$$\mu_{f, A}(y_1 \ast y_2) = \sup_{x \in f^{-1}(y_1 \ast y_2)} \mu_A(x) = \sup_{x \in A_1} \mu_A(x) \ast \sup_{x \in A_2} \mu_A(x) \ast \sup_{x_1 \in A_1, x_2 \in A_2} \mu_A(x_1 \ast x_2) \leq \sup_{x_1 \in A_1, x_2 \in A_2} T(\mu_A(x_1), \mu_A(x_2)).$$

Since $T$ is continuous, for every $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\sup_{x_1 \in A_1} \mu_A(x_1) - x_1^* \leq \delta \text{ and } \sup_{x_2 \in A_2} \mu_A(x_2) - x_2^* \leq \delta,$$ 

then

$$T\left(\sup_{x_1 \in A_1} \mu_A(x_1), \sup_{x_2 \in A_2} \mu_A(x_2)\right) - T(x_1^*, x_2^*) \leq \epsilon.$$ 

Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that

$$\sup_{x_1 \in A_1} \mu_A(x_1) - \mu_A(a_1) \leq \delta \text{ and } \sup_{x_2 \in A_2} \mu_A(x_2) - \mu_A(a_2) \leq \delta.$$
Then
\[ T \left( \sup_{x_1 \in A_1} \mu_A(x_1), \sup_{x_2 \in A_2} \mu_A(x_2) \right) = T(\mu_A(a_1), \mu_A(a_2)) \leq \varepsilon. \]

Consequently
\[
\mu_{\xi, \alpha}(y_1 \ast y_2) \geq \sup_{x_1 \in A_1, x_2 \in A_2} T(\mu_A(x_1), \mu_A(x_2))
\]
\[
\geq T \left( \sup_{x_1 \in A_1} \mu_A(x_1), \sup_{x_2 \in A_2} \mu_A(x_2) \right)
\]
\[
= T \left( \mu_{\xi, \alpha}(y_1), \mu_{\xi, \alpha}(y_2) \right)
\]

which shows that \( f(A) \) is a T-normed fuzzy TM-subalgebra of \( Y \).

**Theorem 15.** Let \( f : X \to Y \) be an epimorphism from a TM-algebra \( X \) onto a TM-algebra \( Y \). If \( A \) is an idempotent T-normed fuzzy TM-subalgebra of \( X \), then the image \( f(A) \) of \( A \) under \( f \) is a T-normed fuzzy TM-subalgebra of \( Y \).

**Proof.** Let \( A \) be an idempotent T-normed fuzzy TM-subalgebra of \( X \). By Theorem 11, \( U(\mu_A ; \alpha) \) is TM-subalgebra of \( X \) for every \( \alpha \in [0, 1] \). Therefore by Theorem 1, \( f(U(\mu_A ; \alpha)) \) is a TM-subalgebra of \( Y \). But \( f(U(\mu_A ; \alpha)) = U(f(\mu_A ; \alpha)) \). Hence \( U(f(\mu_A ; \alpha)) \) is a TM-subalgebra of \( X \) for every \( \alpha \in [0, 1] \). By Theorem 12, \( f(A) \) is a T-normed fuzzy TM-subalgebra of \( Y \).

### 4. PRODUCT OF T-NORMED FUZZY TM-SUBALGEBRAS

We will define a concept called T-product in TM-algebra using a t-norm \( T \), analogue to the pointwise product of functions.

**Definition 12.** Let \( A_1 = (X, \mu_{A_1}) \) and \( A_2 = (X, \mu_{A_2}) \) be two fuzzy sets of a TM-algebra \( X \) and \( T_1 \) be a t-norm. Then the T-product of \( A_1 \) and \( A_2 \) denoted by \( [A_1 \cdot A_2]_T \) is defined by
\[
\mu_{[A_1 \cdot A_2]_T}(x) = T(\mu_{A_1}(x), \mu_{A_2}(x))\]
for all \( x \in X \). Also \( \mu_{[A_1 \cdot A_2]_T} = \mu_{[A_2 \cdot A_1]_T} \).

**Theorem 16.** Let \( A_1 \) and \( A_2 \) be two T-normed fuzzy TM-subalgebras of \( X \). If \( T^n \) is a t-norm such that \( T^n \gg T \), then the \( T^n \)-product of \( A_1 \) and \( A_2 \), \( [A_1 \cdot A_2]_{T^n} \), is a T-normed fuzzy TM-subalgebra of \( X \).

**Proof.** For any \( x, y \in X \), we have
\[
\mu_{[A_1 \cdot A_2]_{T^n}}(x \ast y) = T^n(\mu_{A_1}(x \ast y), \mu_{A_2}(x \ast y))
\]
\[
\geq T^n \left( T(\mu_{A_1}(x), \mu_{A_1}(y)), T(\mu_{A_2}(x), \mu_{A_2}(y)) \right)
\]
\[
\geq T \left( T^n(\mu_{A_1}(x), \mu_{A_2}(x)), T^n(\mu_{A_1}(y), \mu_{A_2}(y)) \right)
\]
\[
= T \left( \mu_{[A_1 \cdot A_2]_{T^n}}(x), \mu_{[A_1 \cdot A_2]_{T^n}}(y) \right).
\]

Hence, \( [A_1 \cdot A_2]_{T^n} \) is a T-normed fuzzy TM-subalgebra of \( X \).

**Corollary 17.** Let \( f : X \to Y \) be an epimorphism of TM-algebras. Let \( T \) and \( T^n \) be t-norms such that \( T^n \gg T \). If \( A_1 \) and \( A_2 \) be two T-normed fuzzy TM-subalgebras of \( Y \), then the pre-images \( f^{-1}(A_1), f^{-1}(A_2) \) and \( f^{-1}[A_1 \cdot A_2]_{T^n} \) are T-normed fuzzy TM-subalgebras of \( X \).

**Proof.** Since every epimorphic pre-image of a T-normed fuzzy TM-subalgebra is again a T-normed fuzzy TM-subalgebra, their \( T^n \)-product is also T-normed fuzzy TM-subalgebra by the previous theorem.

The relationship of \( f^{-1}([A_1 \cdot A_2]_{T^n}) \) with the \( T^n \)-product of \( f^{-1}(A_1) \) and \( f^{-1}(A_2) \) can be viewed by the following theorem:

**Theorem 18.** Let \( f : X \to Y \) be an epimorphism of TM-algebras. Let \( T \) and \( T^n \) be t-norms such that \( T^n \gg T \). Let \( A_1 \) and \( A_2 \) be two T-normed fuzzy TM-subalgebras of \( Y \). If \( [A_1 \cdot A_2]_{T^n} \) is the \( T^n \)-product of \( A_1 \) and \( A_2 \), then
\[
f^{-1} \left[ f^{-1}(A_1) \cdot f^{-1}(A_2) \right]_{T^n} = f^{-1} \left[ f^{-1}(A_1) \cdot f^{-1}(A_2) \right]_{T^n}.
\]

**Proof.** For any \( x \in X \) we get,
\[
f^{-1} \left[ f^{-1}(A_1) \cdot f^{-1}(A_2) \right]_{T^n}(x) = \mu_{[f^{-1}(A_1), f^{-1}(A_2)]_{T^n}}(f(x))
\]
\[
= T^n(\mu_{f^{-1}(A_1)}, \mu_{f^{-1}(A_2)})(f(x))
\]
\[
= T^n_1(\mu_{f^{-1}(A_1)}(x), f^{-1}(\mu_{f^{-1}(A_2)})(x))
\]
\[
= f^{-1} \left[ f^{-1}(A_1) \cdot f^{-1}(A_2) \right]_{T^n}(x).
\]

Hence the proof.

**Remark 1.** Now let us consider about the image of T-product of T-normed fuzzy TM-subalgebras.

Let \( f : X \to Y \) be an epimorphism of TM-algebras. Let \( T \) and \( T^n \) be t-norms such that \( T^n \gg T \), where \( T \) is a continuous t-norm. If \( A_1 \) and \( A_2 \) be two T-normed fuzzy TM-subalgebras of \( X \), then the images \( f(A_1), f(A_2), f([A_1 \cdot A_2]_{T^n}) \), and \( f([A_1 \cdot A_2]_{T^n}) \) are T-normed fuzzy TM-subalgebras of \( Y \) by Theorems 14 and 16.

**Theorem 19.** Let \( T \) and \( T^n \) be t-norms such that \( T^n \gg T \), where \( T \) is a continuous t-norm. Let \( A_1 \) and \( A_2 \) be two T-normed fuzzy TM-subalgebras of a TM-algebra \( X \) and \( f : X \to Y \) be an epimorphism of TM-algebras. Then \( f^{-1}([A_1 \cdot A_2]_{T^n}) \subset [f^{-1}(A_1), f^{-1}(A_2)]_{T^n} \).

**Proof.** For each \( y \in Y \),
\[
\mu_{f^{-1}([A_1 \cdot A_2]_{T^n})}(y) = f \left( \mu_{[A_1 \cdot A_2]_{T^n}} \right)(y)
\]
\[
= \sup_{x \in f^{-1}(y)} \mu_{[A_1 \cdot A_2]_{T^n}}(x)
\]
\[
= \sup_{x \in f^{-1}(y)} T^n(\mu_{A_1}(x), \mu_{A_2}(x))
\]
\[
\leq T^n \left( \sup_{x \in f^{-1}(y)} \mu_{A_1}(x), \sup_{x \in f^{-1}(y)} \mu_{A_2}(x) \right)
\]
\[
= T^n \left( \mu_{f^{-1}(A_1)}(y), \mu_{f^{-1}(A_2)}(y) \right)
\]
\[
= \mu_{[f^{-1}(A_1), f^{-1}(A_2)]_{T^n}}(y).
\]

Next we consider the \( T^n \)-direct product of two T-normed fuzzy TM-subalgebras.
\[ \mu(x) = \begin{cases} \mu_A(x), & \text{if } x \in A \\ \mu_B(x), & \text{if } x \notin A \end{cases} \]

Hence \( A \) is a T-normed fuzzy TM-subalgebra of \( X \).

5. CONCLUSION

The previous works related to fuzzy TM-subalgebra relied on the conventional min/max t-norm/t-conorm dual combinations. But the literature on t-norms suggests that there exists other widely accepted t-norms. In this article, we put forth a new notion of T-normed fuzzy TM-subalgebra of TM-algebra by generalizing the concept of fuzzy TM-subalgebra (defined using minimum t-norm) introduced in [25]. We observed that our generalized concept satisfy most of the various theorems stated in the previous related works. The theorem (Theorem 14 of [25]) which is stated as "A is a fuzzy TM-subalgebra of a TM-algebra X if and only if its level set \( U(\mu_A, \alpha) \) is either empty or a TM-subalgebra for all \( \alpha \in [0,1] \)," is found to be different in our generalized case. Theorems 11 and 12 in this article shows that this may not hold in the case of a T-normed fuzzy TM-subalgebra in general, but the level set can be a T-normed fuzzy TM-subalgebra when the corresponding fuzzy set \( A \) is an idempotent T-normed fuzzy TM-subalgebra. The converse part of the theorem always holds.

Moreover, we studied the properties of image and the inverse image of a T-normed fuzzy TM-subalgebra under a homomorphism. The relationship between the T-direct product and T-product of T-normed fuzzy TM-subalgebras is also obtained. In this paper, we focused on the t-norms and so this can be extended by exploring the analogous observations for the t-conorms in a fuzzy TM-algebra focused on the t-norms and so this can be extended by exploring the analogous observations for the t-conorms in a fuzzy TM-algebra based on the duality between these operators.

CONFLICT OF INTEREST

All authors have no conflict of interest to report.

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