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Recurrence Relations for Moments and Estimation of Parameters of Extended Exponential Distribution Based on Progressive Type-II Right-Censored Order Statistics

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ABSTRACT

In this article we derive the recurrence relations for the single and product moments based on progressively Type-II right-censored order statistics for the extended exponential (EE) distribution. The estimation of the model parameters under progressively Type-II right-censored order statistics are obtained by maximum likelihood method. Furthermore, Monte Carlo simulation study has been carried out to compare the performances of the proposed method. Finally, a real data set has been analyed for illustrative purposes.

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1. INTRODUCTION

The most commonly censoring schemes found in statistics literature are Type-I and Type-II censoring, but the conventional Type-I and Type-II censoring schemes do not have the suppleness of allowing removal of units at points other than the terminal point of the experiment. For this reason; we consider here a more general censoring scheme called the progressive Type-II censoring scheme. Several authors have studied progressive Type-II censoring and properties of order statistics arising from such a progressively censored life test. Some key references are Cohen [1] and Thomas and Wilson [2]. We refer readers to Balakrishnan and Aggarwala [3] for an excellent review of the progressive censoring.

Consider an experiment in which n units are placed on life test. In progressive censoring schemes, the experimenter decides before hand the quantity m, the number of failures to be observed. When the first failure is observed, R_1 of the $n-1$ surviving units are randomly selected and removed. At the second observed failure, R_2 of the $n-2-R_1$ surviving units are randomly selected and removed. The experiment finally terminates at the time of the mth failure when all remaining $R_m = n-m-R_1-R_2-\ldots-R_{m-1}$ surviving units are removed. The censoring numbers $(R_i; i = 1 \ldots m-1)$ are prefixed. We will denote the m ordered failure times thus observed by $X_{1:m:n}, \ldots, X_{m:m:n}$. It is evident that $n = m + \sum_{k=1}^{m} R_k$. The resulting m ordered values which are obtained from this type of censoring are referred to as progressively Type-II right-censored order statistics.

A random variable X is said to follow the extended exponential (EE) distribution with parameters α and λ. if its probability density function (pdf) and cumulative distribution function (cdf) are given as

$$f(x;\alpha,\lambda) = \alpha\lambda\,(1+\lambda x)^{\alpha-1}\,e^{1-(1+\lambda x)^{\alpha}}, x > 0, \alpha, \lambda > 0 \tag{1}$$

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and the corresponding cumulative density function (cdf) is

$$F(x;\alpha,\lambda) = 1 - e^{1-(1+\lambda x)^{\alpha}}, x > 0, \alpha, \lambda > 0. \tag{2}$$

Here α and λ are the shape and scale parameters, respectively. Hereafter, a random variable X that follows the distribution in (1) is denoted by $X \sim EE(\alpha,\lambda)$. An important characteristic of this distribution is that the density function (1) has a decreasing probability function like an exponential distribution but its mode is at zero which has been dealt in detail by Nadarajah and Haghighi [4]. Faster decay of the upper tail. The shape of the hazard rate of this distribution shows increasing, decreasing, and constant like a Weibull or generalized exponential distribution. For $\alpha = 1$, the distribution reduces to standard exponential distribution. Further, this distribution is a particular member of the three-parameter power generalized Weibull distribution, introduced by Nikulin and Haghighi [5]. Lemonte [6] introduced exponentiated Nadarajah and Haghighi (ENH) distribution in the lines of Gupta and Kundu [7] where he provided detailed mathematical properties of the ENH distributions. He also discussed the estimation of the unknown parameters of the distribution by the method of maximum likelihood for complete samples as well as for censored samples. Mirmostafaee *et al.* [8] studied recurrence relations for the single and product moments of record values and associated inference for EE distribution. Kumar *et al.* [9] obtained recurrence relations for the single and product moments of order statistics from the EE distribution. They also obtained the best linear unbiased estimators (BLUEs) for the location and scale parameters.

One can observe from (1) and (2) that

$$f(x) = \alpha \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \lambda^{u+1} x^{u} \left[1 - F(x)\right]. \tag{3}$$

provided that $\alpha \geq 1$ is an integer. This equation will be exploited in order to derive some recurrence relations for the single and product moments of progressive Type-II right-censored order statistics for the EE distribution. This relation will be exploited in this section to derive recurrence relations for the single moments of progressively Type-II right-censored order statistics from the generalized half-logistic distribution.

The joint pdf of the progressively Type-II censored samples $X_1{:}m{:}n$, $X_2{:}m{:}n$, $\cdots$, $Xm{:}m{:}n$, is given by

$$\begin{aligned} f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(x_1,x_2,\cdots,x_m) &= C(n,m-1)\prod_{i=0}^{m} f(x_i)\left[1-F(x_i)\right]^{R_i} \\ &\quad -\infty < x_1 < x_2 < \cdots < x_m < \infty, \end{aligned} \tag{4}$$

where

$$C(n,m-1) = n(n-R_1-1)\cdots(n-R_1-R_2-\cdots-R_{m-1}-m+1). \tag{5}$$

And $f(x)$ and $F(x)$ are given by (1) and (2).

Let the progressively Type-II right-censored sample $X_{1:m:n}^{(R_1,R_2,\cdots,R_m)}$, $X_{2:m:n}^{(R_1,R_2,\cdots,R_m)}, \cdots,$ $X_{m:m:n}^{(R_1,R_2,\cdots,R_m)}$ with censoring scheme $(R_1,R_2,\cdots,R_m)$, $m \leq n$ arise from EE distribution with pdf and cdf given in (1) and (2), respectively.

The single moments of the progressive Type-II censored order statistics for the EE distribution can be written as

$$\begin{aligned} \mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} &= E\left[x_{i:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}}\right] \\ &= C(n,m-1)\int\int\cdots\int_{0<x_1<x_2<\cdots<x_m<\infty} x_i^k f(x_1)\left[1-F(x_1)\right]^{R_1} f(x_2)\left[1-F(x_2)\right]^{R_2} \\ &\quad \times f(x_3)\left[1-F(x_3)\right]^{R_3}\cdots f(x_m)\left[1-F(x_m)\right]^{R_m} dx_2 dx_3 \cdots dx_m, \end{aligned} \tag{6}$$

where $f(.)$ and $F(.)$ are given respectively in (1), (2), and $C(n,m-1)$ as defined in (5).

Means and variances of a distribution can be computed by using recurrence relations for single and product moments for any continuous distribution. Several papers have been published on recurrence relation for progressively Type-II right-censored order statistics for different distributions. Recent works in this area are those of Balakrishnan *et al.* [10], Balakrishnan and Saleh [11], Dey *et al.* [12], and Malik and Kumar [13], and the references cited therein.

The motivation of the paper is two fold: first, we derive recurrence relations for the single and product moments of progressive Type-II right-censored order statistics. These recurrence relations will allow one for the recursive computation of these moments w.r.to. the given censoring scheme, and second is to obtain the maximum likelihood estimators and confidence intervals (CIs) of the unknown parameters of the model. The uniqueness of this study comes from the fact that we provide explicit expressions for single and product moments using progressive Type-II right-censored order statistics along with parameter estimation using Maximum Likelihood Estimates (MLE).

This article unfolds as follows: In Sections 2 and 3, we provide the recurrence relations for single and product moments of progressive Type-II right-censored samples from EE distribution. In Section 4, we discuss the maximum likelihood estimation method of the unknown parameters along with approximate CI. A Monte Carlo simulation study is presented in Section 5 to evaluate the performances of the estimation method discussed in Section 5. Then, in Section 6, we illustrate the methodology developed in this manuscript and the usefulness of the EE based on progressive Type-II right-censored order statistics using a real data example. Finally, Section 7 concludes the paper.

2. RECURRENCE RELATION FOR SINGLE MOMENTS

In this section, we derive several new recurrence relations for the single moments of progressive Type-II censored order statistics for all sample sizes n and all censoring schemes $(R_1, R_2, \cdots, R_m)$, $m \leq n$ from EE distribution.

Theorem 2.1. For $2 \leq m \leq n$ and $k \geq 0$,

$$\mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(u+k+1)}} = \frac{1}{(1+R_1)\,\alpha \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \lambda^{u+1} (u+k+1)^{-1}} \mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} - \frac{(n-R_1-1)}{(1+R_1)} \mu_{1:m-1:n}^{(R_1+1+R_2,\cdots,R_m)^{(u+k+1)}}. \tag{7}$$

Proof: From equations (5) and (6), we have

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} &= C(n,m-1) \int\int\cdots\int_{0<x_1<x_2<\cdots<x_m<\infty} \\ &\times L(x_2) f(x_2) [1-F(x_2]^{R_2} f(x_3) [1-F(x_3]^{R_3} \cdots f(x_m) \\ &\times [1-F(x_m]^{R_m} dx_2 dx_3 \cdots dx_m,\end{aligned} \tag{8}$$

where

$$L(x_2) = \int_0^{x_2} x_1^k f(x_1) [1-F(x_1]^{R_1} dx_1. \tag{9}$$

Using (3) in (9), we get

$$\begin{aligned}L(x_2) &= \int_0^{x_2} x_1^k \left\{ \alpha \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \lambda^{u+1} x_1^u [1-F(x_1] \right\} [1-F(x_1)]^{R_1} dx_1 \\ &= \alpha \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \lambda^{u+1} \int_0^{x_2} x_1^{u+k} [1-F(x_1)]^{R_1+1} dx_1.\end{aligned} \tag{10}$$

Integrating (10) by parts, we get after simplification

$$\begin{aligned}&= \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \frac{\alpha\lambda^{u+1}}{u+k+1} \left[[1-F(x_2)]^{R_1+1} x_2^{u+k+1} + (R_1+1) \int_0^{x_2} x_1^{u+k+1} \right. \\ &\left. \times [1-F(x_1)] \right].\end{aligned} \tag{11}$$

Substituting the value of $L(x_2)$ from (11) in (8) and using (6), we simply have

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} &= \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \frac{\alpha\lambda^{u+1}}{(u+k+1)} \left[\int\int\cdots\int x_2^{u+k+1} (1-F(x_2))^{R_1+1} f(x_2) \right. \\ &\left. \times (1-F(x_2))^{R_2} \cdots f(x_m) (1-F(x_m))^{R_m} + (1+R_1) \mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(u+k+1)}} \right] \\ &= \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \frac{\alpha\lambda^{u+1}}{(u+k+1)} \left[(n-R_1-1) \mu_{1:m-1:n}^{(R_1+1+R_2,\cdots,R_m)^{(u+k+1)}} \right. \\ &\left. + (1+R_1) \mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(u+k+1)}} \right],\end{aligned}$$

upon rearrangement the above equations, yields the relation in (7).

Theorem 2.2. For $m = 1, n = 1, 2, \cdots$ and $k \geq 0$,

$$\mu_{1:1:n}^{(n-1)^{(u+k+1)}} = \frac{1}{n\alpha \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \lambda^{u+1} (u+k+1)^{-1}} \mu_{1:1:n}^{(n-1)^{(k)}}. \tag{12}$$

Proof: Similar to the Proof of Theorem 2.1.

Theorem 2.3. For $2 \leq i \leq m-1$, $m \leq n$ and $k \geq 0$,

$$\begin{aligned}\mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)^{(u+k+1)}} = \frac{1}{1+R_i} \Bigg[& \frac{1}{\alpha \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \lambda^{u+1} (u+k+1)^{-1}} \mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} \\ & - (n - R_1 - R_2 - \cdots - R_i - i)\, \mu_{i:m-1:n}^{(R_1,R_2,\cdots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\cdots,R_m)^{(u+k+1)}} \\ & + (n - R_1 - R_2 - \cdots - R_{i-1} - i + 1) \\ & \times \mu_{i-1:m-1:n}^{(R_1,R_2,\cdots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\cdots,R_m)^{(u+k+1)}} \Bigg].\end{aligned} \tag{13}$$

Proof: Similar to the Proof of Theorem 2.1.

Theorem 2.4. For $2 \leq m \leq n$, and $k \geq 0$,

$$\begin{aligned}\mu_{m:m:n}^{(R_1,R_2,\cdots,R_m)^{(u+k+1)}} = & \frac{1}{(1+R_m)\alpha \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \lambda^{u+1} (u+k+1)^{-1}} \mu_{m:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} \\ & + \mu_{m-1:m-1:n}^{\left(R_1,R_2,\cdots,R_{m-2},R_{m-1}+R_m+1,R_{i+1},\cdots,R_m\right)^{(u+k+1)}}.\end{aligned} \tag{14}$$

Proof: Similar to the Proof of theorem 2.1.

Corollary 2.1. By letting $\alpha = \lambda = 1$ in (7), we can deduce relation for the single moments of progressively Type-II censored order statistics for the standard exponential distribution

$$\mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(k+1)}} = \frac{1}{(1+R_1)} \left[(k+1)\, \mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} - (n - R_1 - 1)\, \mu_{1:m-1:n}^{(R_1+1+R_2,\cdots,R_m)^{(k+1)}} \right], \tag{15}$$

Corollary 2.2. For $\alpha = \lambda = 1$ in (12), we get

$$\mu_{1:1:n}^{(n-1)^{(k+1)}} = \frac{k+1}{(n)} \mu_{1:1:n}^{(n-1)^{(k)}}, \tag{16}$$

Corollary 2.3. For $\alpha = \lambda = 1$ in (13), we get

$$\begin{aligned}\mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)^{(k+1)}} = \frac{1}{1+R_i} \Bigg[& (k+1)\, \mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} - (n - R_1 - R_2 - \cdots - R_i - i) \\ & \times \mu_{i:m-1:n}^{(R_1,R_2,\cdots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\cdots,R_m)^{(k+1)}} + (n - R_1 - R_2 - \cdots - R_{i-1} - i + 1) \\ & \times \mu_{i-1:m-1:n}^{(R_1,R_2,\cdots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\cdots,R_m)^{(k+1)}} \Bigg],\end{aligned} \tag{17}$$

Corollary 2.4. For $\alpha = \lambda = 1$ in (14), we get

$$\mu_{m:m:n}^{(R_1,R_2,\cdots,R_m)^{(k+1)}} = \frac{k+1}{1+R_m} \mu_{m:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} + \mu_{m-1:m-1:n}^{\left(R_1,R_2,\cdots,R_{m-2},R_{m-1}+R_m+1,R_{i+1},\cdots,R_m\right)^{(k+1)}}, \tag{18}$$

Deductions: When $R_1 = R_2 = \cdots = R_m = 0$ so that $m = n$, in which the case of progressive Type-II censored order statistics become the usual order statistics $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$, then

i. From (7): For $k \geq 0$, we get

$$\mu_{1:n}^{(u+k+1)} = \frac{1}{\alpha \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \lambda^{u+1} (u+k+1)^{-1}} \mu_{1:n}^{k} - (n-1)\, \mu_{1:n-1:n}^{(1,0,0,\cdots,0)^{(u+k+1)}} \tag{19}$$

ii. From (13): For $k \geq 0$, we get

$$\mu_{i:n}^{(u+k+1)} = \frac{1}{\alpha \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \lambda^{u+1} (u+k+1)^{-1}} \mu_{i:n}^{(k)} - (n-i)\, \mu_{i:n}^{(u+k+1)} + (n-i+1)\, \mu_{i-1:n}^{(u+k+1)} \tag{20}$$

3. RECURRENCE RELATION FOR PRODUCT MOMENTS

For EE distribution, we can write the $(r,s)^{th}$ product moment of the progressively Type-II right-censored order statistics as

$$\begin{aligned} \mu_{r,s:m:n}^{(R_1,R_2,\cdots,R_m)} &= E\left[x_{r:m:n}^{(R_1,R_2,\cdots,R_m)} x_{s:m:n}^{(R_1,R_2,\cdots,R_m)}\right] \\ &= C(n,m-1) \int\int \ldots \int_{0<x_1<x_2<\cdots<x_m<\infty} x_r x_s f(x_1)\left[1-F(x_1\right]^{R_1} f(x_2) \\ &\quad \times [1-F(x_2)]^{R_2} \cdots f(x_m)\,[1-F(x_m)]^{R_m}\, dx_1 dx_2 dx_3 \cdots dx_m, \end{aligned} \tag{21}$$

where $f(.)$ and $F(.)$ are defined in (1) and (2) and $C(n,m-1)$ is defined in (5).

Theorem 3.1. For $1 \leq i < j \leq m-1$ and $m \leq n$,

$$\begin{aligned} \mu_{i,j:m:n}^{(R_1,R_2,\cdots,R_m)^{(1,u+1)}} &= \frac{1}{R_j+1} \Bigg[\frac{1}{\sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \alpha\lambda^{u+1}(u+1)^{-1}} \mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)} \\ &\quad - \left(n-R_1-1-\cdots-R_j-j\right) \mu_{i,j:m-1:n}^{(R_1,R_2,\cdots,R_{j-1},R_j+R_{j+1}+1,\cdots R_m)^{(1,u+1)}} \\ &\quad + \left(n-R_1-1-\cdots-R_{j-1}-j+1\right) \mu_{i,j-1:m-1:n}^{(R_1,R_2,\cdots,R_{j-1}+R_j+1,\cdots,R_m)^{(1,u+1)}} \Bigg]. \end{aligned} \tag{22}$$

Proof: Using (3) and (6), we have

$$\begin{aligned} \mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)} &= A(n,m-1) \int\int \cdots \int_{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty} \\ &\quad \times \left\{ \int_{x_{j-1}}^{x_{j+1}} \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \alpha\lambda^{u+1} \left[1-F\left(x_j\right)\right]^{R_j+1} dx_j \right\} x_i f(x_1)\left[1-F(x_1\right]^{R_1} \cdots f\left(x_{j-1}\right) \\ &\quad \times \left[1-F\left(x_{j-1}\right]^{R_{j-1}} f\left(x_{j+1}\right)\left[1-F\left(x_{j+1}\right]^{R_{j+1}} \cdots f(x_m) \right. \\ &\quad \times \left[1-F(x_m\right]^{R_m} dx_1 dx_2 \cdots dx_{j-1} dx_{j+1} \cdots dx_m. \end{aligned} \tag{23}$$

By integrating the innermost integral by parts and then substituting into (23), we obtain

$$\begin{aligned} \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \alpha\lambda^{u+1} \int_{x_{j-1}}^{x_{j+1}} [1-F(x_j)]^{R_j+1} dx_j &= \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \alpha\lambda^{u+1} \Bigg[x_{j+1}^{u+1} [1-F(x_{j+1})]^{1+R_j} \\ &\quad - x_{j-1}^{u+1} [1-F(x_{j-1})]^{1+R_j} + \left(1+R_j\right) \\ &\quad \times \int_{x_{j-1}}^{x_{j+1}} \left[1-F\left(x_j\right)\right] \times \int_{x_{j-1}}^{x_{j+1}} [1-F(x_j)]^{R_j} f\left(x_j\right) x_j^{u+1} dx_j \Bigg], \end{aligned}$$

which, when substituted into (23) and using (21), we have

$$\mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)} = \sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\alpha\lambda^{u+1}(u+1)^{-1}\Big[\left(n-R_1-1-\cdots-R_j-j\right) \times\mu_{i,j:m-1:n}^{(R_1,R_2,\cdots,R_{j-1},R_j+R_{j+1}+1,\cdots R_m)^{(1,u+1)}} - \left(n-R_1-1-\cdots-R_{j-1}-j+1\right) \times \mu_{i,j-1:m-1:n}^{(R_1,R_2,\cdots,R_{j-1}+R_j+1,\cdots,R_m)^{(1,u+1)}} + \left(R_j+1\right)\mu_{i,j:m:n}^{(R_1,R_2,\cdots,R_m)^{(1,u+1)}}\Big].$$

Upon rearrangement the above equations, yields the relation in (22).

Theorem 3.2. For $1 \le i \le m-1$ and $m \le n$,

$$\mu_{i,m:m:n}^{(R_1,R_2,\cdots,R_m)^{(1,u+1)}} = \frac{1}{R_m+1}\left[\frac{1}{\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\alpha\lambda^{u+1}(u+1)^{-1}}\mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)} + \left(n-R_1-1-\cdots-R_{m-1}-m+1\right)\mu_{i,m-1:m-1:n}^{(R_1,R_2,\cdots,R_{m-1}+R_m+1,\cdots,R_m)^{(1,u+1)}}\right]. \tag{24}$$

Proof: Similar to the Proof of theorem 3.1.

Corollary 3.1. By letting $\alpha = \lambda = 1$ in (22), we can obtain the recurrence relation for product moments of progressively Type-II censored order statistics for the standard exponential distribution.

$$\mu_{i,j:m:n}^{(R_1,R_2,\cdots,R_m)} = \frac{1}{R_j+1}\Big[\mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)} - \left(n-R_1-1-\cdots-R_j-j\right)\mu_{i,j:m-1:n}^{\left(R_1,R_2,\cdots,R_{j-1},R_j+R_{j+1}+1,\cdots R_m\right)} + \left(n-R_1-1-\cdots-R_{j-1}-j+1\right)\mu_{i,j-1:m-1:n}^{\left(R_1,R_2,\cdots,R_{j-1}+R_j+1,\cdots,R_m\right)}\Big],$$

Corollary 3.2. For $\alpha = \lambda = 1$ in (24), we get

$$\mu_{i,m:m:n}^{(R_1,R_2,\cdots,R_m)} = \frac{1}{R_m+1}\Big[\mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)} + (n-R_1-1-\cdots-R_{m-1}-m+1) \times \mu_{i,m-1:m-1:n}^{(R_1,R_2,\cdots,R_{m-1}+R_m+1,\cdots,R_m)}\Big], \tag{25}$$

Table 1 Means for selected progressive censoring schemes.

m	n	Scheme	$\lambda = 4, \alpha = 2$		Mean
5	2	(0,3)	0.030068	0.067654	
5	2	(3,0)	0.030068	0.180410	
8	2	(6,0)	0.018792	0.169135	
8	2	(0,6)	0.018792	0.040270	
10	2	(8,0)	0.015034	0.165376	
10	2	(0,8)	0.015034	0.031738	
12	2	(10,0)	0.012528	0.162870	
12	2	(0,10)	0.012528	0.026195	
15	2	(13,0)	0.010022	0.160365	
15	2	(0,13)	0.010022	0.020761	
18	2	(16,0)	0.008352	0.158694	
18	2	(0,16)	0.008352	0.017196	
20	2	(18,0)	0.007517	0.157859	
20	2	(0,18)	0.007517	0.015429	
5	3	(2,0,0)	0.030068	0.105239	0.255581
5	3	(0,0,2)	0.030068	0.067654	0.117768

Table 1 | Means for selected progressive censoring schemes.

m	n	Scheme	$\lambda = 4, \alpha = 2$		Mean		
8	3	(5,0,0)	0.018792	0.093963	0.244306		
8	3	(0,0,5)	0.018792	0.040270	0.065327		
10	3	(7,0,0)	0.015034	0.090205	0.240547		
10	3	(0,0,7)	0.015034	0.031738	0.050531		
12	3	(9,0,0)	0.012528	0.087699	0.238041		
12	3	(0,0,9)	0.012528	0.026195	0.041230		
15	3	(12,0,0)	0.010022	0.085193	0.235536		
15	3	(0,0,12)	0.010022	0.020761	0.032326		
18	3	(15,0,0)	0.008352	0.083523	0.233865		
18	3	(0,0,15)	0.008352	0.017196	0.026592		
20	3	(17,0,0)	0.007517	0.082688	0.233030		
20	3	(0,0,17)	0.007517	0.015429	0.023782		
5	4	(1,0,0,0)	0.030068	0.080182	0.155353	0.305695	
5	4	(0,0,0,1)	0.030068	0.067654	0.117768	0.192939	
8	4	(4,0,0,0)	0.018792	0.068906	0.144077	0.294420	
8	4	(0,0,0,4)	0.018792	0.040270	0.065327	0.095395	
10	4	(6,0,0,0)	0.015034	0.065148	0.140319	0.290661	
10	4	(0,0,0,6)	0.015034	0.031738	0.050531	0.072009	
12	4	(8,0,0,0)	0.012528	0.062642	0.137813	0.288155	
12	4	(0,0,0,8)	0.012528	0.026195	0.041230	0.057934	
15	4	(11,0,0,0)	0.010022	0.060136	0.135308	0.285650	
15	4	(0,0,0,11)	0.010022	0.020761	0.032326	0.044854	
18	4	(14,0,0,0)	0.008352	0.058466	0.133637	0.283979	
18	4	(0,0,0,14)	0.008352	0.017196	0.026592	0.036615	
20	4	(16,0,0,0)	0.007517	0.057631	0.132802	0.283144	
20	4	(0,0,0,16)	0.007517	0.015429	0.023782	0.032625	
5	5	(0,0,0,0,0)	0.030068	0.067654	0.117768	0.192939	0.343281
8	5	(3,0,0,0,0)	0.018792	0.056378	0.106492	0.181663	0.332005
8	5	(0,0,0,0,3)	0.018792	0.040270	0.065327	0.095395	0.132981
10	5	(5,0,0,0,0)	0.015034	0.052619	0.102733	0.177904	0.328247
10	5	(0,0,0,0,5)	0.015034	0.031738	0.050531	0.072009	0.097066
12	5	(7,0,0,0,0)	0.012528	0.050114	0.100228	0.175399	0.325741
12	5	(0,0,0,0,7)	0.012528	0.026195	0.041230	0.057934	0.076727
15	5	(10,0,0,0,0)	0.010022	0.047608	0.097722	0.172893	0.323235
15	5	(0,0,0,0,10)	0.010022	0.020761	0.032326	0.044854	0.058522
18	5	(13,0,0,0,0)	0.008352	0.045937	0.096051	0.171223	0.321565
18	5	(0,0,0,0,13)	0.008352	0.017196	0.026592	0.036615	0.047353
20	5	(15,0,0,0,0)	0.007517	0.045102	0.095216	0.170387	0.320730
20	5	(0,0,0,0,15)	0.007517	0.015429	0.023782	0.032625	0.042022

Table 2 | Variances for selected progressive censoring schemes.

m	n	Scheme	$\lambda = 4, \alpha = 2$		Variance	
5	2	(0,3)	0.002140	0.005483		
5	2	(3,0)	0.002140	0.055641		
8	2	(6,0)	0.000835	0.054337		
8	2	(0,6)	0.000835	0.001927		
10	2	(8,0)	0.000535	0.054036		
10	2	(0,8)	0.000535	0.001195		
12	2	(10,0)	0.000371	0.053872		
12	2	(0,10)	0.000371	0.000813		
15	2	(13,0)	0.000237	0.053739		
15	2	(0,13)	0.000237	0.000510		
18	2	(16,0)	0.000165	0.053666		
18	2	(0,16)	0.000165	0.000350		
20	2	(18,0)	0.000133	0.053635		
20	2	(0,18)	0.000133	0.000281		
5	3	(2,0,0)	0.002140	0.015515	0.069016	
5	3	(0,0,2)	0.002140	0.005483	0.011428	
8	3	(5,0,0)	0.000835	0.014211	0.067712	
8	3	(0,0,5)	0.000835	0.001927	0.003413	

(*continued*)

Table 2 Variances for selected progressive censoring schemes. *(Continued)*

m	n	Scheme	$\lambda = 4, \alpha = 2$		Variance		
10	3	(7,0,0)	0.000535	0.013910	0.067411		
10	3	(0,0,7)	0.000535	0.001195	0.002031		
12	3	(9,0,0)	0.000371	0.013746	0.067248		
12	3	(0,0,9)	0.000371	0.000813	0.001348		
15	3	(12,0,0)	0.000237	0.013613	0.067114		
15	3	(0,0,12)	0.000237	0.000510	0.000827		
18	3	(15,0,0)	0.000165	0.013540	0.067041		
18	3	(0,0,15)	0.000165	0.000350	0.000559		
20	3	(17,0,0)	0.000133	0.013509	0.067010		
20	3	(0,0,17)	0.000133	0.000281	0.000447		
5	4	(1,0,0,0)	0.002140	0.008084	0.021459	0.074961	
5	4	(0,0,0,1)	0.002140	0.005483	0.011428	0.024803	
8	4	(4,0,0,0)	0.000835	0.006780	0.020155	0.073657	
8	4	(0,0,0,4)	0.000835	0.001927	0.003413	0.005554	
10	4	(6,0,0,0)	0.000535	0.006479	0.019854	0.073356	
10	4	(0,0,0,6)	0.000535	0.001195	0.002031	0.003123	
12	4	(8,0,0,0)	0.000371	0.006316	0.019691	0.073192	
12	4	(0,0,0,8)	0.000371	0.000813	0.001348	0.002009	
15	4	(11,0,0,0)	0.000237	0.006182	0.019557	0.073059	
15	4	(0,0,0,11)	0.000237	0.000510	0.000827	0.001198	
18	4	(14,0,0,0)	0.000165	0.006109	0.019485	0.072986	
18	4	(0,0,0,14)	0.000165	0.000350	0.000559	0.000797	
20	4	(16,0,0,0)	0.000133	0.006078	0.019453	0.072954	
20	4	(0,0,0,16)	0.000133	0.000281	0.000447	0.000632	
5	5	(0,0,0,0,0)	0.002140	0.005483	0.011428	0.024803	0.078305
8	5	(3,0,0,0,0)	0.000835	0.004179	0.010124	0.023499	0.077001
8	5	(0,0,0,0,3)	0.000835	0.001927	0.003413	0.005554	0.008897
10	5	(5,0,0,0,0)	0.000535	0.003878	0.009823	0.023198	0.076700
10	5	(0,0,0,0,5)	0.000535	0.001195	0.002031	0.003123	0.004609
12	5	(7,0,0,0,0)	0.000371	0.003715	0.009659	0.023035	0.076536
12	5	(0,0,0,0,7)	0.000371	0.000813	0.001348	0.002009	0.002845
15	5	(10,0,0,0,0)	0.000237	0.003581	0.009526	0.022901	0.076402
15	5	(0,0,0,0,10)	0.000237	0.000510	0.000827	0.001198	0.001641
18	5	(13,0,0,0,0)	0.000165	0.003508	0.009453	0.022828	0.076330
18	5	(0,0,0,0,13)	0.000165	0.000350	0.000559	0.000797	0.001069
20	5	(15,0,0,0,0)	0.000133	0.003477	0.009422	0.022797	0.076298
20	5	(0,0,0,0,15)	0.000133	0.000281	0.000447	0.000632	0.000841

4. PARAMETER ESTIMATION UNDER PROGRESSIVE TYPE-II CENSORED ORDER STATISTICS

Let $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ be the ordered m observed failures under Type-II progressively censored sample from $EE(\alpha, \lambda)$ with censoring scheme $(R_1, R_2, \ldots, R_m)$. For notational convenience, we will use X_i in place of $X_{i:m:n}$. Thus the likelihood function is given by

$$L\left(\mathbf{x}|\alpha,\lambda\right) = C(n, m-1)\prod_{i=1}^{m}\left[\alpha\lambda(1+\lambda x_i)^{\alpha-1}\exp\left\{1-(1+\lambda x_i)^{\alpha}\right\}\right] \times\left[\exp\left\{1-(1+\lambda x_i)^{\alpha}\right\}\right]^{R_i}.$$

The corresponding log-likelihood function is given by

$$\ln L\left(\mathbf{x}|\alpha,\lambda\right) = D + m\ln\alpha + m\ln\lambda + (\alpha-1)\sum_{i=1}^{m}\ln(1+\lambda x_i) + m - \sum_{i=1}^{m}(1+\lambda x_i)^{\alpha} + \sum_{i=1}^{m}R_i - \sum_{i=1}^{m}R_i(1+\lambda x_i)^{\alpha}, \tag{26}$$

where $D = \ln\{C(n, m-1)\}$.

By differentiating the log-likelihood function (26) , we obtain the MLEs of α and λ, $\hat{\alpha}$, $\hat{\lambda}$ by solving numerically the following nonlinear equations:

$$\begin{aligned}\frac{\partial \ln L\left(\mathbf{x}|\alpha,\lambda\right)}{\partial\alpha} &= \frac{m}{\alpha}+\sum_{i=1}^{m}\ln\left(1+\lambda x_i\right)-\sum_{i=1}^{m}\ln\left(1+\lambda x_i\right)\left(1+\lambda x_i\right)^{\alpha}\\ &\quad-\sum_{i=1}^{m}\ln\left(1+\lambda x_i\right)R_i\left(1+\lambda x_i\right)^{\alpha}=0\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\partial \ln L\left(\mathbf{x}|\alpha,\lambda\right)}{\partial\lambda} &= \frac{m}{\lambda}+(\alpha-1)\sum_{i=1}^{m}\frac{x_i}{1+\lambda x_i}-\alpha\sum_{i=1}^{m}x_i(1+\lambda x_i)^{\alpha-1}\\ &\quad-\alpha\sum_{i=1}^{m}x_iR_i(1+\lambda x_i)^{\alpha-1}=0\end{aligned} \tag{28}$$

4.1. Approximate CIs

Once the ML estimates of α and λ are obtained, we can apply the asymptotic normality of the MLEs to compute the apoximate CIs for the parameters. The observed variance–covariance matrix for the MLEs of the unknown parameters $\theta=(\alpha,\lambda)$ is

$$\begin{aligned}I^{-1}(\theta) &= \left.\begin{pmatrix}-\frac{\partial^2\log L}{\partial\alpha^2} & -\frac{\partial^2\log L}{\partial\alpha\partial\lambda}\\ -\frac{\partial^2\log L}{\partial\lambda\partial\alpha} & -\frac{\partial^2\log L}{\partial\lambda^2}\end{pmatrix}\right|^{-1}_{(\alpha,\lambda)=(\hat{\alpha},\hat{\lambda})}\\ &= \begin{pmatrix}var(\hat{\alpha}) & cov\left(\hat{\alpha},\hat{\lambda}\right)\\ cov\left(\hat{\lambda},\hat{\alpha}\right) & var\left(\hat{\lambda}\right)\end{pmatrix}.\end{aligned}$$

The derivatives in $I(\theta)$ are given as follows:

$$\begin{aligned}I_{11} &= \frac{\partial^2\ln L\left(\mathbf{x}|\alpha,\lambda\right)}{\partial\alpha^2}=-\frac{m}{\alpha^2}-\textstyle\sum_{i=1}^{m}\left\{\ln\left(1+\lambda x_i\right)\right\}^2\left(1+\lambda x_i\right)^{\alpha}\\ &\quad-\textstyle\sum_{i=1}^{m}\left\{\ln\left(1+\lambda x_i\right)\right\}^2R_i\left(1+\lambda x_i\right)^{\alpha},\end{aligned} \tag{29}$$

$$\begin{aligned}I_{22} &= \frac{\partial^2\ln L\left(\mathrm{x}|\alpha,\lambda\right)}{\partial\lambda^2}=-\frac{m}{\lambda^2}-(\alpha-1)\textstyle\sum_{i=1}^{m}\frac{x_i^2}{1+\lambda x_i}\\ &\quad-\alpha(\alpha-1)\textstyle\sum_{i=1}^{m}x_i^2\left(1+\lambda x_i\right)^{\alpha-2}-\alpha(\alpha-1)\sum_{i=1}^{m}x_i^2R_i\left(1+\lambda x_i\right)^{\alpha-2}\end{aligned} \tag{30}$$

$$\begin{aligned}I_{12} &= \frac{\partial^2\ln L\left(\mathrm{x}|\alpha,\lambda\right)}{\partial\alpha\partial\lambda}=\textstyle\sum_{i=1}^{m}\frac{x_i}{1+\lambda x_i}-\sum_{i=1}^{m}x_i\left(1+\lambda x_i\right)^{\alpha-1}\\ &\quad+\alpha\textstyle\sum_{i=1}^{m}\ln\left(1+\lambda x_i\right)x_i\left(1+\lambda x_i\right)^{\alpha-1}-\sum_{i=1}^{m}R_ix_i\left(1+\lambda x_i\right)^{\alpha-1}\\ &\quad-\alpha\textstyle\sum_{i=1}^{m}\ln\left(1+\lambda x_i\right)R_ix_i\left(1+\lambda x_i\right)^{\alpha-1}.\end{aligned} \tag{31}$$

Therefore, the above approach is used to derive the approximate $100(1-\tau)$ %confidence intervalCIs of the parameters $\theta=(\alpha,\lambda)$ as in the following forms:

$$\hat{\alpha}\pm z_{\frac{\tau}{2}}\sqrt{Var(\hat{\alpha})},\hat{\lambda}\pm z_{\frac{\tau}{2}}\sqrt{Var\left(\hat{\lambda}\right)},$$

Here, $Z_{\frac{\tau}{2}}$ is the upper $\left(\frac{\tau}{2}\right)$th percentile of the standard normal distribution.

We have generated progressive Type-II censored order statistics from EE distribution by using following algorithm given by Balakrishnan and Sandhu [14]:

- Define $\gamma_j=\sum_{i=j}^{m}(R_i+1)$ and generate m independent Beta-distributed random variable $B_1,B_2,\ldots,B_m$ with $B_j\sim Beta\left(\gamma_j,1\right)$.
- Let $V_0=1$; calculate $V_k=B_kV_{k-1},\ k=1,2,\ldots,m$
- Let $U_{r:m:n}=1-V_r\,;\ r=1,2,\ldots,m.$
- Compute $X_{r:m:n}=\frac{1}{\lambda}\left[\left\{1-(1-U_{r:m:n})\right\}^{1/\alpha}-1\right]$ as random progressive Type-II censored ordered sample from EE distribution.

5. SIMULATION

In this section, a Monte Carlo simulation is performed to compute the average estimates of the MLEs to evaluate the performance of the proposed method for different censoring schemes. For simplicity of notation, we denote these censoring schemes, for instance, by $\left(8, 0^{*2}, 3, 0^{*2}, 2\right)$ which represents the censoring scheme $R = (8, 0, 0, 3, 0, 0, 2)$ with $n = 20$ and $m = 7$. The results show that the average estimates of MLEs overestimates the true parameter values in most of the censoring schemes. The censoring schemes with zero removals in middle provide underestimates the true parameter values. We have also obtained 95% CI under various censoring schemes. It shows that all the censoring schemes cover true value of the parameters with 95% confidence. The results are listed in Table 3. In all cases we have used $\alpha = 1.5$ and $\lambda = 0.5$ (for other combinations of α, λ are not reported).

6. ILLUSTRATIVE EXAMPLE

To illustrate the use of the method proposed in this article, we consider a data set from Lee and Wang [15] which corresponds to remission times (in months) of a random sample of 128 bladder cancer patients.

Table 3 The average estimates of the parameters for the MLE and Confidence Intervals (CI) are presented for different sample sizes and different sampling schemes.

n	m	Scheme	α	λ	CI for α		CI for λ	
					Lower	Upper	Lower	Upper
10	2	(8,0)	2.617	0.365	2.392	2.842	0.185	0.545
10	2	(7,1)	2.156	0.417	1.919	2.393	0.235	0.599
10	2	(5,3)	1.445	0.416	1.161	1.729	0.253	0.579
10	2	(3,5)	1.462	0.397	1.056	1.868	0.326	0.468
10	2	(0,8)	1.481	0.419	0.938	2.024	0.364	0.474
10	4	(6,0*3)	1.931	0.496	1.743	2.119	0.284	0.708
10	4	(4,2,0*2)	2.118	0.475	1.991	2.245	0.434	0.516
10	4	(4,0*2,2)	1.681	0.457	1.589	1.773	0.296	0.618
10	4	(2,2,2,0)	1.581	0.483	1.399	1.763	0.279	0.687
10	4	(0*3,6)	1.618	0.436	1.4	1.836	0.336	0.536
10	5	(5,0*4)	1.689	0.517	1.456	1.922	0.329	0.705
10	5	(3,2,0*3)	1.462	0.524	1.235	1.689	0.353	0.695
10	5	(3,0*3,2)	1.499	0.514	1.279	1.719	0.383	0.645
10	5	(2,2,1,0*2)	1.494	0.507	1.257	1.731	0.368	0.646
10	5	(0*4,5)	1.531	0.517	1.3	1.762	0.386	0.648
15	4	(11,0*3)	1.552	0.531	1.307	1.797	0.419	0.643
15	4	(8,3,0*2)	1.576	0.538	1.329	1.823	0.418	0.658
15	4	(6,5,0*2)	1.591	0.562	1.332	1.85	0.421	0.703
15	4	(5,4,2,0)	1.586	0.573	1.335	1.837	0.412	0.734
15	4	(0*3,11)	1.571	0.523	1.314	1.828	0.309	0.737
15	6	(9,0*5)	1.579	0.502	1.295	1.863	0.31	0.694
15	6	(7,2,0*4)	1.572	0.521	1.323	1.821	0.321	0.721
15	6	(6,2,0*3,1)	1.493	0.584	1.268	1.718	0.421	0.747
15	6	(3,2,1,1,1,1)	1.517	0.561	1.288	1.746	0.379	0.743
15	6	(0*5,9)	1.598	0.572	1.371	1.825	0.431	0.713
15	8	(7,0*7)	1.603	0.538	1.37	1.836	0.35	0.726
15	8	(5,2,0*6)	1.573	0.531	1.175	1.971	0.323	0.739
15	8	(4,0,3,0*5)	1.568	0.537	1.339	1.797	0.337	0.737
	8	(3,0*3,2,0*2,2)	1.487	0.521	1.166	1.808	0.341	0.701
15	8	(0*6,1,6)	1.548	0.532	1.181	1.915	0.367	0.697
20	5	(15,0*4)	1.581	0.571	1.277	1.885	0.383	0.759
20	5	(12,3,0*3)	1.589	0.573	1.332	1.846	0.363	0.783
20	5	(10,3,2,0*2)	1.572	0.583	1.245	1.899	0.454	0.712
20	5	(5,5,5,0*2)	1.603	0.567	1.235	1.971	0.373	0.761
20	5	(0*3,1,14)	1.492	0.582	1.153	1.831	0.38	0.784
20	7	(10,3,0*5)	1.533	0.581	1.135	1.931	0.426	0.736
20	7	(8,5,0*5)	1.586	0.595	1.225	1.947	0.452	0.738
20	7	(8,0*2,3,0*2,2)	1.493	0.504	1.105	1.881	0.349	0.659
20	7	(0*5,5,8)	1.521	0.509	1.202	1.84	0.303	0.715
20	7	(0*6,13)	1.597	0.548	1.236	1.958	0.348	0.748
20	10	(10,0*9)	1.603	0.541	1.205	2.001	0.357	0.725
20	10	(6,4,0*8)	1.584	0.538	1.245	1.923	0.336	0.74
20	10	(4,4,0*7,2)	1.587	0.586	1.242	1.932	0.386	0.786
20	10	(0*6,2,2,3,3)	1.544	0.562	1.17	1.918	0.399	0.725
20	10	(0*9,10)	1.53	0.586	1.163	1.897	0.421	0.751

Table 4 The MLEs and the -ln(L), AIC values of different models based on bladder cancer data.

Distribution	Scheme	α	β	λ	$-lnL$	AIC
Extended exponentail	Complete	0.924 −0.149	-	0.123 −0.034	−409.574	823.147
	Censored	0.802 (−0.239)	-	0.182 (−0.099)	−106.502	217.004
Exponentiated exponential	Complete	1.235 (−0.151)	-	0.124 (−0.014)	−410.754	825.508
	Censored	1.035 (−0.235)	-	0.128 (−0.029)	−107.743	219.486
Kumaraswamy exponential	Complete	1.546 (−0.276)	0.262 (−0.026)	0.453 (−0.003)	−409.965	825.931
	Censored	1.037 (−0.217)	0.157 (−0.027)	0.795 (−0.076)	−107.659	221.318
Exponential	Complete			0.109 −0.01	−412.188	826.377
	Censored			0.125 (−0.021)	−107.754	217.508

For the purposes of illustrating the method discussed in this article, a progressively Type-II censoring scheme $R = \left(0^{*48}, 80\right)$ has been used. The results are listed in Table 4. We fitted EE distribution to the data set by using the method of maximum likelihood and the results are compared with the other competitive models namely, exponentiated exponential, Kumaraswamy exponential, and exponential distribution. The statistics -ln(L) where -ln(L) denotes the log-likelihood function evaluated at the maximum likelihood estimates and AICs are listed in Table 4 for the data set. Based on the results displayed in Table 4, we can see that the EE distribution has the lowest Akike information criterion (AIC) value among all other competitive models, therefore, the EE distribution is a suitable model for the proposed data set and can chosen as the best model.

7. CONCLUSION

In this paper, we have established several recurrence relations for single and product moments of progressive Type-II censored order statistics from EE distribution. Since recurrence relations reduce the amount of direct computation and hence reduce the time and labor, therefore the relations under consideration can be useful in computing the moments of higher order from the EE lifetime distribution. ML method of estimation is used for estimation of the parameters of the EE distribution based on progressively Type-II right-censored order statistics. The simulation results provide us some idea to choose the censoring schemes though it is not exhaustive. Finally, the work of this paper can be extended for Bayesian analysis of record values under different loss functions.

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