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Increasing the Reliability of Numerical Data Using Several Methods Under Conditions of Indeterminacy

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***Abstract*—In order to increase data reliability, between two and four methods may be used to solve the same problem, with the results being compared. An estimation of the error in every method must be found. The paper offers a mathematical model for the unobservable component of the error, which may be caused by various factors (for example, software implementation). This gives a numerical characteristic of certainty. The main assumption is that the errors from the different methods may be considered as independent random quantities. Moreover, the probability density of the error has a single maximum at zero. In connection with the indeterminacy of the probability densities, the worst case distribution functions (leading to maximum error probability) must be found. An estimate is obtained of the probability of the existence of an error greater than some given value, under the condition that the results coincide up to the accuracy of the computations.**

Keywords—programmer mistake, undetermined error, computer methods, indeterminacy intervals comparison

I. INTRODUCTION

Great attention is usually paid to the testing of software. There are many approaches and testing methods [1-5]. A programmer's errors can have catastrophic consequences if they affect the simulation and control of different objects such as, for example, space stations. Nevertheless, space stations sometimes fail or lose orientation because of errors in programs or in their applications. As for scientific investigations, an analysis of published numerical results often shows the unreliability of typical methods of error estimation and of the results obtained. Many published numerical results contain errors in digits that are declared to be valid. There are many results in which the real error is between ten and one hundred times greater than the declared error. Invalid results like these for one particular problem are shown in [6-10]. There are examples of problem solutions in which the real accuracy corresponds to the declared accuracy [11-13]. These examples are explicitly considered in [14]. However, as a rule, if one has only a single published result it is not possible to define where this correspondence takes place and where it does not.

This is why at present it is possible to find a correct estimate of the error in published results only if more precise results appear. However, the chronology for obtaining a more precise result can be violated. The problem of justifying the estimates that have been obtained therefore remains relevant for this method of verification too.

It should be noted that widely-accepted mathematical software packages have the same weaknesses. On the one hand, software developers cannot guarantee the reliability of the results obtained by a user. On the other hand, the accompanying documentation has no instructions to justify the results that are obtained. A software product user has to think of a way to do this.

We can get an approximate value $\hat{z}$ of an unknown exact solution z, and estimate the error by applying regularization methods [15-19]. However, the known methods of regularization require some additional a priori information about the required parameter $\hat{z}$. Estimates obtained by such methods are dependent on this additional information, so they can hardly be regarded as error estimates.

Let us assume that some quantity is calculated at a subsequently increasing number of mesh points. We can then use some rules, such as Runge and Richardson's rules [20] or other algorithms [21-25], or their improved variants [13, 14], to obtain an extrapolated result with respect to an increase in the number of mesh points. The difference between the calculated and extrapolated ('standard') values gives the error estimate. We call this the *observable* error. However, we cannot guarantee that there is no additional error from sources such as faults in software implementation.

To increase the certainty of computations, different methods of solving the same problem are often used [26]. If

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there are a few computation results ξ_k with estimates of the observable error $|\xi_k - \xi| \le \varepsilon_k$ (ξ is a standard value), then we can know whether the indeterminacy intervals $[\xi_k - \varepsilon_k, \xi_k + \varepsilon_k]$ overlap. We test whether the inequality

$$|\xi_i - \xi_j| \le \varepsilon_i + \varepsilon_j \quad (1)$$

is fulfilled. If the indeterminacy intervals do not overlap then an additional error x_k exists for one or several results.

If one or more of the inequalities (1) are not fulfilled, then the numerical results contradict each other and consequently there is an error. If a contradiction is detected then the error must be found and eliminated. The methods for searching for an error are not under discussion in this paper. The determination of the fact of the existence of an error is, in our opinion, the main problem. And what do we obtain when all the inequalities (1) are fulfilled? The answer to this question is the subject of further discussion.

The absence of divergence of results, or

$$|\xi_i + x_i - \xi_j - x_j| \le \varepsilon_i + \varepsilon_j \quad (2)$$

means that no additional error is detected, but it does not mean that there is an absence of error. The error can exist in every result and have a value greater than the admissible value. However, the additional errors for the different results have to coincide up to the accuracy of the computations:

$$|x_i - x_j| \le \varepsilon_i + \varepsilon_j + |\xi_i - \xi_j| \le 2(\varepsilon_i + \varepsilon_j). \quad (3)$$

Later on we call the error *unobservable* if it cannot be detected by a comparison of the numerical results because the inequalities (2) are fulfilled.

As is easily seen, the probability of the coincidence of errors in the sense of (3) is rather small in comparison with the probability of the existence of an error in a single result.

The purpose of the further investigation is the estimation of the fiducial probability $P_{fiducial}$ of the unobservable error location lying in some interval

$$P\{|x_i| \le A\} = P_{fiducial} = 1 - P_{error}, \quad (4)$$

where A is the half-width of the fiducial interval.

It is necessary to have a mathematical model of the unobservable component of the error in order to obtain the numerical estimations of the probability $P_{fiducial}$.

II. MATHEMATICAL MODEL OF UNOBSERVABLE ERROR

It is difficult to describe an unobservable error because of the absence of measured or estimated values for it, and there is also a problem in giving it any concrete properties.

Nevertheless, the rejection of all assumptions cannot lead to an acceptable result if there is an opportunity for the intentional and coordinated distortion of the results. We can overcome this problem only by an independent test.

We therefore propose the supposition of unintentional distortions and errors. This allows us, as a first approximation, to consider the errors as independent random events.

The probability model of the error requires the determination of the probability distribution of the error.

There is a definite view in the theory of measurement error on the choice of a distribution law for random errors. As a rule, the following assumptions are used:

- larger errors occur with lower probability;
- the distribution of the error is considered to be symmetric around its expectation; and
- in most cases the normal law for the distribution is acceptable.

In the case of the third assumption, the number of independent computations must be small (between two and four), since each requires a relatively great effort. The law of large numbers is not applicable and the probability distribution of the combination of the computations cannot be assumed to be normal.

In the case of the second assumption, the symmetry of the distribution is often caused by the relatively small size of the error, which cannot be assumed for unobservable error. For this reason the second assumption cannot be accepted.

In the case of the first assumption, the supposition of unintentional error does not allow us to prefer any value for the error besides zero. In other words, larger errors occur with lower probability. This is why the probability density function $f(x)$ cannot have any maxima with the exception of a maximum at zero.

In this connection we suppose that $f(x)$ is unimodal – that is, the maximum of $f(x)$ is at the point x=0. Moreover $f(0) \ge f(x_1) \ge f(x_2)$ for $0 < x_1 < x_2$ and for $x_2 < x_1 < 0$.

Every method k may have its own function $f_k(x_k)$.

III. DETERMINATION OF THE PROBABILITY OF THE COINCIDENCE OF ERRORS

It is necessary to estimate the probability that m independent random quantities x_k coincide up to the accuracy of the computations (that is, that they satisfy the inequalities (3)). In order to simplify the estimation we shall not compare each result with each other result, but will choose the result numbered one as the base result and compare all the other results with it.

Let us write

$$\sigma_k = 2(\varepsilon_k + \varepsilon_1), \ (k = 2, \ldots, m), \ \sigma_1 = 2\varepsilon_1 .$$

It is better to choose the most accurate method as method number one, i.e. $\varepsilon_1 = \min_k \varepsilon_k$. Then

$$\sigma_k \ge 2\sigma_1, \ k = 2, \ldots, m . \quad (5)$$

Further, we shall associate the concept of the coincidence of errors with the fulfillment of the inequalities

$$|x_k - x_1| \le \sigma_k,\ k = 2,..., m\ .$$

Let us denote

$$Q = \left\{ (x_1, x_2, \ldots, x_m) \;\middle|\; x_1 \in (-\infty, \infty);\ |x_k - x_1| \le \sigma_k,\ k = 2,..., m \right\}. \quad (6)$$

According to (5) and by virtue of the independence of the errors, the probability that we are seeking of the coincidence of the unobservable errors is equal to

$$P_{coincidence} = P\{x \in Q\} = \int\int \ldots \int_Q \prod_{k=1}^{m} f_k(x_k)\,dx_1 dx_2 \ldots dx_m\ . \quad (7)$$

Let us put integration limits into equation (7). The integration is carried out in a rectangular domain for the change of variables $y=x_1$, $y_2=x_2-x_1,\ldots, y_m=x_m-x_1$

$$P_{coincidence} = \int_{-\infty}^{\infty} \int_{-\sigma_2}^{\sigma_2} f_2(y_2 + y)\,dy_2 \ldots \int_{-\sigma_m}^{\sigma_m} f_m(y_m + y)\,dy_m\, f_1(y)\,dy\ .$$

Let us denote the probability of the error falling within a finite interval as follows:

$$P_k(y, \sigma_k) = \int_{-\sigma_k}^{\sigma_k} f_k(y_k + y)\,dy_k = \int_{y-\sigma_k}^{y+\sigma_k} f_k(x_k)\,dx_k\ . \quad (8)$$

Then $P_{coincidence} = \int_{-\infty}^{\infty} \prod_{k=2}^{m} P_k(y, \sigma_k) f_1(y)\,dy\ .$

This probability of the coincidence of the results up to the accuracy σ_k is represented as the sum of the probability of the existence of a 'little error' $P_{error1} = P\{|x_1 < A|\}$ and the probability of the existence of a 'great error' $P_{error2} = P\{|x_1 \ge A|\}$:

$$P_{coincidece} = P_{error1} + P_{error2}\ ,$$

$$P_{error1} = \int_{-A}^{A} \prod_{k=2}^{m} P_k(x, \sigma_k) f_1(x)\,dx\ ,$$

$$P_{error2} = \int_{-\infty}^{-A} \prod_{k=2}^{m} P_k(x, \sigma_k) f_1(x)\,dx + \int_{A}^{\infty} \prod_{k=2}^{m} P_k(x, \sigma_k) f_1(x)\,dx\ . \quad (9)$$

We will consider this probability of the existence of a great unobservable error further. Later on we will assume:

$$A \ge 2 \max_k \sigma_k\ . \quad (10)$$

IV. STATEMENT OF THE PROBLEM

In connection with the determinacy of $f_k(x)$ it is interesting to find the pessimistic estimate (i.e., the worst distribution from the point of view of the maximum probability of the error (9)). In this case, the function $f_k(x)$ must satisfy the conditions

$$f_k(x) \ge 0,\ \int_{-\infty}^{\infty} f_k(x)\,dx = 1\ . \quad (11)$$

Note, that

$$P_{error2} = \int_{A}^{\infty} \prod_{k=2}^{m} P_k(-y, \sigma_k) f_1(-y)\,dy + \int_{A}^{\infty} \prod_{k=2}^{m} P_k(y, \sigma_k) f_1(y)\,dy \le$$

$$\le \int_{A}^{\infty} \prod_{k=2}^{m} [P_k(y, \sigma_k) + P_k(-y, \sigma_k)]\,[f_1(y) + f_1(-y)]\,dy\ ,$$

$$\int_{0}^{\infty} [f_k(y) + f_k(-y)]\,dy = 1\ .$$

Thus, in order to obtain the upper estimate it is possible to use a one-sided distribution of the unobservable error probability, which can be found from any other $f_k^*(y)$

$$f_k(y) = \begin{cases} f_k^*(y) + f_k^*(-y), & y \ge 0, \\ 0, & y < 0. \end{cases}$$

We shall consider a unimodal (monotone non-increasing for $x \ge 0$) step function of the probability density (Fig. 1)

$$f_k(x) = \begin{cases} 0, & x < x_{k,0},\ x \ge x_{k,n}, \\ \Delta_{k,i} & x_{k,i} \le x < x_{k,i+1}, \end{cases} \quad i = 0,..., n-1\ , (12)$$

where $x_{k,0} = 0,\ x_{k,i-1} \le x_{k,i}\ ,\ \Delta_{k,n} = 0\ .$

The unimodality of $f_k(x)$ means that the following inequalities are fulfilled:

$$\Delta_{k,i} \le \Delta_{k,i-1},\ \ i = 1,..., n-1, \quad (13)$$

$$\Delta_{k,i} \ge 0\ ,\ \ i = 1,..., n-1. \quad (14)$$

If the number of steps of the functions $f_k(x)$ differs for different k, we can add the deficient nodes by dividing any interval into as many parts as we need.

The function $f_k(x_k)$ can be presented in the form

$$f_k(x) = \sum_{i=0}^{n-1} \Delta_{k,i} I(x, x_{k,i}, x_{k,i+1}), \quad (15)$$

where

$$I(x, x_{k,i}, x_{k,i+1}) = \begin{cases} 0, & x < x_{k,i}, x > x_{k,i+1}, \\ 1, & x_{k,i} \le x \le x_{k,i+1}. \end{cases}$$

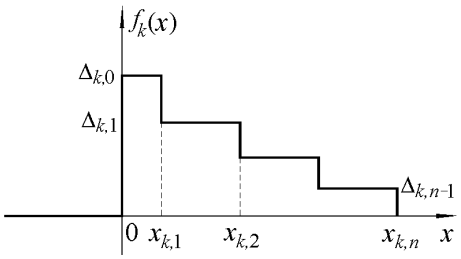

Fig. 1. The step function of the probability density

The integral represents the trapezoidal type dependence

$$J(y, x_{k,i}, x_{k,i+1}, \sigma_k) = \int_{y-\sigma_k}^{y+\sigma_k} I(x, x_{k,i}, x_{k,i+1}) dx. \quad (16)$$

Then (8) has the form

$$P_k(y, \sigma_k) = \sum_{i=0}^{n-1} \Delta_{k,i} J(y, x_{k,i}, x_{k,i+1}, \sigma_k). \quad (17)$$

In view of (11) and (12)

$$\int_0^{\infty} f_k(x_k) dx_k = \sum_{i=0}^{n-1} (x_{k,i+1} - x_{k,i}) \Delta_{k,i} = 1. \quad (18)$$

Solving (18) with respect to $\Delta_{k,0}$ we find

$$\Delta_{k,0} = \frac{1 - \sum_{i=1}^{n-1} (x_{k,i+1} - x_{k,i}) \Delta_{k,i}}{x_{k,1}} \ge \Delta_{k,1}. \quad (19)$$

The equality (19) is used for the restriction instead of (18). Taking into account (19) we change $\Delta_{k,0}$ in (15) and (17)

$$f_k(x) = \sum_{i=1}^{n-1} \Delta_{k,i} \left[I(x, x_{k,i}, x_{k,i+1}) - \frac{x_{k,i+1} - x_{k,i}}{x_{k,1}} I(x, x_{k,0}, x_{k,1}) \right] + \frac{1}{x_{k,1}} I(x, x_{k,0}, x_{k,1}), \quad (20)$$

$$P_k(y, \sigma_k) = \sum_{i=1}^{n-1} \Delta_{k,i} \left[J(y, x_{k,i}, x_{k,i+1}, \sigma_k) - \frac{x_{k,i+1} - x_{k,i}}{x_{k,1}} J(y, x_{k,0}, x_{k,1}, \sigma_k) \right] + \frac{1}{x_{k,1}} J(y, x_{k,0}, x_{k,1}, \sigma_k). \quad (21)$$

Note, $P_k(y, \sigma_k)$ is a *y*-continuous function since it is represented by the linear combination of continuous functions. Moreover, the partial derivative

$$\frac{\partial P_k}{\partial \Delta_{k,q}} = J(y, x_{k,q}, x_{k,q+1}, \sigma_k) - \frac{x_{k,q+1} - x_{k,q}}{x_{k,1}} J(y, x_{k,0}, x_{k,1}, \sigma_k) \quad (22)$$

is also a *y*-continuous function and it does not depend on $\Delta_{k,q}$. For $r \ne k$ the partial derivatives $\partial P_k / \partial \Delta_{r,q} = 0$.

Thus, the problem is reduced to a determination of the maximum $P_{error\,2}$

$$P_{error\,2} = \int_A^{x_{1,n}} \prod_{k=2}^{m} P_k(y, \sigma_k) f_1(y) dy, \quad (23)$$

where $f_1(y)$ and $P_k(y, \sigma_k)$ are determined by the expressions (20) and (21) with the restrictions (13), (14) and (19).

V. UPPER ESTIMATION FOR A UNIMODAL FUNCTION

Lemma 1. Under the hypotheses (20) – (23), for $n>1$ the magnitude $P_{error\,2}$ (23) is a multilinear function of the parameters $\Delta_{k,q}$; in other words, the partial derivatives $\partial P_{error\,2} / \partial \Delta_{k,q}$ for a given l ($1 \le l \le m$) do not depend on $\Delta_{l,q}$ ($q=1,\ldots,n-1$).

The proof of the lemma is based on the following. The integrand function in (23) is continuous with respect to *y* on every partial interval of integration $[x_{1,j}, x_{1,j+1}]$, and the partial derivative of the integrand function $\partial P_l / \partial \Delta_{l,q}$ is also a continuous function with respect to *y* and does not depend on $\Delta_{l,q}$, because of the properties described above. Thus, the theorem conditions for an integral differentiation with respect to the parameter are fulfilled [27]. The partial derivative $\partial J_l / \partial \Delta_{l,q}$ is then calculated by differentiation of $P_l(y, \sigma_l)$ under the integral sign and, by virtue of (22), does not depend on $\Delta_{l,q}$ ($q=1,\ldots,n-1$).

In order to obtain the estimate $P_{error\,2}$ we consider the special case $n=1$, that is, the uniform distribution on [0, $x_{k,1}$]. It follows from the equality (18) that $\Delta_{k,0} = 1/x_{k,1}$. Then

$$f_k(x) = \begin{cases} 0, & x < 0,\ x > x_{k,1}, \\ 1/x_{k,1}, & 0 \le x \le x_{k,1}. \end{cases} \quad (24)$$

The corresponding dependence $P_k(x,\sigma_k)$ (8) in these cases has the form (Fig. 2(b))

$$P_k(x,\sigma_k)=\begin{cases}0, & x\le-\sigma_k,\ x\ge x_{k,1}+\sigma_k,\\ (x+\sigma_k)/x_{k,1}, & x\in[-\sigma_k,\sigma_k],\\ 2\sigma_k/x_{k,1}, & x\in[\sigma_k,x_{k,1}-\sigma_k],\\ (x_{k,1}+\sigma_k-x)/x_{k,1}, & x\in[x_{k,1}-\sigma_k,x_{k,1}+\sigma_k].\end{cases} \quad (25)$$

Lemma 2. Under the assumption of the uniform distribution (n=1), the maximum of $P_{error\,2}$ is achieved for $A<x_{1,1}$, $A\le x_{k,1}-\sigma_k$ (k=2,...,m).

For $A\ge x_{1,1}$ the integral (23) $P_{error\,2}=0$.

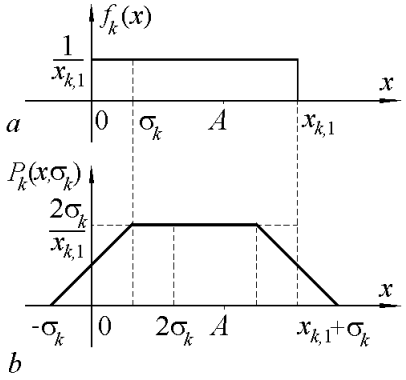

Fig. 2. The integral (8) of the probability density function for n=1

For $A\ge\min_k(x_{k,1}+\sigma_k)$ the integral $P_{error\,2}$ is also equal to zero because of the values $P_k(y,\sigma_k)=0$ for $y\ge A$ in view of (25) for $k=\arg\min(x_{k,1}+\sigma_k)$.

Let us demonstrate that, for $x_{k,1}-\sigma_k<A<x_{k,1}+\sigma_k$ (k>1), the value $P_{error\,2}$ can increase with an increase in $x_{k,1}$, (i.e. $P_{error\,2}$ is not a maximum). In fact, according to (25), (Fig. 2(b))

$$\frac{\partial}{\partial x_{k,1}}P_{error\,2}=\frac{1}{x_{1,1}}\frac{dx^*}{dx_{k,1}}\prod_{\substack{j=2\\ j\ne k}}^{m}P_j(x_{k,1},\sigma_j)\frac{x_{k,1}+\sigma_k-x^*}{x_{k,1}}+$$

$$+\frac{1}{x_{1,1}}\int_A^{x^*}\prod_{\substack{j=2\\ j\ne k}}^{m}P_j(y,\sigma_j)\frac{y-\sigma_k}{(x_{k,1})^2}dy, \quad (26)$$

where

$$x^*=\min_k\{x_{k,1}+\sigma_k\},\ \frac{dx^*}{dx_{k,1}}=\begin{cases}1, & x^*=x_{k,1}+\sigma_k,\\ 0, & x^*\ne x_{k,1}+\sigma_k.\end{cases}$$

Thus, the first term in (26) is always equal to zero and the derivative is nonnegative. A derivative can only be equal to zero if the integral $P_{error\,2}$ is equal to zero.

Lemma 3. For $A\ge2\sigma_k$ under the hypotheses of Lemma 2 the following inequality is valid

$$\max_{x_{k,1}}\int_A^{\infty}\prod_{k=2}^{m}P_k(y,\sigma_k)f_1(y)dy\le(x_{\min}-A)\prod_{k=2}^{m}2\sigma_k\prod_{k=1}^{m}\Delta_{k,0}, \quad (27)$$

where $x_{\min}=\min_k x_{k,1}$.

Note that the meaning of this lemma consists in changing the functions $P_k(y,\sigma_k)$ from a trapezium form (where the integrand is composed from multiplication) to step functions (Fig. 3) – we replace the integrals $P_k(y,\sigma_k)=\int_{x-\sigma_k}^{x+\sigma_k}f_k(y)dy$ by $2\sigma_k\Delta_{k,0}$ for a simplified estimate.

The proof of this conclusion is obvious, because a rectangle area is equal to a trapezium area under the integration of such a function (Fig. 3).

Under integration of the product of such functions

$$P_{error\,2}=\int_A^{\infty}\prod_{\substack{k=2\\ k\ne j}}^{m}\left[\int_{x-\sigma_k}^{x+\sigma_k}f_k(y)dy\right]f_1(x)\left[\int_{x-\sigma_j}^{x+\sigma_j}f_j(y)dy\right]dx$$

the inequality

$$P_{error\,2}\le\int_A^{\infty}\prod_{\substack{k=2\\ k\ne j}}^{m}\left[\int_{x-\sigma_k}^{x+\sigma_k}f_k(y)dy\right]f_1(x)2\sigma_j f_j(x)dx\le$$

$$\le2^{m-1}\prod_{k=2}^{m}\sigma_k\int_A^{\infty}\prod_{k=1}^{m}f_k(x)dx$$

is fulfilled by virtue of the fact that the products $\prod_{\substack{k=2\\ k\ne j}}^{m}\left[\int_{x-\sigma_k}^{x+\sigma_k}f_k(y)dy\right]f_1(x)$ do not increase monotonically.

This proves that (27) is correct.

Lemma 4 (the upper estimate for n=1). For n=1 the following inequality is valid

$$P_{error\,2}\le\frac{1}{m}\left(2\frac{m-1}{m}\right)^{m-1}\prod_{k=2}^{m}\frac{\sigma_k}{A}. \quad (28)$$

Proof. Since $\Delta_{k,0}=1/x_{k,1}$ and according to Lemma 3, we have

$$P_{error\,2}\le(x_{j1}-A)\prod_{k=1}^{m}\frac{1}{x_{k1}}2^{m-1}\prod_{k=2}^{m}\sigma_k\le\frac{x_{j1}-A}{x_{j1}^m}2^{m-1}\prod_{k=2}^{m}\sigma_k=F(x_{j1}),$$

where $x_{j1}=\min_{1\le k\le m}x_{k1}$. We find the stationary point

$$\frac{dF}{dx_{j1}} = 2^{m-1}\prod_{k=2}^{m}\sigma_k \frac{x_{j1} - m\left(x_{j1} - A\right)}{x_{j1}^{m+1}} = 0 .$$

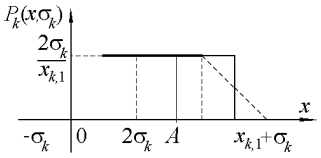

Fig. 3. Replacement of the trapezium by the rectangle

Then

$$P_{error\,2} \le \frac{1}{m}\left(2\frac{m-1}{m}\right)^{m-1}\prod_{k=2}^{m}\frac{\sigma_k}{A} . \qquad (29)$$

So the second derivative at the point $x_{j1} = x_1^*$

$$\frac{d^2F}{dx_{j1}^2} = -\frac{m-1}{\left(\frac{m}{m-1}A\right)^{m+2}} 2^{m-1}\prod_{k=2}^{m}\sigma_k < 0 .$$

Hence for $x_{j,1} = x_1^*$ the maximum of $F\left(x_{j1}\right)$ is achieved, and this is equal to

$$F\left(x_1^*\right) = \frac{1}{m}\left(\frac{m-1}{m}\right)^{m-1}\prod_{k=2}^{m}\frac{2\sigma_k}{A} .$$

This proves the lemma.

Let us consider the upper estimate of $P_{error\,2}$ in an arbitrary case.

Theorem 1 (the upper estimate for arbitrary *n*). For any unimodal step functions of probability density $f_k\left(x_k\right)$ (12) under the condition (10), the estimate (28) is valid.

Proof (by induction). For *n*=1 the estimate (28) is valid. Let us assume that it is valid for *n*–1.

Let us now assume that the maximum of $P_{error\,2}$ is achieved for some values $\Delta_{k,i}$ forming functions $f_k\left(x\right)$ with *n* steps.

Let us consider the partial derivatives $\partial P_{error\,2}/\partial\Delta_{k,q}$ for these values $\Delta_{k,i}$. If $\partial P_{error\,2}/\partial\Delta_{k,q} \ne 0$ for any *k* and *q* and all the inequalities $\Delta_{k,q+1} < \Delta_{k,q} < \Delta_{k,q-1}$, $\Delta_{k,0} > \Delta_{k,1}$, $\Delta_{k,i} > 0$ are strictly fulfilled, then it is not an extremum. If there is even one equality instead of one of these inequalities, then the function $f_k\left(x_k\right)$ has *n*–1 steps.

If $\partial P_{error\,2}/\partial\Delta_{k,q} = 0$, then according to Lemma 1 this derivative value keeps for any values $\Delta_{k,q}$ including boundary ones if one of inequalities above turns into an equality. In this case the value $P_{error\,2}$ for the boundary value $\Delta_{k,q}$ is equal to its maximum.

In the first and in the second case we have functions with *n*–1 steps for which the estimate (28) is valid by assumption.

The theorem is proved by induction. The inequality (28) can be written in the form

$$P_{error\,2} \le k_m \prod_{k=2}^{m}\frac{\sigma_k}{A}, \qquad k_m = \frac{1}{m}\left(2\frac{m-1}{m}\right)^{m-1}, \quad (30)$$

where the value k_m<1 for m≤4.

Thus, the certainty of numerical results increases with the help of a comparison of the indeterminacy intervals. The estimate of the measure of reliability for this comparison is found.

VI. Mathematical Model for the Unobservable Error with Finite Probability of the Absence of Mistakes

Unlike a computational error, an unobservable error has its own singularities. The result of the computations of the magnitude *z* obtained with the help of a concrete method (the implementation of a program) can be distorted by different sources. Considering every source separately, it is possible to assume either that such source is present, the error exists and $x=x_k\ne0$, or that $x=0$. Let the probability of the absence of unobservable error for method number *k* be equal to α_k. The probability density function of errors from this source can be presented as the Dirac δ–function. If we consider a combination of these sources and assume that the influences of these on final result are different, then we give a probability density $f_k(x_k)$ for every x_k. The common function of the probability density of the unobservable error then has the form

$$\varphi_k\left(x_k\right) = \alpha_k\delta\left(x_k\right) + \left(1-\alpha_k\right)f_k\left(x_k\right), \quad \int_{-\infty}^{\infty} f_k\left(x_k\right)dx_k = 1 , \quad (31)$$

where α_k is the probability of the absence of unobservable error for the method number *k*.

The probability of the error lying in a finite interval

$$P_k\left(y,\sigma_k\right) = P\left\{y-\sigma \le x_k \le y+\sigma\right\} = \int_{y-\sigma_k}^{y+\sigma_k}\varphi_k\left(x_k\right)dx_k =$$

$$= \left(1-\alpha_k\right)\int_{y-\sigma_k}^{y+\sigma_k} f\left(x_k\right)dx_k + \begin{cases} 0, & |y| > \sigma, \\ \alpha_k, & |y| \le \sigma. \end{cases}$$

The probability of the coincidence of results up to the accuracy σ_k is the sum of probabilities of the presence of small unobservable errors $P_{error1}=P\{\ 0<|x|<A\ \}$, of large errors $P_{error2}=P\{\ |x|\ge A\ \}$ and of the absence of errors $P_0=P\{x=0\}$.

$$P_{\text{coinsidence}} = \int_{-\infty}^{\infty} \prod_{k=2}^{m} P_k(x,\sigma_k)\rho_1(x)dx = P_0 + P_{error1} + P_{error2},$$

$$P_0 = \alpha_1 \prod_{k=2}^{m}[(1-\alpha_k)J_k(0,\sigma_k)+\alpha_k],\ J_k(y,\sigma_k) = \int_{y-\sigma_k}^{y+\sigma_k} f_k(x)dx,$$

$$P_{error1} = (1-\alpha_1)\int_{-A}^{A} \prod_{k=2}^{m} P_k(x,\sigma_k) f_1(x)dx,$$

$$P_{error2} = \prod_{k=1}^{m}(1-\alpha_k)\times$$

$$\times\left[\int_{-\infty}^{-A} \prod_{k=2}^{m} J_k(x,\sigma_k) f_1(x)dx + \int_{A}^{\infty} \prod_{k=2}^{m} J_k(x,\sigma_k) f_1(x)dx\right].$$

Thus, the probability of there being large unobservable error is determined analogously (9). Therefore the probability error in this case has the form

$$P_{error2} \le k_m \prod_{k=1}^{m}(1-\alpha_k)\prod_{k=2}^{m}\frac{\sigma_k}{A}, \qquad (32)$$

where the values k_m are determined from (30).

Note that the expression (32) takes into account the a priori value of the probability of the absence of unobservable error, in comparison with (30). It is difficult to estimate these probabilities α_k in practice. If an estimated method is verified by different samples, and the programs for the obtained results are debugged by known methods, then it can be asserted that $\alpha_k \ge 1/2$.

In fact, if $\alpha_k < 1/2$ then distrust of the results predominates. This requires additional testing in order to find an error in a given method, so that the error can be localized and removed. If the tests do not detect any discordance, then trust in the results increases. In this case the multiplication is $\prod_{k=1}^{m}(1-\alpha_k) \le 2^{-m}$.

The magnitude of the threshold is chosen to be equal to the required precision ε for the application of the estimate (32). Note that such an approach without additional assumptions requires the real results with an error σ that is smaller than the magnitude of ε by one or two orders. This method therefore represents an increase in certainty by means of excessive accuracy.

We assume σ_k<0.1ε, k=1,...,m. Then the coincidence of the results of two or three not very reliable methods (with a probability of error 1-α_k≈0.5) decreases the probability of the existence of unobservable error exceeding ε down to the magnitude of 0.05-0.0025.

VII. CONCLUSION

The main conclusion of this work is that it is permissible to apply weak methods that do not guarantee the precision or validation of the estimation of the error in a solution if there is a way of estimating the confidence interval and decreasing the probability of incorrect conclusions down to an acceptable value (for the solution to a theoretical problem this value may be equal to 0.01-0.001).

A comparison of the numerical results for solutions to one and the same problem by a few (between two and four) methods can be used as one way to do this.

In this case there are two main approaches to the results and to increasing the reliability of the estimates: increasing the number of methods that are applied, and obtaining results that are more precise.

However, an increase in precision is only required for testing. There is a distinction in principle between the testing of filtration algorithms and the estimation of error from often practical applications of an estimation of the difference between computational results and control data. In our case, the computations algorithm, a concrete programme and also an estimation method for computational error are checked. This is why the number of tests can be decreased along with the improvement in precision. This demonstrates the high efficiency of the given testing method.

The proposed scheme for the computation process and certainty proof is therefore as follows:

- computations by a small number of methods and programs;
- error estimation of every result obtained;
- numerical estimation of certainty of results (the probability that the estimations obtained are valid) on the basis of the comparison;
- additional investigations to increase certainty if one is not satisfied with the numerical estimation of certainty.

In our opinion this scheme can replace the logical proof in real conditions (including numerical computations) when the logical proof cannot be completed. Moreover, this scheme (with the exception of the numerical estimation of certainty, which is not generally accepted) is the base for any scientific investigation. Unlike statistical methods for the verification of hypotheses, the proposed method does not require many experiments. Its basis is finding the worst probabilities distribution for any given case.

We can recommend the parallel application of three programs and a comparison of the results for the control of complex objects such as space stations. If the difference between one result and others is within the definition, this result is removed. The additional cost of taking these actions is much lower than the cost of losing the station.

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