A Novel Method Based on Fuzzy Tensor Technique for Interval-Valued Intuitionistic Fuzzy Decision-Making with High-Dimension Data

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ABSTRACT

To solve the interval-valued intuitionistic fuzzy decision-making problems with high-dimension data, the fuzzy matrix is extended to the fuzzy tensor in this paper. Based on the constructed tensor definition, we propose the generalized interval-valued intuitionistic fuzzy weighted averaging (GIIFWA) and generalized interval-valued intuitionistic fuzzy weighted geometric (GIIFWG) operators. By exploring the properties of GIIFWA and GIIFWG operators, a new algorithm is presented to solve the interval-valued intuitionistic fuzzy multiple attribute group decision-making problem. Two typical application examples are also provided to demonstrate the efficiency and universal applicability of our proposed method.

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1. INTRODUCTION

As an important branch of decision-making fields, the multiple attribute group decision-making has been paid a close attention in past decades. Normally, multiple attribute group decision-making problems is that multiple decision-makers select the optimal alternatives or ranking them from a set of feasible alternatives by the attribute weights and attribute values, for details refers to Xu and Cai [1]. However in some real applications such as Xu and Cai [1], Liu et al. [2], Wang et al. [3], Qin et al. [4], He [5], and Hashemi et al. [6], due to the undetermined decision-making environment, the multi-attribute group decision-making seems to be useless for decision-making. One alternative dealing with this difficulty is the fuzzy set, which was subsequently extended to intuitionistic fuzzy set by Atanassov [7] for applications in various decision-making areas, and Atanassov and Gargov [8] presented the concept and properties of interval-valued intuitionistic fuzzy set based on intuitionistic fuzzy set in 1989, which enriched intuitionistic fuzzy set theory. Especially in recent researches, multiple attribute group decision-making with incorporated interval-valued intuitionistic fuzzy sets has attracted great attentions and yielded plentiful results. For example, Xu [9] developed a method based on distance measure for group decision-making with interval-valued intuitionistic fuzzy matrices. Kabak and Ervural [10] devised a generic conceptual framework and a classification scheme for multiple attribute group decision-making methods. Yang et al. [11] proposed a new method based on dynamic intuitionistic normal fuzzy aggregation operators and VIKOR method with time sequence preference for the dynamic intuitionistic normal fuzzy multi-attribute decision-making problems. Liu [12] proposed the interval-valued intuitionistic fuzzy power Heronian aggregation operator and interval-valued intuitionistic fuzzy power weight Heronian aggregation operator for the multiple attribute group decision-making. Chen and Huang [13] proposed a new multi-attribute decision-making method by the interval-valued intuitionistic fuzzy weighted geometric average (IIFWGA) operator and the accuracy function of interval-valued intuitionistic fuzzy values. Wang and Chen [14] proposed an improved multiple attribute decision-making method by the score function $S_{WC}$ of interval-valued...
intuitionistic fuzzy values and the linear programming methodology. Qiu and Li [15] employed the plant growth simulation algorithm (PGSA) to calculate the optimal preferences of the entire expert group and proposed a new method to solve the multi-attribute group decision-making problem.

However the above mentioned models which are based on matrix frame meet with difficulties in processing higher dimension data and might lose their efficiency. To tackle this problem, we introduce a new developed tensor model which is a generalization of matrix. The concepts of higher-order tensor eigenvalues and eigenvectors were introduced by [16] and [17]. Subsequently, the theory and algorithms of some special tensors and the spectra of tensors with their various applications have attracted wide attention [18–31]. For example, Ding and Wei [18,19] investigated the concepts of some structured multi-linear systems whose coefficient tensor is M-tensor. Qi [20] proved two new spectral properties and a maximum property of the largest H-eigenvalue in a symmetric nonnegative tensor system. Ni et al. [21] obtained an upper bound of different US-eigenvalues and the count of US-eigenpairs corresponding to all nonzero eigenvalues in the symmetric tensors. Ng et al. [22] proposed an iterative method to calculate the largest eigenvalue of an irreducible nonnegative tensor. Rajesh Kannan et al. [23] gained some properties of strong H-tensors and (general) H-tensors. Based on the diagonal product dominance and S diagonal product dominance of tensor, Wang et al. [24] established some new implementable attribute which can be used for identifying nonsingular H-tensor. By studying the general product of two n-dimensional tensors A and B with orders m ⩾ 2 and k ⩾ 1, Shao et al. [25,26] found that the product is a generalization of the usual matrix product and it satisfies the associative law. Bu et al. [27] gave some basic properties for the (left) inverse, rank, and product of tensors. Pumplün [28] studied the tensor product of an associative and a nonassociative cyclic algebra. Giladi et al. [29] studied the volume ratio of the projective tensor products εp ⊗ εq ⊗ εr with 1 ⩽ p ⩽ q ⩽ r ⩽ ∞ and obtained asymptotic formulas that are sharp in almost all cases. Gutiérrez García et al. [30] employed tensor products of complete lattices into fuzzy set theory. Hilberdink [31] studied operators having (infinite) matrix representations and gave such operators infinite tensor products over the primes. Moreover, we have defined the concept of fuzzy tensor and established the general form of the fuzzy synthetic evaluation model for solving multiple attribute group decision-making problems [32].

Based on the research results we have achieved [32], we will propose two new generalized aggregation operators based on interval-valued intuitionistic fuzzy tensor for solving the interval-valued intuitionistic fuzzy multiple attribute group decision-making problem. Specifically, we will first establish the generalized interval-valued intuitionistic fuzzy weighted averaging (GIIFWA) and generalized interval-valued intuitionistic fuzzy weighted geometric (GIIFWG) operators. Then some properties about those new generalized aggregation operators are developed and a new algorithm is presented for the corresponding decision-making problems. Indeed as shown in numerical experiments, the proposed interval-valued intuitionistic fuzzy tensor model does provide a new way for solving multiple attribute group decision-making problems with high-dimension data.

The whole paper is arranged as follows: In Section 2, we introduce some concepts and properties of the fuzzy tensor and interval-valued intuitionistic fuzzy aggregation. Section 3 is devoted to the derivation of the GIIFWA and GIIFWG operators by the product of tensor and vector, and gives some properties of two new generalized aggregation operators. In Section 4, we present an algorithm for solving the interval-valued intuitionistic fuzzy multiple attribute group decision-making problems. In Section 5, two different application examples are shown for illustrating the proposed approach. A conclusion is finally drawn in Section 6.

2. PRELIMINARIES

This section provides basic preliminaries about the fuzzy tensor, interval-valued intuitionistic fuzzy set, and interval-valued intuitionistic fuzzy information aggregation theory.

Let Ω be the real field and F and IVIF be the fuzzy set and interval-valued intuitionistic fuzzy set defined in universe Ω, respectively. The T(m, n), Tp(m, n), and TIVIF(m, n) denote the set of all nth-order n-dimension real tensors, fuzzy tensors, and interval-valued intuitionistic fuzzy tensors, respectively, and |n| = {1, 2, ⋯, n}. F and IVIF denote the n-dimensional fuzzy vector in the F and n-dimensional interval-valued intuitionistic fuzzy vector in the IVIF, respectively.

Definition 2.1. [8] Let X be a finite nonempty set. Then

\[ \hat{A} = \{ (\hat{x}_1, \hat{x}_2, \hat{x}_3) | x \in X \} \]

is called an interval-valued intuitionistic fuzzy set, where \( \hat{x}_1(x) \subseteq [0, 1] \) and \( \hat{x}_2(x) \subseteq [0, 1], x \in X \), with the condition:

\[ \sup \hat{x}_1(x) + \sup \hat{x}_2(x) \leq 1, x \in X \]

Note: For convenience, the interval-valued intuitionistic fuzzy numbers (IVIFNs) [33] can be denoted as

\[ \hat{A} = \left[ \left[ \mu^H_1(x), \mu^H_2(x), \rho^H_1(x), \rho^H_2(x) \right] \right] \]

in this paper, where

\[ \left[ \mu^H_1, \mu^H_2, \rho^H_1, \rho^H_2 \right] \subseteq [0, 1], \left[ \mu^H_1(x), \mu^H_2(x), \rho^H_1(x), \rho^H_2(x) \right] \subseteq [0, 1], \mu^H_1 + \mu^H_2 \leq 1. \]

and \( \left[ \mu^H_1, \mu^H_2, \rho^H_1, \rho^H_2 \right] \) and \( \left[ \mu^L_1, \mu^L_2, \rho^L_1, \rho^L_2 \right] \) represent the supported interval and opposed interval about an evaluation object, respectively.

Definition 2.2. [33] Let \( \hat{\alpha} = \left[ \left[ \mu^H_1, \mu^H_2, \rho^H_1, \rho^H_2 \right], \left[ \mu^L_1, \mu^L_2, \rho^L_1, \rho^L_2 \right] \right] \) denote the complement of \( \hat{\alpha} \).

Then

1. \( \hat{\alpha} = \left[ \left[ \mu^H_1, \mu^H_2, \rho^H_1, \rho^H_2 \right], \left[ \mu^L_1, \mu^L_2, \rho^L_1, \rho^L_2 \right] \right] \), where \( \hat{\alpha} \) is the complement of \( \hat{\alpha} \).

2. \( \hat{\alpha}_1 \wedge \hat{\alpha}_2 = \left[ \min \{ \mu^H_1, \mu^H_2 \}, \min \{ \mu^L_1, \mu^L_2 \}, \max \{ \rho^H_1, \rho^H_2 \}, \max \{ \rho^L_1, \rho^L_2 \} \right] \);

3. \( \hat{\alpha}_1 \vee \hat{\alpha}_2 = \left[ \max \{ \mu^H_1, \mu^H_2 \}, \max \{ \mu^L_1, \mu^L_2 \}, \min \{ \rho^H_1, \rho^H_2 \}, \min \{ \rho^L_1, \rho^L_2 \} \right] \);

4. \( \hat{\alpha}_1 + \hat{\alpha}_2 = \left[ \left[ \mu^H_1(x) + \mu^H_2(x), \mu^L_1(x) - \mu^L_2(x), \rho^H_1(x) + \rho^H_2(x), \rho^L_1(x) - \rho^L_2(x) \right] \right] \).

\[ \left[ \mu^H_1, \mu^H_2, \rho^H_1, \rho^H_2 \right], \left[ \mu^L_1, \mu^L_2, \rho^L_1, \rho^L_2 \right] \]
\[ \mathcal{A}_{\text{IVIF}} = \left( a_{1i2\cdots im}, a_{2i2\cdots im}, \cdots, a_{ni2\cdots im}, a_{ni2\cdots im} \right) \times R \]

where the interval \( \left[ \frac{\mu_{1i2\cdots im}}{\nu_{1i2\cdots im}}, \frac{\mu_{ni2\cdots im}}{\nu_{ni2\cdots im}} \right] \) denotes the supported interval and opposed interval about an evaluation and the interval \( f(\mathcal{A}) \), respectively. Then the supported interval and opposed interval about an evaluation and the interval \( f(\mathcal{A}) \).

**Definition 2.6.** Let \( \mathcal{A} \in T_R (m, n_1 \times n_2 \times \cdots \times n_m) \), and its elements \( a_{1i2\cdots im}, a_{2i2\cdots im}, \cdots, a_{ni2\cdots im}, a_{ni2\cdots im} \) \( \in R \) where \( i_1 \in [n_1], i_2 \in [n_2], \cdots, i_m \in [n_m] \). Then \( \mathcal{A} \) is called a \( m \)-th order tensor.

**Note:** According to the Definition 2.3, we know that the matrix is the 2nd-order tensor.

**Definition 2.4.** Let \( \mathcal{A} \in T_R (m, n_1 \times n_2 \times \cdots \times n_m) \), and its elements \( a_{1i2\cdots im}, a_{2i2\cdots im}, \cdots, a_{ni2\cdots im}, a_{ni2\cdots im} \) \( \in [0, 1] \) where \( i_1 \in [n_1], i_2 \in [n_2], \cdots, i_m \in [n_m] \), then \( \mathcal{A} \) is called a \( m \)-th order fuzzy tensor.

**Definition 2.5.** Let \( \mathcal{A}_{\text{IVIF}} = \left( a_{1i2\cdots im}, a_{2i2\cdots im}, \cdots, a_{ni2\cdots im}, a_{ni2\cdots im} \right) \in T_{\text{IVIF}} \), and its elements \( a_{1i2\cdots im}, a_{2i2\cdots im}, \cdots, a_{ni2\cdots im}, a_{ni2\cdots im} \) \( \in [0, 1] \) satisfy the condition

\[ \left( \frac{\mu_{1i2\cdots im}}{\nu_{1i2\cdots im}}, \frac{\mu_{ni2\cdots im}}{\nu_{ni2\cdots im}} \right) \times R \]

where \( \omega = (\omega_1, \omega_2, \cdots, \omega_n)^T \) is the weight vector of \( \mathcal{A} \), \( (i = 1, 2, \cdots, n) \), with \( \omega_i \in [0, 1] \) \( (i = 1, 2, \cdots, n) \), and \( \sum_{i=1}^{n} \omega_i = 1 \), then the function IIFWA is called an interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator.

**Definition 2.7.** Let IIFWG: \( F_{\text{IVIF}} \rightarrow F_{\text{IVIF}} \). If

\[ IIFWG_{\omega}(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n) = \mathcal{A}_1^{\omega_1} \cdot \mathcal{A}_2^{\omega_2} \cdot \cdots \cdot \mathcal{A}_n^{\omega_n} \]

then the function IIFWG is called an interval-valued intuitionistic fuzzy weighted geometric (IIFWG) operator.

**Definition 2.8.** Let \( \mathcal{A} = \left( \left[ \mu_{1i2\cdots im}, \mu_{ni2\cdots im} \right], \left[ \nu_{1i2\cdots im}, \nu_{ni2\cdots im} \right] \right) \) be an IVIFN. Then we call

\[ s(\mathcal{A}) = \frac{1}{2} \left( \frac{\mu_{1i2\cdots im} - \nu_{1i2\cdots im}}{\mu_{ni2\cdots im} - \nu_{ni2\cdots im}} \right) \]

the score of \( \mathcal{A} \), where \( s \) is the score function of \( \mathcal{A} \), \( s(\mathcal{A}) \in [-1, 1] \).

**Definition 2.9.** The accuracy function of an IVIFN \( \mathcal{A} \) is defined as

\[ h(\mathcal{A}) = \frac{1}{2} \left( \mu_{1i2\cdots im} + \mu_{ni2\cdots im} - \nu_{1i2\cdots im} - \nu_{ni2\cdots im} \right) \]

where \( h(\mathcal{A}) \in [0, 1] \).
3. GENERALIZED INTERVAL-VALUED INTUITIONISTIC FUZZY AGGREGATION OPERATORS BASED ON FUZZY TENSOR TECHNIQUE

Since the interval-valued intuitionistic fuzzy information aggregation is helpful for dealing with fuzzy multiple attribute decision-making problem, we will first develop, in this section, the GIIFWA and GIIFWG operators by the product of the 2nd-order fuzzy tensor with vector. Then both the GIIFWA and the GIIFWG operators are proved to having properties of idempotency and boundedness, which lays a theoretical foundation for the algorithm to solve the fuzzy multiple attribute group decision-making problems in next section.

Theorem 3.1. Let \( \tilde{A}_{IIF} = (a_{i1j2\ldots m}) \) \( T_{IIF}(m, n_1 \times n_2 \times \cdots \times n_m) \) be a mth-order interval-valued intuitionistic fuzzy tensor, and its elements \( a_{i1j2\ldots m} = \left( \left[ \mu_{1i1j2\ldots m}, \nu_{1i1j2\ldots m} \right] \right) \). Then the aggregated value by using Equation (1) is

GIIFWA
\[
\left( \tilde{A}_{IIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right) = \left(\begin{array}{c}
\left( 1 - \prod_{i=1}^{n_1} \prod_{l=1}^{n_2} \prod_{i=1}^{n_3} \left( 1 - \mu_{1i1j2\ldots m} \right) \right)^{2 \times m}
\left( 1 - \prod_{i=1}^{n_1} \prod_{l=1}^{n_2} \prod_{i=1}^{n_3} \left( 1 - \mu_{1i1j2\ldots m} \right) \right)^{2 \times m}
\prod_{l=1}^{n_2} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} \prod_{i=1}^{n_3} \left( \nu_{1i1j2\ldots m} \right)^{2 \times m}
\end{array}\right),
\]

where \( X_2 = \left( x_2^1, x_2^2, \ldots, x_2^m \right)^T, \ldots, X_m = \left( x_m^1, x_m^2, \ldots, x_m^m \right)^T \) are the weight vectors of \( a_{i1j2\ldots m} \) (\( i_2 = \cdots = i_m = 1, 2, \ldots, n_m \)) respectively, and \( \sum_{i=1}^{n_2} x_2^2 = 1, x_2^2 \geq 0; \ldots; \sum_{i=1}^{n_m} x_m^2 = 1, x_m^2 \geq 0 \).

Proof. We prove the Theorem 3.1 by using mathematical induction on \( n_2, \ldots, n_m \).

1. When \( n_2 = \cdots = n_m = 1 \), we have

GIIFWA
\[
\left( \tilde{A}_{IIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right) = a_{i1j2\cdots m} \left( x_2^1 \cdots x_2^m \right)
\]

2. Let \( I_1 = \{1, 2, \ldots, m\} \) and \( I_2 = \{n_2, n_3, \ldots, n_m\} \) be indicator sets. When at least one element in the indicator set \( I_2 \) add to “1,” then we consider the following cases:

(a) When \( j \in I_1 \) and \( n_j = 2 \), then we have

GIIFWA
\[
\left( \tilde{A}_{IIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right) = \left( 1 - \prod_{i=1}^{n_1} \prod_{l=1}^{n_2} \prod_{i=1}^{n_3} \left( 1 - \mu_{1i1j2\ldots m} \right) \right)^{2 \times m}
\]
3. Suppose that $n_2 = K_2, n_3 = K_3, \cdots, n_m = K_m$, the Theorem 3.1 holds, that is,

\[
\text{GIHFWA} \left( A_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right) = \sum_{i_2=1}^{K_2} \cdots \sum_{i_m=1}^{K_m} a_{1i_2 \cdots i_m} x_{i_2}^2 \cdots x_{i_m}^m
\]

\[
= \left[ 1 - \prod_{i_2=1}^{K_2} \prod_{i_m=1}^{K_m} (1 - \mu_{i_2 \cdots i_m}) x_{i_2}^2 \cdots x_{i_m}^m \right],
\]

\[
\prod_{i_2=1}^{K_2} \prod_{i_m=1}^{K_m} (\psi_{i_2 \cdots i_m}) x_{i_2}^2 \cdots x_{i_m}^m,
\]

\[
\prod_{i_2=1}^{K_2} \prod_{i_m=1}^{K_m} (\mu_{i_2 \cdots i_m}) x_{i_2}^2 \cdots x_{i_m}^m.
\]

Therefore, according to the above analysis, when at least one element in the indicator set $I_2$ add to "1," the Theorem 3.1 holds.

(c) When $j_1, j_2, \cdots, j_l \in I_1$ ($j_1 \neq j_2 \neq \cdots \neq j_l$) and $n_1 = K_{j_1} + 1, n_2 = K_{j_2} + 1, \cdots, n_l = K_{j_l} + 1$, then we have

\[
\text{GIHFWA} \left( A_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right) = \sum_{i_1=1}^{K_{j_1}+1} \sum_{i_2=1}^{K_{j_2}+1} \cdots \sum_{i_m=1}^{K_{j_m}+1} a_{1i_2 \cdots i_m} x_{i_1}^{j_1} x_{i_2}^2 \cdots x_{i_m}^m
\]

\[
= \left[ 1 - \prod_{i_2=1}^{K_2} \prod_{i_m=1}^{K_m} (1 - \mu_{i_2 \cdots i_m}) x_{i_2}^2 \cdots x_{i_m}^m \right],
\]

\[
\prod_{i_2=1}^{K_2} \prod_{i_m=1}^{K_m} (\psi_{i_2 \cdots i_m}) x_{i_2}^2 \cdots x_{i_m}^m,
\]

\[
\prod_{i_2=1}^{K_2} \prod_{i_m=1}^{K_m} (\mu_{i_2 \cdots i_m}) x_{i_2}^2 \cdots x_{i_m}^m.
\]
Let the Corollary 3.1. holds from (1), (2), and (3). This completes the proof of
\[
\prod_{i=1}^{n_2} \prod_{i_3=1}^{n_3} \prod_{i_m=1}^{n_m} \left( \mu_{i_1 i_2 \ldots i_m}^{x_1 x_2 \ldots x_m} \right).
\]

(d) When all the elements in the indicator \( I_3 \) add to “1,” that is, \( n_2 = K_2 + 1, n_3 = K_3 + 1, \ldots, n_m = K_m + 1 \), then we have

\[
\text{GIIFWA} \left( \tilde{A}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right)
= \sum_{i_2=1}^{K_2+1} \cdots \sum_{i_m=1}^{K_m+1} a_{i_2 \ldots i_m} x_2 \ldots x_m,
\]

\[
= \left[ \prod_{i_1=1}^{K_1+1} \prod_{i_2=1}^{K_2+1} \prod_{i_3=1}^{K_3+1} \prod_{i_m=1}^{K_m+1} \left( 1 - \mu_{ij_1 \ldots j_m}^{i_2 \ldots i_m} \right) \right] x_2 \ldots x_m,
\]

\[
= \left[ \prod_{i_1=1}^{K_1+1} \prod_{i_2=1}^{K_2+1} \prod_{i_3=1}^{K_3+1} \prod_{i_m=1}^{K_m+1} \left( 1 - \mu_{ij_1 \ldots j_m}^{i_2 \ldots i_m} \right) \right] x_2 \ldots x_m.
\]

Therefore, for any \( n_2, n_3, \ldots, n_m \), the Theorem 3.1 holds from (1), (2), and (3). This completes the proof of Theorem 3.1.

**Corollary 3.1.** [1] Let \( \tilde{A}_{IVIF} \in T_{IVIF}(2, n \times m) \) be an interval-valued intuitionistic fuzzy matrix, and \( \tilde{A}_{IVIF} = (a_{ij})_{n \times m} \where a_{ij} = \left( \left[ \mu_{ij}^x, \mu_{ij}^y \right], [v_{ij}^x, v_{ij}^y] \right) \). then their aggregated value by using the GIIFWA operator is also an IVIFN and

\[
\text{GIIFWA} \left( \tilde{A}_{IVIF} \circ X \right)
= \left[ 1 - \prod_{j=1}^{m} \left( 1 - \mu_{ij_1 \ldots j_m}^{i_2 \ldots i_m} \right) \right] x_1 \ldots x_m.
\]

**Remark 3.1.**

Clearly, the Theorem 3.1 is the extension of Corollary 3.1 which is the Theorem 2.3.1 in Xu [1].

**Theorem 3.2.** Let \( \tilde{A}_{IVIF} = \left( a_{i_1 \ldots i_m} \right)_{n_1 \times n_2 \times \cdots \times n_m} \in T_{IVIF}(m, n_1 \times n_2 \times \cdots \times n_m) \where its elements \( a_{i_1 \ldots i_m} = \left( \left[ \mu_{i_1 \ldots i_m}^x, \mu_{i_1 \ldots i_m}^y \right], [v_{i_1 \ldots i_m}^x, v_{i_1 \ldots i_m}^y] \right) \). Then the aggregated value by using Equation (2) is

\[
\text{GIIFWG} \left( \tilde{A}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right)
= \left[ 1 - \prod_{i_1=1}^{n_2} \prod_{i_3=1}^{n_3} \prod_{i_m=1}^{n_m} \left( 1 - \mu_{i_1 \ldots i_m}^{i_2 \ldots i_m} \right) \right] x_2 \ldots x_m.
\]

**Proof.** The proof of the Theorem 3.2 is similar to the proof of Theorem 3.1.

**Corollary 3.2.** [1] Suppose that \( \tilde{A}_{IVIF} \in T_{IVIF}(2, n \times m) \) is an interval-valued intuitionistic fuzzy matrix, and \( \tilde{A}_{IVIF} = (a_{ij})_{n \times m} \where a_{ij} = \left( \left[ \mu_{ij}^x, \mu_{ij}^y \right], [v_{ij}^x, v_{ij}^y] \right) \). then their aggregated value by using the GIIFWG operator is also an IVIFN, and

\[
\text{GIIFWG} \left( \tilde{A}_{IVIF} \circ X \right)
= \left[ 1 - \prod_{j=1}^{m} \left( 1 - \mu_{ij_1 \ldots j_m}^{i_2 \ldots i_m} \right) \right] x_1 \ldots x_m.
\]

where \( X = (x_1, \ldots, x_j, \ldots, x_m)^T \) is the exponential vector of \( a_{ij} \ (j = 1, 2, \ldots, m) \), with \( x_j \in [0, 1] \) and \( \sum_{j=1}^{m} x_j = 1 \).

**Remark 3.2.**

The Theorem 3.2 is the general form of Corollary 3.2 which is the Theorem 2.3.2 in Xu [1].

**Theorem 3.3.** The operational results in Theorems 3.1 and 3.2 are \( n_1 \)-dimension IVIF vectors.

**Proof.** By the Theorems 3.1 and 3.2, we have

\[
\text{GIIFWA} \left( \tilde{A}_{IVIF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right)
= \left[ 1 - \prod_{i_1=1}^{n_2} \prod_{i_3=1}^{n_3} \prod_{i_m=1}^{n_m} \left( 1 - \mu_{i_1 \ldots i_m}^{i_2 \ldots i_m} \right) \right] x_2 \ldots x_m.
\]

\[
= \left[ 1 - \prod_{i_1=1}^{n_2} \prod_{i_3=1}^{n_3} \prod_{i_m=1}^{n_m} \left( 1 - \mu_{i_1 \ldots i_m}^{i_2 \ldots i_m} \right) \right] x_2 \ldots x_m.
\]
and
\[
\text{GIFWG} \left( \tilde{A}_{IVF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right) = \left( \prod_{i=1}^{n_1} \prod_{i=2}^{n_2} \ldots \prod_{i=m}^{n_m} \left( \mu_{12 \ldots i}^{1 \ldots n_1} x_{12 \ldots i}^{1 \ldots n_1} \right), \left( 1 - \mu_{12 \ldots i}^{1 \ldots n_1} \right) x_{12 \ldots i}^{1 \ldots n_1} - \mu_{12 \ldots i}^{1 \ldots n_1} x_{12 \ldots i}^{1 \ldots n_1} \right),
\]
and \(i_1 \in [n_1]\), then both GIFWA (\(\tilde{A}_{IVF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\)) and GIFWG (\(\tilde{A}_{IVF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\)) are \(n_1\)-dimensional IVF vectors.

Therefore, the operational results in the Theorems 3.1 and 3.2 are \(n_1\)-dimension IVF vectors.

**Theorem 3.4.** Let \(\tilde{A}_{IVF} = \left( a_{112 \ldots i}^{1 \ldots n_1} \right)_{n_1 \times n_2 \times \cdots \times n_m} \in T_{IVF}(m, n_1 \times n_2 \times \cdots \times n_m)\) be a \(m\)th-order interval-valued intuitionistic fuzzy tensor, and \(X_2 = \left( x_{12}^{1}, x_{12}^{2}, \ldots, x_{12}^{m} \right)^{T}, \ldots, X_m = \left( x_{1m}^{1}, x_{1m}^{2}, \ldots, x_{1m}^{m} \right)^{T}\) are the weight vectors of \(a_{112 \ldots i}^{1 \ldots n_1} \) (\(i_2 = 1, 2, \ldots, n_2\)), \(\ldots, a_{112 \ldots i}^{1 \ldots n_1} \) (\(i_m = 1, 2, \ldots, n_m\)), respectively, that is, \(\sum_{i_2=1}^{n_2} x_{i_2}^{2} = 1, x_{i_2}^{2} \geq 0; \ldots; \sum_{i_m=1}^{n_m} x_{i_m}^{m} = 1, x_{i_m}^{m} \geq 0\). Then GIFWA (\(\tilde{A}_{IVF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\)) and GIFWG (\(\tilde{A}_{IVF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\)) are fuzzy mappings.

**Proof.** \(\tilde{A}_{IVF} \in T_{IVF}(m, n_1 \times n_2 \times \cdots \times n_m)\) is a \(m\)th-order interval-valued intuitionistic fuzzy tensor.

According to the Definition 2.5, we have
\[
\tilde{A}_{IVF} = \left( \left[ \mu_{112 \ldots i}^{1 \ldots n_1}, \mu_{112 \ldots i}^{1 \ldots n_1}, \nu_{112 \ldots i}^{1 \ldots n_1} \right] \right)_{n_1 \times n_2 \times \cdots \times n_m}
\]
for arbitrary \(\left[ \mu_{112 \ldots i}^{1 \ldots n_1}, \mu_{112 \ldots i}^{1 \ldots n_1}, \nu_{112 \ldots i}^{1 \ldots n_1} \right] \in [0,1], \left[ \nu_{112 \ldots i}^{1 \ldots n_1}, \nu_{112 \ldots i}^{1 \ldots n_1} \right] \in [0,1]\) and \(\mu_{112 \ldots i}^{1 \ldots n_1} + \nu_{112 \ldots i}^{1 \ldots n_1} \leq 1\).

Owing to \(X_2 = \left( x_{12}^{1}, x_{12}^{2}, \ldots, x_{12}^{m} \right)^{T}, \ldots, X_m = \left( x_{1m}^{1}, x_{1m}^{2}, \ldots, x_{1m}^{m} \right)^{T}\) are the weight vectors of \(a_{112 \ldots i}^{1 \ldots n_1} \) (\(i_2 = 1, 2, \ldots, n_2\)), \(\ldots, a_{112 \ldots i}^{1 \ldots n_1} \) (\(i_m = 1, 2, \ldots, n_m\)), respectively, that is, \(\forall x_{i_2}^{2} \in [0,1], \forall x_{i_m}^{m} \in [0,1].\) Then we obtain \(X_2 \in [0,1]^{t_2}, \ldots, X_m \in [0,1]^{t_m}\).

On the basis of the Theorem 3.3, we get GIFWA (\(\tilde{A}_{IVF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\)) \(\in IVF^{n_1}\) and GIFWG (\(\tilde{A}_{IVF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\)) \(\in IVF^{n_1}\)

Thus GIFWA (\(\tilde{A}_{IVF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\)) and GIFWG (\(\tilde{A}_{IVF} \circ X_2 \circ X_3 \circ \cdots \circ X_m\)) are fuzzy mappings from \(\left[0,1\right]^{t_2 \times t_3 \times \cdots \times t_m}\) to \(IVF^{n_1}\) by the Definition 2.12.

**Theorem 3.5.** Let \(\tilde{A}_{IVF} = \left( a_{112 \ldots i}^{1 \ldots n_1} \right)_{n_1 \times n_2 \times \cdots \times n_m} \in T_{IVF}(m, n_1 \times n_2 \times \cdots \times n_m)\) be a \(m\)th-order interval-valued intuitionistic fuzzy tensor, where
Since for any \( i_1, i_2, \ldots, i_m \), we have
\[
\begin{align*}
\min_{i_1 \neq i_2 \neq \cdots \neq i_m} \{ \mu_{i_1} \} & \leq \mu_{i_1} \leq \max_{i_1 = 1} \{ \mu_{i_1} \}, \\
\min_{i_1 \neq i_2 \neq \cdots \neq i_m} \{ \mu_{i_1} \} & \leq \mu_{i_1} \leq \max_{i_1 = 1} \{ \mu_{i_1} \}, \\
\min_{i_1 \neq i_2 \neq \cdots \neq i_m} \{ \nu_{i_1} \} & \leq \nu_{i_1} \leq \max_{i_1 = 1} \{ \nu_{i_1} \}, \\
\min_{i_1 \neq i_2 \neq \cdots \neq i_m} \{ \nu_{i_1} \} & \leq \nu_{i_1} \leq \max_{i_1 = 1} \{ \nu_{i_1} \}.
\end{align*}
\]
Then

\[
\begin{align*}
1 - \prod_{i_2 = 1}^{n_2} \prod_{i_m = 1}^{n_m} (1 - \mu_{i_1})^{x_{i_2} \cdots x_{i_m}} & \leq 1 - \prod_{i_2 = 1}^{n_2} \prod_{i_m = 1}^{n_m} (1 - \min_{i_1 \neq i_2 \neq \cdots \neq i_m} \{ \mu_{i_1} \})^{x_{i_2} \cdots x_{i_m}} \\
& \leq 1 - \prod_{i_2 = 1}^{n_2} \prod_{i_m = 1}^{n_m} (1 - \max_{i_1 = 1} \{ \mu_{i_1} \})^{x_{i_2} \cdots x_{i_m}} \\
& = 1 - \prod_{i_2 = 1}^{n_2} \prod_{i_m = 1}^{n_m} (1 - \mu_{i_1})^{x_{i_2} \cdots x_{i_m}}.
\end{align*}
\]

\[
\begin{align*}
1 - \prod_{i_2 = 1}^{n_2} \prod_{i_m = 1}^{n_m} (1 - \nu_{i_1})^{x_{i_2} \cdots x_{i_m}} & \leq 1 - \prod_{i_2 = 1}^{n_2} \prod_{i_m = 1}^{n_m} (1 - \min_{i_1 \neq i_2 \neq \cdots \neq i_m} \{ \nu_{i_1} \})^{x_{i_2} \cdots x_{i_m}} \\
& \leq 1 - \prod_{i_2 = 1}^{n_2} \prod_{i_m = 1}^{n_m} (1 - \max_{i_1 = 1} \{ \nu_{i_1} \})^{x_{i_2} \cdots x_{i_m}} \\
& = 1 - \prod_{i_2 = 1}^{n_2} \prod_{i_m = 1}^{n_m} (1 - \nu_{i_1})^{x_{i_2} \cdots x_{i_m}}.
\end{align*}
\]
and
\[
\prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} \left( \nu_{i_1,i_2,\ldots,i_m}^\mu \right)^2 x_{i_2}^{\nu_{i_1,i_2,\ldots,i_m}^\mu} \leq \prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} \left( \min_{i_1=1} \nu_{i_1,i_2,\ldots,i_m}^{\mu_1} \right) \prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} x_{i_2}^{\nu_{i_1,i_2,\ldots,i_m}^{\mu_1}} = \max_{i_1,i_2,\ldots,i_m} \nu_{i_1,i_2,\ldots,i_m}^\mu.
\]

Without loss of generality, for \( \forall i_1 \in [n_1] \), let
\[
\text{GIFWFA} \left( \hat{A}_{IVF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right)_{i_1} = \alpha
\]
where \( \alpha = (\mu^l, \mu^u, [\nu^l, \nu^u]) \). By the Definitions 2.8 and 2.10, we get
\[
s(\alpha) = \frac{1}{2} \left( \mu^l - \nu^l + \mu^u - \nu^u \right)
\]
and
\[
s(\alpha) = \frac{1}{2} \left( \mu^l - \nu^l + \mu^u - \nu^u \right)
\]
Next, we will consider the following three cases:

i. When \( s(\alpha) < s(\alpha^+) \) and \( s(\alpha^+) > s(\alpha^-) \), the conclusion (2) in Theorem 3.5 holds.

ii. When \( s(\alpha) = s(\alpha^+) \), we have \( \alpha = \alpha^+ \), that is,
\[
\mu^l = \max_{i_1,i_2,\ldots,i_m} \nu_{i_1,i_2,\ldots,i_m}^{\mu_1}, \quad \mu^u = \max_{i_1,i_2,\ldots,i_m} \mu_{i_1,i_2,\ldots,i_m}^{\mu_1}, \quad \nu^l = \min_{i_1,i_2,\ldots,i_m} \nu_{i_1,i_2,\ldots,i_m}^{\mu_1}, \quad \nu^u = \min_{i_1,i_2,\ldots,i_m} \nu_{i_1,i_2,\ldots,i_m}^{\mu_1}.
\]

Hence, by the Definition 2.9, we get
\[
h(\alpha) = \frac{1}{2} \left( \mu^l + \mu^u + \nu^l + \nu^u \right)
\]

iii. When \( s(\alpha) = s(\alpha^-) \), we have \( \alpha = \alpha^- \), that is,
\[
\mu^l = \min_{i_1,i_2,\ldots,i_m} \nu_{i_1,i_2,\ldots,i_m}^{\mu_1}, \quad \mu^u = \min_{i_1,i_2,\ldots,i_m} \mu_{i_1,i_2,\ldots,i_m}^{\mu_1}, \quad \nu^l = \max_{i_1,i_2,\ldots,i_m} \nu_{i_1,i_2,\ldots,i_m}^{\mu_1}, \quad \nu^u = \max_{i_1,i_2,\ldots,i_m} \nu_{i_1,i_2,\ldots,i_m}^{\mu_1}.
\]

In this case, according to the Theorem 3.1 and Definition 2.10, we obtain GIFWFA \( \left( \hat{A}_{IF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right)_{i_1} = \alpha^+ \).

Due to the arbitrariness of \( i_1 \), we get
\[
\text{GIFWFA} \left( \hat{A}_{IF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right) = (\alpha^+, \cdots, \alpha^+) \in \text{IVIF}^{n_1}.
\]

Proof. The proof of the Theorem 3.6 is similar to the proof of Theorem 3.5.
4. ALGORITHM

In this section, we will employ the generalized GIfWA and GIIFWG operators to devise a new approach for solving the multiple attribute group decision-making problems with high-dimension data. The concrete steps of the algorithm are listed as follows:

Step 1. The interval-valued intuitionistic fuzzy decision matrices are transformed into interval-valued intuitionistic fuzzy tensor $\mathcal{A}_{IWF}$:

Step 2. According to the Theorems 3.1 or 3.2, we utilize the GIfWA operator:

$$\hat{c}_{i} = GIfWA \left( \mathcal{A}_{IWF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right)_{i}$$

or the GIIFWG operator:

$$\tilde{c}_{i} = GIIFWG \left( \mathcal{A}_{IWF} \circ X_2 \circ X_3 \circ \cdots \circ X_m \right)_{i}$$

to aggregate all the elements $a_{i_1i_2\cdots i_m}$ ($i_1 \in [n_1], i_2 \in [n_2], \cdots, i_m \in [n_m]$) of the interval-valued intuitionistic fuzzy tensor $\mathcal{A}_{IWF}$ and get the values $\hat{c}_{i}$ (or $\tilde{c}_{i}$) corresponding to the alternatives $A_i$ ($i_1 \in [n_1]$);

Step 3. Calculate the sores $s(\hat{c}_{i})$ (or $s(\tilde{c}_{i})$) and the accuracy degrees $h(\hat{c}_{i})$ (or $h(\tilde{c}_{i})$) ($i_1 \in [n_1]$) by the Definitions 2.8 and 2.9.

Step 4. Rank the alternatives $A_i$ ($i_1 \in [n_1]$) by the Definition 2.10, and then obtain the best desirable alternative.

5. APPLICATION EXAMPLES AND DISCUSSION

5.1. Interval-Valued Intuitionistic Fuzzy Multiple Attribute Group Decision-Making

In this subsection, we apply the GIfWA and GIIFWG operators to solving the interval-valued intuitionistic fuzzy multiple attribute group decision-making problem with the numerical example used in Qiu [15].

5.1.1. Numerical example

In this example, let us assume that someone intends to buy a car and consults a set of experts. The car supplier $x_{i}$ ($i_1 = 1, 2, \cdots, 5$) are evaluated by four decision-makers $e_{i_2}$ ($i_2 = 1, 2, 3, 4$), and each decision-maker evaluates the alternatives based on five different characteristics $c_{i_3}$ ($i_3 = 1, 2, \cdots, 5$). The interval-valued intuitionistic fuzzy decision matrix proposed by $e_{i_2}$ ($i_2 = 1, 2, 3, 4$) are listed in the Tables 1–4, and the weighted vector of the four experts is $X_2 = (0.3, 0.2, 0.3, 0.2)^T$, and the weighted vector of the five characteristics is $X_3 = (0.2, 0.15, 0.2, 0.3, 0.15)^T$. Due to space limitations, the original interval-valued intuitionistic fuzzy decision matrices are omitted in this paper. For a detailed description, please see Qiu [15].

We now implement our algorithm to solve this problem.
Theorem 3.2. to rank the IVIFs $x_i (i = 5)$, we calculate the scores $s(x_i) (i = 5)$ by the Definition 2.8. $s(x_1) = 0.466, s(x_2) = 0.414, s(x_5) = 0.369, s(x_4) = 0.474, s(x_3) = 0.438$.

Step 4. By the scores $s(x_i)$ result, the ranking order of all the alternatives is generated as $x_4 > x_1 > x_5 > x_2 > x_3$. Therefore, the best car supplier is $x_4$.

We can also replace the GIIFWA with the GIIFWG to resolve this problem. The difference starts from step 2.

Step 2'. By the Theorem 3.2, we have

$$
\text{GIIFWG} \left( A_{\text{IVIF}} \circ X_2 \circ X_3 \right) = \left( \prod_{i=1}^{5} \prod_{j=1}^{5} \left( \mu_{i,j}^{\text{IVIF}} \right)^{2^{3} \cdot 2^{3}} \right)
$$

that is, $x_1 = (0.551, 0.651), (0.000, 0.267)$,

$x_2 = (0.460, 0.657), (0.000, 0.290)$,

$x_3 = (0.431, 0.570), (0.000, 0.264)$,

$x_4 = (0.511, 0.661), (0.000, 0.224)$,

$x_5 = (0.487, 0.645), (0.000, 0.255)$.
Located in Central China and the middle reaches of the Changjiang (Yangtze) River, Hubei Province is distributed in a transitional belt where physical conditions and landscapes are on the transition from north to south and from east to west. Thus, Hubei Province is well known as a land of rice and fish since the region enjoys some of the favorable physical conditions, with a diversity of natural resources and the suitability for growing various crops. At the same time, however, there are also some restrictive factors for developing agriculture, such as a tight man-land relation between a constant degradation of natural resources and a growing population pressure on land resource reserve. Despite cherishing a burning desire to promote their standard of living, people living in the area are frustrated because they have no ability to enhance their power to accelerate economic development because of a dramatic decline in quantity and quality of natural resources and a deteriorating environment. Based on the distinctness and differences in environment and natural resources, Hubei Province can be roughly divided into seven agroecological regions: Y1–Wuhan-Ezhou-Huanggang; Y2–Northeast of Hubei; Y3–Southeast of Hubei; Y4–Jianghan region; Y5–North of Hubei; Y6–Northwest of Hubei; Y7–Southwest of Hubei. In order to prioritize these agroecological regions Yj (j = 1, 2, 3), and constructs, respectively, the interval-valued intuitionistic fuzzy decision matrices R (λj) (l = 1, 2, 3) (here, λ1 denotes the year “2004,” λ2 denotes the year “2005,” and λ3 denotes the year “2006”) as listed in Tables 6–14. Let \( \omega = (1/6, 2/6, 3/6)^T \) be the weight vector of the years \( \lambda_k \) (k = 1, 2, 3), \( \lambda = (0.5, 0.2, 0.3)^T \) be the weight vector of the experts \( E_k \) (l = 1, 2, 3), and \( \xi = (0.3, 0.4, 0.3)^T \) be the weight vector of the attributes \( G_j \) (j = 1, 2, 3).

Step 1. If the interval-valued intuitionistic fuzzy tensor and the GIIFWA operator are employed for expressing data in Tables 6–14, then \( A_{IVF} = \left(a_{i_{1}j_{2}k_{3}}\right)_{7 \times 3 \times 3} \in T_{IVF}(4, 7 \times 3 \times 3 \times 3) \), where its elements \( a_{i_{1}j_{2}k_{3}} = (\mu_{i_{1}j_{2}k_{3}}; \nu_{i_{1}j_{2}k_{3}}) \), and \( a_{i_{1}j_{2}k_{3}} \) represent seven agroecological regions, \( a_{i_{1}} \) (i in [7]) represent three years, \( a_{i_{2}} \) (i in [3]) represent three experts, and \( a_{i_{3}} \) (i in [3]) represent three attributes. The details are as follows:

\[
\begin{align*}
   a_{1111} &= ([0.0, 0.9], [0.0, 0.1]), a_{1112} = ([0.7, 0.8], [0.1, 0.2]), \\
   a_{1113} &= ([0.6, 0.8], [0.0, 0.2]), a_{1121} = ([0.5, 0.6], [0.2, 0.3]), \\
   a_{1122} &= ([0.2, 0.6], [0.1, 0.2]), a_{1123} = ([0.3, 0.6], [0.2, 0.3]), \\
   a_{1131} &= ([0.3, 0.6], [0.1, 0.3]), a_{1132} = ([0.2, 0.5], [0.2, 0.5]), \\
   a_{1133} &= ([0.2, 0.5], [0.3, 0.4]), a_{1211} = ([0.7, 0.8], [0.1, 0.2]), \\
   a_{1212} &= ([0.8, 0.9], [0.0, 0.1]), a_{1213} = ([0.7, 0.9], [0.0, 0.1]), \\
   a_{1221} &= ([0.2, 0.6], [0.3, 0.4]), a_{1222} = ([0.2, 0.5], [0.3, 0.4]), \\
   a_{1223} &= ([0.4, 0.5], [0.2, 0.5]), a_{1231} = ([0.4, 0.6], [0.1, 0.3]), \\
   a_{1232} &= ([0.2, 0.6], [0.1, 0.2]), a_{1233} = ([0.2, 0.5], [0.2, 0.4]), \\
   a_{1311} &= ([0.6, 0.7], [0.1, 0.3]), a_{1312} = ([0.7, 0.9], [0.0, 0.1]), \\
   a_{1313} &= ([0.8, 0.9], [0.0, 0.1]), a_{1321} = ([0.4, 0.6], [0.2, 0.3]), \\
\end{align*}
\]

5.2. Dynamic Interval-Valued Intuitionistic Fuzzy Multiple Attribute Group Decision-Making

In this subsection, we will use a practical example which is a slightly revised version of Case illustration in Xu and Yager [35] to illustrate the efficiency and universal applicability of the presented algorithm.
The comparison among the results of the GIIFWA and GIIFWG operators in this paper and the results of Qiu.

<table>
<thead>
<tr>
<th>Method</th>
<th>Results</th>
<th>Sort Function Values</th>
<th>Preference order</th>
<th>The Optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Qiu’s [15]</td>
<td>$x_1 = ([0.350, 0.774], [0.226, 0.349])$</td>
<td>$s(x_1) = 0.203$</td>
<td>$x_4 &gt; x_5 &gt; x_1 &gt; x_2 &gt; x_3$</td>
<td>$x_4$</td>
</tr>
<tr>
<td>GIIFWA operator</td>
<td>$x_2 = ([0.423, 0.692], [0.171, 0.227])$</td>
<td>$s(x_2) = 0.181$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>GIIFWG operator</td>
<td>$x_3 = ([0.318, 0.698], [0.272, 0.302])$</td>
<td>$s(x_3) = 0.169$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_4 = ([0.259, 0.740], [0.191, 0.200])$</td>
<td>$s(x_4) = 0.235$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_5 = ([0.392, 0.646], [0.185, 0.193])$</td>
<td>$s(x_5) = 0.222$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6 Interval-valued intuitionistic fuzzy decision matrix $R (r_1^1)$.

<table>
<thead>
<tr>
<th>G_1</th>
<th>G_2</th>
<th>G_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y_1</td>
<td>([0.809], [0.0001])</td>
<td>([0.708], [0.102])</td>
</tr>
<tr>
<td>Y_2</td>
<td>([0.607], [0.203])</td>
<td>([0.507], [0.203])</td>
</tr>
<tr>
<td>Y_3</td>
<td>([0.405], [0.204])</td>
<td>([0.506], [0.203])</td>
</tr>
<tr>
<td>Y_4</td>
<td>([0.708], [0.102])</td>
<td>([0.608], [0.001])</td>
</tr>
<tr>
<td>Y_5</td>
<td>([0.507], [0.103])</td>
<td>([0.708], [0.102])</td>
</tr>
<tr>
<td>Y_6</td>
<td>([0.203], [0.506])</td>
<td>([0.305], [0.405])</td>
</tr>
<tr>
<td>Y_7</td>
<td>([0.405], [0.304])</td>
<td>([0.205], [0.305])</td>
</tr>
</tbody>
</table>

Table 7 Interval-valued intuitionistic fuzzy decision matrix $R (r_2^1)$.

<table>
<thead>
<tr>
<th>G_1</th>
<th>G_2</th>
<th>G_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y_1</td>
<td>([0.506], [0.203])</td>
<td>([0.206], [0.102])</td>
</tr>
<tr>
<td>Y_2</td>
<td>([0.405], [0.103])</td>
<td>([0.206], [0.104])</td>
</tr>
<tr>
<td>Y_3</td>
<td>([0.505], [0.203])</td>
<td>([0.708], [0.102])</td>
</tr>
<tr>
<td>Y_4</td>
<td>([0.405], [0.203])</td>
<td>([0.206], [0.103])</td>
</tr>
<tr>
<td>Y_5</td>
<td>([0.305], [0.203])</td>
<td>([0.306], [0.103])</td>
</tr>
<tr>
<td>Y_6</td>
<td>([0.306], [0.203])</td>
<td>([0.207], [0.102])</td>
</tr>
<tr>
<td>Y_7</td>
<td>([0.406], [0.203])</td>
<td>([0.405], [0.102])</td>
</tr>
</tbody>
</table>

Table 8 Interval-valued intuitionistic fuzzy decision matrix $R (r_3^1)$.

<table>
<thead>
<tr>
<th>G_1</th>
<th>G_2</th>
<th>G_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_1x</td>
<td>([0.306], [0.103])</td>
<td>([0.205], [0.203])</td>
</tr>
<tr>
<td>a_2x</td>
<td>([0.305], [0.205])</td>
<td>([0.305], [0.304])</td>
</tr>
<tr>
<td>a_3x</td>
<td>([0.406], [0.103])</td>
<td>([0.304], [0.203])</td>
</tr>
<tr>
<td>a_4x</td>
<td>([0.305], [0.103])</td>
<td>([0.305], [0.203])</td>
</tr>
<tr>
<td>a_5x</td>
<td>([0.206], [0.102])</td>
<td>([0.205], [0.104])</td>
</tr>
<tr>
<td>a_6x</td>
<td>([0.305], [0.203])</td>
<td>([0.305], [0.102])</td>
</tr>
<tr>
<td>a_7x</td>
<td>([0.407], [0.103])</td>
<td>([0.207], [0.203])</td>
</tr>
</tbody>
</table>

a_1322 = ([0.3, 0.6], [0.1, 0.3]), a_1323 = ([0.3, 0.6], [0.2, 0.4]),

a_1331 = ([0.3, 0.5], [0.2, 0.4]), a_1332 = ([0.3, 0.5], [0.1, 0.2]),

a_2111 = ([0.3, 0.6], [0.2, 0.3]), a_2112 = ([0.5, 0.7], [0.2, 0.3]),

a_2121 = ([0.4, 0.5], [0.1, 0.3]), a_2122 = ([0.2, 0.6], [0.1, 0.4]),

a_2132 = ([0.3, 0.3], [0.3, 0.4]), a_2211 = ([0.5, 0.7], [0.1, 0.3]),

a_2212 = ([0.4, 0.5], [0.1, 0.3]), a_2221 = ([0.6, 0.7], [0.1, 0.3]),

a_2311 = ([0.4, 0.5], [0.3, 0.5]), a_2322 = ([0.3, 0.5], [0.2, 0.3]),

a_2332 = ([0.1, 0.7], [0.2, 0.3]), a_3222 = ([0.2, 0.7], [0.1, 0.2]),
### Table 12  Interval-valued intuitionistic fuzzy decision matrix $R \left( R_{13} \right)$.

<table>
<thead>
<tr>
<th>$R_{13}$</th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_1$</td>
<td>$(0.6,0.7)$, $(0.1,0.3)$</td>
<td>$(0.7,0.9)$, $(0.0,0.1)$</td>
<td>$(0.8,0.9)$, $(0.0,0.1)$</td>
</tr>
<tr>
<td>$Y_2$</td>
<td>$(0.4,0.6)$, $(0.1,0.2)$</td>
<td>$(0.5,0.7)$, $(0.1,0.2)$</td>
<td>$(0.6,0.7)$, $(0.1,0.3)$</td>
</tr>
<tr>
<td>$Y_3$</td>
<td>$(0.2,0.4)$, $(0.2,0.3)$</td>
<td>$(0.3,0.5)$, $(0.2,0.3)$</td>
<td>$(0.4,0.6)$, $(0.2,0.4)$</td>
</tr>
<tr>
<td>$Y_4$</td>
<td>$(0.7,0.9)$, $(0.0,0.1)$</td>
<td>$(0.8,0.9)$, $(0.1,0.3)$</td>
<td>$(0.9,0.9)$, $(0.1,0.3)$</td>
</tr>
<tr>
<td>$Y_5$</td>
<td>$(0.5,0.6)$, $(0.2,0.3)$</td>
<td>$(0.4,0.5)$, $(0.1,0.2)$</td>
<td>$(0.6,0.7)$, $(0.2,0.3)$</td>
</tr>
<tr>
<td>$Y_6$</td>
<td>$(0.2,0.3), (0.5,0.6)$</td>
<td>$(0.3,0.5), (0.3,0.4)$</td>
<td>$(0.3,0.6), (0.2,0.4)$</td>
</tr>
<tr>
<td>$Y_7$</td>
<td>$(0.5,0.6), (0.3,0.4)$</td>
<td>$(0.2,0.3), (0.4,0.5)$</td>
<td>$(0.7,0.8), (0.1,0.2)$</td>
</tr>
</tbody>
</table>

### Table 13  Interval-valued intuitionistic fuzzy decision matrix $R \left( R_{13} \right)$.

<table>
<thead>
<tr>
<th>$R_{13}$</th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_1$</td>
<td>$(0.4,0.6), (0.2,0.3)$</td>
<td>$(0.3,0.5), (0.1,0.2)$</td>
<td>$(0.3,0.6), (0.2,0.5)$</td>
</tr>
<tr>
<td>$Y_2$</td>
<td>$(0.2,0.3), (0.1,0.2)$</td>
<td>$(0.2,0.7), (0.1,0.2)$</td>
<td>$(0.5,0.6), (0.1,0.3)$</td>
</tr>
<tr>
<td>$Y_3$</td>
<td>$(0.5,0.7), (0.2,0.3)$</td>
<td>$(0.5,0.6), (0.1,0.3)$</td>
<td>$(0.4,0.5), (0.1,0.2)$</td>
</tr>
<tr>
<td>$Y_4$</td>
<td>$(0.1,0.7), (0.2,0.3)$</td>
<td>$(0.2,0.7), (0.1,0.2)$</td>
<td>$(0.3,0.6), (0.1,0.2)$</td>
</tr>
<tr>
<td>$Y_5$</td>
<td>$(0.4,0.5), (0.1,0.3)$</td>
<td>$(0.2,0.6), (0.1,0.4)$</td>
<td>$(0.1,0.7), (0.2,0.3)$</td>
</tr>
<tr>
<td>$Y_6$</td>
<td>$(0.5,0.6), (0.1,0.3)$</td>
<td>$(0.4,0.6), (0.2,0.4)$</td>
<td>$(0.2,0.6), (0.1,0.3)$</td>
</tr>
<tr>
<td>$Y_7$</td>
<td>$(0.2,0.7), (0.1,0.2)$</td>
<td>$(0.2,0.8), (0.1,0.2)$</td>
<td>$(0.1,0.8), (0.1,0.2)$</td>
</tr>
</tbody>
</table>

### Table 14  Interval-valued intuitionistic fuzzy decision matrix $R \left( R_{13} \right)$.

<table>
<thead>
<tr>
<th>$R_{13}$</th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{2323} = [0.5,0.6], [0.1,0.3]$</td>
<td>$a_{3311} = [0.3,0.7], [0.2,0.3]$</td>
<td>$a_{4232} = [0.1,0.7], [0.2,0.3]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3323} = [0.3,0.5], [0.1,0.4]$</td>
<td>$a_{2333} = [0.2,0.5], [0.2,0.4]$</td>
<td>$a_{3411} = [0.7,0.8], [0.0,0.1]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3111} = [0.4,0.5], [0.2,0.4]$</td>
<td>$a_{3112} = [0.5,0.6], [0.2,0.3]$</td>
<td>$a_{4312} = [0.8,0.9], [0.0,0.1]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3113} = [0.4,0.6], [0.1,0.2]$</td>
<td>$a_{3112} = [0.4,0.5], [0.2,0.3]$</td>
<td>$a_{4322} = [0.1,0.7], [0.2,0.3]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3222} = [0.7,0.8], [0.1,0.2]$</td>
<td>$a_{3221} = [0.3,0.6], [0.1,0.2]$</td>
<td>$a_{4232} = [0.3,0.6], [0.1,0.4]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3223} = [0.4,0.5], [0.1,0.3]$</td>
<td>$a_{3223} = [0.4,0.5], [0.1,0.3]$</td>
<td>$a_{3411} = [0.7,0.8], [0.0,0.1]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3212} = [0.4,0.5], [0.1,0.2]$</td>
<td>$a_{3211} = [0.3,0.5], [0.1,0.3]$</td>
<td>$a_{4312} = [0.8,0.9], [0.0,0.1]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3221} = [0.4,0.5], [0.1,0.2]$</td>
<td>$a_{3222} = [0.2,0.4], [0.2,0.3]$</td>
<td>$a_{4322} = [0.1,0.7], [0.2,0.3]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3223} = [0.2,0.7], [0.1,0.2]$</td>
<td>$a_{3221} = [0.2,0.4], [0.2,0.3]$</td>
<td>$a_{4232} = [0.3,0.6], [0.1,0.4]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3323} = [0.5,0.7], [0.1,0.2]$</td>
<td>$a_{4111} = [0.7,0.8], [0.1,0.2]$</td>
<td>$a_{4311} = [0.7,0.8], [0.0,0.1]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3321} = [0.4,0.6], [0.2,0.4]$</td>
<td>$a_{4112} = [0.6,0.8], [0.0,0.1]$</td>
<td>$a_{4312} = [0.8,0.9], [0.0,0.1]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3322} = [0.5,0.7], [0.2,0.3]$</td>
<td>$a_{4121} = [0.4,0.5], [0.2,0.3]$</td>
<td>$a_{4122} = [0.2,0.6], [0.1,0.3]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3313} = [0.4,0.6], [0.2,0.4]$</td>
<td>$a_{4122} = [0.2,0.6], [0.1,0.3]$</td>
<td>$a_{4123} = [0.2,0.8], [0.1,0.2]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3312} = [0.5,0.6], [0.1,0.3]$</td>
<td>$a_{4131} = [0.3,0.5], [0.1,0.3]$</td>
<td>$a_{4132} = [0.3,0.5], [0.2,0.3]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3311} = [0.4,0.7], [0.2,0.3]$</td>
<td>$a_{4132} = [0.3,0.5], [0.2,0.3]$</td>
<td>$a_{4211} = [0.6,0.7], [0.1,0.2]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3333} = [0.5,0.7], [0.1,0.2]$</td>
<td>$a_{4111} = [0.7,0.8], [0.1,0.2]$</td>
<td>$a_{4212} = [0.7,0.8], [0.1,0.2]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3332} = [0.5,0.7], [0.2,0.3]$</td>
<td>$a_{4212} = [0.2,0.7], [0.1,0.3]$</td>
<td>$a_{4213} = [0.5,0.7], [0.1,0.3]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3331} = [0.4,0.6], [0.2,0.4]$</td>
<td>$a_{4221} = [0.2,0.7], [0.1,0.3]$</td>
<td>$a_{4221} = [0.2,0.7], [0.1,0.3]$</td>
<td></td>
</tr>
<tr>
<td>$a_{3322} = [0.2,0.7], [0.1,0.3]$</td>
<td>$a_{4222} = [0.3,0.6], [0.2,0.4]$</td>
<td>$a_{4222} = [0.2,0.7], [0.1,0.3]$</td>
<td></td>
</tr>
</tbody>
</table>
\[ a_{733} = \{0.2, 0.8\}, \{0.1, 0.2\}, a_{7332} = \{0.4, 0.5\}, \{0.2, 0.3\}, \]
\[ a_{7333} = \{0.1, 0.6\}, \{0.2, 0.4\} \].

**Step 2.** By Theorem 3.1, \( \tilde{A}^{c}_{I/F} \in T_{IVF} (4, 7 \times 3 \times 3 \times 3) \). Let \( X_2 = \omega \) (the years weight), \( X_3 = \lambda \) (the decision-makers weight), and \( X_4 = \xi \) (the attributes weight), we have

\[
\text{GIIFWA} (\tilde{A}^{c}_{I/F} \circ X_2 \circ X_3 \circ X_4) = \left( \begin{array}{c}
1 - \frac{3}{\sum_{i=1}^{3} \left( \frac{1}{2} - \mu_{15,1}^{1i} \right) + 1}^2 \frac{3}{\sum_{i=1}^{3} \left( \frac{1}{2} - \mu_{15,1}^{1i} \right) + 1}^4 \\
1 - \frac{3}{\sum_{i=1}^{3} \left( \frac{1}{2} - \mu_{15,1}^{1i} \right) + 1}^2 \frac{3}{\sum_{i=1}^{3} \left( \frac{1}{2} - \mu_{15,1}^{1i} \right) + 1}^4 \\
1 - \frac{3}{\sum_{i=1}^{3} \left( \frac{1}{2} - \mu_{15,1}^{1i} \right) + 1}^2 \frac{3}{\sum_{i=1}^{3} \left( \frac{1}{2} - \mu_{15,1}^{1i} \right) + 1}^4 \\
1 - \frac{3}{\sum_{i=1}^{3} \left( \frac{1}{2} - \mu_{15,1}^{1i} \right) + 1}^2 \frac{3}{\sum_{i=1}^{3} \left( \frac{1}{2} - \mu_{15,1}^{1i} \right) + 1}^4 \\
\end{array} \right) \]

Then, we get
\[
Y_1 = \{0.444, 0.684\}, \{0.107, 0.249\}, \{0.369, 0.610\}, \{0.147, 0.302\}, \{0.334, 0.562\}, \{0.165, 0.299\}, \{0.346, 0.724\}, \{0.104, 0.231\}, \{0.332, 0.610\}, \{0.145, 0.282\}, \{0.258, 0.514\}, \{0.285, 0.419\}, \{0.290, 0.575\}, \{0.214, 0.338\} \]

Then, we get
\[
Y_1 = \{0.444, 0.684\}, \{0.107, 0.249\}, \{0.369, 0.610\}, \{0.147, 0.302\}, \{0.334, 0.562\}, \{0.165, 0.299\}, \{0.346, 0.724\}, \{0.104, 0.231\}, \{0.332, 0.610\}, \{0.145, 0.282\}, \{0.258, 0.514\}, \{0.285, 0.419\}, \{0.290, 0.575\}, \{0.214, 0.338\} \]

**Step 3.** To rank the IVIFNs \( Y_{i} \) (\( i \in [7] \)), we calculate the scores \( s \{ Y_{i} \} (i \in [7]) \) by the **Definition 2.8**. Then, we get
\[
s(Y_1) = 0.386, s(Y_2) = 0.266, s(Y_3) = 0.216, s(Y_4) = 0.367, s(Y_5) = 0.258, s(Y_6) = 0.304, s(Y_7) = 0.156 \]

**Step 4.** By the scores \( s \{ Y_{i} \} \) result, the ranking order of all the alternatives is generated as \( Y_1 > Y_4 > Y_2 > Y_5 > Y_3 > Y_7 > Y_6 \). Therefore, the agroecological region with the most comprehensive functions is also \( Y_1 \)-Wuhan-Ezhou-Huanggang.

**5.2.2. Discussion**

1. The comparison of the results is shown in Table 15. By using the same data and weight information, we get the same results calculated by the GIIFWA and GIIFWG operators. That is, the agroecological region with the most comprehensive functions is \( Y_1 \)-Wuhan-Ezhou-Huanggang.

2. The GIIFWA and GIIFWG operators proposed in this paper can effectively solve the dynamic multiple attribute group decision-making problem (four-dimensional data) through analyzing the above practical decision-making problem. Therefore, in order to solve the actual decision problem of high-dimensional data, the proposed methods have better adaptability. For example, it can effectively deal with multiple attribute group decision-making problem (three-dimensional data), dynamic multiple attribute group decision-making problem (four-dimensional data), and practical decision problems with higher dimension data.

**6. CONCLUSION**

As a generalization of fuzzy decision matrix, this paper has presented the concept of \( m \)-th order interval-valued intuitionistic fuzzy tensor and related properties. The GIIFWA and GIIFWG operators by the product of tensor with vector have been obtained and found effective to deal with the multiple attribute group decision-making and dynamic multiple attribute group decision-making problems in an interval-valued intuitionistic condition. Two typical examples have also been provided to demonstrate the efficiency and universal applicability of the proposed method.
The comparison between the results of the GIIFWA and GIIFWG operators in this paper.

<table>
<thead>
<tr>
<th>Method</th>
<th>Results</th>
<th>Sort Function Values</th>
<th>Preference Order</th>
<th>The Agroecological Region with the Most Comprehensive Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>GIIFWA operator</td>
<td>$Y_1 = (0.556, 0.754], [0.000, 0.207]$</td>
<td>$s(Y_1) = 0.552$</td>
<td>$Y_1 &gt; Y_4 &gt; Y_2 &gt; Y_5 &gt; Y_3 &gt; Y_7 &gt; Y_6$</td>
<td>$Y_1$</td>
</tr>
<tr>
<td></td>
<td>$Y_2 = (0.415, 0.630], [0.134, 0.283]$</td>
<td>$s(Y_2) = 0.314$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y_3 = (0.363, 0.581], [0.153, 0.290]$</td>
<td>$s(Y_3) = 0.251$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y_4 = (0.479, 0.749], [0.000, 0.208]$</td>
<td>$s(Y_4) = 0.510$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y_5 = (0.391, 0.632], [0.135, 0.273]$</td>
<td>$s(Y_5) = 0.307$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y_6 = (0.279, 0.542], [0.233, 0.384]$</td>
<td>$s(Y_6) = 0.102$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y_7 = (0.349, 0.626], [0.183, 0.304]$</td>
<td>$s(Y_7) = 0.244$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>GIIFWG operator</td>
<td>$Y_1 = (0.444, 0.684], [0.107, 0.249]$</td>
<td>$s(Y_1) = 0.386$</td>
<td>$Y_1 &gt; Y_4 &gt; Y_2 &gt; Y_5 &gt; Y_3 &gt; Y_7 &gt; Y_6$</td>
<td>$Y_1$</td>
</tr>
<tr>
<td></td>
<td>$Y_2 = (0.369, 0.610], [0.147, 0.302]$</td>
<td>$s(Y_2) = 0.266$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y_3 = (0.334, 0.562], [0.165, 0.299]$</td>
<td>$s(Y_3) = 0.216$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y_4 = (0.346, 0.724], [0.104, 0.231]$</td>
<td>$s(Y_4) = 0.367$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y_5 = (0.332, 0.610], [0.145, 0.282]$</td>
<td>$s(Y_5) = 0.258$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y_6 = (0.258, 0.514], [0.285, 0.419]$</td>
<td>$s(Y_6) = 0.034$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Y_7 = (0.290, 0.575], [0.214, 0.338]$</td>
<td>$s(Y_7) = 0.156$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

GIIFWA, generalized interval-valued intuitionistic fuzzy weighted averaging; GIIFWG, generalized interval-valued intuitionistic fuzzy weighted geometric.

ACKNOWLEDGMENT

The work was supported in part by the National Natural Science Foundation of China (Grant nos. 11571292 and 11747104), the General Project of Hunan Provincial Natural Science Foundation of China (2016J043 and 2019J0125), the Postgraduate Innovation Project of Hunan Province of China (CJ2017B263), Since the Guangxi Municipality Project for the Basic Ability Enhancement of Young and Middle-Aged Teachers (KY2016YB532) has been completed, the project is deleted in the final version.

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