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Statistical Inference for Topp–Leone-generated Family of Distributions Based on Records

Mohd Arshad* and Qazi Azhad Jamal

Department of Statistics and Operations Research, Aligarh Muslim Univeristy, Aligarh, Uttar Pradesh, India

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ABSTRACT

In this paper, we consider a general family of distributions generated by Topp–Leone distribution (known as TL family of distributions) proposed by Rezaei *et al.* [1]. We consider the problem of estimation of the shape parameter, scale parameter, and reliability function based on record data from TL family of distributions. We derive the maximum likelihood estimator (MLE) for shape parameter, scale parameter, and reliability function. We have also obtained UMVUE (uniformly minimum-variance unbiased estimator) for reliability function when scale parameter is known. A Bayesian study is carried out under symmetric and asymmetric loss functions in order to find the Bayes estimators for unknown parameters and reliability function. Further, we have predicted future record values using Bayesian approach. A numerical comparison of various estimators is also reported.

1. INTRODUCTION

Topp–Leone (TL) distribution was first introduced by Topp and Leone [2]. Later Nadarajah and Kotz [3] discussed this distribution elaborately by obtaining explicit algebraic expressions such as hazard rate function and n^{th} moment, and so on. The density and distribution function (df) of TL distribution is given by

$$f(x) = 2\alpha\,(1-x)\,[x\,(2-x)]^{\alpha-1}\,,\ \ \alpha > 0, 0 < x < 1,$$
$$F(x) = [x\,(2-x)]^{\alpha}\,,\ \ \alpha > 0, 0 < x < 1.$$

Since its emergence, many authors have studied different properties of TL distribution. We mention reliability measures and stochastic orderings Ghitany *et al.* [4]; distributions of sums, products, and ratios Zhou *et al.* [5]; behavior of kurtosis Kotz and Seier [6]; record values Zghoul [7]; moments of order statistics Genc [8].

Though probability distributions are very useful in practical problems but in some situations the available distributions do not support our problem appropriately. Then it becomes necessary to either define a new distribution or modify some existing distributions, so that they can be useful for various practical problems. This modification of probability distribution gives boost to generalization of distribution. From the last couple of years, we see that several authors have proposed various generated family of distributions. TL distribution is very useful and widely applicable distribution. But due to fact that it has only one parameter and its support is restricted to $(0, 1)$, it is not flexible. It cannot be used for lifetime modeling. So, the generalization of TL distribution is needed. One of the generalization of TL distribution is discussed by Al-Shomrani *et al.* [9]. They have considered $G(x)$ as the baseline df in TL distribution and obtained moments and hazard rate of the new TL family of distributions. Rezaei *et al.* [1] also generalized this TL distribution by using $[G(x)]^{\theta}$ as the baseline distribution called TL-generated (TLG) family of distributions. The authors have explained some special cases of this distribution and also derived expressions of maximum likelihood estimators (MLEs) for unknown parameters.

In this paper, we consider TLG family of distributions proposed by Rezaei *et al.* [1]. For this generated family of distribution, we consider baseline distribution $G(x/\theta)$ where θ denotes an unknown scale parameter. Several well-known distributions can be used for the baseline distribution, for example, exponential distribution with df $G(x/\theta) = 1 - e^{-x/\theta}, \theta > 0, x > 0$, Rayleigh distribution with df $G(x/\theta) = 1 - e^{-x^2/2\theta^2}, \theta > 0, x > 0$, and so on.

*Correspondence author. Email: arshad.iitk@gmail.com

The density and df of TLG family of distributions is given by

$$\begin{aligned} f(x,\alpha,\theta) &= \frac{2\alpha}{\theta} g(x/\theta)\left[1-G(x/\theta)\right]\left[G(x/\theta)\left(2-G(x/\theta)\right)\right]^{(\alpha-1)}, \alpha>0, \theta>0 \quad x\in\mathbb{R}, \\ F(x;\alpha,\theta) &= \left[G(x/\theta)\left(2-G(x/\theta)\right)\right]^{\alpha}, \alpha>0, \theta>0, x\in\mathbb{R}. \end{aligned} \tag{1}$$

Nowadays several researchers are interested in study of record data (extreme values) because of its application in various fields, such as in sports, the longest winning streak of a team, the highest runs of a player, lowest run given by a bowler in an over. In field of marketing; lowest stock market figure, minimum cost of a certain product in market. Medical sciences; most number of people affected by a disease at a particular place, and so on. In all these field of research, record data is widely used.

Chandler [10] introduced the idea of record values and studied some of its basic properties. After that many authors have worked in this field and gave their valuable inputs. For excellent understanding of records, one may refer to books written by Ahsanullah [11], Ahsanullah [12], and Arnold [13]. For application of record values in various disciplines, one may refer to Minimol and Thomas [14], Ahsanullah [15], itekhan2016umvu, Bdair and Raqab [16], MirMostafaee *et al.* [17], Ahsanullah and Nevzorov [18], Arshad and Jamal [19], Arshad and Baklizi [20], Anwar [21], and Arshad and Jamal [22]. Now we discuss the mathematical definition of records and its distribution.

Definition 1.1. Let $\{X_i : i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with an absolutely continuous df $F(x)$ and probability density function (pdf) $f(x)$. An observation X_j is called a lower record if its value precedes all previous observations, that is, X_j is a lower record if $X_j < X_i$ for every $j > i$. Let $R_1, R_2, \dots, R_n$ be n lower records and let $r_1, r_2, \dots, r_n$ denote the observed values of $R_1, R_2, \dots, R_n$, respectively. The density of nth record is given by

$$f_{R_n}(r_n) = \frac{1}{(n-1)!}\left(-\ln\left[F(r_n)\right]\right)^{(n-1)} f(r_n), \quad -\infty < r_n < \dots < r_2 < r_1 < \infty.$$

The joint density of pth and q^{th} lower record is given by $(p<q)$

$$f_{R_p,R_q}(r_p, r_q) = \frac{(-\ln[F(r_p)])^{(p-1)}}{(p-1)!}\frac{(\ln[F(r_p)]-\ln[F(r_q)])^{(q-p-1)}}{(q-p-1)!}\frac{f(r_p)f(r_q)}{F(r_p)}, \quad -\infty < r_n < \cdots < r_q < r_p \cdots < r_2 < r_1 < \infty.$$

The joint density of $\underline{R} = (R_1, R_2, \dots, R_n)$ is given by

$$f_{\underline{R}}(r_1, r_2, \dots, r_n) = \prod_{i=1}^{n-1}\frac{f(r_i)}{F(r_i)} f(r_n), \quad -\infty < r_n < \dots < r_2 < r_1 < \infty. \tag{2}$$

The remainder of the paper is as follows. In Section 2, we derive the expressions for finding out the MLEs for the unknown parameters. An example for finding out MLE is also provided. In Section 3, uniformly minimum-variance unbiased estimator (UMVUE) of the reliability function is derived when the scale parameter is known. In Section 4, a Bayesian study is carried out for obtaining the Bayes estimators for scale parameter, shape parameter, and reliability function under symmetric (squared error) and asymmetric (LINEX and entropy) loss functions. In Section 5, we provide Bayesian prediction interval for future records. Finally, in Section 6, a numerical study is provided to illustrate the results.

2. MAXIMUM LIKELIHOOD ESTIMATION

The likelihood function based on the lower records observed form TLG family of distributions is given by

$$L(\alpha,\theta;\underline{r}) = \left(\frac{2\alpha}{\theta}\right)^n \prod_{i=1}^{n-1}\left(\frac{g(r_i/\theta)\left[1-G(r_i/\theta)\right]}{G(r_i/\theta)\left[2-G(r_i/\theta)\right]}\right)\left[G(r_n/\theta)\left[2-G(r_n/\theta)\right]\right]^{\alpha}.$$

Now taking log both sides, we get

$$\ln L(\alpha,\theta;\underline{r}) = n\ln(2\alpha) - n\ln(\theta) + \sum_{i=1}^{n-1}\ln\left(\frac{g(r_i/\theta)\left[1-G(r_i/\theta)\right]}{G(r_i/\theta)\left[2-G(r_i/\theta)\right]}\right) + \alpha\ln\left[G(r_n/\theta)\left[2-G\left(\frac{r_n}{\theta}\right)\right]\right]. \tag{3}$$

Differentiating Eq. (3) with respect to α, we get

$$\frac{\partial \ln L(\alpha,\theta;\underline{r})}{\partial\alpha} = \frac{n}{\alpha} + \ln\left[G(r_n/\theta)(2-G(r_n/\theta))\right].$$

In order to find MLE of α, we will equate the above equation to 0 and we have

$$\alpha \ln\left[G\left(r_n/\theta\right)\left(2-G\left(r_n/\theta\right)\right)\right]+n=0. \tag{4}$$

Similarly, differentiating Eq. (3) with respect to θ and equating to 0, we get

$$-\frac{n}{\theta}-2\alpha\left(\frac{r_n}{\theta^2}\right)\frac{g\left(r_n/\theta\right)\left[1-G\left(r_n/\theta\right)\right]}{G\left(r_n/\theta\right)\left(2-G\left(r_n/\theta\right)\right)}-\sum_{i=1}^{n}\left(\frac{r_i}{\theta^2}\right)\frac{\left[g'\left(r_i/\theta\right)-g'\left(r_i/\theta\right)G\left(r_i/\theta\right)-g^2\left(r_i/\theta\right)\right]}{g\left(r_i/\theta\right)\left(1-G\left(r_i/\theta\right)\right)}$$
$$+\sum_{i=1}^{n}\left(\frac{2r_i}{\theta^2}\right)\frac{\left[g\left(r_i/\theta\right)\left(1-G\left(r_i/\theta\right)\right)\right]}{G\left(r_i/\theta\right)\left(2-G\left(r_i/\theta\right)\right)}=0. \tag{5}$$

The MLE $\left(\hat{\alpha},\hat{\theta}\right)$ of (α,θ) is a solution of the Eqs. (4) and (5). Because of the nonlinear nature of these equations, it is very cumbersome to obtain the numerical values of unknown parameters explicitly. So, we will use numerical computation techniques to obtain the MLEs for both the parameters and the reliability function, based on lower records obtained from TLG family of distributions. The corresponding MLE of the reliability function $R(t)$ is obtained, after replacing α and θ, respectively, by their MLEs $\hat{\alpha}$ and $\hat{\theta}$, obtained after solving Eqs. (4) and (5), that is, the MLE of the reliability function $R(t)$ is given by

$$\hat{R}(t)=1-\left[G\left(t/\hat{\theta}\right)\left(2-G\left(t/\hat{\theta}\right)\right)\right]^{\hat{\alpha}}.$$

Example 2.1.

Let TLG family of distributions has baseline distribution as exponential distribution with df

$$G\left(x/\theta\right)=1-e^{-x/\theta},\quad x>0,\ \theta>0.$$

Therefore, X has Topp-Leone exponential (TL-Exp) distribution. From Eqs. (4) and (5), we have

$$\alpha\ln\left(1-e^{-r_n/\theta}\right)+n=0, \tag{6}$$

$$-\frac{n}{\theta}+\frac{n}{\theta^2}\frac{r_n e^{-2r_n/\theta}}{\left(1-e^{-2r_n/\theta}\right)\ln\left(1-e^{-2r_n/\theta}\right)}+\sum_{i=1}^{n}\frac{r_i}{\theta^3}-\sum_{i=1}^{n}\frac{r_i e^{-2r_i/\theta}}{\theta^3\left(1-e^{-2r_i/\theta}\right)}=0. \tag{7}$$

For MLE of θ, Eq. (7) has to be solved, then MLE of α can be obtained from Eq. (6), after putting value of $\hat{\theta}$ obtained from Eq. (7). The numerical computation of the MLE of α and θ and $R(t)$ is illustrated in Section 6 (see Example 6.1).

3. UMVUE OF RELIABILITY FUNCTION

In this section, we derive the UMVUE of $R(t)$ when the scale parameter θ is known (WLOG, assume $\theta=1$). For this, we need the following lemma. The proof of lemma is straightforward and is omitted. This lemma can be obtained from the Lemma 3.1 of Khan and Arshad [23].

Lemma 3.1. Let $R_1,R_2,\ldots R_n$ be the first n lower records having joint pdf given in Eq. (2). Define $Z=G(R_n)(2-G(R_n))$. Then, for $z\in(0,1)$, the conditional distribution of R_1 given $Z=z$

$$f_{R_1/Z}\left(r_1/z\right)=\begin{cases}\dfrac{2(n-1)}{(-\ln(z))}\left[1-\dfrac{\ln\left[G\left(r_1\right)\left(2-G\left(r_1\right)\right)\right]}{\ln(z)}\right]^{n-2}\dfrac{g\left(r_1\right)\left(1-G\left(r_1\right)\right)}{G\left(r_1\right)\left(2-G\left(r_1\right)\right)}, & \text{if } G^{-1}\left(1-\sqrt{1-z}\right)<r_1<\infty\\ 0, & \text{otherwise.}\end{cases}$$

Now we shall derive the UMVUE of $R(t)=1-F(t)$. Since Z is a complete sufficient statistic for α, it follows from the Lehmann–Scheffé theorem that the UMVUE of $R(t)$ can be obtained as

$$\mathfrak{R}(t)=E\left[J\left(R_1,t\right)|Z=z\right],$$

where

$$J\left(R_1,t\right)=\begin{cases}1, & \text{if } R_1>t\\ 0, & \text{if } R_1\le t.\end{cases}$$

Using Lemma 3.1, we have

$$\begin{aligned}\mathfrak{R}(t) &= \int_t^{\infty} f_{R_1/Z}(r_1/z)dr_1 \\ &= \int_{\max\left\{t, G^{-1}(1-\sqrt{1-z})\right\}}^{\infty} \frac{2(n-1)}{\left(-\ln(z)\right)}\left[1-\frac{\ln G(r_1)(2-G(r_1))}{\ln(z)}\right]^{n-2}\frac{g(r_1)(1-G(r_1))}{G(r_1)(2-G(r_1))}dr_1 \\ &= 1-\left[\max\left\{0, \left(1-\frac{\ln G(t)(2-G(t))}{\ln(z)}\right)\right\}\right]^{n-1}.\end{aligned}$$

The UMVUE of $R(t)$ is

$$\mathfrak{R}(t) = \begin{cases} 1-\left[1-\dfrac{\ln\left[G(t)(2-G(t))\right]}{\ln(z)}\right]^{n-1}, & \text{if } z < G(t)(2-G(t)) \\ 1, & \text{if } z \geq G(t)(2-G(t)). \end{cases} \tag{8}$$

Example 2.1 continued The UMVUE of $R(t)$ for the TL-Exp distribution is

$$\mathfrak{R}(t) = \begin{cases} 1-\left(1-\dfrac{\ln\left(1-e^{-2t}\right)}{\ln\left(1-e^{-2r_n}\right)}\right)^{n-1}, & \text{if } t > r_n \\ 1, & \text{if } t < r_n. \end{cases} \tag{9}$$

4. BAYESIAN ESTIMATION

In this section, we consider the problem of estimation under Bayesian view point. For this, we consider one symmetric and two asymmetric loss functions. Under these loss functions, Bayes estimators for both the parameters and reliability function are obtained. Squared error loss function is taken as symmetric loss function, it gives equal weight to overestimation as well as underestimation. For asymmetric loss function, linear exponential (LINEX) loss function is used, which was proposed by Varian [24] (also see Zellner [25]) and entropy loss function is also taken, which was proposed by James and Stein [26].

In TLG family of distributions, it is not possible to find a mathematically tractable continuous joint prior distribution for both unknown parameters α and θ. To choose a joint prior distribution for (α, θ) that incorporate uncertainty about both unknown parameters, we adopt the method proposed by Soland [27]. This method is also used by several researchers (see Asgharzadeh and Fallah [28]).

Assume that the scale parameter θ is restricted to a finite number of values $\theta_1, \theta_2, \ldots, \theta_k$ with prior probabilities $p_1, p_2, \ldots, p_k$, respectively, that is, the prior distribution for θ is given by

$$\pi\left(\theta_j\right) = P\left(\theta = \theta_j\right) = p_j, \quad j = 1, 2, \ldots, k.$$

Further, we are assuming that the conditional prior distribution for α given $\theta = \theta_j$ has gamma distribution with parameters a_j and b_j, that is,

$$\pi\left(\alpha|\theta_j\right) = \frac{b_j^{a_j}\alpha^{a_j-1}e^{-\alpha b_j}}{\Gamma\left(a_j\right)}, \quad \alpha > 0, a_j > 0, b_j > 0. \tag{10}$$

The joint density of records $\underline{R} = (R_1, R_2, \ldots, R_n)$ is given by

$$f_{\underline{R}}\left(\underline{r}|\alpha, \theta\right) = \left(\frac{2\alpha}{\theta}\right)^n \prod_{i=1}^{n}\left(\frac{g(r_i/\theta)\left[1-G(r_i/\theta)\right]}{G(r_i/\theta)\left[2-G(r_i/\theta)\right]}\right)\left(G(r_n/\theta)\left[2-G(r_n/\theta)\right]\right)^{\alpha}, \underline{r} = (r_1, r_2, \ldots, r_n). \tag{11}$$

Using Eqs. (10) and (11), we get the conditional posterior density of α given $\theta = \theta_j$ as

$$\begin{aligned}\pi\left(\alpha \mid \theta_j; \underline{r}\right) &= \frac{\pi\left(\alpha \mid \theta_j\right) f_{\underline{R}}\left(\underline{r} \mid \alpha, \theta_j\right)}{\displaystyle\int_0^{\infty} \pi(\alpha \mid \theta_j) f_{\underline{R}}(\underline{r} \mid \alpha, \theta_j)d\alpha} \\ &= \frac{\alpha^{\left(n+a_j-1\right)}e^{-\alpha I(r_n, \theta_j)}}{\displaystyle\int_0^{\infty} \alpha^{\left(n+a_j-1\right)}e^{-\alpha I(r_n, \theta_j)}d\alpha} \\ &= \frac{\left[I\left(r_n, \theta_j\right)\right]^{\left(n+a_j\right)}}{\Gamma\left(n+a_j\right)}\alpha^{\left(n+a_j-1\right)}e^{-\alpha I(r_n, \theta_j)}, \quad \alpha > 0,\end{aligned} \tag{12}$$

where

$$I\left(r_n,\theta_j\right)=\left[b_j-\ln\left[G\left(r_n/\theta_j\right)\left(2-G\left(r_n/\theta_j\right)\right)\right]\right].$$

The joint prior distribution can be obtained by multiplying $\pi\left(\alpha\,|\,\theta_j\right)$ and $\pi\left(\theta_j\right), j=1,2,\ldots,k$. Then the joint posterior distribution for (α,θ) is given by

$$\pi\left(\alpha,\theta_j\,|\,\underline{r}\right)=\frac{f_{\underline{R}}\left(\underline{r}\,|\,\alpha,\theta_j\right)\pi\left(\alpha\,|\,\theta_j\right)\pi\left(\theta_j\right)}{\int_0^\infty\sum_{j=1}^k f_{\underline{R}}\left(\underline{r}\,|\,\alpha,\theta_j\right)\pi\left(\alpha\,|\,\theta_j\right)\pi\left(\theta_j\right)d\alpha}. \tag{13}$$

Now, we will first solve the denominator integral of above equation, that is,

$$\begin{aligned}
I&=\int_0^\infty\sum_{j=1}^k f_{\underline{R}}\left(\underline{r}\,|\,\alpha,\theta_j\right)\pi\left(\alpha\,|\,\theta_j\right)\pi\left(\theta_j\right)d\alpha\\
&=\sum_{j=1}^k\int_0^\infty\frac{b_j^{a_j}\alpha^{a_j-1}e^{-\alpha b_j}}{\Gamma\left(a_j\right)}p_j\left(\frac{2\alpha}{\theta_j}\right)^\theta\prod_{i=1}^n\left(\frac{g\left(r_i/\theta_j\right)\left[1-G\left(r_i/\theta_j\right)\right]}{G\left(r_i/\theta_j\right)\left[2-G\left(r_i/\theta_j\right)\right]}\right)(G(r_n/\theta_j)[2-G(r_n/\theta_j)])^\alpha d\alpha\\
&=\sum_{j=1}^k p_j\frac{b_j^{a_j}}{\Gamma\left(a_j\right)}\left(\frac{2}{\theta_j}\right)^n\prod_{i=1}^n\left(\frac{g\left(r_i/\theta_j\right)\left[1-G\left(r_i/\theta_j\right)\right]}{G\left(r_i/\theta_j\right)\left[2-G\left(r_i/\theta_j\right)\right]}\right)\\
&\qquad\times\int_0^\infty\alpha^{n+a_j-1}e^{-\alpha[b_j-\ln[G(r_n/\theta_j)][2-G(r_n/\theta_j)]]}d\alpha\\
&=\sum_{j=1}^k p_j\frac{b_j^{a_j}}{\Gamma\left(a_j\right)}\left(\frac{2}{\theta_j}\right)^n\prod_{i=1}^n\left(\frac{g\left(r_i/\theta_j\right)\left[1-G\left(r_i/\theta_j\right)\right]}{G\left(r_i/\theta_j\right)\left[2-G\left(r_i/\theta_j\right)\right]}\right)\frac{\Gamma\left(n+a_j\right)}{[b_j-\ln[G(r_n/\theta_j)(2-G(r_n/\theta_j))]]^{(n+a_j)}}
\end{aligned} \tag{14}$$

Using Eqs. (13) and (14), the joint posterior density is

$$\pi\left(\alpha,\theta_j|\underline{r}\right)=\frac{p_jm_jb_j^{a_j}\alpha^{n+a_j-1}}{\theta_j^n\Gamma\left(a_j\right)G}e^{-\alpha\left[I\left(r_n,\theta_j\right)\right]},\quad\alpha>0,\ j=1,2,\ldots,n.$$

where

$$G=\sum_{j=1}^k\frac{m_jp_jb_j^{a_j}}{\theta_j^n\Gamma\left(a_j\right)}\frac{\Gamma\left(n+a_j\right)}{\left[I\left(r_n,\theta_j\right)\right]^{\left(n+a_j\right)}}\quad\text{and}\quad m_j=\prod_{i=1}^n\frac{g\left(r_i/\theta_j\right)\left(1-G\left(r_i/\theta_j\right)\right)}{G\left(r_i/\theta_j\right)\left(2-G\left(r_i/\theta_j\right)\right)},\quad j=1,2,3,\ldots,k.$$

The marginal posterior density of θ_j is

$$\begin{aligned}
P_j&=\int_0^\infty\pi(\alpha,\theta_j|\underline{r})d\alpha\\
P_j&=\int_0^\infty\frac{p_jm_jb_j^{a_j}\alpha^{n+a_j-1}}{\Gamma(a_j)G}e^{-\alpha\left[I(r_n,\theta_j)\right]}d\alpha\\
P_j&=\frac{\Gamma(n+a_j)}{\left[I(r_n,\theta_j)\right]^{(n+a_j)}}\frac{p_jm_jb_j^{a_j}}{\theta_j^n\Gamma(a_j)G},\quad j=1,2,3,\ldots,k.
\end{aligned}$$

Now we will derive the Bayes estimators of unknown quantities under various loss functions.

4.1. Squared Error Loss Function

The squared error loss function is defined as

$$L\left(\delta,\lambda\right)=\left(\delta-\lambda\right)^2,\quad\delta\in\mathbb{D},\ \lambda\in\Theta,$$

where $\mathbb{D}$ is decision space and Θ is the parameter space. Clearly, the Bayes estimator under squared error loss function is the posterior mean, then the Bayes estimator for α is

$$\alpha^*_{\mathrm{BS}} = \int_0^\infty \alpha \sum_{j=1}^k P_j \pi\left(\alpha|\theta_j; \underline{r}\right) d\alpha = \sum_{j=1}^k \frac{\left(n+a_j\right) P_j}{I\left(r_n, \theta_j\right)}.$$

Similarly, the Bayes estimator for θ is given by

$$\theta^*_{\mathrm{BS}} = \int_0^\infty \sum_{j=1}^k \theta_j P_j \pi\left(\alpha|\theta_j; \underline{r}\right) d\alpha = \sum_{j=1}^k P_j \theta_j.$$

The Bayes estimator for reliability function $R(t)$ is

$$\begin{aligned} R(t)^*_{\mathrm{BS}} &= \int_0^\infty \sum_{j=1}^k \left[1 - F(t;\alpha,\theta_j)\right] P_j \pi(\alpha|\theta_j; \underline{r}) d\alpha \\ &= \sum_{j=1}^k P_j \left[1 - \left[1 - \frac{\ln\left[G(t/\theta_j)\left(2 - G(t/\theta_j)\right)\right]}{I(r_n,\theta_j)}\right]^{-(n+a_j)}\right]. \end{aligned} \tag{15}$$

4.2. Entropy Loss Function

The entropy loss function is given by

$$L(\delta,\lambda) = (\delta/\lambda) - \ln(\delta/\lambda) - 1, \quad \delta \in \mathbb{D},\ \lambda \in \Theta.$$

The Bayes estimator under this loss function is

$$\delta^*_{\mathrm{BE}} = \left(E\left(\delta^{-1}\right)\right)^{-1}.$$

So, Bayes estimator of α is

$$\alpha^*_{\mathrm{BE}} = \left(E\left(\alpha^{-1}\right)\right)^{-1}.$$

$$\begin{aligned} E\left(\frac{1}{\alpha}\right) &= \int_0^\infty \frac{\pi(\alpha|\theta_j; \underline{r})\pi(\theta_j|\underline{r})}{\alpha} d\alpha \\ &= \sum_{j=1}^k P_j \int_0^\infty \frac{1}{\alpha} \frac{\left[I(r_n,\theta_j)\right]^{(n+a_j)}}{\Gamma(n+a_j)} \alpha^{(n+a_j-1)} e^{-\alpha I(r_n,\theta_j)} d\alpha \\ &= \sum_{j=1}^k P_j \frac{\left[I(r_n,\theta_j)\right]^{(n+a_j)}}{\Gamma(n+a_j)} \frac{\Gamma(n+a_j-1)}{\left[I(r_n,\theta_j)\right]^{(n+a_j-1)}} \\ &= \sum_{j=1}^k P_j \frac{\left[I(r_n,\theta_j)\right]}{(n+a_j-1)}. \end{aligned}$$

Hence

$$\alpha^*_{\mathrm{BE}} = \frac{1}{\sum_{j=1}^k P_j \frac{\left[I(r_n,\theta_j)\right]}{\left(n+a_j-1\right)}}.$$

Similarly, the Bayes estimator for θ is

$$\theta^*_{\mathrm{BE}} = \left[\int_0^\infty \sum_{j=1}^k P_j \frac{1}{\theta_j} \pi\left(\alpha|\theta_j; \underline{r}\right) d\alpha\right]^{-1} = \left[\sum_{j=1}^k \frac{P_j}{\theta_j}\right]^{-1}.$$

The Bayes estimator of reliability function is given by

$$R^*(t)_{\mathrm{BE}} = \left[\sum_{j=1}^k P_j \int_0^\infty \left[1 - F\left(t;\alpha,\theta_j\right)\right] \pi\left(\alpha|\theta_j; \underline{r}\right) d\alpha\right]^{-1}.$$

Using the binomial expansion in the above expression, we have

$$\begin{aligned} R^{*}(t)_{\mathrm{BE}} &= \Big[\sum_{j=1}^{k} P_j \int_0^{\infty} \sum_{s=0}^{\infty} \left[F(t;\alpha,\theta_j)\right]^s \pi(\alpha|\theta_j;\underline{r})d\alpha\Big]^{-1} \\ &= \left[\sum_{j=1}^{k}\sum_{s=0}^{\infty} P_j \left(1 - \frac{s\ln\left[G(t/\theta_j)\left(2-G(t/\theta_j)\right)\right]}{I(r_n,\theta_j)}\right)^{-(n+a_j)}\right]^{-1}. \end{aligned}$$

4.3. LINEX Loss

The LINEX loss function is

$$L(\delta,\lambda) = e^{c(\delta-\lambda)} - c(\delta-\lambda) - 1 \quad c \neq 0,\ \delta \in \mathbb{D},\ \lambda \in \Theta,$$

where $c \neq 0$ is the parameter of loss function. The Bayes estimator under this loss function is

$$\delta^{*}_{\mathrm{BE}} = -\frac{1}{c}\ln\left(E\left(e^{-c\delta}\right)\right), \quad c \neq 0.$$

The Bayes estimator for α is

$$\begin{aligned} \hat{\alpha}_{\mathrm{BL}} &= -\frac{1}{c}\ln\Big[\sum_{j=1}^{k} P_j \int_0^{\infty} e^{-c\alpha} \frac{\left[I(r_n,\theta_j)\right]^{(n+a_j)}}{\Gamma(n+a_j)} \alpha^{n+a_j-1} e^{-\alpha I(r_n,\theta_j)} d\alpha\Big] \\ &= -\frac{1}{c}\ln\Big[\sum_{j=1}^{k} P_j \left(1 + \frac{c}{I(r_n,\theta_j)}\right)^{-(n+a_j)}\Big]. \end{aligned}$$

The Bayes estimator for θ is

$$\begin{aligned} \theta^{*}_{\mathrm{BL}} &= -\frac{1}{c}\ln\left(E\left(e^{-c\theta}\right)\right) \\ &= -\frac{1}{c}\ln\left[\sum_{j=1}^{k} P_j e^{-c\theta_j}\right]. \end{aligned}$$

The Bayes estimator of reliability function is given by

$$R(t)^{*}_{\mathrm{BL}} = -\frac{1}{c}\ln\left(E\left(e^{-cR(t)}\right)\right).$$

To solve this, we will use exponential series expansion

$$\begin{aligned} e^{-cR(t)} &= e^{-c\left(1-F(t;\alpha,\theta_j)\right)} \\ &= e^{-c}e^{cF(t;\alpha,\theta_j)} \\ &= e^{-c}\sum_{s=0}^{\infty} \frac{c^s\left[F(t;\alpha,\theta_j)\right]^s}{s!}. \end{aligned}$$

Hence

$$R(t)^{*}_{\mathrm{BL}} = -\frac{1}{c}\ln\left[e^{-c}\sum_{j=1}^{k}\sum_{s=0}^{\infty} \frac{c^s P_j}{s!}\left(1 - \frac{s\ln\left[G(t/\theta_j)(2-G(t/\theta_j))\right]}{I(r_n,\theta_j)}\right)^{-\left(n+a_j\right)}\right].$$

5. PREDICTION INTERVAL

In this section, we will predict the future lower record R_s while already having $R_1, R_2, \ldots, R_n$ for $n < s$. For this problem, we will use Bayesian procedure and Markovian property of record statistics. The conditional distribution of R_s given R_n is obtained by using Markovian property (see Arnold *et al.* [13]).

$$f_{R_s/R_n}(r_s/r_n;\alpha,\theta) = \frac{\left[H(r_s)-H(r_n)\right]^{s-n-1}}{\Gamma(s-n)} \frac{f(r_s;\alpha,\theta)}{F(r_n;\alpha,\theta)}, \quad -\infty < r_s < r_n < \infty,$$

where $H(\cdot) = -\ln F(\cdot)$. For TLG family of distribution, with pdf given by Eq. (1), the function $f_{R_s/R_n}(r_s/r_n;\alpha,\theta)$ is given by

$$f_{R_s/R_n}(r_s/r_n;\alpha,\theta) = \frac{2}{\theta}\frac{\alpha^{s-n}}{\Gamma(s-n)}\left[\ln\left(\frac{G(r_n/\theta)(2-G(r_n/\theta))}{G(r_s/\theta)(2-G(r_s/\theta))}\right)\right]^{s-n-1} \times \frac{g(r_s/\theta)(1-G(r_s/\theta))}{G(r_s/0a)(2-G(r_s/\theta))}\left[\frac{G(r_s/\theta)(2-G(r_s/\theta))}{G(r_n/\theta)(2-G(r_n/\theta))}\right]^{\alpha}, -\infty < r_s < r_n < \infty.$$

The Bayes predictive density function of R_s given $R_n = r_n$ is given by

$$f(r_s/r_n) = \int_{\alpha} f_{R_s/R_n}(r_s/r_n;\alpha,\theta)\sum_{j=1}^{k} P_j \pi\left(\alpha|\theta_j;\underline{r}\right) d\alpha. \tag{16}$$

Using Eq. (12) in Eq. (16), we get

$$\begin{aligned} f(r_s/r_n) &= \sum_{j=1}^{k} P_j \int_0^{\infty} \frac{2}{\theta_j}\frac{\alpha^{s-n}}{\Gamma(s-n)}\left[\ln\left(\frac{G(r_n/\theta_j)\left(2-G(r_n/\theta_j)\right)}{G(r_s/\theta_j)\left(2-G(r_s/\theta_j)\right)}\right)\right]^{s-n-1} \\ &\quad \times \frac{g\left(r_s/\theta_j\right)\left(1-G\left(r_s/\theta_j\right)\right)}{G\left(r_s/\theta_j\right)\left(2-G\left(r_s/\theta_j\right)\right)}\left[\frac{G\left(r_s/\theta_j\right)\left(2-G\left(r_s/\theta_j\right)\right)}{G\left(r_n/\theta_j\right)\left(2-G\left(r_n/\theta_j\right)\right)}\right]^{\alpha} \\ &\quad \times \frac{\left[I\left(r_n,\theta_j\right)\right]^{\left(n+a_j\right)}}{\Gamma\left(n+a_j\right)}\alpha^{\left(n+a_j-1\right)}e^{-\alpha I\left(r_n,\theta_j\right)}dr_s \\ &= \sum_{j=1}^{k}\frac{2P_j}{\theta_j B(n+a_j,s-n)}\left[\ln\left(\frac{G(r_n/\theta_j)\left(2-G(r_n/\theta_j)\right)}{G(r_s/\theta_j)\left(2-G(r_s/\theta_j)\right)}\right)\right]^{s-n-1} \\ &\quad \times \frac{g\left(r_s/\theta_j\right)\left(1-G\left(r_s/\theta_j\right)\right)}{G\left(r_s/\theta_j\right)\left(2-G\left(r_s/\theta_j\right)\right)}\frac{\left[I\left(r_n,\theta_j\right)\right]^{n+a_j}}{\left[I\left(r_s,\theta_j\right)\right]^{s+a_j}}, \quad -\infty < r_s < r_n < \infty, \end{aligned}$$

where $B(a,b)$ is the complete beta function. Now we will find the lower and upper $100(1-\alpha)\%$ prediction bounds for R_s. First, we will find the predictive survival function $P\left(R_s \geq d|r_n\right)$ for some positive constant d

$$\begin{aligned} P(R_s \geq d|r_n) &= \int_d^{r_n} f(r_s/r_n)dr_s \\ &= \sum_{j=1}^{k} P_j\left(1-\frac{\mathrm{IB}(n+a_j,s-n,\chi)}{\mathrm{B}(n+a_j,s-n)}\right), \end{aligned}$$

where $\chi = \dfrac{I\left(r_n,\theta_j\right)}{I\left(d,\theta_j\right)}$, and $\mathrm{IB}\left(n+a_j,s-n,\chi\right)$ is the incomplete beta function defined by

$$\mathrm{IB}(a,b,\chi) = \int_0^{\chi} u^{a-1}(1-u)^{b-1}\,du.$$

Let $L(r_n)$ and $U(r_n)$ be two constants such that

$$P[R_s > L(r_n)|r_n] = 1-\frac{\alpha}{2} \quad \text{and} \quad P[R_s > U(r_n)|r_n] = \frac{\alpha}{2}. \tag{17}$$

Using Eq. (17), we obtain two-sided $100(1-\alpha)\%$ predictive bounds for R_s as $(L(r_n), U(r_n))$, that is,

$$P[L(r_n) < R_s < U(r_n)] = 1-\alpha.$$

We are considering here a special case when $s = n+1$, which is of our interest practically because after getting n records we want the next record $n+1$. The predictive survival function of R_{n+1} is given as

$$P\left(R_{n+1} \geq d|r_n\right) = \sum_{j=1}^{k} P_j\left[1-\left(\frac{I\left(r_n,\theta_j\right)}{I\left(d,\theta_j\right)}\right)^{n+a_j}\right].$$

Here we are assuming the case when the scale parameter is known (WLOG, $\theta = 1$). For this case, predictive survival function can be written as

$$P\left(R_{n+1} \geq d|r_n\right) = 1 - \left(\frac{b - \ln\left(G\left(r_n\right)\right)\left(2 - G\left(r_n\right)\right)}{b - \ln\left(G\left(d\right)\right)\left(2 - G\left(d\right)\right)}\right)^{n+a}. \tag{18}$$

From Eqs. (17) and (18) we have lower and upper limits as

$$\begin{aligned} L(r_n) &= G^{-1}\left\{1 \pm \sqrt{1 - \exp\left[b - \left(b - \ln\left[G(r_n)\left(2 - G(r_n)\right)\right]\left(\frac{\alpha}{2}\right)^{-1/(n+a)}\right)\right]}\right\} \\ U(r_n) &= G^{-1}\left\{1 \pm \sqrt{1 - \exp\left[b - \left(b - \ln\left[G(r_n)\left(2 - G(r_n)\right)\right]\left(1 - \frac{\alpha}{2}\right)^{-1/(n+a)}\right)\right]}\right\} \end{aligned}$$

6. NUMERICAL COMPUTATIONS

In this section, a simulation study is conducted to illustrate all the estimation and prediction methods described in the preceding sections. We consider exponential distribution with df

$$G\left(x;\theta\right) = 1 - e^{-x/\theta}, \quad x > 0, \theta > 0,$$

as a special case for the baseline df in the model (1), named TL-Exp distribution.

Example 6.1.

We generate lower records of size $n = 9$ from TL-Exp distribution for $\alpha = 5$ and $\theta = 3$. The lower record values are

$$7.0752, 4.6823, 4.0686, 3.9577, 3.3374, 1.6600, 1.5436, 1.1236, 0.6410.$$

The MLE for α and θ are 4.014078 and 2.885522, respectively, obtained by solving nonlinear Eqs. (6) and (7), in R software by Newton–Raphson method. Using these estimates we get the MLE of reliability function at $t = 0.5$ as $\hat{R}\left(0.5\right) = 0.9928$ and $t = 1$ as $\hat{R}\left(1\right) = 0.9381$. Here we assume that scale parameter θ, takes finite values as 2.0 (0.1) 2.9, with equal probability 0.1 for each $\theta_j, j = 1, 2, \ldots, 10$.

For obtaining the Bayes estimators for different parameters, first it is necessary to obtain the hyper-parameters $\left(a_j, b_j\right)$ for each $\theta_j, j = 1, 2, \ldots, 10$. The hyper parameters $\left(a_j, b_j\right)$ can be obtained based on the expected value of the reliability function $R\left(t\right)$ conditional on $\theta = \theta_j$, using

$$\begin{aligned} E_{\alpha|\theta_j}\left[R(t)|\theta = \theta_j\right] &= \int_0^\infty \left(1 - (1 - e^{-2t/\theta_j})^\alpha\right)\frac{b_j^{a_j}\alpha^{a_j - 1}e^{-\alpha b_j}}{\Gamma(a_j)}d\alpha \\ &= 1 - \left(\frac{1 - \ln\ (1 - e^{-2t/\theta_j})}{b_j}\right)^{-a_j}. \end{aligned} \tag{19}$$

For the two values of $\left(R\left(t_1\right), t_1\right)$ and $\left(R\left(t_2\right), t_2\right)$, the values of a_j and b_j for each value of θ can be obtained numerically from Eq. (19). A nonparametric approach $\tilde{R}\left(t_i = R_i\right) = \left(n - i + 0.625\right)/\left(n + 0.25\right), \ i = 1, 2, 3 \ldots, n$ can be used to estimate any two different values of the reliability function $R\left(t_1\right)$ and $R\left(t_2\right)$ (see Martz and Waller [29]). In this case, we use $\tilde{R}\left(5.9577\right) = 0.6081081$ and $\tilde{R}\left(1.1236\right) = 0.1756757$. These two values are substituted into Eq. (19), where a_j and b_j are solved numerically for each $\theta_j, j = 1, 2, \ldots, 10$, using the Newton–Raphson method. After that, posterior probabilities are calculated for each θ_j, and presented in Table 1. The MLEs, Bayes estimators, and reliability function (for different $t = 0.5, 1, 1.5$) are also calculated and presented in Tables 2 and 3.

Table 1 Prior information and posterior probabilities.

j	1	2	3	4	5
θ	2	2.1	2.2	2.3	2.4
p	0.1	0.1	0.1	0.1	0.1
a	5.273678	5.069663	4.893300	4.739474	4.604236
b	1.0022315	0.9941453	0.9855453	0.9765224	0.9671572
u	$1.347749e^{-14}$	$3.602815e^{-14}$	$8.744126e^{-14}$	$1.952425e^{-13}$	$4.054791e^{-13}$
P	0.07127739	0.08399823	0.09474197	0.10297468	0.10845833
j	6	7	8	9	10
θ	2.5	2.6	2.7	2.8	2.9
p	0.1	0.1	0.1	0.1	0.1
a	4.484489	4.377778	4.282129	4.195944	4.117911
b	0.9575209	0.9476757	0.9376761	0.9275691	0.9173958
u	$7.904341e^{-13}$	$1.457474e^{-12}$	$2.558572e^{-12}$	$4.299937e^{-12}$	$6.951137e^{-12}$
P	0.11121071	0.11144265	0.10949040	0.10575451	0.10065113

Table 2 Estimates of α and θ.

	$^{(*)}ML$	$^{(*)}BS$	$^{(*)}BE$	$^{(*)}BL$				
				c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
α	4.014078	5.307494	4.900998	5.937348	4.831392	4.454377	4.145928	3.887385
θ	2.885522	2.475928	2.444745	2.494692	2.456996	2.438097	2.419431	2.401179

ML, maximum likelihood estimator; BS, Bayes estimator under squared error loss; BE, Bayes estimator under entropy loss; BL, Bayes estimator under LINEX loss.

Table 3 Estimates of reliability for different t.

t	$^{(*)}UMVUE$	$^{(*)}ML$	$^{(*)}BS$	$^{(*)}BE$	$^{(*)}BL$				
					c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
0.5	0.9902	0.9928	0.9836	0.9830	0.9838	0.9835	0.9834	0.9832	0.9831
1	0.6737	0.9381	0.9064	0.9007	0.9075	0.9052	0.9041	0.9028	0.9016
1.5	0.3092	0.8265	0.7699	0.7543	0.7725	0.7672	0.7645	0.7617	0.7589

UMVUE, uniformly minimum-variance unbiased estimator; ML, maximum likelihood estimator; BS, Bayes estimator under squared error loss; BE, Bayes estimator under entropy loss; BL, Bayes estimator under LINEX loss.

Using the prediction procedure described in Section 5, the 95% prediction interval for the next lower record R_8 is $(0.10889, 0.2756101)$.

The mean squared error (MSEs) and risks of estimators and reliability function are compared according to following steps:

1. Samples of lower records with different size of $n \in \{6, 7, 8\}$ are generated from the TL-Exp distribution for $\theta = 3$ and different $\alpha \in \{1.5, 1.8, 2\}$.
2. The values of a_j and b_j for a given value of $\theta_j, j = 1, 2, 3, \ldots, 10$ are obtained using the procedure discussed.
3. Estimates of α, θ and $R(t)$ are obtained.
4. Above steps are repeated 10,000 times to evaluate the MSEs of these estimates and also estimated risks are compared under different loss functions using

$$\mathrm{ER}(\delta) = \frac{1}{m}\sum_{i=1}^{m} L(\delta_i, \lambda).$$

All these the result are presented in Tables 4 to 8.

From Tables 4 through 6, we observe that Bayes estimates for asymmetric loss functions are performing better than Bayes estimates for symmetric loss function and MLEs. UMVUE of reliability function is better than Bayes estimates of reliability and MLE. From Tables 7 and 8, comparison of risk for Bayes estimates of α, θ and reliability function can be seen, and it is clear that, estimators for asymmetric loss function are again performing better than estimators for symmetric loss function.

Table 4 MSE of the MLEs and UMVUE for $(\theta, t) = (3, 0.5)$.

n	α	MSE $(\hat{\alpha})$	MSE $\left(\hat{\theta}\right)$	MSE $\left(\hat{R(t)}\right)$	MSE $(\mathfrak{R}(t))$
	1.5	0.6073	7.4651	0.6497	0.0199
6	1.8	0.5958	7.4469	0.6484	0.0199
	2	2.1409	7.7985	0.7996	0.005
	1.5	0.4598	7.3734	0.637	0.0199
7	1.8	1.1648	7.5737	0.7389	0.0088
	2	1.8346	7.6939	0.7917	0.005
	1.5	0.3375	7.2909	0.6245	0.0199
8	1.8	1.0045	7.5234	0.7311	0.0088
	2	1.6336	7.6652	0.7856	0.005

MLE, maximum likelihood estimation; UMVUE, uniformly minimum-variance unbiased estimator; MSE, mean squared error.

Table 5 MSEs of the Bayes estimates of α and θ.

			$(\alpha, \theta) = (1.5, 3)$				
n	MSE$(\alpha)_{BS}$	MSE$(\alpha)_{BE}$	MSE$(\alpha)_{BL}$				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.1873	0.1981	0.1845	0.1901	0.193	0.1958	0.1986
7	0.1872	0.198	0.1844	0.1901	0.1929	0.1957	0.1985
8	0.1872	0.1979	0.1843	0.19	0.1928	0.1956	0.1985
n	MSE$(\theta)_{BS}$	MSE$(\theta)_{BE}$	MSE$(\theta)_{BL}$				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.5442	0.5825	0.5205	0.5666	0.5878	0.6077	0.6263
7	0.5733	0.6098	0.5507	0.5947	0.6147	0.6334	0.6508
8	0.6007	0.6354	0.5791	0.6209	0.6398	0.6573	0.6736
			$(\alpha, \theta) = (1.8, 3)$				
n	MSE$(\alpha)_{BS}$	MSE$(\alpha)_{BE}$	MSE$(\alpha)_{BL}$				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.537	0.5551	0.5322	0.5417	0.5465	0.5513	0.556
7	0.5368	0.5549	0.532	0.5416	0.5464	0.5511	0.5559
8	0.5371	0.5547	0.5323	0.5413	0.5462	0.5509	0.5557
n	MSE$(\theta)_{BS}$	MSE$(\theta)_{BE}$	MSE$(\theta)_{BL}$				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.5516	0.5897	0.5281	0.5739	0.595	0.6146	0.633
7	0.5811	0.6173	0.5586	0.6023	0.6221	0.6405	0.6577
8	0.6098	0.6434	0.5886	0.6293	0.6479	0.6652	0.6811
			$(\alpha, \theta) = (2, 3)$				
n	MSE$(\alpha)_{BS}$	MSE$(\alpha)_{BE}$	MSE$(\alpha)_{BL}$				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.8701	0.8931	0.8639	0.8761	0.8822	0.8882	0.8943
7	0.8709	0.8928	0.864	0.8758	0.8819	0.888	0.894
8	0.8726	0.8918	0.8675	0.8746	0.8809	0.8872	0.8933
n	MSE$(\theta)_{BS}$	MSE$(\theta)_{BE}$	MSE$(\theta)_{BL}$				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.554	0.5919	0.5305	0.5762	0.5972	0.6168	0.635
7	0.5877	0.6209	0.5629	0.606	0.6257	0.644	0.661
8	0.6218	0.6482	0.5975	0.6332	0.6518	0.669	0.6849

MSE, mean squared error.

Table 6 MSEs of the estimates of $R(t)$.

n	$\text{MSE}(R(t))_{BS}$	$\text{MSE}(R(t))_{BE}$	$\text{MSE}(R(t))_{BL}$				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
$(\alpha, \theta) = (\mathbf{1.5, 3, 0.5})$							
6	0.0352	1.1539	0.035	0.0354	0.0357	0.0359	0.0361
7	0.036	1.1162	0.0358	0.0363	0.0365	0.0367	0.0369
8	0.0367	1.0875	0.0365	0.0369	0.0371	0.0374	0.0376
$(\alpha, \theta, t) = (1.8, 3, 0.5)$							
6	0.0554	1.0537	0.0551	0.0556	0.0559	0.0562	0.0565
7	0.0566	1.0107	0.0563	0.0568	0.0571	0.0574	0.0577
8	0.0576	0.9769	0.0573	0.0578	0.0581	0.0584	0.0586
$(\alpha, \theta, t) = (2, 3, 0.5)$							
6	0.0669	1.0006	0.0666	0.0672	0.0675	0.0678	0.0681
7	0.0682	0.9591	0.0679	0.0685	0.0688	0.0691	0.0694
8	0.07	0.9341	0.0723	0.0709	0.07	0.0701	0.0704
$(\alpha, \theta, t) = (1.5, 3, 1)$							
6	0.0532	0.0059	0.0529	0.0535	0.0538	0.0541	0.0544
7	0.0545	0.0048	0.0542	0.0548	0.0551	0.0554	0.0557
8	0.0559	0.0038	0.0556	0.0562	0.0565	0.0567	0.057
$(\alpha, \theta, t) = (1.8, 3, 1)$							
6	0.0883	0.0003	0.0879	0.0887	0.0891	0.0895	0.0898
7	0.0902	0.0002	0.0898	0.0906	0.0909	0.0913	0.0917
8	0.0919	0.0003	0.0918	0.0923	0.0926	0.093	0.0933
$(\alpha, \theta, t) = (2, 3, 1)$							
6	0.1116	0.0011	0.1112	0.1121	0.1125	0.113	0.1134
7	0.1139	0.0019	0.1141	0.1146	0.1147	0.1151	0.1155
8	0.1161	0.0025	0.1168	0.1163	0.1165	0.1169	0.1173

MSE, mean squared error.

Table 7 Estimated risks for Bayes estimates of α and θ.

n	$\text{ER}(\alpha_{BS})$	$\text{ER}(\alpha_{BE})$	$\text{ER}(\alpha_{BL})$				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
$(\alpha, \theta) = (\mathbf{1.5, 3})$							
6	0.1873	0.0553	0.0248	0.0221	0.0838	0.1787	0.3015
7	0.1872	0.0552	0.0248	0.0221	0.0838	0.1786	0.3013
8	0.1872	0.0552	0.0248	0.0221	0.0837	0.1785	0.3012
n	$\text{ER}(\theta_{BS})$	$\text{ER}(\theta_{BE})$	$\text{ER}(\theta_{BL})$				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.5438	0.8625	0.0737	0.0627	0.2313	0.4799	0.7881
7	0.5738	0.8625	0.0783	0.0657	0.2406	0.4969	0.8126
8	0.6006	0.8625	0.0825	0.0683	0.2491	0.5121	0.8345
$(\alpha, \theta) = (1.8, 3)$							
n	$\text{ER}(\alpha_{BS})$	$\text{ER}(\alpha_{BE})$	$\text{ER}(\alpha_{BL})$				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.537	0.1204	0.0754	0.0601	0.2167	0.4421	0.7164
7	0.5368	0.1203	0.0754	0.0601	0.2167	0.442	0.7162
8	0.537	0.1203	0.0754	0.0601	0.2166	0.4418	0.7161

(*continued*)

Table 7 Estimated risks for Bayes estimates of α and θ. *(Continued)*

			$(\alpha, \theta) = (1.5, 3)$				
n	ER (θ_{BS})	ER (θ_{BE})	ER (θ_{BL})				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.5508	0.8625	0.0747	0.0634	0.2334	0.4837	0.7937
7	0.5812	0.8625	0.0795	0.0664	0.2431	0.5014	0.8191
8	0.6113	0.8625	0.0841	0.0693	0.2522	0.5177	0.8427
			$(\alpha, \theta) = (2, 3)$				
n	ER (α_{BS})	ER (α_{BE})	ER (α_{BL})				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.8701	0.1671	0.1269	0.0943	0.3302	0.6569	1.0422
7	0.87	0.167	0.1269	0.0942	0.3301	0.6568	1.0419
8	0.873	0.1669	0.1286	0.0942	0.3296	0.656	1.041
n	ER (θ_{BS})	ER (θ_{BE})	ER (θ_{BL})				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.554	0.8625	0.0752	0.0637	0.2344	0.4857	0.7966
7	0.5854	0.8625	0.0801	0.0668	0.2443	0.5035	0.8223
8	0.6224	0.8625	0.0866	0.0697	0.253	0.5194	0.8451

ER, estimated risk.

Table 8 Estimated risk for Bayes estimates of $R(t)$.

			$(\alpha, \theta, t) = (1.5, 3, 0.5)$				
n	ER$(R\,(t)_{BS})$	ER$(R\,(t)_{BE})$	ER$(R\,(t)_{BL})$				
			c = −0.5	c = 0.5	c = 1	c = 1.5	c = 2
6	0.0351	0.4487	0.0045	0.0043	0.0172	0.0381	0.0662
7	0.036	0.4364	0.0046	0.0044	0.0176	0.0389	0.0675
8	0.0367	0.4263	0.0047	0.0045	0.0179	0.0395	0.0687
			$(\alpha, \theta, t) = (1.8, 3, 0.5)$				
6	0.0554	0.3808	0.0072	0.0067	0.0264	0.0576	0.0993
7	0.0565	0.3692	0.0073	0.0068	0.0269	0.0587	0.101
8	0.0576	0.3592	0.0076	0.007	0.0279	0.0614	0.1074
			$(\alpha, \theta, t) = (2, 3, 0.5)$				
6	0.0554	0.3808	0.0072	0.0067	0.0264	0.0576	0.0993
7	0.0565	0.3692	0.0073	0.0068	0.0269	0.0587	0.101
8	0.0576	0.3592	0.0076	0.007	0.0279	0.0614	0.1074
			$(\alpha, \theta, t) = (1.5, 3, 1)$				
6	0.0531	0.0063	0.0069	0.0064	0.0254	0.0557	0.0961
7	0.0545	0.0051	0.007	0.0066	0.026	0.057	0.0982
8	0.0558	0.0041	0.0072	0.0067	0.0266	0.0581	0.1001
			$(\alpha, \theta, t) = (1.8, 3, 1)$				
6	0.0883	3e-04	0.0115	0.0106	0.0409	0.0884	0.1508
7	0.0902	2e-04	0.0118	0.0108	0.0417	0.0901	0.1535
8	0.092	3e-04	0.012	0.011	0.0424	0.0915	0.1559
			$(\alpha, \theta, t) = (2, 3, 1)$				
6	0.1117	0.001	0.0147	0.0133	0.051	0.1094	0.1855
7	0.1138	0.0015	0.015	0.0135	0.0518	0.1112	0.1884
8	0.1165	0.0023	0.0158	0.0138	0.0541	0.1164	0.1985

ER, estimated risk.

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