Further Complete Solutions to Four Open Problems on Filter of Logical Algebras

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ABSTRACT
This paper focuses on the investigation of filters of pseudo BCK-algebra and BL-algebra, important and popular generic commutative and non-commutative logical algebras. By characterizing Boolean filter and implicative filter in pseudo BCK-algebra, the essential equivalent relation between these two filters is revealed. An open problem that “In pseudo BCK-algebra or bounded pseudo BCK-algebra, is the notion of implicative pseudo-filter equivalent to the notion of Boolean filter?” is solved. Based on this, this paper explores the essential relations between the implicative (Boolean) filter and implicative pseudo BCK-algebra. A complete solution to an open problem that “Prove or negate that pseudo BCK-algebras is implicative BCK-algebras if and only if every filter of them is implicative filters (or Boolean filter)” is derived. This paper further characterizes the fantastic filter and normal filter in BL-algebra, then gets the equivalent relation between the two filters, and completely solves two open problems regarding the relationship between these two filters: 1. Under what suitable condition a normal filter becomes a fantastic filter? and 2. (Extension property for a normal filter) Under what suitable condition extension property for normal filter holds?

1. INTRODUCTION
The rapid development of computer science, technology, and mathematical logic put forward many new requirements, thus contributing to the nonclassical logic and the rapid development of modern logic [1]. The study of fuzzy logic has become a hot topic with scientific information and artificial intelligence, which makes fuzzy logic study of algebra and logic inseparable. Fuzzy logic is both the scientific information and artificial intelligence, which makes fuzzy logic both the mathematical basis of the artificial intelligence and fuzzy reasoning. Based on the actual background, different forms of fuzzy logic system are proposed.

Logical algebras are the algebraic counterparts of the nonclassical logic and the algebraic foundation of reasoning mechanism in information sciences, computer sciences, theory of control, artificial intelligence, and other important fields. For example, BCK-algebra, BL-algebra, pseudo MTL-algebras, and noncommutative residuated lattice are algebraic counterparts of BCK Logic, Basic Logic, monoidal t-norm-based logic, and monoid logic, respectively [1–3].

Hájek introduced BL-algebra as algebraic structure for his Basic Logic [3, 5]. Di Nola generalized BL-algebra in a noncommutative form and introduced the notion of pseudo BL-algebra as a common extension of BL-algebra in order to express the noncommutative reasoning [4, 6].

In 1966, BCK-algebra was introduced by Iséki and Imai from BCK/BCI Logic [7–9]. Iorgulescu established the connections between BCK-algebra and BL-algebra in [10]. Afterward, Georgescu and Iorgulescu introduced the notion of pseudo BCK-algebra as an extension of BCK-algebra to express the noncommutative reasoning [11, 12]. Iorgulescu established the connections between pseudo BL-algebra and pseudo BCK-algebra [12]. In [13], Wang and Zhang presented the necessary and sufficient conditions for residuated lattice and bounded pseudo BCK-algebra to be Boolean algebra.

Filter theory plays a vital role not only in studying of algebraic structure, but also in nonclassical logic and computer science [14, 15]. From logical point of view, various filters correspond to various sets of provable formulae [16, 17]. For example, based on filter and prime filter in BL-algebra, Hájek proved the completeness of Basic Logic [3]. In [18], Turunen proposed the notions of implicative filter and Boolean filter and proved that implicative filter is equivalent to Boolean filter in BL-algebra. In [19], some types of filters in BL-algebra were proposed. In [20–22], filters of pseudo MV-algebra, pseudo BL-algebra, pseudo effect algebra, and pseudo hoops were further studied. Literatures [5, 18, 19, 23–28] further studied filters of BL-algebra, lattice implication algebra, pseudo BL-algebra,
The role of filters are important not only in pseudo BCK-algebra, but also in related domains. In [26], there are two open problems: 1. Under what suitable condition a normal filter becomes a fantastic filter?" and 2. "(Extension property for a normal filter) Under what suitable condition extension property for normal filter holds?"

In our previous work, we have characterized the fuzzy fantastic filter and normal filter of BL-algebra, and discussed the relation between them, then partly solved the two open problems [30]. But so far we have not got sufficient and necessary condition for a normal filter to be fantastic. Based on this, we further obtained the relation between them and completely solved the two open problems.

This paper is organized as follows: In Section 2, we present some basic definitions and results in BL-algebra and pseudo BCK-algebra. In Section 3, we focus on the relation between implicative filter and Boolean filter of pseudo BCK-algebra or bounded pseudo BCK-algebra and give a complete solution to an open problem. In Section 4, based on the result we obtained in Section 3, we investigate the relation between implicative filter (Boolean filter) and implicative pseudo BCK-algebra and give a complete solution to another open problem. In Section 5, we recall the concept of filter and the corresponding properties of filter in BL-algebra and we propose complete solutions to two open problems of filter in BL-algebra.

2. PRELIMINARIES

Here we recall some definitions and results which will be needed. Reader can refer to [9, 11, 12, 19, 34–38].

**Definition 1.** (Birkhoff [32]) Suppose L is a nonempty set with two binary operations ∧ and ∨. L is called a lattice if for x, y, z ∈ L, the following conditions hold

1. \( x \land x = x, x \lor x = x \),
2. \( x \land y = y \land x, x \lor y = y \lor x \),
3. \( (x \land y) \land z = x \land (y \land z), (x \lor y) \lor z = x \lor (y \lor z) \),
4. \( (x \land y) \lor x = x, (x \lor y) \land x = x \).

**Definition 2.** (Balbes and Dwyer [31]) A lattice L is called a distributive lattice if for x, y, z ∈ L, the following conditions hold

1. \( x \land (y \lor z) = (x \land y) \lor (x \land z) \),
2. \( x \lor (y \land z) = (x \lor y) \land (x \lor z) \).

In lattices, (1) and (2) are equivalent.

Suppose L be a lattice. A binary relation ≤ is defined as for x, y ∈ L, \( x \leq y \) if \( x \land y = x \) or \( x \lor y = y \). Then we can find that binary relation ≤ is a partially ordered relation.

**Definition 3.** (Meng and Jun [9]) An algebraic structure \( (A, \land, 1) \) is called an BCK-algebra if for all \( x, y, z \in A \)

1. \( (z \land x) \land y = (y \land x) \leq (y \land z) \),
2. \( (y \land x) \land x \leq y \),
3. \( x \leq x \),
4. \( x \leq y \) and \( y \leq z \) imply \( x = z \),
5. \( x \land 1 = 1 \).

where \( x \leq y \) means \( x = y \).

**Definition 4.** (Georgescu and Iorgulescu [11]) A (reversed left-) pseudo BCK-algebra is a structure \( (A, \land, \lor, 0) \), where \( \land \) is a binary relation on A, \( \land \) and \( \lor \) are binary operations on A and 1 is an element of A, verifying for all \( x, y, z \in A \), the axioms

1. \( (z \land x) \land y = (y \land x) \leq (y \land z) \),
2. \( (y \land x) \land x \leq y \),
3. \( x \leq x \),
4. \( 1 \leq x \),
5. \( x \leq y \) and \( y \leq x \) imply \( x = y \).
6. \( x \leq y \) if \( y \land x = 1 \) if \( y \land x = 1 \).

**Example 1.** (Jun, Kim and Neggers [38]) Let \( X = [0, \infty] \) and let ≤ be the usual order on X. Define \( \land \) and \( \lor \) on X as follows:

\( x \land y = 0 \) if \( y \leq x \) or \( x \land y = \frac{2y}{\pi} \arctan \left( \frac{y}{\pi} \right) \) (if \( x < y \)) and

\( x \lor y = 0 \) if \( y \leq x \) or \( x \lor y = ye^{-\frac{\pi x}{2y}} \) (if \( x < y \)),

for all \( x, y \in X \). Then \( (X, \leq, \land, \lor, 0) \) is a pseudo BCK-algebra.

**Definition 5.** (Iorgulescu [12]) A pseudo BCK-algebra \( (A, \land, \lor, 0) \) is called bounded if there exits unique element 0 such that \( 0 \land x = x \lor 0 = x \) for any \( x \in A \).

In a pseudo-BCK-algebra A we can define \( x' = x \land 0, x'' = x \lor 0 \) for any \( x \in A \).

**Proposition 1.** (Iorgulescu [35]) Let \( (A, \lor, \land, 0) \) be a pseudo BCK-algebra. Then the following properties hold for any \( x, y, z \in A \):

1. \( x \leq y \) implies \( y \leq z \leq x \) and \( y \leq x \leq z \),
2. \( x \leq y \) implies \( x \land z \leq z \leq x \land z \).

3. \( x \land y \leq z \) implies \( x \land z \leq z \leq x \land z \),
4. \( x \land y \leq z \) implies \( z \leq x \land z \).

5. \( x \land y \leq z \) implies \( x \land z \leq z \leq x \land z \),
6. \( x \land y \leq z \) implies \( z \leq x \land z \).
Definition 7. “1” instead of towards the operations $(\lor, \land, \oslash)$.

In the sequel, we shall agree that the operations $(\lor, \land, \oslash, \rightarrow, \hookrightarrow, 1)$ have priority over the operations $\lor, \land, \oslash, \land, \hookrightarrow, 1$.

**Theorem 2.** (Ciungu [34]) Let $(A, \geq, \rightarrow, \hookrightarrow, 1)$ be a pseudo BCK-algebra with condition (pP), $x \lor y$ is defined as $\min \{z \mid x \rightarrow y \rightarrow z\}$, then the following hold in $A$.

1. $(x \lor y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
2. $(y \lor x) \hookrightarrow z = y \hookrightarrow (x \hookrightarrow z)$,
3. $(x \rightarrow y) \lor x \leq x, y, x \lor (x \hookrightarrow y) \leq x, y$.

Proposition 4. (Zhang [37]) In a BCK-algebra $A$, the following properties hold for all $x, y, z \in A$.

1. $y \rightarrow (x \rightarrow z) = x \rightarrow (y \rightarrow z)$,
2. $1 \rightarrow x = x$,
3. $x \leq y$ if $x \rightarrow y = 1$,
4. $x \lor y = ((x \rightarrow y) \rightarrow y) \land ((y \rightarrow x) \rightarrow x)$,
5. $x \leq y \Rightarrow y \leq z \leq x$,
6. $x \leq y \Rightarrow z \leq x \leq y$,
7. $z \leq (x \rightarrow y) \rightarrow (z \rightarrow y)$,
8. $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
9. $x \leq (x \rightarrow y) \rightarrow y$,
10. $x \rightarrow (x \rightarrow y) = x \land y$.

We shall agree that the operations $\lor, \land, \oslash, \land, \hookrightarrow, 1$ have priority over the operations $\rightarrow$.

**3. THE RELATION BETWEEN IMPLICATIVE FILTER AND BOOLEAN FILTER OF PSEUDO BCK-ALGEBRA OR BOUNDED PSEUDO BCK-ALGEBRA**

In this section, we recall the definitions of filter, positive implicative pseudo filter, Boolean filter, normal filter, and implicative filter of pseudo BCK-algebra.

**Definition 12.** (Zhang [29]) A nonempty subset $F$ of pseudo BCK-algebra $A$ is called a pseudo BCK-algebra $A$ is a filter of $A$ if and only if it satisfies

1. $F \subset F$, $y \in F$, $x \leq y \Rightarrow y \in F$,
2. $x \in F$, $x \rightarrow y \in F$ or $x \hookrightarrow y \in F \Rightarrow y \in F$.

**Theorem 5.** (Wang [1]) A nonempty subset $F$ of a pseudo BCK-algebra $A$ is a filter of $A$ if and only if $F$ satisfies

1. $F \subset F$,
2. $x \in F$, $x \rightarrow y \in F$ or $x \hookrightarrow y \in F \Rightarrow y \in F$.

In example 1, we can find $\{0\}$ is a filter.

**Theorem 6.** (Wang [1]) A nonempty subset $F$ of a pseudo BCK-algebra $A$ with condition (pP) is a filter of $A$ if and only if it satisfies
(F5) $x \in F, y \in F \Rightarrow x \odot y \in F$,

(F6) $x \in F, y \in A, x \leq y \Rightarrow y \in F$.

**Definition 13.** (Zhang [29]) Let $F$ be a filter of $A$, for all $x, y \in A$, $F$ is called (an)

1. **Boolean filter** if $(x \rightarrow y) \Rightarrow x \in F$ and $(x \Leftarrow y) \rightarrow x \in F$, then $x \in F$,
2. **Prime filter** if $x \vee y \in F$ implies $x \in F$ or $y \in F$,
3. **Maximal filter** if $x \in F$ or $x' \in F$ and $(x' \rightarrow x) \in F$,
4. **Loyal filter** if $x \not\in F$ and $y \not\in F$ implies $x \rightarrow y \in F$ and $x \not\Rightarrow y \in F$,
5. **Normal filter** if $x \rightarrow y \in F$ and $(x \rightarrow y) \Rightarrow x \in F$,
6. **Implicational filter** if $(x \rightarrow y) \rightarrow x \in F$ and $(x \Leftarrow y) \Rightarrow x \in F$, then $x \in F$.

**Definition 14.** (Zhang and Jun [13]) A nonempty subset $F$ of a pseudo $BCK$-algebra $A$ is called a positive implicational filter of $A$ if it satisfies (F1) and for all $x, y \in A$.

(F7) $x \Rightarrow (y \rightarrow z) \in F, x \Rightarrow y \in F$ implies $x \Rightarrow z \in F$,

(F8) $x \Rightarrow (y \rightarrow z) \in F, x \Rightarrow y \in F$ implies $x \rightarrow z \in F$.

Note that any filter of a $BCK$-algebra is normal.

**Theorem 7.** (Zhang [29]) Let $(A; \leq, \rightarrow, \Rightarrow, 0, 1)$ be a bounded pseudo $BCK$-algebra and $F$ be a positive implicational filter of $A$. Then

1. $\forall x \in A, (x \rightarrow 0) \Rightarrow x \in F$, that is, $(x \rightarrow x) \Rightarrow x \in F$,
2. $\forall x \in A, ((x \rightarrow x) \Rightarrow x) \rightarrow x \in F$, that is, $(x \rightarrow x) \rightarrow x \in F$,
3. $\forall x, y \in A, ((x \rightarrow y) \Rightarrow x) \Rightarrow x \in F$,
4. $\forall x, y \in A, ((x \rightarrow y) \Rightarrow x) \rightarrow x \in F$,
5. $\forall x, y \in A, ((x \rightarrow y) \rightarrow y \Rightarrow x \in F$, then $(y \rightarrow x) \Rightarrow x \in F$,
6. $\forall x, y \in A, if (x \Leftarrow y) \Rightarrow y \in F$, then $(y \Leftarrow x) \Rightarrow x \in F$,
7. $\forall x, y \in A, if x \not\Rightarrow y \in F$, then $(y \rightarrow x) \Rightarrow y \in F$,
8. $\forall x, y \in A, if x \not\Rightarrow y \in F$, then $(y \rightarrow x) \Rightarrow y \in F$.

In [29], there is an open problem: "In pseudo $BCK$-algebra or bounded pseudo $BCK$-algebra, is the notion of implicational pseudo filter equivalent to the notion of Boolean filter?"

To solve the open problem, we recall the results of the relation between the two filters and then get a new solution to the open problem in pseudo $BCK$-algebra.

**Theorem 8.** (Zhang [29]) Let $(A; \geq, \rightarrow, \Rightarrow, 1)$ be a pseudo $BCK$-algebra and $F$ be a normal pseudo filter of $A$. Then $F$ is implicational if and only if $F$ is Boolean.

With the help of the equivalent conditions of fuzzy normal filter of pseudo $BCK$-algebra (pF), [1, 39] get the following results and partly solve the open problem.

**Theorem 9.** (Wang [1]) In bounded pseudo $BCK$-algebra, every implicational pseudo filter is a Boolean filter. In pseudo $BCK$-algebra (pF), every Boolean filter is an implicational pseudo filter.

We further investigate the properties of Boolean filter and implicational filter which make the relation between the two filters much clear, and get the solution for the open problem.

**Theorem 10.** Implicational pseudo filter is Boolean filter in pseudo $BCK$-algebra.

**Proof.** Let $F$ be an implicational pseudo filter of $A$. Then $\forall x \in A$, suppose $(x \Rightarrow y) \Rightarrow x \in F$,

from $x \leq ((x \rightarrow y) \Rightarrow x) \Rightarrow y \Rightarrow x$, so $((x \rightarrow y) \Rightarrow x) \Rightarrow x \leq x \rightarrow x$,

and $(((x \rightarrow y) \Rightarrow x) \Rightarrow x) \Rightarrow y \Rightarrow (x \rightarrow y) \Rightarrow 1$.

On the other hand, $x \rightarrow y \leq ((x \rightarrow y) \Rightarrow x) \Rightarrow x$,

so we get $(((x \rightarrow y) \Rightarrow x) \Rightarrow y) \Rightarrow ((x \rightarrow y) \Rightarrow x) \Rightarrow x$.

Then $(((x \rightarrow y) \Rightarrow x) \Rightarrow x) \Rightarrow y \Rightarrow (x \rightarrow y) \Rightarrow (x \rightarrow y) \Rightarrow x = 1 \in F$,

and $((x \rightarrow y) \Rightarrow x) \Rightarrow x \in F$, since $F$ is an implicational filter.

Combine that $(x \rightarrow y) \Rightarrow x \in F$, according to the definition of filter, we get $x \in F$.

Similarly, suppose $(x \Rightarrow y) \Rightarrow x \in F$,

from $x \leq ((x \Rightarrow y) \Rightarrow x) \Rightarrow x$,

so $((x \Rightarrow y) \Rightarrow x) \Rightarrow x \leq x \Rightarrow y \Rightarrow x$,

and $(((x \Rightarrow y) \Rightarrow x) \Rightarrow x) \Rightarrow y \Rightarrow (x \Rightarrow y) \Rightarrow 1$.

By $x \Rightarrow y \leq ((x \Rightarrow y) \Rightarrow x) \Rightarrow x$,

so we get $(((x \Rightarrow y) \Rightarrow x) \Rightarrow x) \Rightarrow y \Rightarrow (x \Rightarrow y) \Rightarrow x$.

Then $(((x \Rightarrow y) \Rightarrow x) \Rightarrow x) \Rightarrow y \Rightarrow (x \Rightarrow y) \Rightarrow x = 1 \in F$,

and $((x \Rightarrow y) \Rightarrow x) \Rightarrow x \in F$, since $F$ is an implicational filter.

Combine that $(x \Rightarrow y) \Rightarrow x \in F$, according to the definition of filter, then we get $x \in F$.

According to the definition, then $F$ is a Boolean filter of $A$.

Similarly, we can get

**Theorem 11.** In pseudo $BCK$-algebra, every Boolean filter is an implicational pseudo filter.

**Proof.** Let $F$ be an Boolean filter of $A$. Then $\forall x \in A$, suppose $(x \Rightarrow y) \Rightarrow x \in F$,

from $x \leq ((x \Rightarrow y) \Rightarrow x) \Rightarrow x$,

so $((x \Rightarrow y) \Rightarrow x) \Rightarrow x \leq x \rightarrow y$,

and $(((x \Rightarrow y) \Rightarrow x) \Rightarrow x) \Rightarrow (x \Rightarrow y) \Rightarrow 1$.

On the other hand, $x \rightarrow y \leq ((x \Rightarrow y) \Rightarrow x) \Rightarrow x$,

so we get $(((x \Rightarrow y) \Rightarrow x) \Rightarrow x) \Rightarrow y \Rightarrow ((x \Rightarrow y) \Rightarrow x) \Rightarrow x$.

Then $(((x \Rightarrow y) \Rightarrow x) \Rightarrow x) \Rightarrow y \Rightarrow (x \Rightarrow y) \Rightarrow (x \Rightarrow y) \Rightarrow x = 1 \in F$ and $((x \Rightarrow y) \Rightarrow x) \Rightarrow x \in F$, since $F$ is an implicational filter.

Combine that $(x \Rightarrow y) \Rightarrow x \in F$, according to the definition of filter, then we get $x \in F$.

Similarly, suppose $(x \Rightarrow y) \Rightarrow x \in F$,
from \( x \leq ((x \rightarrow y) \rightarrow x) \rightarrow x \),
so \( (((x \rightarrow y) \rightarrow x) \rightarrow y) \leq x \leq y \),
and \( (((x \rightarrow y) \rightarrow x) \rightarrow y) \rightarrow (x \rightarrow y) = 1 \).

On the other hand, \( x \leq y \leq ((x \rightarrow y) \rightarrow x) \rightarrow x \),
so we get \( (((x \rightarrow y) \rightarrow x) \rightarrow y) \rightarrow (x \rightarrow y) \leq (((x \rightarrow y) \rightarrow x) \rightarrow x) = 1 \).

Then \( (((x \rightarrow y) \rightarrow x) \rightarrow y) \rightarrow (((x \rightarrow y) \rightarrow x) \rightarrow y) = 1 \in F \) and \((x \rightarrow y) \rightarrow x) \rightarrow x \rightarrow x = 1 \) since \( F \) is a BCK filter.

Combine that \( (x \rightarrow y) \rightarrow x \in F \), according to the definition of filter, we get \( x \in F \).

Thus \( F \) is an implicative pseudo filter of \( A \).

From the above results, we can get the following results as a solution for the open problem.

**Theorem 12.** In pseudo BCK-algebra or bounded pseudo BCK-algebra, the notion of implicative pseudo filter is equivalent to the notion of Boolean filter.

**Remark 1.** The equivalent relation between the implicative filter and Boolean filter is of importance in the study of logical algebras. For example, when studying of the pseudo BCK-algebra, implicative filter and Boolean filter can reflect the algebraic structure of the pseudo BCK-algebra. When we get the equivalent relation between them, and based on some other results we obtained [1, 16, 30], we can completely solve some other problems like this.

### 4. THE RELATION BETWEEN IMPLICATIVE FILTER (BOOLEAN FILTER) AND IMPLICATIVE PSEUDO BCK-ALGEBRA

The filters play a vital role in representing the algebras, such as in a pseudo BL-algebra \( A \),

\( A \) is pseudo MV-algebra if and only if every filter of \( A \) is a pseudo MV filter,

\( A \) is a Gödel-algebra if and only if every filter of \( A \) is a pseudo G filter,

\( A \) is a Boolean algebra if and only if every filter of \( A \) is a Boolean filter.

The similar relation between implicative or Boolean filter and implicative pseudo BCK-algebra is not obtained, yet. For this reason, [25] set it as an open problem.

Prove or negate that pseudo BCK-algebras is implicative BCK-algebras if and only if every filter of them is implicative filters (or Boolean filter).

[1] partly solve the open problem. Based on this, we can further get the following results as a solution for the open problem.

**Proposition 13.** Let \( A \) be a pseudo BCK algebra. Then the following statements are equivalent:

1. \( A \) is an implicative BCK-algebras,
2. Every filter of \( A \) is an implicative filters (or Boolean filter),
3. \( 1 \) is an implicative filters (or Boolean filter).

**Proof.** (1) \( \Rightarrow \) (2) Based on the results of [9] and the previous result, pseudo BCK-algebra \( A \) is implicative BCK-algebra if and only if \( A \) is a 1-type (or 2-type) implicative pseudo BCK-algebra. Then for every filter of them, if \( (x \rightarrow y) \rightarrow x, (x \rightarrow y) \rightarrow x \in F \Rightarrow x \Rightarrow F, \)

\( (x \rightarrow y) \rightarrow x, (x \rightarrow y) \rightarrow x \in F \Rightarrow x \in F \). Then every pseudo filters of them is implicative pseudo filters (Boolean filter), so necessity is obvious.

(2) \( \Rightarrow \) (3) obvious.

(3) \( \Rightarrow \) (1) Now suppose every pseudo filter of a pseudo BCK-algebra \( A \) is an implicative pseudo filters (Boolean filter), then pseudo filter \( 1 \) is an implicative pseudo filters (Boolean filter).

For any \( x, y \in A \), from \( x \leq ((x \rightarrow y) \rightarrow x) \rightarrow x \) (by Theorem (7)),
we get \( (((x \rightarrow y) \rightarrow x) \rightarrow x) \rightarrow x \rightarrow x \) (by Theorem (1)),
then \( (((x \rightarrow y) \rightarrow x) \rightarrow x) \rightarrow x \rightarrow x \rightarrow x = 1 \) (by Definition (6)).

From \( x \leq y \leq ((x \rightarrow y) \rightarrow x) \rightarrow x \) (by Definition (2)),
we get \( (((x \rightarrow y) \rightarrow x) \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow x) \rightarrow x \leq ((x \rightarrow y) \rightarrow x) \rightarrow x \) (by Theorem (2)),
then \( (((x \rightarrow y) \rightarrow x) \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow x) \rightarrow x = 1 \in [1] \), (by Definition (4)),
we get \( ((x \rightarrow y) \rightarrow x) \rightarrow x = 1 \in [1] \), since \( 1 \) is an implicative pseudo filters, that is, \( (x \rightarrow y) \rightarrow x \leq x \) (by Definition (6)).

On the other hand, \( x \leq (x \rightarrow y) \rightarrow x \), then \( (x \rightarrow y) \rightarrow x = x \).

For the same reason, from \( x \leq ((x \rightarrow y) \rightarrow x) \rightarrow x \),
we get \( (((x \rightarrow y) \rightarrow x) \rightarrow x) \rightarrow y \leq x \rightarrow y \) and \( (((x \rightarrow y) \rightarrow x) \rightarrow y) \leq (x \rightarrow y) = 1 \). And \( x \rightarrow y \leq (((x \rightarrow y) \rightarrow x) \rightarrow y) \leq (x \rightarrow y) \rightarrow x \), so we get \( (((x \rightarrow y) \rightarrow x) \rightarrow y) \leq (x \rightarrow y) \rightarrow x \), then \( ((x \rightarrow y) \rightarrow x) \rightarrow y \rightarrow x \rightarrow (x \rightarrow y) \rightarrow x \rightarrow x \rightarrow x = 1 \in [1] \),
then \( (x \rightarrow y) \rightarrow x \rightarrow x = x \in [1] \), that is, \( (x \rightarrow y) \rightarrow x \leq x \). On the other hand, \( x \leq (x \rightarrow y) \rightarrow x \leq x \rightarrow x = x \).

From above results, we find that \( A \) is a 1-type implicative pseudo BCK-algebra, then \( A \) is implicative BCK-algebra.

### 5. THE RELATION BETWEEN FANTASTIC FILTER AND NORMAL FILTER IN BL-ALGEBRA

Here we recall some kinds of filters in BL-algebra. Similar with the pseudo BCK-algebra, here we recall some definitions and results which will be needed. Reader can refer to [3, 16, 30, 37, 40–42].

**Definition 15.** A filter of a BL-algebra \( A \) is a nonempty subset \( F \) of \( A \) such that for all \( x, y \in A \),

(1) \( x \in F \),
(2) \( x \cup y \in F \),
(3) \( x \in F \) if and only if the following conditions hold

1. \( 1 \in F \),
2. \( x, x \rightarrow y \in F \) implies \( y \in F \).
A filter $F$ of a BL-algebra $A$ is proper if $F \neq A$, that is, $0 \not\in A$.

In example 2, we can find $\left(\frac{1}{2}, 1\right)$ is a filter.

**Definition 16.** A proper filter $F$ is prime if for any $x, y, z \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$.

**Theorem 15.** A proper filter $F$ is prime if for any $x, y \in A, x \rightarrow y \in F$ or $y \rightarrow x \in F$.

**Definition 17.** A filter $F$ of $A$ is called normal if for any $x, y, z \in A$, $z \rightarrow ((y \rightarrow x) \rightarrow x) \in F$ and $z \in F$ implies $(x \rightarrow y) \rightarrow y \in F$.

**Definition 18.** Let $F$ be a nonempty subset of a BL-algebra $A$. Then $F$ is called a fantastic filter of $A$ if for all $x, y, z \in A$, the following conditions hold:

(1) $1 \in F$;

(2) $z \rightarrow (y \rightarrow x) \in F, z \in F$ implies $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$.

**Theorem 19.** Let $F$ be a nonempty subset of $A$. Then $F$ is called an implicative filter if for all $x, y, z \in A$, the following conditions hold:

(1) $1 \in F$;

(2) $x \rightarrow (y \rightarrow z) \in F, x \rightarrow y \in F$ imply $x \rightarrow z \in F$.

**Definition 20.** Let $F$ be a filter of $A$.

(1) $F$ is a Boolean filter if and only if one of the followings holds for all $x$:

(a) $F$ is a Boolean filter of $A$, and $x \in F$ implies $x \vee x' \in F$ for any $x \in A$.

(b) $x \rightarrow (y \rightarrow z) \in F, x \rightarrow y \in F$ imply $y \in F$.

**Theorem 21.** Let $F$ be a filter of $A$. Then $F$ is a normal filter if and only if

$\forall x \in F, \exists n \in A$ such that $x^n \in F$.

In order to investigate the essential relations among the filters, and based on the past work [1, 16, 30], we characterize the following filters.

**Theorem 16.** Let $F$ be a filter of $A$. If $F$ is a normal filter then $F$ is a Boolean filter if and only if one of the following holds for all $x, y \in A$:

(1) $y \rightarrow x \in F$ implies $x \rightarrow y \in F$.

(2) $x' \in F$ implies $x \in F$.

(3) $x' \rightarrow x \in F$ implies $x \in F$.

**Theorem 17.** Let $F$ be a filter of $A$. Then the followings are equivalent for all $x, y, z \in A$:

(1) $F$ is a Boolean filter of $A$;

(2) $x \rightarrow y \in F$ implies $x \in F$;

(3) $x' \rightarrow x \in F$.

**Theorem 22.** Let $F$ be a filter of $A$. Then for all $x, y \in A$ the followings are equivalent:

(1) $F$ is a Boolean filter, and $x' \rightarrow y \in F$ implies $x \rightarrow (y \rightarrow x) \in F$.

(2) $x \rightarrow y \in F$ implies $x \rightarrow (y \rightarrow x) \in F$.

(3) $x' \rightarrow y \in F$ implies $x \rightarrow (y \rightarrow x) \in F$.

By the above results, we can get the following results.

**Corollary 21.** If $F$ is a Boolean filter, then $F$ is a maximal filter.

**Corollary 22.** In an MV-algebra, every filter is a nonproper filter.

**Corollary 23.** In a Gödel algebra, every filter is a nonproper filter.

Based on this, we get some relations among the filters in BL-algebras.

**Theorem 24.** Each Boolean filter is equivalent to a normal filter in BL-algebras.

**Theorem 25.** Each ultra filter is equivalent to an normal filter in BL-algebras.

**Theorem 26.** Each ultra filter $f$ of a BL-algebra $A$ is a fantastic filter.

**Proof.** Suppose $F$ is an ultra filter. Then for all $x \in A$, we have $x \leq x' \rightarrow x' \rightarrow x \rightarrow 0 \in F$. Therefore, $F$ is an implicative and fantastic filter.

**Theorem 27.** Let $F$ be a filter of $A$. Then $F$ is a Boolean filter if and only if it is an implicative and fantastic filter.

**Proof.** If $F$ is a Boolean filter, then by Theorem 17, 19, and 21, we know that $F$ is an implicative and fantastic filter.

**Theorem 28.** Let $F$ be a filter of $A$. Then $F$ is a Boolean filter if and only if it is an implicative and fantastic filter.

**Proof.** If $F$ is a Boolean filter, then by Theorem 17, 19, and 21, we know that $F$ is an implicative and fantastic filter.

**Theorem 29.** Let $F$ be a filter of $A$. Then $F$ is an implicative and fantastic filter if and only if it is a Boolean filter.

**Proof.** If $F$ is a Boolean filter, then by Theorem 17, 19, and 21, we know that $F$ is an implicative and fantastic filter.

**Theorem 30.** Let $F$ be a filter of $A$. Then $F$ is a Boolean filter if and only if it is an implicative and fantastic filter.

**Proof.** If $F$ is a Boolean filter, then by Theorem 17, 19, and 21, we know that $F$ is an implicative and fantastic filter.

**Theorem 31.** Let $F$ be a filter of $A$. Then $F$ is a Boolean filter if and only if it is an implicative and fantastic filter.

**Proof.** If $F$ is a Boolean filter, then by Theorem 17, 19, and 21, we know that $F$ is an implicative and fantastic filter.

**Theorem 32.** Let $F$ be a filter of $A$. Then $F$ is a Boolean filter if and only if it is an implicative and fantastic filter.

**Proof.** If $F$ is a Boolean filter, then by Theorem 17, 19, and 21, we know that $F$ is an implicative and fantastic filter.
1. Under what suitable condition a normal filter becomes a fantastic filter?

2. (Extension property for a normal filter) Under what suitable condition extension property for normal filter holds?

[30] proposed solutions for the two open problems by the fuzzy filters, respectively as follows:

1. Suitable condition should be
   (1) $BL$-algebra is an $MV$-algebra,
   (2) $F$ is a normal and implicative filter of $A$,
   (3) $F$ is a normal and obisinate filter of $A$,
   (4) $F$ is a normal and ultra filter of $A$.

2. Under the condition
   (1) If $A$ is an $MV$-algebra,
   (2) $F$ is a normal and implicative filter of $A$,
   (3) $F$ is a normal and obisinate filter of $A$,
   (4) $F$ is a normal and ultra filter of $A$.

Extension property for normal filter holds.

According to the above theorem and corollary, we can get the equivalent relation between the two filters and give answers to the open problems.

The suitable condition should be
(1) A normal filter is equivalent to a fantastic filter.
(2) Extension property for a normal filter holds.

We further characterized the filters in $BL$-algebra. Compared with the solutions in [30], the condition that the filter is an obisinate filter or the filter is an ultra filter is redundant.

6. CONCLUSION

We discuss the properties of implicative filters and Boolean filters in pseudo $BCK$-algebra. Based on the results and previous work, we completely solve an open problem which is important to deep study of the algebraic structure of pseudo $BCK$-algebra. Based on this, we prove that pseudo $BCK$-algebra is implicative $BCK$-algebra if and only if every filter of them is implicative filter (or Boolean filter).

We further characterize the filters in $BL$-algebra. Compared with the solutions in [30], the condition that the filter is an obisinate filter or the filter is an ultra filter is redundant.

In the future work, we will extend the corresponding filter theory to different algebraic structures, and study the congruence relations induced by the filters.

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