

On the Lindley Record Values and Associated Inference

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ABSTRACT

In this paper, we discuss the record values arising from the Lindley distribution. We compute the means, variances and covariances of the record values. These values are used to compute the best linear unbiased estimators (BLUEs) and the best linear invariant estimators (BLIEs) of the location and scale parameters. By using the BLUEs and BLIEs, we construct confidence intervals for the location and scale parameters through Monte Carlo simulations. Prediction for the future records is also discussed.

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1. INTRODUCTION

Let X_1, X_2, \dots be a sequence of independent and identically distributed (IID) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. An observation X_j is an upper record value of this sequence if it exceeds in value all preceding observations, i.e., if $X_j > X_i, \forall i < j$. Lower records are analogously defined. Generally, if $\{U(n), n \geq 1\}$ is defined by

$$U(1) = 1, \quad U(n) = \max \{j : j < U(n-1), X_j > X_{U(n-1)}\}, \quad (1)$$

then the sequence $\{X_{U(n)}, n \geq 1\}$ provides a sequence of upper record statistics. The sequence $\{U(n), n \geq 1\}$ represents the upper record times. From the above definition, the sequence of record statistics can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of the observations. Note that from a sequence of n IID continuous random variables, only about $\log(n)$ records are expected, see Houchens [1].

Chandler [2] defined the model of record statistics as a model for successive extremes in a sequence of IID random variables. These statistics are of interest and important in many real life applications involving data relating to weather, economics, sport and life testing studies. For more details and applications regarding record values, see Ahsanullah [3], Arnold *et al.* [4] and Nevzorov [5].

Recently, the Lindley distribution has received a considerable attention in the statistical literature. It was first proposed by Lindley [6] in the context of Bayesian inference. The pdf of the Lindley distribution is given by

$$f(x) = \frac{\theta^2}{\theta + 1} (1 + x) e^{-\theta x}, \quad x > 0, \quad \theta > 0, \quad (2)$$

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and the corresponding cdf is

$$F(x) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x}, \quad x > 0, \quad \theta > 0. \quad (3)$$

A distribution that is close in form to Eq. (2) is the well-known exponential distribution with pdf

$$f(x) = \theta e^{-\theta x}, \quad x > 0, \quad \theta > 0.$$

Ghitany *et al.* [7] showed in many ways that the Lindley distribution is a better model than one based on the exponential distribution. The shape of the hazard rate function of the Lindley distribution is an increasing function. This distribution belongs to an exponential family and it can be written as a mixture of an exponential and a gamma distribution with shape parameter 2. In recent years, this distribution has been studied and extended by many authors [7; 8; 9; 10; 11; 12; 13; 14].

Recently, some work has been done on inferential procedures for the Lindley distribution based on complete and censored data [15; 16; 17; 18; 19]. Recently, Asgharzadeh *et al.* [20] discussed the maximum likelihood and Bayesian estimation of the shape parameter of the Lindley distribution based on upper records. In this article, we consider the upper record values from the Lindley distribution. We compute the means, variances and covariances of the upper record values. Then, we use these moments to calculate best linear unbiased estimators (BLUEs) and best linear invariant estimators (BLIEs) for the location and scale parameters of the Lindley distribution. Prediction for the future records is also discussed. Ahsanullah [21] and Dunsmore [22] discussed the BLUEs and prediction of future record values from a two-parameter exponential distribution. Some work in this direction has been done for the logistic distribution by Balakrishnan *et al.* [23], for the normal distribution by Balakrishnan and Chan [24], for the generalized exponential distribution by Raqab [25], for the gamma distribution by Sultan *et al.* [26] and for the Nadarajah-Haghighi distribution by MirMostafaei *et al.* [27].

In this paper, we consider the upper record values from the Lindley distribution. In Section 2, we compute the means, variances and covariances of record values up to sample size 6. Next, in Section 3, we obtain the BLUEs and BLIEs of the location and scale parameters of the Lindley distribution. The BLUEs and BLIEs are then used to construct the confidence intervals (CIs) for the location and scale parameters. In Section 4, we discuss point and interval predictions for future records. In Section 5, two numerical examples are given to illustrate the estimation and prediction methods discussed in this paper. In Section 6, Monte Carlo simulations are performed to compare the proposed CIs of the location and scale parameters and also to compare the prediction intervals (PIs) of the future records.

2. MOMENTS OF THE UPPER RECORD VALUES

Let $X_{U(1)}, X_{U(2)} \dots, X_{U(n)}$ be the first n upper record values from the Lindley distribution. Then the pdf of $X_{U(n)}$ is given by [4]

$$f_n(x) = \frac{1}{\Gamma(n)} [-\log\{1 - F(x)\}]^{n-1} f(x), \quad x > 0, \quad n = 1, 2, \dots$$

where $f(\cdot)$ and $F(\cdot)$ are given by Eqs. (2) and (3), respectively. The joint pdf of $X_{U(m)}$ and $X_{U(n)}$ is given by

$$f_{m,n}(x, y) = \frac{1}{\Gamma(m)\Gamma(n-m)} [-\log\{1 - F(x)\}]^{m-1} \frac{f(x)}{1 - F(x)} \times [-\log\{1 - F(y)\} + \log\{1 - F(x)\}]^{n-m-1} f(y), \\ 0 < x < y < \infty, \quad m = 1, 2, \dots, \quad m < n.$$

Then, the r th single moment of $X_{U(n)}$ denoted by $\mu_n^{(r)}$ ($r = 0, 1, 2, \dots$) is given by

$$\begin{aligned} \mu_n^{(r)} &= E(X_{U(n)}^r) = \int_0^\infty x^r f_n(x) dx \\ &= \int_0^\infty x^r \frac{\theta^2(1+x)}{\Gamma(n)(\theta+1)} \left[-\log\left(\frac{\theta+1+\theta x}{\theta+1}\right) + \theta x \right]^{n-1} e^{-\theta x} dx \\ &= \frac{1}{(n-1)!} \sum_{i=0}^{n-1} \binom{n-1}{i} g(i, n, r, \theta), \end{aligned}$$

where

$$g(i, n, r, \theta) = \frac{\theta^2}{\theta + 1} \int_0^\infty x^r [-\log\{1 + \theta x/(\theta + 1)\}]^i (\theta x)^{n-i-1} (1+x) e^{-\theta x} dx.$$

Note that using the transformation $y = 1 + \theta x/(\theta + 1)$, we can write $g(i, n, r, \theta)$ as

$$\begin{aligned} g(i, n, r, \theta) &= (\theta + 1)^{n-i+r-1} \theta^{-r} e^{\theta+1} \int_1^\infty [-\log y]^i (y-1)^{n-i+r-1} e^{-(\theta+1)y} [(\theta+1)y-1] dy \\ &= (\theta + 1)^{n-i+r} \theta^{-r} e^{\theta+1} \\ &\quad \times \left[\sum_{q=1}^{n-i+r+1} \binom{n-i+r}{q-1} (-1)^{n+r-q+1} \int_1^\infty [\log y]^i y^{q-1} e^{-(\theta+1)y} dy \right. \\ &\quad \left. + \frac{\theta}{\theta+1} \sum_{q=1}^{n-i+r} \binom{n-i+r-1}{q-1} (-1)^{n+r-q} \int_1^\infty [\log y]^i y^{q-1} e^{-(\theta+1)y} dy \right] \\ &= (\theta + 1)^{n-i+r} \theta^{-r} e^{\theta+1} \\ &\quad \times \left[\sum_{q=1}^{n-i+r+1} \binom{n-i+r}{q-1} (-1)^{n+r-q+1} \frac{\partial^i}{\partial q^i} \{(\theta+1)^{-q} \Gamma(q, \theta+1)\} \right. \\ &\quad \left. + \frac{\theta}{\theta+1} \sum_{q=1}^{n-i+r} \binom{n-i+r-1}{q-1} (-1)^{n+r-q} \frac{\partial^i}{\partial q^i} \{(\theta+1)^{-q} \Gamma(q, \theta+1)\} \right], \end{aligned}$$

where the last equality is calculated by considering the Eq. (4.358.1) in Gradshteyn and Ryzhik [28]. Here $\Gamma(\cdot, \cdot)$ denotes the incomplete gamma function given by

$$\Gamma(v, \lambda) = \int_\lambda^\infty x^{v-1} e^{-x} dx.$$

Let us now consider the double moments of the upper record values $X_{U(m)}$ and $X_{U(n)}$, $m < n$. The double (r, s) th moment of $X_{U(n)}$ and $X_{U(m)}$ is given by

$$\mu_{m,n}^{(r,s)} = E(X_{U(m)}^r X_{U(n)}^s) = \int_0^\infty \int_x^\infty x^r y^s f_{m,n}(x, y) dy dx,$$

$$0 < x < y < \infty, \quad m = 1, 2, \dots, \quad m < n, \quad r, s = 0, 1, 2, \dots$$

For the Lindley distribution, we obtain

$$\begin{aligned} \mu_{m,n}^{(r,s)} &= \frac{\theta^4}{\Gamma(m) \Gamma(n-m) (\theta+1)} \int_0^\infty \int_x^\infty x^r y^s \left[\log \left(\frac{\theta+1+\theta x}{\theta+1+\theta y} \right) + \theta(y-x) \right]^{n-m-1} \\ &\quad \times \left\{ -\log \left(\frac{\theta+1+\theta x}{\theta+1} \right) + \theta x \right\}^{m-1} (1+x)(1+y) \frac{e^{-\theta y}}{\theta+1+\theta x} dy dx. \end{aligned} \quad (4)$$

In general, the double moment in Eq. (4) cannot be obtained in a closed form, but for the special case when $n = m + 1$, we can obtain a simplified closed form (see Appendix A).

Through numerical integration, we can determine the means, μ_n , variances $\sigma_n^2 = \mu_n^{(2)} - \mu_n^2$ ($n = 1, 2, \dots$), and covariances $\sigma_{m,n} = \mu_{m,n} - \mu_m \mu_n$ ($m < n$); of record statistics, respectively. Table 1 presents the values of μ_n for $n = 1(1)6$ for different values of the shape parameter θ . Table 2 provides the variances and covariances of record statistics for $m = 1(1)6$; $n = m + 1(1)6$ and $\theta = 0.5(0.5)4.5$. The values of means, variances and covariances for $n \geq 7$ are computed but not presented here. All the computations were done by using Maple 16. It may be observed from Table 1, that the mean decreases when θ increases, also it increases when n increases. From Table 2, the variance and covariance decrease when θ increases. Also, the variances increase when n increases.

Table 1 Means of record statistics.

n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
1	3.333	1.500	0.933	0.666	0.514	0.416	0.349	0.300	0.262
2	6.069	2.819	1.785	1.289	1.002	0.816	0.686	0.591	0.518
3	8.577	4.052	2.594	1.886	1.473	1.204	1.015	0.876	0.770
4	10.969	5.236	3.375	2.466	1.933	1.584	1.338	1.157	1.018
5	13.288	6.389	4.138	3.034	2.384	1.957	1.657	1.434	1.263
6	15.559	7.520	4.888	3.594	2.830	2.327	1.972	1.709	1.507

Table 2 Variances and covariances of record statistics.

m	n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
1	1	7.5556	1.7500	0.7288	0.3888	0.2383	0.1597	0.1138	0.0850	0.0657
1	2	6.8964	1.6323	0.6904	0.3724	0.2300	0.1550	0.1110	0.0832	0.0645
1	3	6.5612	1.5655	0.6668	0.3616	0.2244	0.1518	0.1090	0.0818	0.0635
1	4	6.3579	1.5227	0.6509	0.3541	0.2203	0.1493	0.1075	0.0808	0.0628
1	5	6.2208	1.4929	0.6395	0.3486	0.2172	0.1475	0.1062	0.0800	0.0628
1	6	6.1216	1.4709	0.6309	0.3444	0.2148	0.1460	0.1053	0.0793	0.0618
2	2	12.639	3.0513	1.3096	0.7138	0.4444	0.3012	0.2167	0.1629	0.1266
2	3	12.027	2.9267	1.2648	0.6932	0.4334	0.2949	0.2127	0.1603	0.1248
2	4	11.656	2.8467	1.2346	0.6788	0.4256	0.2902	0.2097	0.1582	0.1234
2	5	11.406	2.7910	1.2130	0.6682	0.4196	0.2865	0.2074	0.1566	0.1223
2	6	11.225	2.7500	1.1967	0.6600	0.4150	0.2836	0.2055	0.1553	0.1213
3	3	17.192	4.2143	1.8333	1.0102	0.6344	0.4331	0.3133	0.2366	0.1846
3	4	16.663	4.0993	1.7896	0.9892	0.6228	0.4261	0.3089	0.2336	0.1825
3	5	16.306	4.0193	1.7583	0.9737	0.6142	0.4208	0.3054	0.2312	0.1808
3	6	16.047	3.9603	1.7347	0.9619	0.6074	0.4166	0.3026	0.2293	0.1794
4	4	21.546	5.3189	2.3300	1.2919	0.8155	0.5592	0.4060	0.3075	0.2406
4	5	21.085	5.2152	2.2893	1.2717	0.8037	0.5519	0.4012	0.3042	0.2384
4	6	20.752	5.1387	2.2586	1.2562	0.7952	0.5466	0.3978	0.3019	0.2366
5	5	25.800	6.3933	2.8120	1.5650	0.9912	0.6816	0.4962	0.3767	0.2952
5	6	25.393	6.2996	2.7744	1.5460	0.9803	0.6742	0.4913	0.3733	0.2927
6	6	29.994	7.4496	3.2850	1.8327	1.1635	0.8017	0.5846	0.4445	0.3489

3. LINEAR ESTIMATORS

Let $Y_{U(1)}, Y_{U(2)} \dots, Y_{U(n)}$ be the first n upper record values from the three parameter Lindley distribution with pdf

$$f_Y(y) = \frac{\theta^2}{\sigma^2(\theta+1)} (\sigma + y - \mu) e^{-\theta\left(\frac{y-\mu}{\sigma}\right)}, \quad y > \mu, \sigma > 0, \theta > 0, \quad (5)$$

and let $X_{U(i)} = \frac{Y_{U(i)} - \mu}{\sigma}$, $i = 1, 2, \dots, n$ be the corresponding upper record values from the standard Lindley distribution with pdf given in Eq. (2). Following the generalized least-squares approach, the BLUEs of μ and σ can be derived as [29]

$$\hat{\mu}_{BLU} = \sum_{i=1}^n a_i Y_{U(i)}, \quad \hat{\sigma}_{BLU} = \sum_{i=1}^n b_i Y_{U(i)},$$

where

$$\mathbf{a} = (a_1, \dots, a_n) = \frac{\alpha' \beta^{-1} \alpha \mathbf{1}' \beta^{-1} - \alpha' \beta^{-1} \mathbf{1} \alpha' \beta^{-1}}{(\alpha' \beta^{-1} \alpha)(\mathbf{1}' \beta^{-1} \mathbf{1}) - (\alpha' \beta^{-1} \mathbf{1})^2},$$

and

$$\mathbf{b} = (b_1, \dots, b_n) = \frac{\mathbf{1}' \beta^{-1} \mathbf{1} \alpha' \beta^{-1} - \mathbf{1}' \beta^{-1} \alpha \mathbf{1}' \beta^{-1}}{(\alpha' \beta^{-1} \alpha)(\mathbf{1}' \beta^{-1} \mathbf{1}) - (\alpha' \beta^{-1} \mathbf{1})^2},$$

$\alpha' = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is the moment vector with $\alpha_i = E(X_{U(i)})$ and $\beta = (\beta_{ij})$, $1 \leq i \leq j \leq n$ is the covariance matrix with $\beta_{ij} = \text{Cov}(X_{U(i)}, X_{U(j)})$, and $\mathbf{1}' = (1, 1, \dots, 1)_{1 \times n}$.

Furthermore, the variances of these BLUEs are given by

$$\text{Var}(\hat{\mu}_{BLU}) = \left[\frac{\alpha' \beta^{-1} \alpha}{(\alpha' \beta^{-1} \alpha)(\mathbf{1}' \beta^{-1} \mathbf{1}) - (\alpha' \beta^{-1} \mathbf{1})^2} \right] \sigma^2 = V_1 \sigma^2,$$

$$\text{Var}(\hat{\sigma}_{BLU}) = \left[\frac{\mathbf{1}' \beta^{-1} \mathbf{1}}{(\alpha' \beta^{-1} \alpha)(\mathbf{1}' \beta^{-1} \mathbf{1}) - (\alpha' \beta^{-1} \mathbf{1})^2} \right] \sigma^2 = V_2 \sigma^2,$$

and

$$\text{Cov}(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU}) = \left[\frac{-\alpha' \beta^{-1} \mathbf{1}}{(\alpha' \beta^{-1} \alpha)(\mathbf{1}' \beta^{-1} \mathbf{1}) - (\alpha' \beta^{-1} \mathbf{1})^2} \right] \sigma^2 = V_3 \sigma^2.$$

The coefficients a_i 's, b_i 's, $1 \leq i \leq n$ and the values of V_1 , V_2 and V_3 are computed and presented in Tables 3–5, respectively. From Table 5, we note that the variances of the BLUEs decrease as n increases.

Based on the BLUEs of the location and scale parameters, the CIs for μ and σ can be constructed through the pivotal quantities given by

$$R_1 = \frac{\hat{\mu}_{BLU} - \mu}{\hat{\sigma}_{BLU} \sqrt{V_1}} \quad \text{and} \quad R_2 = \frac{\hat{\sigma}_{BLU} - \sigma}{\sigma \sqrt{V_2}}.$$

Constructing such CIs requires then percentage the points of R_1 and R_2 which can be computed by using the BLUEs $\hat{\mu}_{BLU}$ and $\hat{\sigma}_{BLU}$ via Monte Carlo method. In Table 6, we have determined the percentage points of R_1 and R_2 based on 10 000 runs and different values of n and θ . Based on these simulated percentage points, we can determine a $100(1 - \alpha)\%$ CI for μ through the pivotal quantity R_1 as follows

$$P\left(\hat{\mu}_{BLU} - \hat{\sigma}_{BLU} \sqrt{V_1} R_1(1 - \alpha/2) \leq \mu \leq \hat{\mu}_{BLU} - \hat{\sigma}_{BLU} \sqrt{V_1} R_1(\alpha/2)\right) = 1 - \alpha, \quad (6)$$

where $R_1(\gamma)$ is the left percentage point of R_1 at γ , i.e., $P(R_1 < R_1(\gamma)) = \gamma$.

Similarly, a $100(1 - \alpha)\%$ CI for σ can be constructed through the pivotal quantity R_2 as follows

$$P\left(\frac{\hat{\sigma}_{BLU}}{1 + \sqrt{V_2} R_2(1 - \alpha/2)} \leq \sigma \leq \frac{\hat{\sigma}_{BLU}}{1 + \sqrt{V_2} R_2(\alpha/2)}\right) = 1 - \alpha. \quad (7)$$

Now, let us consider the BLIEs of μ and σ . Based on the results of Mann [30], the BLIEs for μ and σ are (see also [4], p. 143)

$$\hat{\mu}_{BLI} = \hat{\mu}_{BLU} - \frac{V_3}{1 + V_2} \hat{\sigma}_{BLU} \quad \text{and} \quad \hat{\sigma}_{BLI} = \frac{\hat{\sigma}_{BLU}}{1 + V_2},$$

Table 3 Coefficients for the BLUEs of μ .

n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	2.2184	2.1369	2.0949	2.0698	2.0536	2.0428	2.0352	2.0295	2.0245
	-1.2184	-1.1369	-1.0949	-1.0698	-1.0536	-1.0428	-1.0352	-1.0295	-1.0245
3	1.6086	1.5680	1.5472	1.5351	1.5268	1.5207	1.5173	1.5148	1.5133
	0.0564	0.0404	0.0310	0.0228	0.0188	0.0168	0.0133	0.0106	0.0074
	-0.6650	-0.6085	-0.5774	-0.5580	-0.5456	-0.5375	-0.5307	-0.5254	-0.5207
4	1.3957	1.3721	1.3604	1.3536	1.3487	1.3452	1.3433	1.3421	1.3410
	0.0526	0.0367	0.0267	0.0198	0.0165	0.0146	0.0109	0.0077	0.0059
	0.0223	0.0173	0.0136	0.0113	0.0086	0.0060	0.0070	0.0078	0.0073
	-0.4707	-0.4262	-0.4008	-0.3848	-0.3739	-0.3659	-0.3613	-0.3577	-0.3543
5	1.2850	1.2710	1.2646	1.2611	1.2585	1.2563	1.2556	1.2550	1.2532
	0.0506	0.0347	0.0250	0.0183	0.0149	0.0128	0.0105	0.0070	0.0030
	0.0223	0.0171	0.0132	0.0107	0.0090	0.0065	0.0067	0.0074	0.0062
	0.0108	0.0090	0.0075	0.0064	0.0040	0.0047	0.0009	0.0017	0.0151
	-0.3688	-0.3320	-0.3104	-0.2967	-0.2865	-0.2803	-0.2737	-0.2712	-0.2776

Table 4 Coefficients for the BLUEs of σ .

n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	-0.3655	-0.7579	-1.1731	-1.6048	-2.0491	-2.5031	-2.9647	-3.4317	-3.9016
	0.3655	0.7579	1.1731	1.6048	2.0491	2.5031	2.9647	3.4317	3.9016
3	-0.1889	-0.3881	-0.5968	-0.8131	-1.0352	-1.2608	-1.4910	-1.7236	-1.9602
	-0.0036	-0.0074	-0.0107	-0.0131	-0.0151	-0.0183	-0.0187	-0.0203	-0.0174
	0.1925	0.3956	0.6076	0.8263	1.0503	1.2791	1.5098	1.7439	1.9776
4	-0.1292	-0.2639	-0.4041	-0.5486	-0.6964	-0.8466	-1.0002	-1.1554	-1.3115
	-0.0025	-0.0051	-0.0072	-0.0088	-0.0109	-0.0132	-0.0119	-0.0110	-0.0119
	-0.0002	-0.0009	-0.0020	-0.0032	-0.0037	-0.0036	-0.0071	-0.0099	-0.0101
	0.1320	0.2700	0.4134	0.5607	0.7111	0.8636	1.0194	1.1764	1.3335
5	-0.0989	-0.2013	-0.3071	-0.4159	-0.5269	-0.6395	-0.7540	-0.8711	-0.9915
	-0.0020	-0.0039	-0.0055	-0.0067	-0.0077	-0.0089	-0.0106	-0.0085	-0.0012
	-0.0002	-0.0007	-0.0015	-0.0023	-0.0045	-0.0047	-0.0063	-0.0087	-0.0060
	0.0003	0.0003	-0.00004	-0.0006	0.0011	0.0001	0.0026	0.0025	-0.0137
	0.1008	0.2057	0.3143	0.4257	0.5381	0.6530	0.7684	0.8858	1.0125

Table 5 Variances and covariances of the BLUEs of μ and σ in terms of σ^2 and V_4 .

n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	18.6661	4.0041	1.6012	0.8333	0.5030	0.3335	0.2358	0.1750	0.1068
	0.8553	0.8828	0.9050	0.9215	0.9351	0.9454	0.9536	0.9597	0.9636
	-3.0922	-1.4135	-0.8897	-0.6406	-0.4978	-0.4056	-0.3413	-0.2941	-0.2577
	2.2462	1.1335	0.7385	0.5434	0.4292	0.3537	0.3005	0.2592	0.2277
3	13.5075	2.9438	1.1861	0.6201	0.3750	0.2489	0.1762	0.1310	0.1009
	0.4226	0.4347	0.4453	0.4539	0.4607	0.4663	0.4706	0.4748	0.4779
	-1.5981	-0.7242	-0.4529	-0.3248	-0.2514	-0.2042	-0.1715	-0.1480	-0.1298
	1.0724	0.5430	0.3581	0.2660	0.2109	0.1743	0.1487	0.1297	0.1147
4	11.7063	2.5784	1.0444	0.5476	0.3318	0.2206	0.1561	0.1161	0.0895
	0.2810	0.2880	0.2946	0.3000	0.3045	0.3083	0.3112	0.3139	0.3161
	-1.0929	-0.4927	-0.3067	-0.2192	-0.1692	-0.1373	-0.1150	-0.0990	-0.0868
	0.6940	0.3516	0.2323	0.1743	0.1390	0.1156	0.0981	0.0860	0.0801
5	10.7697	2.3899	0.9718	0.5107	0.3099	0.2061	0.1461	0.1087	0.0833
	0.2110	0.2157	0.2201	0.2240	0.2272	0.2298	0.2323	0.2342	0.2347
	-0.8369	-0.3759	-0.2332	-0.1662	-0.1281	-0.1037	-0.0869	-0.0746	-0.0645
	0.5101	0.2584	0.1733	0.1290	0.1029	0.0858	0.0737	0.0640	0.0537

Note: For each n ($n = 2, \dots, 5$), the first, second, third and the fourth lines represent $V_1 = \frac{1}{\sigma^2} \text{Var}(\hat{\mu}_{BLU})$, $V_2 = \frac{1}{\sigma^2} \text{Var}(\hat{\sigma}_{BLU})$, $V_3 = \frac{1}{\sigma^2} \text{Cov}(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU})$ and V_4 , respectively.

where $V_1 = \frac{1}{\sigma^2} \text{Var}(\hat{\mu}_{BLU})$, $V_2 = \frac{1}{\sigma^2} \text{Var}(\hat{\sigma}_{BLU})$ and

$$V_3 = \frac{1}{\sigma^2} \text{Cov}(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU}).$$

Furthermore the variances of these BLIEs are given by (see Arnold *et al.* [4], p. 143)

$$\text{Var}(\hat{\mu}_{BLI}) = \sigma^2 \left(V_1 - \frac{V_3^2 (2 + V_2)}{(1 + V_2)^2} \right) \quad \text{and} \quad \text{Var}(\hat{\sigma}_{BLI}) = \frac{\sigma^2 V_2}{(1 + V_2)^2}.$$

Table 6 Simulated percentage points of R_1 and R_2 .

θ	n	R_1				R_2			
		2.5%	5%	95%	97.5%	2.5%	5%	95%	97.5%
0.5	2	−0.7440	−0.7155	14.492	26.198	−1.0478	−1.0213	1.9622	2.6028
	3	−0.8658	−0.8346	4.8155	7.6618	−1.3099	−1.2264	1.9176	2.4617
	4	−0.9355	−0.9016	3.9282	5.5012	−1.4486	−1.3088	1.8902	2.3525
	5	−0.9773	−0.9332	3.2299	4.5157	−1.5113	−1.3567	1.8260	2.3621
1	2	−0.7280	−0.7038	13.305	30.067	−1.0378	−1.0041	1.9768	2.5454
	3	−0.8464	−0.8227	5.2436	7.8190	−1.3088	−1.2210	1.8889	2.4898
	4	−0.9030	−0.8733	3.6417	5.4939	−1.4414	−1.2896	1.8967	2.3361
	5	−0.9404	−0.9018	3.1786	4.7170	−1.5085	−1.3516	1.8264	2.2896
1.5	2	−0.7174	−0.6948	13.769	25.300	−1.0205	−0.9960	1.9763	2.6424
	3	−0.8291	−0.8042	4.6772	7.4613	−1.2891	−1.2059	1.9158	2.4790
	4	−0.8848	−0.8631	3.8001	5.5215	−1.4325	−1.3015	1.8816	2.3670
	5	−0.9198	−0.8902	3.1424	4.5661	−1.4954	−1.3501	1.8362	2.3643
2	2	−0.7110	−0.6905	12.911	29.571	−1.0165	−0.9825	1.9599	2.5509
	3	−0.8249	−0.8003	4.8515	7.8426	−1.2871	−1.1963	1.9217	2.4887
	4	−0.8781	−0.8474	3.8368	5.4495	−1.4125	−1.3013	1.8700	2.4487
	5	−0.9055	−0.8821	3.3815	4.5120	−1.4831	−1.3580	1.8812	2.2959
2.5	2	−0.7061	−0.6846	13.532	24.754	−1.0049	−0.9816	1.9843	2.6678
	3	−0.8142	−0.7911	4.6038	7.3439	−1.2746	−1.1912	1.9176	2.4850
	4	−0.8689	−0.8476	3.7523	5.3652	−1.4201	−1.2933	1.8914	2.3739
	5	−0.8992	−0.8737	3.0930	4.5540	−1.4836	−1.3446	1.8430	2.3753
3	2	−0.7042	−0.6832	12.743	26.604	−1.0038	−0.9713	1.9557	2.5370
	3	−0.8134	−0.7910	4.7970	7.5683	−1.2730	−1.1860	1.9253	2.4916
	4	−0.8659	−0.8354	3.7999	5.4028	−1.4223	−1.2896	1.8683	2.4701
	5	−0.8927	−0.8694	3.3642	4.5019	−1.4798	−1.3543	1.8802	2.3110
3.5	2	−0.7006	−0.6797	11.786	24.528	−0.9955	−0.9666	2.0320	2.6852
	3	−0.8074	−0.7856	4.6534	7.2653	−1.2652	−1.1790	1.9438	2.4921
	4	−0.8627	−0.8371	3.7762	5.2676	−1.4081	−1.2911	1.9073	2.3811
	5	−0.8905	−0.8639	3.1165	4.5332	−1.4773	−1.3310	1.8715	2.3824
4	2	−0.6986	−0.6781	13.615	28.989	−0.9966	−0.9713	1.9175	2.5425
	3	−0.8077	−0.7839	5.1292	7.5166	−1.2635	−1.1831	1.9524	2.4930
	4	−0.8590	−0.8376	3.5764	5.3673	−1.3959	−1.2822	1.9260	2.4645
	5	−0.8863	−0.8140	1.9541	4.4724	−1.4698	−1.3271	1.8644	2.3214
4.5	2	−0.6974	−0.6768	11.708	24.395	−0.9905	−0.9620	2.0339	2.6996
	3	−0.8029	−0.7804	4.6256	7.2221	−1.2576	−1.1724	1.9455	2.4970
	4	−0.8561	−0.8314	3.7530	5.3086	−1.3999	−1.2862	1.9036	2.3828
	5	−0.8866	−0.8609	3.1101	4.5284	−1.4740	−1.3323	1.8860	2.3939

Based on the BLIEs, we can again construct CIs for the location and scale parameters through pivotal quantities given by

$$R_3 = \frac{\hat{\mu}_{BLI} - \mu}{\hat{\sigma}_{BLI} \sqrt{V_1 - \frac{V_3^2(2+V_2)}{(1+V_2)^2}}} \quad \text{and} \quad R_4 = \frac{\hat{\sigma}_{BLI} - \sigma}{\sigma \sqrt{\frac{V_2}{1+V_2}}}.$$

Table 7 presents the percentage points of R_3 and R_4 based on 10 000 runs and different choices of n and θ . With the BLIEs and the use of Table 7, we can determine a $100(1 - \alpha)\%$ CI for μ through the pivotal quantity R_3 as

$$P\left(\hat{\mu}_{BLI} - \hat{\sigma}_{BLI} \sqrt{V_1 - \frac{V_3^2(2+V_2)}{(1+V_2)^2}} R_3(1 - \alpha/2) \leq \mu \leq \hat{\mu}_{BLI} - \hat{\sigma}_{BLI} \sqrt{V_1 - \frac{V_3^2(2+V_2)}{(1+V_2)^2}} R_3(\alpha/2)\right) = 1 - \alpha, \quad (8)$$

Table 7 Simulated percentage points of R_3 and R_4 .

θ	n	R_3				R_4			
		2.5%	5%	95%	97.5%	2.5%	5%	95%	97.5%
0.5	2	-0.7192	-0.6654	28.847	62.296	-1.9750	-1.9440	1.0862	1.7186
	3	-0.8521	-0.8038	8.1365	12.851	-1.9613	-1.8648	1.3133	1.8644
	4	-0.9076	-0.8575	5.3955	8.0337	-1.9595	-1.8304	1.3756	1.8680
	5	-0.9494	-0.8981	4.1167	5.8680	-1.9598	-1.8096	1.4121	1.8921
1	2	-0.7166	-0.6676	29.839	61.833	-1.9773	-1.9486	0.9652	1.6058
	3	-0.8298	-0.7854	8.4338	12.824	-1.9744	-1.8783	1.2793	1.7441
	4	-0.8847	-0.8414	5.33826	7.7013	-1.9625	-1.8453	1.3580	1.8268
	5	-0.9195	-0.8769	4.25628	6.0973	-1.9563	-1.8040	1.3949	1.8661
1.5	2	-0.7145	-0.6753	27.652	55.661	-1.9727	-1.9445	1.0212	1.6648
	3	-0.8232	-0.7870	8.0750	12.324	-1.9601	-1.8716	1.2314	1.7524
	4	-0.8707	-0.8317	5.1743	7.9021	-1.9593	-1.8315	1.3280	1.8161
	5	-0.9036	-0.8643	4.2503	5.9628	-1.9602	-1.7976	1.4296	1.9293
2	2	-0.7137	-0.6689	29.571	61.791	-1.9764	-1.9493	0.9459	1.5909
	3	-0.8188	-0.7820	7.8548	12.669	-1.9664	-1.8770	1.2148	1.7904
	4	-0.8659	-0.8324	5.2732	7.5969	-1.9651	-1.8405	1.3254	1.8318
	5	-0.8971	-0.8627	4.1960	5.8944	-1.9663	-1.8263	1.4045	1.8859
2.5	2	-0.7103	-0.6684	28.695	56.244	-1.9737	-1.9464	1.0746	1.6639
	3	-0.8173	-0.7803	7.5279	12.031	-1.9717	-1.8797	1.2505	1.7248
	4	-0.8637	-0.8334	5.2897	7.9068	-1.9700	-1.8504	1.2999	1.7596
	5	-0.8874	-0.8573	4.2208	5.9416	-1.9688	-1.8205	1.3881	1.8779
3	2	-0.7095	-0.6700	27.772	59.816	-1.9730	-1.9452	0.9777	1.6110
	3	-0.8156	-0.7806	7.8232	12.635	-1.9624	-1.8745	1.2103	1.7864
	4	-0.8567	-0.8272	5.2928	7.7807	-1.9664	-1.8263	1.3073	1.8655
	5	-0.8878	-0.8573	4.0903	5.9822	-1.9646	-1.8219	1.3738	1.9174
3.5	2	-0.7119	-0.6713	28.026	59.938	-1.9737	-1.9483	1.0317	1.7035
	3	-0.8122	-0.7744	7.7792	12.530	-1.9656	-1.8776	1.2310	1.8115
	4	-0.8607	-0.8278	5.2559	7.8655	-1.9572	-1.8424	1.3075	1.8556
	5	-0.8879	-0.8564	4.1511	5.9445	-1.9560	-1.8153	1.4402	1.9335
4	2	-0.7135	-0.6704	29.338	61.613	-1.9763	-1.9510	0.9379	1.5628
	3	-0.8109	-0.7820	7.9785	12.891	-1.9664	-1.8858	1.2750	1.8004
	4	-0.8568	-0.8233	5.2535	7.6812	-1.9626	-1.8359	1.3215	1.8401
	5	-0.8831	-0.8545	4.2507	5.9219	-1.9621	-1.8221	1.4286	1.9605
4.5	2	-0.7128	-0.6725	28.003	59.957	-1.9737	-1.9488	1.0271	1.6993
	3	-0.8113	-0.7753	8.1742	12.527	-1.9622	-1.8771	1.2054	1.7388
	4	-0.8588	-0.8283	5.0210	7.5207	-1.9526	-1.8453	1.3567	1.8113
	5	-0.8854	-0.8554	4.0604	5.9901	-1.9475	-1.8141	1.3953	1.9071

Similarly, we can determine a $100(1 - \alpha)\%$ CI for σ , through the pivotal quantity R_4 as

$$P\left(\frac{\hat{\sigma}_{BLI}}{1 + \frac{\sqrt{V_2}}{1+V_2}R_4(1 - \alpha/2)} \leq \sigma \leq \frac{\hat{\sigma}_{BLI}}{1 + \frac{\sqrt{V_2}}{1+V_2}R_4(\alpha/2)}\right) = 1 - \alpha. \quad (9)$$

Now, let us compare the BLUEs and BLIEs using the relative efficiency criterion (REC). Since the mean squared errors (MSEs) of BLUEs are equal to their corresponding variances, we have

$$MSE(\hat{\mu}_{BLU}) = \sigma^2 V_1 \quad \text{and} \quad MSE(\hat{\sigma}_{BLU}) = \sigma^2 V_2.$$

On the other hand, the MSEs of BLIEs of μ and σ can be obtained as

$$MSE(\hat{\mu}_{BLI}) = \sigma^2 \left(V_1 - \frac{V_3^2}{1 + V_2} \right) \quad \text{and} \quad MSE(\hat{\sigma}_{BLI}) = \frac{\sigma^2 V_2}{1 + V_2}.$$

Therefore, we can readily obtain the RECs of the BLIEs of μ and σ with respect to their corresponding BLUEs as follows

$$REC(\hat{\mu}_{BLI}, \hat{\mu}_{BLU}) = \frac{MSE(\hat{\mu}_{BLU})}{MSE(\hat{\mu}_{BLI})} = \frac{V_1}{V_1 - \frac{V_3^2}{1 + V_2}} \geq 1, \quad REC(\hat{\sigma}_{BLI}, \hat{\sigma}_{BLU}) = \frac{MSE(\hat{\sigma}_{BLU})}{MSE(\hat{\sigma}_{BLI})} = 1 + V_2 \geq 1.$$

Therefore, both of the BLIEs of μ and σ perform better than the corresponding BLUEs in terms of MSEs.

4. LINEAR PREDICTORS

Suppose we have observed the first n upper records $\mathbf{Y}' = (Y_{U(1)}, Y_{U(2)} \dots, Y_{U(n)})$ from the three parameter Lindley distribution with pdf given in Eq. (5) and our aim is to predict the next upper record $Y = Y_{U(n+1)}$. The best linear unbiased predictor (BLUP) of Y is given by (Arnold *et al.* [4], p. 150)

$$\hat{Y}_{BLUP} = \hat{\mu}_{BLU} + \hat{\sigma}_{BLU} \alpha_{n+1} + \omega' \beta^{-1} (\mathbf{Y} - \hat{\mu}_{BLU} \mathbf{1} - \hat{\sigma}_{BLU} \alpha),$$

where

$$\omega' = (Cov(X_{U(1)}, X_{U(n+1)}), \dots, Cov(X_{U(n)}, X_{U(n+1)})),$$

in which $X_{U(i)} = \frac{Y_{U(i)} - \mu}{\sigma}$, $i = 1, 2, \dots, n + 1$. Moreover, the mean squared prediction error (MSPE) of \hat{Y}_{BLUP} is given by (Burkschat [31])

$$\begin{aligned} MSPE(\hat{Y}_{BLUP}) &= E[(\hat{Y}_{BLUP} - Y_{U(n+1)})^2] \\ &= \sigma^2 [(1 - \omega' \beta^{-1} \mathbf{1})^2 V_1 + (\alpha_{n+1} - \omega' \beta^{-1} \alpha)^2 V_2 - \omega' \beta^{-1} \omega \\ &\quad + 2(1 - \omega' \beta^{-1} \mathbf{1})(\alpha_{n+1} - \omega' \beta^{-1} \alpha) V_3 + Var(X_{U(n+1)})]. \end{aligned}$$

Let us now consider the best linear invariant predictor (BLIP) of the next upper record value. From the results of Mann [30], the BLIP of Y can be obtained based on the BLUP of Y as follows (see also [4], p. 153)

$$\hat{Y}_{BLIP} = \hat{Y}_{BLUP} - \left(\frac{V_4}{1 + V_2} \right) \hat{\sigma}_{BLU},$$

where \hat{Y}_{BLUP} is the BLUP of $Y_{U(n+1)}$ and

$$V_4 = (1 - \omega' \beta^{-1} \mathbf{1}) V_3 + (\alpha_{n+1} - \omega' \beta^{-1} \alpha) V_2.$$

In Table 5, we reported the values of V_4 for different values of n and θ .

The MSPE of \hat{Y}_{BLIP} is given by (Burkschat [31])

$$\begin{aligned} MSPE(\hat{Y}_{BLIP}) &= E[(\hat{Y}_{BLIP} - Y_{U(n+1)})^2] \\ &= \sigma^2 \left[\frac{\alpha' \beta^{-1} \alpha + 1}{\Delta} (1 - \omega' \beta^{-1} \mathbf{1})^2 - \omega' \beta^{-1} \omega + \frac{\mathbf{1}' \beta^{-1} \mathbf{1}}{\Delta} (\alpha_{n+1} - \omega' \beta^{-1} \alpha)^2 \right. \\ &\quad \left. + Var(X_{U(n+1)}) - 2 \frac{\alpha' \beta^{-1} \mathbf{1}}{\Delta} (1 - \omega' \beta^{-1} \mathbf{1})(\alpha_{n+1} - \omega' \beta^{-1} \alpha) \right], \end{aligned}$$

where

$$\Delta = (\alpha' \beta^{-1} \alpha + 1) (\mathbf{1}' \beta^{-1} \mathbf{1}) - (\alpha' \beta^{-1} \mathbf{1})^2.$$

Now we compare the BLUP and BLIP of Y using the REC. The REC of \hat{Y}_{BLIP} relative to \hat{Y}_{BLUP} is

$$REC(\hat{Y}_{BLIP}, \hat{Y}_{BLUP}) = \frac{MSPE(\hat{Y}_{BLUP})}{MSPE(\hat{Y}_{BLIP})}.$$

Table 8 The REC of \hat{Y}_{BLIP} with respect to \hat{Y}_{BLUP} .

n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	1.3041	1.3126	1.2850	1.2667	1.2565	1.2503	1.2454	1.2363	1.2296
3	1.1105	1.1138	1.1026	1.0948	1.0892	1.0846	1.0840	1.0826	1.0813
4	1.0578	1.0594	1.0538	1.0490	1.0455	1.0440	1.0398	1.0408	1.0643
5	1.0357	1.0367	1.0328	1.0301	1.0278	1.0259	1.0259	1.0232	1.0048

In Table 8, we presented the REC of \hat{Y}_{BLIP} relative to \hat{Y}_{BLUP} for different choices of n and θ . From Table 8, we observe that the BLIP works better than BLUP in terms of MSPE.

Suppose we are now interested in PIs for $Y_{U(n+1)}$. The PIs can be constructed using the pivotal quantities [24]

$$T_1 = \frac{Y_{U(n+1)} - Y_{U(n)}}{\hat{\sigma}_{BLU}},$$

and

$$T_2 = \frac{Y_{U(n+1)} - Y_{U(n)}}{\hat{\sigma}_{BLI}}.$$

Constructing such PIs requires the percentage points of T_1 and T_2 . In Table 9, we presented the simulated percentage points of T_1 and T_2 using Monte Carlo method based on 10 000 runs and different choices of n and θ . Using the pivotal quantity T_1 , a $100(1 - \alpha)\%$ PI for $Y = Y_{U(n+1)}$ is given by

$$P\left(Y_{U(n)} + \hat{\sigma}_{BLU} T_1(1 - \alpha/2) \leq Y \leq Y_{U(n)} + \hat{\sigma}_{BLU} T_1(\alpha/2)\right) = 1 - \alpha. \quad (10)$$

Similarly, using the pivotal quantity T_2 , a $100(1 - \alpha)\%$ PI for Y is given by

$$P\left(Y_{U(n)} + \hat{\sigma}_{BLI} T_2(1 - \alpha/2) \leq Y \leq Y_{U(n)} + \hat{\sigma}_{BLI} T_2(\alpha/2)\right) = 1 - \alpha. \quad (11)$$

Table 9 Simulated percentage points of T_1 and T_2 .

θ	n	T_1				T_2			
		2.5%	5%	95%	97.5%	2.5%	5%	95%	97.5%
0.5	2	0.0712	0.1413	39.647	76.692	0.1322	0.2622	73.555	142.290
	3	0.0601	0.1207	14.392	20.527	0.0768	0.1837	22.327	32.855
	4	0.0614	0.1257	10.878	15.6182	0.0782	0.1568	14.165	20.079
	5	0.0516	0.1104	9.6559	12.550	0.0705	0.1456	11.467	15.348
1	2	0.0339	0.0681	20.278	42.555	0.0609	0.1228	40.294	88.631
	3	0.0334	0.0660	7.1150	10.249	0.0399	0.0851	10.712	16.057
	4	0.0312	0.0626	5.4481	7.5059	0.0384	0.0780	7.2854	10.282
	5	0.0280	0.0555	4.7133	6.2944	0.0356	0.0740	5.7481	7.7365
1.5	2	0.0225	0.0451	13.594	26.469	0.0435	0.0874	25.909	54.927
	3	0.0206	0.0452	4.9984	7.7554	0.0332	0.0633	7.2247	11.462
	4	0.0205	0.0412	3.6181	5.0762	0.0281	0.0555	4.8975	6.6060

(continued)

Table 9 | Simulated percentage points of T_1 and T_2 . (Continued)

θ	n	T_1				T_2			
		2.5%	5%	95%	97.5%	2.5%	5%	95%	97.5%
2	5	0.0190	0.0389	3.1497	4.2039	0.0236	0.0481	3.9040	5.2271
	2	0.0154	0.0328	10.581	21.302	0.0264	0.0594	21.275	43.214
	3	0.0145	0.0310	3.8087	3.8087	0.0229	0.0458	5.4379	8.4731
	4	0.0147	0.0304	2.6935	3.6855	0.0208	0.0415	3.5587	5.0637
	5	0.0144	0.0297	2.3256	3.1242	0.0202	0.0376	2.9245	3.9670
2.5	2	0.0129	0.0258	8.0905	16.145	0.0264	0.0515	16.379	34.770
	3	0.0110	0.0235	2.9614	4.6853	0.0183	0.0359	4.3968	7.0489
	4	0.0116	0.0224	2.2877	3.0986	0.0155	0.0311	2.9313	4.0433
	5	0.0115	0.0223	1.9293	2.5587	0.0141	0.0278	2.3385	3.2454
3	2	0.0107	0.0208	6.7242	14.437	0.0188	0.0393	13.919	30.501
	3	0.0105	0.0208	2.3687	3.5040	0.0129	0.0280	3.8110	5.7746
	4	0.0096	0.0193	1.9203	2.7296	0.0112	0.0245	2.4994	3.5728
	5	0.0091	0.0184	1.6067	2.1887	0.0111	0.0223	1.9888	2.7872
3.5	2	0.0089	0.0184	5.7860	12.1082	0.0185	0.0359	11.736	24.746
	3	0.0082	0.0179	2.1454	3.3560	0.0132	0.0271	3.2235	5.1909
	4	0.0081	0.0175	1.5263	2.1420	0.0114	0.0226	2.0896	2.8773
	5	0.0079	0.0164	1.3732	1.8560	0.0108	0.0207	1.6497	2.3159
4	2	0.0078	0.0154	5.4120	11.3077	0.0137	0.0288	10.4154	23.0863
	3	0.0074	0.0151	1.9758	2.7955	0.0102	0.0217	2.8633	4.4407
	4	0.0071	0.0140	1.3406	1.8174	0.0090	0.0190	1.8185	2.5268
	5	0.0070	0.0144	1.1706	1.5569	0.0079	0.0171	1.4869	2.0726
4.5	2	0.0071	0.0139	4.6528	9.6756	0.0141	0.0274	9.1362	18.999
	3	0.0058	0.0122	1.6271	2.4055	0.0093	0.0191	2.4354	3.6441
	4	0.0069	0.0137	1.2583	1.7334	0.0083	0.0168	1.6336	2.3774
	5	0.0067	0.0128	1.0487	1.3822	0.0077	0.0157	1.3140	1.7646

5. ILLUSTRATIVE EXAMPLES

In this section, we present two numerical examples for illustrative purposes.

5.1. Example 1 (Real Data)

Here, we consider the total annual rainfall (in inches) during March recorded at Los Angeles Civic Center from 1972 to 2006 (see the website of Los Angeles Almanac: www.laalmanac.com/weather/we08aa.htm). From these data, we observe the upper record values as follows:

$$2.70, 3.78, 4.83, 8.02, 8.37.$$

A simple plot of these five upper record values against the expected values in Table 1 for $\theta = 1$ indicates a very strong correlation (correlation coefficient as high as 0.972). Hence, the assumption that these record values come from a Lindley distribution with $\theta = 1$ is quite reasonable.

Based on these data and Tables 3 and 4, we find the BLUEs of μ and σ .

$$\begin{aligned}\hat{\mu}_{BLU} &= a_1 Y_{U(1)} + a_2 Y_{U(2)} + a_3 Y_{U(3)} + a_4 Y_{U(4)} + a_5 Y_{U(5)} \\ &= (1.2710 \times 2.70) + (0.0347 \times 3.78) + (0.0171 \times 4.83) + (0.0090 \times 8.02) - (0.3320 \times 8.37) \\ &= 0.9388,\end{aligned}$$

$$\begin{aligned}\hat{\sigma}_{BLU} &= b_1 Y_{U(1)} + b_2 Y_{U(2)} + b_3 Y_{U(3)} + b_4 Y_{U(4)} + b_5 Y_{U(5)} \\ &= (-0.2013 \times 2.70) + (-0.0039 \times 3.78) + (-0.0007 \times 4.83) + (0.0003 \times 8.02) + (0.2057 \times 8.37) \\ &= 1.1625.\end{aligned}$$

The corresponding variances and covariances of $\hat{\mu}_{BLU}$ and $\hat{\sigma}_{BLU}$ (see Table 5) are computed to be:

$$Var(\hat{\mu}_{BLU}) = 2.3899\sigma^2, \quad Var(\hat{\sigma}_{BLU}) = 0.2157\sigma^2, \quad Cov(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU}) = -0.3759\sigma^2.$$

The BLIEs of the location and scale parameters are given by $\hat{\mu}_{BLI} = 1.2982$ and $\hat{\sigma}_{BLI} = 0.9562$. The variances of $\hat{\mu}_{BLI}$ and $\hat{\sigma}_{BLI}$ are

$$Var(\hat{\mu}_{BLI}) = 2.2943\sigma^2, \quad Var(\hat{\sigma}_{BLI}) = 0.1459\sigma^2.$$

From Eqs. (6) and (8) and use of Tables 6 and 7, the 95% CIs for μ based on R_1 and R_3 are $(-6.3534, 2.3926)$ and $(-7.5330, 2.6300)$, respectively. Also from Eqs. (7) and (9), the 95% CIs for σ based on R_2 and R_4 are $(0.5633, 3.8827)$ and $(0.6786, 4.6015)$, respectively.

Suppose that we want to find the BLUP of the next record $Y_{U(6)}$ ($n+1=6$) based on the first $n=5$ observed records. From Table 1 we have $\alpha_{n+1} = \alpha_6 = 7.520$ when $\theta = 1$. From Table 2 for $\theta = 1$, we have

$$\begin{aligned}\omega' &= (Cov(X_{U(1)}, X_{U(6)}), \dots, Cov(X_{U(5)}, X_{U(6)})) \\ &= (1.4709, 2.7500, 3.9603, 5.1387, 6.2996),\end{aligned}$$

the vector of observed records is $\mathbf{Y}' = (2.70, 3.78, 4.83, 8.02, 8.37)$ and the vector of standard means is (from Table 1)

$$\alpha' = (1.500, 2.819, 4.052, 5.236, 6.389).$$

$\beta_{5 \times 5}$ is the variance-covariance matrix of the first five standard records which can be obtained from Table 2. The BLUP of $Y_{U(6)}$ is $\hat{Y}_{BLUP} = 9.6840$. Also the BLIP of $Y_{U(6)}$ is $\hat{Y}_{BLIP} = 9.4368$. In addition, using Eqs. (10) and (11) and use of Table 9, the 95% PIs for the next upper record $Y_{U(6)}$ based on the pivotal quantity T_1 and T_2 are computed as

$$(L_1(Y_{U(6)}), U_1(Y_{U(6)})) = (8.4025, 15.687),$$

and

$$(L_2(Y_{U(6)}), U_2(Y_{U(6)})) = (8.4040, 15.768),$$

respectively.

5.2. Example 2 (Simulated Data)

For given values of $\theta = 1.5$, $\mu = 0$ and $\sigma = 1$, we generated $n = 5$ upper record values from the Lindley distribution as follows:

$$1.0773, 3.1755, 3.4962, 4.5126, 4.6015.$$

The BLUEs of μ and σ are computed to be $\hat{\mu}_{BLU} = 0.0934$ and $\hat{\sigma}_{BLU} = 1.0925$. The corresponding variances and covariances of $\hat{\mu}_{BLU}$ and $\hat{\sigma}_{BLU}$ are computed to be:

$$Var(\hat{\mu}_{BLU}) = 0.9718, \quad Var(\hat{\sigma}_{BLU}) = 0.2201, \quad Cov(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU}) = 0.2332.$$

The BLIEs and the corresponding variances are

$$\hat{\mu}_{BLI} = 0.3022, \quad \hat{\sigma}_{BLI} = 0.8954$$

$$Var(\hat{\mu}_{BLI}) = 0.9352, \quad Var(\hat{\sigma}_{BLI}) = 0.1478.$$

The 95% CIs for μ based on R_1 and R_3 are $(-4.8243, 1.0841)$ and $(-4.8614, 1.847)$, respectively. Also, the 95% CIs for σ based on R_2 and R_4 are $(0.5179, 3.6608)$ and $(0.6272, 4.4363)$, respectively.

Let us now consider the BLUP and BLIP of the next record, $Y_{U(6)}$. The BLUP and BLIP are $\hat{Y}_{BLUP} = 5.4264$ and $\hat{Y}_{BLIP} = 5.2712$, respectively. Moreover, the 95% PIs for the next upper record $Y_{U(6)}$ based on the pivotal quantities T_1 and T_2 are computed as

$$(L_1(Y_{U(6)}), U_1(Y_{U(6)})) = (4.6223, 9.1944),$$

and

$$(L_2(Y_{U(6)}), U_2(Y_{U(6)})) = (4.6226, 9.2820),$$

respectively.

6. SIMULATION

In this section, we carry out an intensive Monte Carlo simulation to compare different CIs presented in Section 3. In this simulation, we have randomly generated 10 000 upper record sample $Y_{U(1)}, Y_{U(2)} \dots, Y_{U(n)}$ from the standard Lindley distribution with different choices of n and θ . We then computed the 95% CIs for the location parameter μ based on the pivotal quantities R_1 and R_3 . We also computed the 95% CIs for the scale parameter μ based on the pivotal quantities R_2 and R_4 . Table 10 presents the average confidence lengths and the corresponding coverage probabilities over 10 000 replications.

From Table 10, it is observed that the average lengths of CIs for μ and σ using BLUEs (the pivotal quantities R_1 and R_2) are smaller than the corresponding average lengths obtained using BLIEs (the pivotal quantities R_3 and R_4). Also, the coverage probabilities of all CIs are quite close to the nominal level 95%. Also, all the CIs lengths decrease as n increases.

We also computed the 95% PIs for $Y = Y_{U(n+1)}$ based on the upper record sample $Y_{U(1)}, Y_{U(2)} \dots, Y_{U(n)}$, by using the pivotal quantities T_1 and T_2 . For various choices of n and θ , Table 11 presents the means and coverage probabilities of the lengths of the PIs. From Table 11, we note that the average lengths of the PIs using BLUP (the pivotal quantity T_1) are smaller than the corresponding lengths obtained using BLIP (the pivotal quantity T_2). Also, the lengths of PIs decrease as n increases.

Table 10 | Average length (ALs) and coverage probabilities (CPs) of 95% CIs based on R_1, R_2, R_3 and R_4 .

θ	n	μ				σ			
		R_1		R_3		R_2		R_4	
		AL	CP	AL	CP	AL	CP	Average Length	Coverage Probability
0.5	2	116.968	0.9541	133.501	0.9490	32.1525	0.9533	63.0546	0.9454
	3	31.4461	0.9515	33.4645	0.9499	6.3723	0.9527	9.1189	0.9490
	4	22.1436	0.9526	23.0607	0.9500	3.8845	0.9541	4.7927	0.9507
	5	17.9984	0.9533	18.1236	0.9501	2.7875	0.9566	3.3262	0.9468
1	2	60.5284	0.9596	61.3064	0.9498	39.1433	0.9532	74.4361	0.9436
	3	14.7719	0.9469	15.4661	0.9499	6.8716	0.9515	10.1473	0.9521
	4	10.2411	0.9481	10.4097	0.9499	3.9601	0.9481	4.9239	0.9544
	5	8.7077	0.9509	8.8175	0.9510	2.8429	0.9485	3.4934	0.9450
1.5	2	33.2921	0.9498	34.8740	0.9514	34.3637	0.9477	66.8064	0.9550
	3	9.0784	0.9544	9.4398	0.9481	6.7328	0.9488	9.9251	0.9479
	4	6.5841	0.9496	6.8003	0.9492	4.0801	0.9505	5.0701	0.9515
	5	5.4468	0.9511	5.4691	0.9500	2.8973	0.9502	3.5035	0.9457
2	2	27.1318	0.9499	27.8236	0.9511	40.2612	0.9501	79.1642	0.9483
	3	6.8626	0.9499	7.0130	0.9512	7.1925	0.9503	10.7277	0.9519
	4	4.6924	0.9498	4.7060	0.9472	3.9994	0.9501	5.2650	0.9438
	5	3.8978	0.9500	3.9028	0.9503	2.8952	0.9500	3.6018	0.9537

(continued)

Table 10 | Average length (ALs) and coverage probabilities (CPs) of 95% CIs based on R_1, R_2, R_3 and R_4 . (Continued)

θ	n	μ				σ			
		R_1		R_3		R_2		R_4	
		AL	CP	AL	CP	AL	CP	Average Length	Coverage Probability
2.5	2	18.2254	0.9498	19.3539	0.9482	35.4389	0.9501	72.0573	0.9504
	3	5.0056	0.9500	5.1745	0.9492	7.0564	0.9500	11.4473	0.9518
	4	3.6115	0.9499	3.8093	0.9540	4.2128	0.9502	5.4784	0.9550
	5	3.0177	0.9481	3.0396	0.9500	2.9284	0.9498	3.6385	0.9458
3	2	15.8027	0.9499	16.1069	0.9489	40.6387	0.9501	74.1516	0.9499
	3	4.1521	0.9484	4.2726	0.9488	7.2285	0.9503	11.1147	0.9467
	4	2.9453	0.9510	2.9559	0.9476	4.3355	0.9533	5.4733	0.9553
	5	2.4669	0.9458	2.4707	0.9482	2.9882	0.9454	3.6971	0.9566
3.5	2	12.3725	0.9499	14.1363	0.9500	35.9585	0.9501	72.8842	0.9510
	3	3.3948	0.9499	3.6863	0.9470	7.2159	0.9500	11.6175	0.9536
	4	2.4362	0.9502	2.5758	0.9534	4.2576	0.9500	5.3757	0.9537
	5	2.0773	0.9499	2.0885	0.9514	3.0120	0.9501	3.6965	0.9555
4	2	12.1945	0.9500	12.5072	0.9506	41.1703	0.9496	72.9495	0.9451
	3	3.0108	0.9498	3.2115	0.9479	7.3866	0.9501	11.6581	0.9485
	4	2.1256	0.9498	2.1618	0.9509	4.1769	0.9501	5.5760	0.9502
	5	1.7783	0.9499	1.7858	0.9502	3.0125	0.9501	3.7677	0.9483
4.5	2	9.2964	0.9499	10.6092	0.9504	36.1874	0.9500	74.7037	0.9490
	3	2.5535	0.9499	2.7794	0.9493	7.3018	0.9500	11.6565	0.9506
	4	1.8548	0.9500	1.8588	0.9492	4.2612	0.9501	5.4964	0.9504
	5	1.5655	0.9500	1.5823	0.9509	3.0399	0.9502	3.6726	0.9527

Table 11 | Average length (ALs) and coverage probabilities (CPs) of 95% CIs based on T_1 and T_2 .

θ	n	T_1		T_2	
		AL	CP	AL	CP
0.5	2	77.144	0.9543	77.145	0.9522
	3	20.512	0.9419	23.092	0.9449
	4	15.647	0.9431	15.699	0.9435
	5	12.397	0.9467	12.513	0.9463
1	2	42.748	0.9471	47.2928	0.9455
	3	10.248	0.9532	11.199	0.9534
	4	7.5315	0.9449	7.9066	0.9449
	5	6.2920	0.9485	6.3604	0.9467
1.5	2	26.548	0.9574	28.921	0.9466
	3	7.7232	0.9535	7.8957	0.9440
	4	5.0579	0.9579	5.0833	0.9560
	5	4.1784	0.9553	4.2582	0.9491
2	2	21.192	0.9483	22.376	0.9442
	3	3.7841	0.9437	5.7966	0.9441
	4	3.6817	0.9473	3.8702	0.9423
	5	3.096	0.9455	3.2107	0.9477

(continued)

Table 11 | Average length (ALs) and coverage probabilities (CPs) of 95% CIs based on T_1 and T_2 . (Continued)

θ	n	T_1		T_2	
		AL	CP	AL	CP
2.5	2	16.0882	0.9425	17.9052	0.9466
	3	4.7601	0.9433	4.8020	0.9456
	4	3.0764	0.9465	3.0770	0.9467
	5	2.5529	0.9480	2.6390	0.9480
3	2	14.512	0.9570	15.762	0.9554
	3	3.4745	0.9564	3.6993	0.9578
	4	2.7251	0.9522	2.7274	0.9509
	5	2.1801	0.9485	2.2579	0.9482
3.5	2	12.1655	0.9426	12.7269	0.9464
	3	3.3610	0.9458	3.5346	0.9484
	4	2.1383	0.9577	2.1902	0.9546
	5	1.8356	0.9521	1.8580	0.9522
4	2	11.3436	0.9532	11.8191	0.9535
	3	2.7905	0.9555	3.0067	0.9529
	4	1.8135	0.9460	1.9197	0.9450
	5	1.5562	0.9446	1.6797	0.9462
4.5	2	9.6661	0.9527	9.6663	0.9557
	3	2.4182	0.9464	2.4784	0.9458
	4	1.7283	0.9457	1.8020	0.9463
	5	1.3776	0.9444	1.4251	0.9417

COMPETING INTERESTS

The authors declare that they have no conflict of interests.

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APPENDIX A. DERIVATION OF EQ. (2) WHEN $n = m + 1$.

$$\begin{aligned}
 \mu_{m,m+1}^{(r,s)} &= \frac{\theta^4}{\Gamma(m)(\theta+1)} \int_0^\infty x^r \left\{ -\log\left(\frac{\theta+1+\theta x}{\theta+1}\right) + \theta x \right\}^{m-1} \frac{1+x}{\theta+1+\theta x} \\
 &\quad \times \int_x^\infty y^s (1+y) e^{-\theta y} dy dx \\
 &= \frac{\theta^3}{\Gamma(m)(\theta+1)} \int_0^\infty x^r \left\{ -\log\left(\frac{\theta+1+\theta x}{\theta+1}\right) + \theta x \right\}^{m-1} \frac{1+x}{\theta+1+\theta x} \\
 &\quad \times \left[\frac{x^{s+1}}{s+1} + \sum_{j=0}^s \frac{\Gamma(s)[1+s/\theta]}{\theta^{s-j}j!} x^j \right] e^{-\theta x} dx \\
 &= \frac{(\theta+1)^{r-1} e^{\theta+1}}{\Gamma(m)\theta^{r-1}} \int_1^\infty (u-1)^{r-1} \left\{ -\log u + (\theta+1)(u-1) \right\}^{m-1} [\theta + (\theta+1)(u-1)] \\
 &\quad \times \left[\frac{[(\theta+1)(u-1)/\theta]^{s+1}}{s+1} + \sum_{j=0}^s \frac{\Gamma(s)[1+s/\theta]}{\theta^s j!} [(\theta+1)(u-1)]^j \right] e^{-(\theta+1)u} du \\
 &= \frac{(\theta+1)^{r-1} e^{\theta+1}}{\Gamma(m)\theta^{r-1}} \sum_{i=0}^{m-1} \binom{m-1}{i} (-1)^i (\theta+1)^{m-i-1} \\
 &\quad \times \int_1^\infty (u-1)^{r+m-2-i} \{ -\log u \}^i [\theta + (\theta+1)(u-1)] \\
 &\quad \times \left(\frac{[(\theta+1)(u-1)/\theta]^{s+1}}{s+1} + \sum_{j=0}^s \frac{\Gamma(s)[1+s/\theta]}{\theta^s j!} [(\theta+1)(u-1)]^j \right) e^{-(\theta+1)u} du \\
 &= \frac{(\theta+1)^{r-1} e^{\theta+1}}{\Gamma(m)\theta^{s+r-1}} \sum_{i=0}^{m-1} \binom{m-1}{i} (-1)^i (\theta+1)^{m-i} \\
 &\quad \times \left\{ \frac{(\theta+1)^s}{s+1} \sum_{q=1}^{r+m-i+s} \binom{r+m-1-i+s}{q-1} (-1)^{r+m+s} h(q, i, \theta) \right. \\
 &\quad + \frac{(\theta+1)^{s+1}}{(s+1)\theta} \sum_{q=1}^{r+m-i+s+1} \binom{r+m-i+s}{q-1} (-1)^{r+m+s+1} h(q, i, \theta) \\
 &\quad + \sum_{j=0}^s (\theta+1)^{j-1} \frac{\Gamma(s)[\theta+s]}{j!} \\
 &\quad \times \sum_{q=1}^{r+m-1+j-i} \binom{r+m-2+j-i}{q-1} (-1)^{r+m-1+j} h(q, i, \theta) \\
 &\quad \left. + \sum_{j=0}^s (\theta+1)^j \frac{\Gamma(s)[1+s/\theta]}{j!} \sum_{q=1}^{r+m+j-i} \binom{r+m-1+j-i}{q-1} (-1)^{r+m+j} h(q, i, \theta) \right\},
 \end{aligned}$$

where

$$h(q, i, \theta) = (-1)^{-i-q} \frac{\partial^i}{\partial q^i} \{ (\theta+1)^{-q} \Gamma(q, \theta+1) \}.$$