A Study of the First-Order Continuous-Time Bilinear Processes Driven by Fractional Brownian Motion

Abdelouahab Bibi¹, and Fateh Merahi¹,²,*

¹Department of Mathematics, UMC(1), Constantine, Algeria
²Department of Mathematics, UMC(1), Constantine, Algeria

ABSTRACT

The continuous-time bilinear (COBL) process has been used to model non linear and/or non Gaussian datasets. In this paper, the first-order continuous-time bilinear COBL (1, 1) model driven by a fractional Brownian motion ($fBm$ for short) process is presented. The use of $fBm$ processes with certain Hurst parameter permits to obtain a much richer class of possibly long-range dependent property which are frequently observed in financial econometrics, and thus can be used as a power tool for modelling irregularly series having memory. So, the existence of Itô’s solutions and there chaotic spectral representations for time-varying COBL (1, 1) processes driven by $fBm$ are studied. The second-order properties of such solutions are analyzed and the long-range dependency property are studied.

1. COBL(1,1) Driven by Fractional Brownian Motion

In discrete-time series analysis, the assumption of linearity and/or Gaussianity is frequently made. Unfortunately these assumption lead to models that fail to capture certain phenomena commonly observed in practice such as limit cycles, asymmetric distribution, leptokurtosis, etc. Motived by these deficiencies, non linear parametric modelling of time series has attracted considerable attention in recent years. Indeed, one of the most useful class of non-linear time series models is the bilinear specification obtained by adding to an Autoregressive moving average (ARMA) model one or more interaction components between the observed series and the innovations. However, it is observed that these models are not be able to give full information about some datasets exhibit unequally spaced observations and hence the resort to a continuous-time bilinear process has been used to model non linear and/or non Gaussian datasets. In this paper, we consider a continuous-time bilinear (COBL) processes

\[ dX(t) = (\alpha(t)X(t) + \mu(t))dt + (\beta(t) + \gamma(t)X(t))dW^h(t) \quad t \geq t_0, \quad X(t_0) = X_0 \]  

(1)

denoted hereafter COBL (1, 1). The parameters $\alpha(t)$, $\mu(t)$, $\gamma(t)$ and $\beta(t)$ are differentiable complex deterministic functions subject to the following assumption

Condition 1

A1 For all $T > t_0$, $\int_t^T |\alpha(t)|dt < \infty$, $\int_t^T |\mu(t)|dt < \infty$, $\int_t^T |\gamma(t)|dt < \infty$. A2 $\alpha(t)$, $\mu(t)$, $\beta(t) \in \mathbb{C}$ and $Re\{\gamma(t)\} = 0$ and $Re\{\alpha(t)\} < 0$, for all $t \geq t_0$.

In Eq. (1) ($W^h(t)$)$_{t \geq 0}$ is a real $fBm$ with Hurst parameter $h \in [0, \frac{1}{2}]$ defined on a basic given filtered stochastic probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, its covariance kernel is $Cov(W^h(t), W^h(s)) = \frac{\kappa(h)}{2} (\langle t^h s^{h+1} + s^h t^{h+1} - t^h s^{h+1} \rangle)$, for all $t, s \geq 0$, where $\kappa(h) =$

*Corresponding author. Email: merahfateh@yahoo.fr
The SDE (complex $\alpha$ SDE) in-depth detailed mathematical framework of the pertinent properties of the Markovian or a semimartingales processes and hence the usual calculus cannot be used, so a different calculus is required. This non Markovian process, so other particular models can be deduced.

Therefore, from the SDE Eq. (2) and the Itô formula Eq. (3) we obtain

$$dY(t) = \frac{\partial U}{\partial t} (t, X(t)) dt + \frac{\partial U}{\partial x} (t, X(t)) dW^h(t).$$

So, the Itô’s solution of the SDE Eq. (1) is given by
Theorem 2.1. Under the assumption 1, the unique Itô's solution of SDE Eq. (1) in $L_2 \left( \mathcal{F}^{(b)} \right)$ is given by

$$
X(t) = \Phi_b(t, t_0) \left\{ X(t_0) + \int_{t_0}^{t} \Phi_b^{-1}(s, t) \mu(s) \, ds + \int_{t_0}^{t} \Phi_b^{-1}(s, t) \beta(s) \, dW^b(s) \right\}, \quad t \geq t_0
$$

where $\Phi_b(t, t_0) = \exp \left\{ \int_{t_0}^{t} \alpha(s) \, ds + \int_{t_0}^{t} \gamma(s) \, dW^b(s) \right\}$ with $\Phi_b(t_0, t_0) = 1$ and the stochastic integral $\int_{t_0}^{t} \gamma(s) \, dW^b(s)$ is defined in Riemann's sense in probability.

Proof. First it is no difficult to see that $\Phi_b(t, t_0)$ is the unique solution of SDE

$$
d\Phi_b(t, t_0) = \alpha(t) \Phi_b(t, t_0) \, dt + \gamma(t) \Phi_b(t, t_0) \, dW^b(t).
$$

Now, set $\Phi_b(t, t_0) = \exp \left\{ Y(t) \right\}$, $Z(t) = X(0) + \int_{t_0}^{t} e^{-\int_{t_0}^{s} \mu(u) \, du} \beta(s) \, dW^b(s)$ and let $X(t) = U(Y(t), Z(t))$, where $U$ is the function defined by $U(x,y) = e^x$. The fractional Itô formula Eq. (3) and the expression Eq. (4) gives

$$
dX(t) = \frac{dU}{dx} (Y(t), Z(t)) \, dY(t) + \frac{dU}{dy} (Y(t), Z(t)) \, dZ(t)
$$

$$
= \gamma(t) \, dW^b(t) + \frac{dU}{dy} (Y(t), Z(t)) \, dZ(t)
$$

$$
= X(t) \, dY(t) + \frac{dU}{dy} (Y(t), Z(t)) \, dZ(t)
$$

and hence the result follows.

Remark 1. If $\beta(t) = 0$, then the Itô solution of SDE Eq. (1) reduces to

$$
X(t) = \Phi_b(t, t_0) \left\{ X(t_0) + \int_{t_0}^{t} \Phi_b^{-1}(s, t) \mu(s) \, ds \right\}, \quad t \geq t_0
$$

and when $\gamma(t) = 0$ and $\beta(t) \neq 0$, this is provides a solution of Gaussian OU process, therefore if we are interested in non-Gaussian solution of Eq. (1), it is necessary to assume that $\mu(t)^T + \beta(t)^T > 0$ and $\gamma(t) \neq 0$.

Remark 2. In time-invariant case, with $\Re \{ \gamma \} = 0$ and $\Re \{ \alpha \} < 0$, then the fractional Itô solution of SDE Eq. (1) can be written as

$$
X(t) = \mu \left\{ \int_{-\infty}^{t} \exp \left\{ \alpha(t-s) + i\gamma \left( W^b(t) - W^b(s) \right) \right\} \, ds + \beta \right\} \int_{-\infty}^{t} \exp \left\{ \alpha(t-s) \right\} \, dW^b(s).
$$

Remark 3. For any $t \geq t_0$, let $-\xi(t) = \int_{t_0}^{t} \alpha(s) \, ds + \int_{t_0}^{t} \gamma(s) \, dW^b(s)$ and $\eta(t) = \int_{t_0}^{t} \mu(s) \, ds + \int_{t_0}^{t} \beta(s) \, dW^b(s)$, then the solution process Eq. (5) may be rewritten as

$$
X(t) = e^{-\xi(t)} \left\{ X(t_0) + \int_{t_0}^{t} e^{\eta(s)} \, dW^b(s) \right\}, \quad t \geq t_0
$$

is the solution process of generalized Ornstein-Uhlenbeck (GOU) process driven by an fBm defined by $dX(t) = -\xi(t) \, X(t) \, dt + d\eta(t)$, $t \geq t_0$, $X(t_0) = X_0$.

2.2. The Frequency Approach

In this subsection, we discuss a second approach to solve the SDE Eq. (1) based on the spectral representation. Indeed, it is now well known that for any regular second-order process $X(t)_{t \geq 0}$ (i.e., $X(t)$ is $\mathcal{F}^{(b)}$-measurable not necessary stationary, belonging to $L_2 \left( \mathcal{F}^{(b)} \right)$) admits the so-called Wiener-Itô (or Stratonovich) spectral representation, i.e.,

$$
X(t) = g_0(0) + \sum_{r \geq 1} \frac{1}{r!} \int_{\mathcal{R}} g_r (\lambda r) \, e^{i\lambda t} \, dW^b \left( \lambda r \right) + \int_{\mathcal{R}} \int_{\mathcal{R}} \exp \left\{ i \lambda \gamma \left( \lambda r \right) \right\} \, dZ (\lambda r) \Big( \lambda r \Big).
$$
defined by Eq. (11) determines a second-order process are

First, applying of the diagram formula for the nonlinear term proofs and that state that for all

Let us recall here the so-called the diagram formula for Wiener–Itô representation Eq. (8) which play an important role in some subsequent

The spectral representation of the solution process of SDE Eq. (1) is given in the following theorem

Theorem 2.2. Assume that the process \((X(t))_{t \geq 0}\) generated by the SDE Eq. (1) has a regular second-order solution. Then, the evolutionary symmetrized transfer functions \(\{g_r(\Delta_{r \theta}(\cdot))\}_{r \geq 0}, r \in \mathbb{N}\) of such solution are given by the symmetrization of the solution of the following first order ordinary differential equations

\[
\begin{align*}
\dot{g}^{(1)}(\Delta_{r \theta}(\cdot)) &= \begin{cases}
\sigma(t) g_r(0) + \mu(t) & (10) \\
\left(\sigma(t) - \beta_r(\cdot)\right) g_r(\Delta_{r \theta}(\cdot)) + r \Xi_{r \theta(n)}(\cdot) \beta(t) + \gamma(t) \left(rg_r(\Delta_{r \theta}(\cdot)) + \frac{r}{2r} \int g_r(\Delta_{r \theta}(\cdot)) \lambda_{r+1} \lambda_{2r+1} d\lambda_{r+1}\right) & r \geq 1
\end{cases}
\end{align*}
\]

where the superscript \(^{(i)}\) denotes \(j\text{-}folds differentiation with respect to \(t\) and where \(\Sigma_{r \theta} = \sum_{\alpha \leq r} \lambda_{\alpha} \).

Proof. First, applying of the diagram formula for the nonlinear term \(\frac{dW^h(t)}{dt}\) we get

\[
X(t) \frac{dW^h(t)}{dt} = \begin{cases}
\sigma(t) g_r(0) + \mu(t) \frac{d}{dt} \left(rg_r(\Delta_{r \theta}(\cdot)) + \frac{r}{2r} \int g_r(\Delta_{r \theta}(\cdot)) \lambda_{r+1} \lambda_{2r+1} d\lambda_{r+1}\right) & (11)
\end{cases}
\]

Second, we insert the spectral representation Eq. (8) of the process \((X(t))_{t \geq 0}\) and the last expression of \(X(t) \frac{dW^h(t)}{dt}\) in the Eq. (1) the results follows.

Remark 4. The existence and uniqueness of the solution Eq. (10) is ensured by general results on linear ordinary differential equations, so

\[
\begin{align*}
g_r(\Delta_{r \theta}(\cdot)) &= \begin{cases}
\phi_t(0) \left(\sum_{\alpha \leq r} \phi_{r \alpha}^{-1}(0) \left(\mu(\alpha) + \gamma(\alpha) \frac{d}{d\alpha} g_r(\alpha) \lambda_{r} \lambda_{2r} d\lambda_{r}\right) ds\right), r = 0 \\
\phi_t(\cdot) \left(\sum_{\alpha \leq r} \phi_{r \alpha}^{-1}(\cdot) \left(\beta(\alpha) + \gamma(\alpha) \frac{d}{d\alpha} g_r(\alpha) \lambda_{r} \lambda_{2r} d\lambda_{r}\right) ds\right), r = 1 \\
\phi_t(\Delta_{r \theta}(\cdot)) \left(\sum_{\alpha \leq r} \phi_{r \alpha}^{-1}(\Delta_{r \theta}(\cdot)) \gamma(\alpha) \left(rg_r(\Delta_{r \theta}(\cdot)) + \frac{r}{2r} \int g_r(\Delta_{r \theta}(\cdot)) \lambda_{r+1} \lambda_{2r+1} d\lambda_{r+1}\right) ds\right), r \geq 2
\end{cases}
\end{align*}
\]

in which \(\phi_t(\cdot) = \exp \left\{ 1 \int_0^t \left(\sigma(s) - \beta_r(\cdot)\right) ds\right\}\).

Remark 5. Noting that beside the condition Eq. (9) a necessary conditions for that the evolutionary transfer functions \(\{g_r(\Delta_{r \theta}(\cdot)), r \in \mathbb{N}\}\) defined by Eq. (11) determines a second-order process are

\[
\int g_r(\Delta_{r \theta}(\cdot)) \lambda_{r+1} \lambda_{2r+1} d\lambda_{r+1} < +\infty \text{ and } \int g_r(\Delta_{r \theta}(\cdot)) \lambda_{r+1} \lambda_{2r+1} d\lambda_{r+1} < +\infty
\]
for all $t \geq t_0$. These conditions are extremely difficult to be verified, except in time-invariant case when an explicit formula for the transfer functions are given (see for instance [9]). It is worth noting that if $\text{Re} \{\gamma (t)\} \neq 0$, the SDE Eq. (1) may be haven't a second-order solution, but it does if $\gamma (t)$ is purely imaginary. So in what follows, we consider the particular SDE

$$dX(t) = (\alpha (t) X(t) + \mu (t)) dt + i\gamma (t) X(t) dW^h (t), \ t \geq t_0, X(t_0) = X_0$$

(12)

and assume that

A3. $\alpha (t), \mu (t) \in \mathbb{C}, \gamma (t) \in \mathbb{R}$ and $\text{Re} \{\alpha (t)\} < 0, \gamma (t) \neq 0$ for all $t \geq t_0$.

Under the condition A3, the Itô’s solution of Eq. (12) reduces to

$$X(t) = \Phi_h(t, t_0) X(t_0) + \int_{t_0}^{t} \Phi_h^{-1}(s, t_0) \mu(s) ds$$

(13)

in which the function $\gamma (t)$ is replaced by $i\gamma (t)$. The spectral representation of Eq. (12) is given in the following lemma

Lemma 1. Assume that the process $(X(t))_{t \geq t_0}$ generated by the model Eq. (12) has a regular second-order solution. Then, the symmetrized evolutionary transfer functions $(\mathring{g} (\lambda_{ir})))_{r \in \mathbb{N}}$ of such solution may be obtained by the symmetrization of the following functions

$$g (\lambda_{ir}) = \begin{cases}
\varphi_r (0) \left[ g_u (0) + \int_{t_0}^{t} \varphi_r^{-1} (s, 0) \left( \mu(s) + i\gamma (s) \int_{\mathbb{R}} \frac{g_u (\lambda) d\lambda}{|\lambda|^2} \right) ds \right], & r = 0 \\
\varphi_r (\lambda_{ir}) \left[ g_u (\lambda_{ir}) + \int_{t_0}^{t} \varphi_r^{-1} (s, \lambda_{ir}) \gamma (s) \left( r g_u (\lambda_{ir+1}) + \int_{\mathbb{R}} \frac{g_u (\lambda_{ir+1}) d\lambda_{ir+1}}{|\lambda_{ir+1}|^2} \right) ds \right], & r \geq 1
\end{cases}$$

(14)

Lemma 2. In time-invariant case we obtain

$$g (\lambda_{ir}) = \begin{cases}
g (\lambda_{ir}) = -\frac{1}{\alpha} \left[ \mu + \frac{i\gamma}{2\pi} \int_{\mathbb{R}} g_u (\lambda) |\lambda|^{-2} d\lambda \right] & \text{if } r = 0 \\
-\frac{i\gamma}{(\alpha - i\lambda_{ir})} \left[ \int_{\mathbb{R}} \frac{g (\lambda_{ir+1}) d\lambda_{ir+1}}{|\lambda_{ir+1}|^2} \right] & \text{if } r \geq 1
\end{cases}$$

so, its symmetrized version may be written as

$$\mathring{g} (\lambda_{ir}) = \text{Sym} \{g (\lambda_{ir})\} = \mu (i\alpha) \prod_{r=1}^{\infty} \left( au - \frac{\gamma^2}{2} k (h) a^{2h+1} \right) \int_{0}^{\infty} \frac{1 - e^{-iu}}{iu} du.$$

3. THE MOMENTS PROPERTIES AND THE SECOND-ORDER STRUCTURE

In this section, we analyze the spectrum, i.e., the second-order structure of the process $(X(t))_{t \geq t_0}$ solution of the SDE Eq. (1). For this purpose let $(\Psi_h (t, t_0))_{t \geq t_0}$ be the mean function of the process $(\Phi_h (t, t_0))_{t \geq t_0}$, and set $W_h (t, u, v) = h(2h + 1)k(h) \int_{0}^{v} \gamma (v_1) \frac{r(v_2)}{v_2} dv_1 - v_2^{2h+1} dv_2 dv_2$, $u \leq t, v \leq s$. Then, we have

Lemma 3. Under the conditions of 1, we have the following assertions

1. $\Psi_h (t, t_0) = \prod_{r=0}^{\infty} \left( au - \frac{\gamma^2}{2} k (h) a^{2h+1} \right) \int_{0}^{\infty} \frac{1 - e^{-iu}}{iu} du$.
2. $E \{\Phi_h (t, t_0) \Phi_h^{-1} (u, t_0)\} = \Psi_h (t, u)$ for $t \geq u$.
3. $E \{\Phi_h (t, t_0) \Phi_h (s, t_0)\} = \Psi_h (t, t_0) \Psi_h (s, t_0) \exp \{W_h (t, t_0, s, t_0)\}$ for $t \geq s$.
4. $E \{\Phi_h (t, t_0) \Phi_h^{-1} (v, t_0)\} = \Psi_h (t, t_0) \Psi_h^{-1} (s, v) \exp \{W_h (t, t_0, s, v)\}$ for $t \geq s \geq v$.
5. $E \{\Phi_h (t, t_0) \Phi_h^{-1} (u, t_0) \Phi_h^{-1} (v, t_0)\} = \Psi_h (t, u) \Psi_h^{-1} (v, t_0) \exp \{W_h (t, u, s, v)\}$ for $t \geq s \geq v$. 
The assertions of Lemma 3 follows upon observation that by using the expectation of exponential Gaussian process, we have

$$
\Psi_h(t, t_0) = \exp \left\{ \int_{t_0}^{t} a(v_1) dv_1 + \frac{1}{2} \mathbb{E} \left\{ \int_{t_0}^{t} \gamma(v_1) dW^h(v_1) \right\} \right\}
$$

$$
= \exp \left\{ \int_{t_0}^{t} a(v_1) dv_1 + h (2h + 1) \frac{\kappa(h)}{2} \int_{t_0}^{t} \gamma(v_1) \gamma(v_2) dv_1 - v_2^{2h-1} dv_1 v_2 \right\}
$$

and for $t \geq u$

$$
E \left\{ \Phi_h(t, t_0) \Phi_h^{-1}(u, t_0) \right\} = \exp \left\{ \int_{t_0}^{t} a(v_1) dv_1 + \frac{1}{2} \mathbb{E} \left\{ \int_{t_0}^{t} \gamma(v_1) dW^h(v_1) \right\} \right\}
$$

$$
= \exp \left\{ \int_{t_0}^{t} a(v_1) dv_1 + h (2h + 1) \frac{\kappa(h)}{2} \int_{u}^{t} \gamma(v_1) \gamma(v_2) dv_1 - v_2^{2h-1} dv_1 v_2 \right\}
$$

and so on the rest are immediate.

Lemma 4. Under the condition of Lemma 3, the mean function ($m_h(t) = E \{X(t)\}$) is given by

$$
m_h(t) = \Psi_h(t, t_0) m(t_0) + \int_{t_0}^{t} \Psi_h(t, u) \mu(u) du, t \geq t_0.
$$

and the covariance function ($R_h(t, s) = E \{(X(t) - m_h(t))(X(s) - m_h(s))\}$ is given by

$$
R_h(t, s) = \Psi_h(t, t_0) \Psi_h(s, t_0) \exp \{ W_h(t, t_0, s, t_0) \} R(t_0) + \Psi_h(t, t_0) \Psi_h(s, t_0) \left[ \exp \{ W_h(t, t_0, s, t_0) \} - 1 \right] m(t_0) t^2
$$

$$
+ m(t_0) \int_{t_0}^{t} \Psi_h(t, t_0) \Psi_h(s, v) \left[ \exp \{ W_h(t, t_0, s, v) \} - 1 \right] \mu(v) dv
$$

$$
+ m(t_0) \int_{t_0}^{t} \Psi_h(s, t_0) \Psi_h(t, u) \left[ \exp \{ W_h(t, u, s, t_0) \} - 1 \right] \mu(u) du
$$

$$
+ \int_{t_0}^{t} \int_{t_0}^{t} \Psi_h(t, u) \Psi_h(s, v) \left[ \exp \{ W_h(t, u, s, v) \} - 1 \right] \mu(v) \mu(u) dv du
$$

$$
+ h (2h + 1) \kappa(h) \int_{t_0}^{t} \int_{t_0}^{t} \Psi_h(t, u) \Psi_h(s, v) \exp \{ W_h(t, u, s, v) \} \beta(v) \beta(u) du - v_2^{2h-1} dv_1 dv_2.
$$

Proof. From the Itô’s solution Eq. (5), we can obtain

$$
m_h(t) = E \{X(t)\} = E \{ \Phi_h(t, t_0) X(t_0) \} + \int_{t_0}^{t} E \{ \Phi_h(t, t_0) \Phi_h^{-1}(u, t_0) \} \mu(u) du
$$

$$
= \Psi_h(t, t_0) m(t_0) + \int_{t_0}^{t} \Psi_h(t, u) \mu(u) du.
$$
Since $W^h(t)$ independent of $X(t_0)$, then $E \{ G_t (t_0, t) X(t_0) \} = E \{ G_t (t_0, t) \} E \{ X(t_0) \} = \Psi_{\Phi} (t, t_0) m_t (t_0)$. In order to evaluate the expression of $R_t (t, s)$ we use the Itô’s solution Eq. (5) to obtain

$$E \{ X(t) \} = E \{ X(t_0) \} + m_t (t_0) \int_0^t E \{ \Phi_t (t_0, t_0) \} \Phi_t (s, t_0) E \{ \Phi_t^\dagger (v, t_0) \} \mu (v) dv$$

$$+ \int_0^t E \{ \Phi_t (t_0, t_0) \} \Phi_t (s, t_0) E \{ \Phi_t^\dagger (v, t_0) \} \mu (v) \mu (u) dv du$$

$$+ \int_0^t \int_0^t E \{ \Phi_t (t_0, t_0) \} \Phi_t (s, t_0) E \{ \Phi_t^\dagger (v, t_0) \} \mu (v) \mu (u) dv du,$$

In other hand

$$m_t (t_0) m_t (s) = E \{ \Phi_t (t_0, t_0) \} \Psi_t (s, t_0) m_t (t_0),$$

$$+ \int_0^t \int_0^t E \{ \Phi_t (t_0, t_0) \} \Psi_t (s, t_0) \mu (v) dv du.$$

the fact that $R_t (t, s) = E \{ X(t) \} E \{ X(s) \}$ the expression for $R_t (t, s)$ the expression for $R_t (t, s)$ follows.

**Lemma 5.** Consider the time-invariant process $(X(t))_{t \geq 0}$ generated by SDE Eq. (1). Then under the condition 1, the mean and covariance functions of the solution process $(X(t))_{t \geq 0}$ are given by

$$m_t = \mu \int_0^\infty K_h (u) du,$$

$$R_t (\tau) = |\mu|^2 \int_0^\infty K_h (u_1) K_h (u_2) \left\{ \exp \left\{ - \frac{\tau^2}{2} \kappa (h) W^h (u_1, u_2) \right\} - 1 \right\} du_1 du_2$$

$$+ |\beta|^2 h (2h + 1) \kappa (h) \int_0^\infty K_h (u_1) K_h (u_2) \exp \left\{ - \frac{\tau^2}{2} \kappa (h) W^h (u_1, u_2) \right\} du_1 du_2,$$

where

$$W^h (u_1, u_2) = |\tau|^{2h + 1} - |r - u_1|^{2h + 1} - |r - u_2|^{2h + 1} + |r - u_1 + u_2|^{2h + 1},$$

and $K_h (t) = \exp \left\{ - \frac{\tau^2}{2} \kappa (h) t^{2h + 1} \right\}$.

**Proof.** Straightforward and hence omitted.

**Corollary 1.** Consider the time-invariant version of the SDE Eq. (12), then $\lim_{c \to 0} R (\tau) = 1$ for some constant $c$ and $0 < \delta < 1$, this means that the solution process exhibits long range dependence. In this case the dependence between $X(t)$ and $X(t + \tau)$ decays slowly as $\tau \to +\infty$ and $\int_{-\infty}^\infty R (\tau) d\tau = \infty$.

**Proof.** First we have

$$\exp \left\{ - \frac{\tau^2}{2} \kappa (h) \left( |\tau|^{2h + 1} - |r - u_1|^{2h + 1} - |r - u_2|^{2h + 1} + |r - u_1 + u_2|^{2h + 1} \right) \right\}$$
\[
\text{It can be shown that } (\lambda g) \text{ is a second-order stationary process which is orthogonal to the first two terms. The symmetrized transfer functions }
\]
\[
\gamma_2 = \mu (\lambda_1) e^{i\lambda_1} (t) + \mu (\lambda_2) e^{i\lambda_2} (t) + \xi (t).
\]

where \( \xi (t) \) is a second-order stationary process which it is orthogonal to the first two terms. The symmetrized transfer functions \( \tilde{g} (\lambda_1) \) and \( \tilde{g} (\lambda_2) \) are given by

\[
g (\lambda_1) = \mu (\mu) \int_0^\infty K_0 (u) \frac{1 - e^{-iu\lambda_1}}{iu\lambda_1} du \text{ and } g (\lambda_1, \lambda_2) = \mu (\mu) \int_0^\infty K_0 (u) \prod_{j=1}^2 \frac{1 - e^{-iu\lambda_j}}{iu\lambda_j} du.
\]

It can be shown that

\[
\text{Let } \delta = -(2h - 1), \text{ it is clear } 0 < \delta < 1 \text{ because } 0 < h < \frac{1}{2}, \text{ then we have}
\]

\[
\lim_{t \to +\infty} \frac{2}{\gamma_1^2} \kappa (h) \eta (u_1, u_2) - 1 = \lim_{t \to +\infty} \exp \left\{ \frac{2}{\gamma_1^2} \kappa (h) \eta (u_1, u_2) \right\} - 1 = \frac{2}{\gamma_1^2} \kappa (h) \eta (u_1, u_2).
\]

\[
\text{It follows that}
\]

\[
\lim_{t \to +\infty} R (\gamma) = \int_0^\infty \left( \int_0^\infty \left( \int_0^\infty \frac{2}{\gamma_1^2} \kappa (h) \eta (u_1, u_2) \right) - 1 \right) du_1 du_2
\]

\[
= \frac{2}{\gamma_1^2} \kappa (h) \eta (u_1, u_2) \int_0^\infty \left( \int_0^\infty \left( \int_0^\infty \frac{1}{\gamma_1^2} \kappa (h) \eta (u_1, u_2) \right) - 1 \right) du_1 du_2
\]

\[
= \frac{2}{\gamma_1^2} \kappa (h) \eta (u_1, u_2) \int_0^\infty \left( \int_0^\infty \left( \int_0^\infty \frac{1}{\gamma_1^2} \kappa (h) \eta (u_1, u_2) \right) - 1 \right) du_1 du_2
\]

\[
\text{Hence, the process } (X(t))_{t \geq 0} \text{ generated by the SDE Eq. (12) with time-invariant parameters is a long memory process.}
\]

3.1. Third-Order Structure of COBL(1,1) Process

For the sake of convenience and simplicity, we shall consider the time-invariant version of the SDE Eq. (1). Moreover, we assume the process solution admits the spectral representation Eq. (8) in which the symmetrized version of transfer functions \( g (\Delta) \) may be written as

\[
g (\Delta) = \mu (\mu) \int_0^\infty K_0 (u) \frac{1 - e^{-iu\lambda}}{iu\lambda} du, \forall \lambda \geq 0.
\]

Then using the representation Eq. (8) we can obtain the following approximation

\[
X (t) = g (0) + \int_\mathbb{R} \left( g (\lambda_1) e^{i\lambda_1} dZ (\lambda_1) + \int_\mathbb{R} g (\lambda_2) e^{i\lambda_2} dZ (\lambda_2) + \xi (t) \right)
\]

\[
= X^{(1)} (t) + X^{(2)} (t) + \xi (t),
\]

where \( \xi (t) \) is a second-order stationary process which it is orthogonal to the first two terms. The symmetrized transfer functions \( \tilde{g} (\lambda_1) \) and \( \tilde{g} (\lambda_2) \) are given by

\[
g (\lambda_1) = \mu (\mu) \int_0^\infty K_0 (u) \frac{1 - e^{-iu\lambda_1}}{iu\lambda_1} du \text{ and } g (\lambda_1, \lambda_2) = \mu (\mu) \int_0^\infty K_0 (u) \prod_{j=1}^2 \frac{1 - e^{-iu\lambda_j}}{iu\lambda_j} du.
\]

It can be shown that

\[
C_\omega (s, u) = E \left\{ (X(t) - g(0)) ((X(t + s) - g(0)) ((X(t + u) - g(0)) \right)
\]

\[
= E \left\{ X^{(1)} (t) X^{(1)} (t + s) X^{(2)} (t + u) \right\} + E \left\{ X^{(1)} (t) X^{(2)} (t + s) X^{(1)} (t + u) \right\}
\]

\[
+ E \left\{ X^{(2)} (t) X^{(1)} (t + s) X^{(1)} (t + u) \right\} + O(1).
\]
We calculate $E \{ X_1(t)X_1(t+s)X_1(t+u) \}$, and the other terms can be obtained by symmetry. First we observe that

$$E \{ X_1(t)X_1(t+s)X_1(t+u) \}$$

$$= 2 \int \text{sym} \left\{ e^{ijkl} \phi (\lambda_1 + \phi (\lambda_2 + \phi (\lambda_3 + \lambda_4)} \right\} \text{d}\lambda_1 \text{d}\lambda_2 \text{d}\lambda_3 \text{d}\lambda_4$$

$$= 2 \int \text{sym} \left\{ e^{ijkl} \phi (\lambda_1 + \phi (\lambda_2 + \phi (\lambda_3 + \lambda_4)} \right\} \text{d}\lambda_1 \text{d}\lambda_2 \text{d}\lambda_3 \text{d}\lambda_4$$

Moreover we have

$$E \{ X_1(t)X_2(t+s)X_1(t+u) \}$$

$$= 2 \int \text{sym} \left\{ e^{ijkl} \phi (\lambda_1 + \phi (\lambda_2 + \phi (\lambda_3 + \lambda_4)} \right\} \text{d}\lambda_1 \text{d}\lambda_2 \text{d}\lambda_3 \text{d}\lambda_4$$

It remains to compute $E \{ X_2(t)X_1(t+s)X_1(t+u) \}$, then

$$E \{ X_2(t)X_1(t+s)X_1(t+u) \}$$

$$= 2 \int \text{sym} \left\{ e^{ijkl} \phi (\lambda_1 + \phi (\lambda_2 + \phi (\lambda_3 + \lambda_4)} \right\} \text{d}\lambda_1 \text{d}\lambda_2 \text{d}\lambda_3 \text{d}\lambda_4$$
Hence

\[ C_h(s, u) = 2 \int_{\mathbb{R}^2} g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2) \text{sym} \left\{ e^{i(\lambda_1-u)\lambda_1} + e^{i(-\lambda_1-u)\lambda_2} + e^{i(\lambda_1-\lambda_2)\lambda_1} \right\} d\lambda_1 d\lambda_2 (2\pi)^2. \]

By taking Fourier transforms (omitting the terms of \( O(1) \)), the bispectral density function \( f(\lambda_1, \lambda_2) \) can be shown to be

\[ f(\lambda_1, \lambda_2) = \frac{2}{(2\pi)^2} \{ S(\lambda_1, \lambda_2) + S(\lambda_1, -\lambda_1 - \lambda_2) + S(\lambda_1, -\lambda_1 - \lambda_2) \} \]

where \( S(\lambda_1, \lambda_2) = g(\lambda_1)g(\lambda_2)g(-\lambda_1, -\lambda_2) \). It is clear from the above that the bispectrum is zero for all frequencies \( \lambda_1 \) and \( \lambda_2 \) if and only if the process is linear (\( \gamma = 0 \)) (and Gaussian).

4. CONCLUSION

This paper describes some basic probabilistic properties of COBL process driven by an \( fBm \). Our main aim was focused firstly on the existence of the solution in time-frequency domain and secondary to prove that the use of \( fBm \) as innovation we led to a long-range dependency property.

COMPETING INTEREST

Stochastic differential equation (SDE), Estimation of SDE, Asymptotic Inference.

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