Adaptive Motion/Force Tracking Control for a Class of Nonholonomic Mechanical Systems

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Abstract—The motion/force tracking control of nonholonomic mechanical systems with affine constraints is investigated in this paper. By flexibly using the algebra processing technique, constraint forces are successfully canceled in the dynamic equations, and then an integral feedback compensation strategy and an adaptive scheme are applied to identify the dynamic uncertainty. The proposed controller ensures that the state of the closed-loop system asymptotically tracks the desired trajectory and the force tracking error has a controllable bound.

Key words—Tracking control; mechanical system; holonomic constraints; affine constraints.

I. INTRODUCTION

Nonholonomic constraints arise in many mechanical systems when there is a rolling or sliding contact, such as wheeled mobile robots, n-trailer systems, space robots, underwater vehicles, multi-fingered robotic hands, and so on. Although great progress[1–5] has been made for nonholonomic systems during the last decades, controller design for these systems still has a challenge to control engineers.

It is worth pointing out that most existing results[6–9] of nonholonomic systems aimed at the classic nonholonomic linear constraints. In fact, there is another large class of constraints which are affine in velocities, called affine constraints[10–11], such as a boat on a running river with the varying stream, ball on rotating table with invariable angular velocity, underactuated mechanical arm, etc. In [10], T. Kai defined rheonomous affine constraints and explained a geometric representation method for them, and derived a necessary and sufficient condition for complete nonholonomicity of the rheonomous affine constraints. But the problems of controllability and stabilizability on the nonholonomic kinematic mechanical systems with affine constraints have not been systematically analysed up to now.

The tracking problem for mechanical systems, as a much more interesting issue in practice, is to make the entire state of the closed-loop system track to a given desired trajectory. It is also important to note that the literatures on the tracking problem of the nonholonomic systems with affine constraints are sparse at present. Hence, researching the tracking problem for such nonholonomic mechanical systems is an innovatory and significative work. In this paper, we establish the dynamical model of the nonholonomic control systems with affine constraints. Based on the asymptotic tracking idea for uncertain multi-input nonlinear systems, and the compensatory strategy, an adaptive tracking controller is designed such that the trajectory tracking error asymptotically tends to zero and the force tracking error is bounded with a controllable bound.

II. SYSTEM DESCRIPTION AND CONTROL DESIGN

A. Dynamics Model

According to Euler-Lagrangian formulation, equations of nonholonomic mechanical systems are described by

\[
M(q)\ddot{q} + V(q, \dot{q})\dot{q} + G(q) = f + B(q)\tau ,
\]

(1)

where \( q = [q_1, \cdots, q_n]^T \) is the generalized coordinates, and \( \dot{q}, \ddot{q} \in \mathbb{R}^n \) represent the generalized velocity vector, acceleration vector, respectively; \( M(q) \in \mathbb{R}^{n \times n} \) is the inertia matrix; \( V(q, \dot{q})\dot{q} \in \mathbb{R}^n \) presents the vector of centripetal, Coriolis forces; \( G(q) \in \mathbb{R}^n \) represents the vector of gravitational forces; \( \tau \) denotes the \( r \)-vector of generalized control inputs; \( B(q) \in \mathbb{R}^{n \times r} \) is a known input transformation matrix\( (r < n) \) with full rank; \( f \in \mathbb{R}^n \) denotes the vector of constraint forces.

Consider the situation where kinematic constraints are imposed, which represented by analytical relations between the generalized coordinates \( q \) and velocity vector \( \dot{q} \), it is can be described by

\[
J^T(q)\dot{q} = A(q),
\]

(2)

where \( J(q) = [j_1(q), \cdots, j_m(q)] \in \mathbb{R}^{m \times n} \) is full of constraint matrix, \( A(q) = [a_1(q), \cdots, a_m(q)]^T \in \mathbb{R}^m \) is known.

B. Reduced Dynamics and State Transformation

It’s easy to find a fullrank matrix \( S \in \mathbb{R}^{n \times (n-m)} \) satisfying

\[
J^T(q)S(q) = 0.
\]

(3)

Noticing that \( S(q) \) is full of rank, there must exist a full-rank matrix \( S_1(q) \in \mathbb{R}^{(n-m) \times n} \) satisfying \( S_1(q)S(q) = I \), where \( I \) is an identity matrix. If defining \( \xi(t) = [q, -\dot{t}]^T \), then (2) can be expressed concisely as

\[
\begin{bmatrix}
J^T(q) & A(q)
\end{bmatrix} \dot{\xi} = 0.
\]

(4)

For the sake of convenience, we define

\[
E(q) = \begin{bmatrix}
S(q) & \eta(q) \\
0 & -1
\end{bmatrix} \in \mathbb{R}^{(n+1) \times (n-m+1)},
\]

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where $\eta(q) \in \mathbb{R}^n$ satisfies $J^T(q)\eta(q) = A(q)$. One can deduce that $E$ is a full of rank and satisfies

$$[ J^T(q) \quad A(q) ] E(q) = 0. \quad (5)$$

From (4) and (5), we know that there exists an $(n - m + 1)$-dimensional vector $\hat{z}$, such that

$$\hat{\hat{z}} = E \hat{z} = \begin{bmatrix} S(q) & \eta(q) \\ 0 & -1 \end{bmatrix} \hat{z}, \quad (6)$$

where $\hat{z} = [z^T, \hat{z}_{n-m+1}]^T, z = [z_1, \ldots, z_{n-m}]^T$.

In view of the relationship (6), one can obtain $\hat{z}_{n-m+1} = 1$, and the generalized velocity vectors can be written as

$$\hat{q} = S(q)\hat{z} + \eta(q). \quad (7)$$

$z$ corresponds to the internal state variable, and $(q, z)$ is sufficient to describe the constrained motion.

Substituting (7) into (1), the system (1) and (2) can be described clearly as

$$M(q)\hat{\hat{z}} + \hat{V}(q, \hat{z})\hat{z} + \hat{G}(q, \hat{z}) = J(q)\lambda + B(q)\tau, \quad (8)$$

where $M(q) = M(q)S(q)$ and $\hat{V}(q, \hat{z}) = M(q)\hat{S}(q) + V(q, \hat{z})S(q), \hat{G}(q, \hat{z}) = M(q)\eta(q) + V(q, \hat{z})\eta(q) + G(q)$.

### C. Error System Development

In practice, the complexity and unpredictability of the structure of uncertainties usually appear in the dynamics of the mechanical systems, we assume that $M(q), V(q, \hat{z})$ and $G(q, \hat{z})$ are expressed in the form

$$\hat{M}(q)\hat{\hat{z}} + \hat{V}(q, \hat{z})\hat{z} + \hat{G}(q, \hat{z}) = \Phi_u(\cdot) + \Phi_v(q, \hat{z}) + \Phi_G(q, \hat{z}) \in \mathbb{R}^{n-m}. \quad (9)$$

where $\Phi_u(\cdot) = \nabla M(q)\hat{\hat{z}} + \nabla V(q, \hat{z})\hat{z} + \nabla G(q, \hat{z}) \in \mathbb{R}^{n-m}$.

Pre-multiplying $S^T(q)$ on both sides of (9), and noting $J^T(q)S(q) = 0$, the following transformed system is received:

$$M_1(q)\hat{\hat{z}} + V_1(q, \hat{z})\hat{z} + G_1(q, \hat{z}) = \Phi_1(q, \hat{z}) \in \mathbb{R}^{n-m} \quad (10)$$

where $M_1(q) = S^T(q)M_0(q), V_1(q, \hat{z}) = S^T(q)V_0(q, \hat{z}), G_1(q, \hat{z}) = S^T(q)G_0(q, \hat{z}), \Phi_1(q, \hat{z}) = S^T(q)\Phi_u(q, \hat{z})$. According to Masahiro Oya’s statement [8], there exists a coordinate transformation $q = \Psi(z)$ such that $\Phi_1(z, \hat{z}) = \Phi_1(q, \hat{z}) |_{q = \Psi(z)}$. Let $\Phi_1$ replace $\Phi_1$ in above equation, the following is obtained:

$$M_1(q)\hat{\hat{z}} + V_1(q, \hat{z})\hat{z} + G_1(q, \hat{z}) = \Phi_1(z, \hat{z}) \in \mathbb{R}^{n-m} \quad (11)$$

The control objective of this paper is specified as: A given desired trajectory $z_d(t)$ satisfying $z_d(t), i = 0, \ldots, 4$ exist and are bounded, a desired constraint force $f_d(t)$ or a desired multiplier $\lambda_d(t)$, determine a adaptive control law for system (1), such that: (i) All the states of the closed-loop system are globally bounded. (ii) The position and velocity tracking error $z(t) - z_d(t), \hat{z}(t) - \hat{z}_d(t)$ converge to zero as $t \to \infty$, respectively.

The subsequent development is based on the assumption that $\Phi_1$ is an $C^2$ nonlinear vector function. In order to solve the previous problem, we make the following assumptions:

**Assumption 1** [12] The matrix $M_1$ is symmetric, positive definite and satisfies

$$a\|x\|^2 \leq x^TM_1(x)x \leq \bar{a}(\|x\|)\|x\|^2,$$

where $a$ is a known positive constant, $\bar{a}(x)$ is a known positive function.

**Assumption 2** If $q(t) \in \mathcal{L}_\infty$, then $\partial M_1(q)/\partial q$ exists and is bounded. Moreover, if $q(t), \hat{q}(t), \hat{\hat{q}}(t) \in \mathcal{L}_\infty$, then $V_1(q, \hat{z})$ and $\partial V_1(q, \hat{z})/\partial q$ exist and are all bounded.

Next, we develop the following error system which will be used in the subsequent controller design and stability analysis

$$e_1 = z_d - \hat{z}, \quad (11)$$
$$e_\lambda = \lambda - \lambda_d. \quad (12)$$

where $e_1 \in \mathbb{R}^{n-m}, e_\lambda \in \mathbb{R}^m$. To achieve the desired control objective, the following filtered tracking errors, denoted by $e_2, \rho \in \mathbb{R}^{n-m}$, are defined as

$$\begin{cases}
e_2 = \hat{e}_1 + \alpha_1 e_1, \\
\rho = \hat{e}_2 + \alpha_2 \hat{e}_2. \quad (13)
\end{cases}$$

where $\alpha_1 > 0, \alpha_2 > 0$ are designed constants.

In view of (9), (11) and (13), pre-multiplying $M_0$ on both sides of the second formula of (13), the following expression can be arrived at:

$$M_0\rho = M_0\hat{\hat{z}} + V_0\hat{z} + G_0 + \Phi_u - \Phi_1 - \Phi_1M_1\hat{\hat{z}} \in \mathbb{R}^{n-m}. \quad (14)$$

By the expression (14), a control torque input is designed as:

$$B\tau = M_0\hat{\hat{z}} + V_0\hat{z} + G_0 - J(q)\lambda - S_1\mu, \quad (15)$$

where the force term $\lambda$ is defined as $\lambda_c = \lambda_d - k_\lambda e_\lambda, k_\lambda$ is a constant of force control feedback gain, and $\mu(t) \in \mathbb{R}^{n-m}$ denotes a subsequently designed control term. Substituting (15) into (14), we can further get

$$M_0\rho = M_0\hat{\hat{z}} + V_0\hat{z} + G_0 - J(q)\lambda - S_1\mu + \alpha_1 M_0\hat{\hat{z}} \in \mathbb{R}^{n-m}. \quad (16)$$

After pre-multiplying $S^T(q)$, noting $S^T(q)J(q) = 0$ and $S_1(q)S(q) = I$, the above equation becomes

$$M_1\rho = \Phi_1 - \mu + \alpha_1 M_1\hat{\hat{z}} + \alpha_2 M_1\hat{\hat{z}} - V_1\hat{\hat{z}}. \quad (17)$$

To facilitate the design of $\mu(t)$, differentiating (17) yields:

$$M_1\dot{\rho} = \dot{\Phi}_1 - \dot{\mu} + \dot{\Phi}_1M_1\hat{\hat{z}} + \alpha_1 M_1\hat{\hat{z}} - \dot{V}_1\hat{\hat{z}}. \quad (18)$$

Based on the method of compensation for uncertain dynamic[13], $\mu(t)$ is designed as follows:

$$\mu(t) = (k_\alpha + 1)e_2(t) - (k_\alpha + 1)e_2(0) + \int_0^t (k_\alpha + 1)\alpha_2 \hat{e}_2(s) \, ds.
where the auxiliary function $Γ(t)$ is defined as

$$Γ(t) = (k_s + 1)ρ(t) + Θ(t)sgn(e_2(t)).$$

(20)

Substituting (21) into (18) results in

$$M_1\ddot{ρ} = -(k_s + 1)ρ - Θ(t)sgn(e_2) - \frac{1}{2}M_1ρ - e_2 + Γ, \quad (22)$$

where $Γ(z, \dot{z}, t) = Φ_1 + \dot{Γ} - \frac{1}{2}M_1ρ + e_2 ∈ R^{n-m}$. Defining

$$Γ_d = \frac{∂Φ_1}{∂zd} \dot{z}_d + \frac{∂Φ_1}{∂z_d} \dot{z}_d + \frac{∂Φ_1}{∂x} e^{(3)}. \quad (3)$$

Noting that $Φ_1$ is an $C^2$ vector function and $z_d^{(i)} i = 0, \cdots, 4$ are all bounded, there must exist two unknown positive constants $B_1$ and $B_2$, such that

$$||Γ_d|| ≤ B_1, \quad ||Γ_d|| ≤ B_2.$$

III. MAIN RESULTS

Theorem 1 Consider the nonholonomic mechanical system described by (1) and (2), subject to Assumptions 1 and 2. Given a desired trajectory $z_d(t)$ which satisfies the constraint equation (2), using the control laws (15), (19) and (20), the following hold: (i) All the states of the closed-loop system are globally bounded. (ii) Tracking error $e_1$ and $e_1$ converge to zero as $t → ∞$.

Proof. Let $D ∈ R^{3(n-m)+2}$ be a domain containing $y(t) = 0$, where $y(t) ∈ R^{3(n-m)+2}$ is defined as $y(t) = [x^T(t), \dot{θ}(t), \sqrt{P(t)}]^T$, $x(t) ∈ R^{3(n-m)}$ is defined as $x(t) = [e_1^T, e_2^T, ρ^T]^T$, and $Θ(t) = Θ - Θ(t)$ represents the parameter estimation error. $P(t) ∈ R$ is defined as

$$P(t) = Θ||e_2(0)|| - e_2(0)^TΓ_d(0) - \int_0^T L(s)ds,$$

where the auxiliary function $L(t)$ is defined as

$$L(t) = ρ^T(Γ_d(t) - Θsgn(e_2)).$$

Selecting $Θ = B_2 + \frac{1}{2}B_2 + 1$, by taking the same manipulations as Appendix A in [13], there is

$$\int_0^T L(s)ds ≤ Θ||e_2(0)|| - e_2(0)^TΓ_d(0).$$

Hence, $P(t) ≥ 0$.

Now, choose a candidate lyapunov function

$$V(y, t) = e_1^T e_1 + \frac{1}{2}e_2^T e_2 + \frac{1}{2}ρ^TM_1ρ + P + \frac{γ}{2}Θ^2. \quad (23)$$

Taking the time derivative of $V$ along solutions of (10), noting the definition of $Θ$ and substituting (11), (13) and (22) into it, we immediately get

$$\dot{V} ≤ -2α_1 ||e_1||^2 - 2α_2 ||e_2||^2 - (k_s + 1)||ρ||^2 + 2e_1^T e_2 + ρ^TΓ.$$

Since $Γ(t)$ is continuously differentiable, by mean value theorem, one can acquire the upper bound of $Γ$ as follows [13]:

$$||Γ|| ≤ \varphi(||x||)||x||,$$

where $\varphi : R^+ → R^+$ is an appropriate $K$ function. By using the fact that $2e_1^T e_2 ≤ ||e_1||^2 + ||e_2||^2$, $V$ can be simplified as

$$\dot{V} ≤ -λ||x||^2 - k_s||ρ||^2 + \varphi(||x||)||ρ|| ||x||, \quad (24)$$

where $λ = \min{2α_1 - 1, α_2 - 1, 1}$, and $α_1, α_2$ must be chosen to satisfy $α_1 > \frac{1}{2}, α_2 > 1$.

Completing the squares for the third term in (24) gives

$$\varphi(||x||)||ρ|| ||x|| ≤ k_s ||ρ||^2 + \frac{2\varphi(2(||x||)||x||)^2}{4k_s},$$

then the following expression can be obtained

$$\dot{V} ≤ -λ||x||^2 + \frac{2\varphi(2(||x||)||x||)^2}{4k_s}.$$

(25)

Now, we define a compact set:

$$N_1 = \{y ∈ R^{3(n-m)+2} ||y|| ≤ ϕ^{-1}\left(2\sqrt{λk_s}\right)\}.$$

The inequality (25) shows $V(t) ≤ V(0)$ in $N_1$, hence, all the the signals $e_1, e_2, ρ, Θ$ on the right-hand side of function (23) are bounded in $N_1$. From the definition of $e_1, e_2, ρ, Θ, ϕ$, we know $e_1 = e_2 - α_1e_1, e_2 = ρ - α_2e_2, Θ = Θ - Θ$, therefore, we can further get $e_1, e_2, Θ ∈ L_∞$ in $N_1$. The assumption that $z_d, \dot{z}_d, \ddot{z}_d$ are bounded can be used to conclude that $z, \dot{z}, \ddot{z} ∈ L_∞$ in $N_1$. With $M_1, V_1, G_1$ are all known and bounded in $N_1$. Thereby $τ_1, τ_2 ∈ L_∞$ in $N_1$ can be further obtained.

Then, let $N_2 ∈ N_1$ denotes a set defined as follows:

$$N_2 = \{y(t) ∈ N_1 ||δ_2(y(t)||y||^2 < δ_1 · \left(ϕ^{-1}(2\sqrt{λk_s})^2\right)\},$$

where $δ_1 = \frac{1}{2}\min\{1, α\}$, $δ_2(y) = \max\{1, \frac{1}{2}α(y)\}$, and the definitions of $α$ and $α(y)$ have been given in Assumption 1. From expression (25), one obtains that there must exist an appropriate positive semidefinite function $U(y) = c||x(t)||^2$, such that $V ≤ -U(y)$. With Invariance-like Theorem (Theorem 8.4 of [14]) in mind, one can further get $U(y) = c||x(t)||^2 → 0 as t → ∞$. Based on the definitions of $x(t)$, one can finally gain $e_1(t), e_2(t), ρ(t) → 0 as t → ∞, ∀y(0) ∈ N_2$. From (13), we then know $e_1(t), \dot{e}_2(t) → 0 as t → ∞, ∀y(0) ∈ N_2$.

On the other hand, from (17), it is evident that if $ρ(t), e_2(t)$ and $e_1(t)$ are all bounded, then $μ(t) - Φ_1$ is bounded. According to the boundedness of $S_1(q), S_2(μ(t) - Φ_1)$ is bounded. Substituting the control laws (15) and (19) into reduced order dynamic model (9) yields

$$J(q)(λ - λ_c) = S_1(Φ_1 - μ(t)) - M_0\ddot{e}_1 - V_0\dot{e}_1 = ω(·).$$

ω(·) be a bounded function vector. Therefore, the force tracking error $(f - f_d)$ is bounded and can be adjusted by changing the feedback gain $k_3$. Thus, the theorem is proved completely.
Objective is to determine an adaptive feedback control so that $J$ of the simulation are shown in Figs 2-5. Fig. 2 shows the trajectories of $\hat{\Theta}(t)$, $e_{11}(t)$, and $e_{12}(t)$.

IV. Simulation

Consider a boat with payload on a running river. The $x$-axis and $y$-axis denote the transverse direction and the downstream direction of the river, respectively. Here, we suppose the stream of the river only depends on transverse position $x$ in the simulation. The affine constraints can be obtained as follows:

$$\cos \theta \dot{y} - \sin \theta \dot{x} = C(x) \cos \theta.$$  

We assume that the traveling direction velocity and the angular velocity of the boat can be controlled. The standard forms are given as follows:

$$M(q) = \begin{bmatrix} m + m_0 & 0 & 0 \\ 0 & m + m_0 & 0 \\ 0 & 0 & I + I_0 \end{bmatrix},$$

$$V(q, \dot{q}) = 0, \quad G(q) = 0,$$

$$J^T(q) \dot{q} = \begin{bmatrix} \cos q_3, \ -\sin q_3, \ 0 \end{bmatrix}, \quad A(q) = C(q_2) \cos q_3,$$

where $m$ is the mass of the boat and $I$ is the inertia of the boat, $m_0$ denotes the unknown mass of the payload and $I_0$ denotes the unknown inertia of the payload. For the sake of simplicity, select $m = 1$, $I = 1$, $C(q_2) = q_2$.

For the given $J(q)$, $S(q)$ and $\eta(q)$, the desired trajectory $q_d = [\sin t - \cos t, \sin t, \frac{\pi}{2}]^T$ satisfies kinematic constraint $J^T(q_d) \dot{q}_d = A(q_d)$ and diffeomorphism transform $\dot{q}_d = S(q_d) \dot{z}_d + \eta(q_d)$ with $z_d = [\sqrt{2} \sin t + 2, \frac{\pi}{4}]^T$. The control objective is to determine an adaptive feedback control so that the trajectory $z$ follows $z_d$, and $\lambda$ is bounded.

In the simulation, suppose $m_0 = 0.1$, $I_0 = 0.1$, chose $\alpha_1 = 1$, $\alpha_2 = 2$, $k_s = 1$, $k_\lambda = 2$, $\gamma = 10$, and select $z_1(0) = z_2(0) = 0.5$, $\dot{z}_1(0) = \dot{z}_2(0) = 0.5$, $\hat{\Theta}(0) = 1$. The results of the simulation are shown in Figs 2-5. Fig. 2 shows the position tracking errors of $z(t) - z_d(t)$ converge to zero, Fig. 3 shows the velocity tracking errors of $\dot{z}(t) - \dot{z}_d(t)$ converge to zero. Fig. 4 shows both state $\dot{\hat{\Theta}}(t)$ and the tracking error of $e_{11}$ are bounded.

V. Conclusions

In this paper, the motion and force tracking problem is addressed for a class of uncertain nonholonomic mechanical systems. The controller guarantees that the configuration state of the system semi-global asymptotically tracks to the desired trajectory and the force tracking error is bounded with a controllable bound.

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