About Extension of Differential Realization of the Countable Beam of Nonlinear Processes Input-Output in a Hilbert Space

Rusanov V.A., Lakeyev A.V.
Institute for System Dynamics and Control Theory, Siberian Branch of RAS
Irkutsk, Russia
e-mail: v.rusanov@mail.ru, lakeyev@icc.ru

Linke Yu.È.
Irkutsk State Transport University
Irkutsk, Russia
e-mail: linkeyurij@gmail.com

Abstract—The study of algebraic extension of a countable family of controlled nonlinear dynamic processes having differential realization in the class of ordinary quasi-linear differential equations (with software-positional control and without) in a separable Hilbert space was conducted.

Keywords—nonlinear processes “input-output”, nonlinear differential realization, nonstationary \((A,B,B^T)\)-model.

Further \((X;\|\cdot\|_X),(Y;\|\cdot\|_Y),(Z;\|\cdot\|_Z)\) – real separable Hilbert spaces (pre-Hilbert [1, p. 64] define norms \(\|\cdot\|_X,\|\cdot\|_Y,\|\cdot\|_Z\), \(U:=X\times Y\times Z\) – Hilbert space with the norm \(\|\cdot\|_U:=\|x\|_X+\|y\|_Y+\|z\|_Z\)^{1/2}\); \(L(Y,X)\) – Banach space with the operator norm \(\|\cdot\|_{L(Y,X)}\) all linear continuous operators from the space \(Y\) to \(X\) (similar \((L(X,Y),\|\cdot\|_{L(X,Y)})\) and \((L(Z,X),\|\cdot\|_{L(Z,X)})\), \(T:=\{t_0,t_1\}\) – segment of the real line \(R\) with the Lebesque measure \(\mu\) and \(\phi\)-algebra of all \(\mu\)-measurable subsets of the interval \(T\). If below \((B,\|\cdot\|)\) – some Banach space, then as usual through \(L_2(T,\mu,B)\) we will denote Banach quotient space of classes \(\mu\)-equivalence of all integrable functions \(m\) on the interval \(T\) with respect to \(\mu\) measure with values in the space \(X\), moreover \(\Pi:=AC(T,X)\times L_2(T,\mu,T)\times AC(T,\mu,Z)\).

Now we will distinguish for consideration controlled differential models of the form

\[
dx(t)/dt = A x(t) + Bu(t) + B^* u^*(x(t)),
\]

where \((x,u,u^*(x))\in \Pi, x = \text{Carathéodory solution (C-solution)}, u \text{ and } u^*(x) = \text{software and positional control, } (A,B,B^T)\in L_2(T,\mu,L(X))\times AC(T,\mu,L(Y))\times L_2(T,\mu,L(Z)), \text{in purposes of terminological convenience triple of vector-functions } (x,u,u^*(x)) \text{ we will also call C-solution of equation (1) and triple of operator-functions } (A,B,B^T), \text{adhering the terminology from [2, 3] we will call } (A,B,B^T)-\text{model of differential system (1).}

The task of elementary (singleton) extension of differential realization of the beam of dynamic processes: for a given (possibly nonlinear) law \(x\mapsto u^*(x): AC(T,X)\to L_2(T,\mu,Z)\) and fixed families \(N, N^*\) of processes “input–output” such that \(N, N^*\subset \{(x,u,q)\in \Pi: (x,u,q)=(x,u,u^*(x))\}, 1 \leq Card N \leq \aleph_0\) (aleph-zero), Card \(N^*=1, N^*\subset N\), where \(N, N^*\) have differential realizations (1) to determine analytical conditions under which \(N\cap N^*\) – family of C-solutions of some equation (1).

We endow the space \(H_2:=L_2(T,\mu,X)\times L_2(T,\mu,Y)\times L_2(T,\mu,Z)\) with the topology of the norm \(\left\langle \int_T \|g(t),w(t),q(t)\|_U^2 \mu(dt) \right\rangle_{1/2}, (g,w,q)\in H_2; H_2 - \text{Hilbert space [1, p. 39]; we differ the element } (x,u,u^*(x))\in \Pi \text{ in the notations as class of equivalence (i.e. element } H_2)) \text{ from the specific representative (vector-function) } (x(t),u(t),u^*(x(t))) \text{ from this class.}

We will denote through \(G_E\) arbitrary (but fixed and numbered) algebraic basis in \(E:=\text{Span } N \text{ and let } \{(x^*,u^*,u^*(x^*))\} := N^*, \text{ while } (x^*,u^*,u^*(x^*))\in E\). It is obvious that at any point \(t\in T\) expansion in the Hilbert space of \(U\) vector \((x(t),u(t),u^*(x(t)))\) is possible on the projection in \(\text{Span } \{(x(t),u(t),u^*(x(t)))\}, (x,u,u^*(x))\in G_E, i=1, 2, \ldots, \text{which is denoted by } (x^*(t),u^*(t),u^*(x^*(t))) \text{ and addition } (x^*(t),u^*(t),u^*(x^*(t))) = (x(t),u(t),u^*(x(t))) \in \Pi \text{-equivalence class.}

Lemma 1. Vector-functions
\[
n\mapsto (x^*_n(t),u^*_n(t),u^*_n(x^*_n(t))): T\to U, n\mapsto (x^*_n(t),u^*_n(t),u^*_n(x^*_n(t))): T\to U
\]
\(\mu\)-measurable.

(By the separability of \(U\) weak and strong measurabilities coincide [1, p. 130]). □

Lemma 2. Representation
\[
(x^*,u^*,u^*(x))=(x^*_{-u^*},u^*_{-u^*}(x))+(x^*_{\#-u^*},u^*_{\#-u^*}(x)) \text{ doesn't depend on the choice of algebraic basis } G_E, \text{ while }\]
\[
(x^*_{-u^*},u^*_{-u^*}(x)), (x^*_{\#-u^*},u^*_{\#-u^*}(x))\in H_2. \quad \square
\]

We denote through \(\Omega_E\) and \(\Omega^*_E\) circuits in the space \(H_2\) respectively to linear manifolds \(\text{Span } \{\chi(x,u,u^*(x))\}, \chi\in F, (x,u,u^*(x))\in E\) and \(\text{Span } \{\chi(x^*,u^*,u^*(x))\}, \chi\in F\), where \(F\subset L(T,\mu,R) \text{ – family of equivalence classes (mod } \mu \text{) of all characteristic functions induced by elements of } \sigma\text{-algebra } \mathcal{G}_\mu.\)

Lemma 3. Subspaces \(\Omega_E, \Omega^*_E\) are orthogonal, i.e. \(\Omega_E \perp \Omega^*_E\). □

Remark 1. Everywhere further for two closed subspaces from the space \(H_2\), such that their intersection is \(\{0\} \subset H_2\), and the vector sum is closed in \(H_2\) we agree to denote the
sign of their vector addition through $\oplus$, in particular, Theorem 14.C [4, p. 28] and Lemma 3 make note $\Omega_2 \oplus \Omega_1$ correctly.

We ask the question: what are the analytical conditions imposed on the sets of controlled dynamic processes $N$ and $\{(x', u', u''(x'))\}$, “extended” family of processes $N \cup \{(x', u', u''(x'))\}$ has a differential realization (1)? On one of the ways of geometric solution of this problem is construction of characteristic feature (see below Theorem 1) defining equality

$$\Omega_2 + \Omega = \Omega_2 \ominus \Omega_1, \quad (2)$$

where $\Omega -$ closure in the space $H_2$ of linear manifold $Span\{\chi(x', u', u''(x')) : \chi \in F\}$, while a particular form of equation (2), namely, of the type

$$\Omega_2 \ominus \Omega = \Omega_2 \ominus \Omega_1, \quad (3)$$

positively responds to the aforesaid issue about the realization of the expanded beam $N \cup \{(x', u', u''(x'))\}$ in the context of approach to geometric solution of the task of expansion of differential realization based on the Theorem 14.C [4, p. 28] and theorem (3) [3] below Theorem 2 detects one characteristic property of equality (3).

Further, $T_0 = \{t \in T : (x_{(t)}(t), u_{(t)}(t), u''(x))(t) = 0\}, \nu^*, \nu^* -$ Lebesque replenishments of measures

$$\int_0^1 \|x'(t), u'(t), u''(x)(t)\|^2 \mu(dt), \nu \in \nu^*,$$

$$\int_0^1 \|x'(t), u'(t), u''(x)(t)\|^2 \mu(dt), \nu \in \nu^*.$$

**Theorem 1.** $\Omega_2 + \Omega = \Omega_2 \ominus \Omega_1$ only if $L_T(T, V) = \chi^* L_2(T, V, R)$, where $\chi^* -$ characteristic function of the set $TT_0$.

Proof of Theorem 1 we reduce to the establishment of Lemmas 4 and 5.

**Lemma 4.** $\Omega_2 + \Omega \subseteq \Omega_2 \ominus \Omega_1$.

Proof. Let $\omega \in \Omega$, then according to Lemma 4 [3] will be

$$\omega = \lambda^* \lambda^* = \lambda^*(x', u', u''(x')) + \lambda^*(x', u', u''(x')),$$

where $\lambda^* \in L_2(T, V, R)$. Further, since for each function $\lambda \in L_2(T, V, R)$ we have

$$\lambda^*(t) \|x'(t), u'(t), u''(x)(t)\|^2_{U} \geq \lambda^2(t),$$

then the following embedding of functional spaces is true $L_2(T, V) \subset L_2(T, V, R)$, where $\lambda^*(x', u', u''(x')) \in \Omega_1$ (based on the analytical structure of the subspace $\Omega_2$, given in Lemma 4 [3]). Thus, by the arbitrariness of the choice of the element $\omega \in \Omega$, the lemma will be proved as soon as we discover:

$$\lambda^*(x', u', u''(x')) \in \Omega_2.$$

For this it is sufficient to show (Corollary [1, p. 109]) that $<\lambda^*(x', u', u''(x'))$, $\omega\omega'_{H_2} = 0$, where $<\cdot, \cdot>_{H_2}$ - scalar product in $H_2$, for all $\omega' \in H_2$, such that $<\omega', \omega'>_{H_2} = 0$, for any $\omega \in span\{\chi(x, u, u'(x)) : \chi \in F, (x, u, u'(x)) \in E\}$, which is equivalent to install:

$$\omega^*(t) \perp \text{span}\{(x(t), u(t), u''(x(t)) ; (x, u, u''(x)) \in E_T, i = 1, 2, \ldots \mu\text{-almost everywhere in } T, \text{here } \perp \text{- relation of orthogonality in the structure of space } U.$$

We expand vector-function $\omega^*(\cdot)$ in each point $t \in T$ in the sum of

$$\omega^*(\cdot) + \omega^*(\cdot) = \omega^*(\cdot),$$

where $\omega^*(\cdot) \in \text{span}\{(x(t), u(t), u''(x(t)) : (x, u, u''(x)) \in E_T, i = 1, 2, \ldots \} \text{ and } \omega^*(\cdot) - \text{ is orthogonal to } \text{span}\{(x(t), u(t), u''(x(t)) : (x, u, u''(x)) \in E_T, i = 1, 2, \ldots \}. \text{ Then if } \omega^*(\cdot) \neq 0, \text{ there exists such set } S \in \nu^*, \mu(S) > 0, \text{ that } \omega^*(\cdot) \neq 0, \forall t \in S^\ast, \text{ while in the basis } E_T \text{ there is such vector } (x, u, u''(x)), \text{ that is } (x(t), u(t), u''(x(t))) \neq 0 \mu\text{-almost everywhere in } S^\ast; \text{ otherwise for } \mu\text{-almost all } t \in S^\ast \text{ equalities will be “realized”}

$$\text{span}\{(x(t), u(t), u''(x(t)) : (x, u, u''(x)) \in E_T, i = 1, 2, \ldots \} = \{0\},$$

and therefore $\omega^*(\cdot) = 0$ should be performed in this position.

Now we denote through $S^\ast$, and $S^\ast$ subsets (partition) $S$ equal

$$S^\ast = \{t \in S^\ast ; <\omega^*(\cdot), (x(t), u(t), u''(x(t))) \geq \|x(t), u(t), u''(x(t))\|^2_{U} \geq 0\} = S^\ast = \{t \in S^\ast ; <\omega^*(\cdot), (x(t), u(t), u''(x(t))) < 0\}.$$

It is obvious that at least one of the sets $S^\ast$, or $S^\ast$ has a nonzero measure. Let $S^\ast$ acts as such set. Then $\chi^*(x, u, u''(x)) \in \text{span}\{(x, u, u''(x)) : \chi \in F, (x, u, u''(x)) \in E\}$ and $<\omega^*(\cdot), \chi^*(x, u, u''(x)) >_{H_2} > 0$, where $\chi^* -$ characteristic function of a set $S^\ast$. It is clear that we obtain $<\omega^*(\cdot), \chi^*(x, u, u''(x)) >_{H_2} > 0$ whereby we arrive at a contradiction with the conditions defined above the construction of the functional $\omega^*$. □

The above proof provides a useful clarification:

**Corollary 1.** $L_2(T, V, R) \subset L_2(T, V, R)$. □

**Lemma 5.** $\Omega_2 + \Omega \supseteq \Omega_2 \ominus \Omega_1 \iff L_2(T, V, R) = \chi^* L_2(T, V, R)$.

**Proof.** (⇒). Let $\lambda^0 \in L_2(T, V, R)$ and $\omega := \lambda^0(x', u', u''(x'))$, where (Lemma 4 [3]) $\omega \in \Omega_2 + \Omega_1$, means (assumption ⇒) $\omega \in \Omega_2 + \Omega_1$. Then by $\omega \in \Omega_2 + \Omega_1$ vector $\omega$ has an expansion of (unique) form $\omega = \omega^* + \lambda^0(x', u', u''(x'))$, where $\omega^* \in \Omega_2$, at this effect $\omega \in \Omega_2 + \Omega_1$ representation is true:

$$\omega = \omega^* + \lambda^0(x', u', u''(x')) = \lambda^0(x', u', u''(x')) + \lambda^0(x', u', u''(x')),$$

where $\omega \in \Omega_2$, $\lambda^0 \in L_2(T, V, R)$. Since (reasonings are similar to the proof of Lemma 4 [3]) the inclusions take place $\lambda^0(x', u', u''(x')) \in \Omega_2$, $\lambda^0(x', u', u''(x')) \in \Omega_1$, then $\omega^* = \lambda^0(x', u', u''(x')) + \lambda^0(x', u', u''(x')) \in \lambda^0(x', u', u''(x')) + \lambda^0(x', u', u''(x'))$. Thus, taking into account the presence of a linear isometry between $L_2(T, V, R)$ and $\Omega_1$ (Lemma 4 [3]) will be $\lambda^0 = \chi^* \lambda^0$, where in the end by the arbitrariness of the choice of function $\lambda^0$, we obtain $L_2(T, V, R) \subset \chi^* L_2(T, V, R)$ or taking into account Corollary 1 $L_2(T, V, R) = \chi^* L_2(T, V, R)$. (⇐). Let $\omega \in \Omega_2 + \Omega_1$. Then $\omega = \omega^* + \lambda^0(x', u', u''(x'))$, where $\omega^* \in \Omega_2$, $\lambda^0 \in L_2(T, V, R)$, and therefore $\neq 0$. Since (assumption ⇐)$\lambda^0 = \chi^* L_2(T, V, R)$, then we have a bunch of equalities.
\[ \omega' + \lambda_\omega(x_\ast, u_\ast^\ast, u_\ast^\ast(x)) = \omega' + \lambda_\omega(x_\ast, u_\ast^\ast, u_\ast^\ast(x)) +
+ \lambda_\omega(x_\ast, u_\ast^\ast, u_\ast^\ast(x)) = \omega' - \lambda_\omega(x_\ast, u_\ast^\ast, u_\ast^\ast(x)) + \lambda_\omega(x_\ast, u_\ast^\ast, u_\ast^\ast(x)), \]

therefore, \(\omega \in \Omega_e \cup \Omega^\ast\) taking into account \((\omega' - \lambda_\omega(x_\ast, u_\ast^\ast, u_\ast^\ast(x))) \in \Omega_e, \lambda_\omega(x_\ast, u_\ast^\ast, u_\ast^\ast(x)) \in \Omega^\ast. \square\]

Now we present a variant of characteristic conditions of equality (3).

**Theorem 2.** If we implement \(T_0 = \emptyset\) (mod \(\mu\)) offer is valid:

\[ \Omega_e \cup \Omega^\ast = \Omega_e \cup \Omega^\ast \]

**Proof.** That \(\Omega_e + \Omega^\ast = \Omega_e \cup \Omega^\ast \) is a direct statement of Theorem 1. On the other hand, confirmation of equality \(\Omega_e \cap \Omega^\ast = \emptyset\) follows from the assumption \(\{t \in T : (x_\ast(t), u_\ast^\ast(t), u_\ast^\ast(x_\ast(t))) = 0\} \subset \emptyset\) (mod \(\mu\)) and Corollary of Mazur’s Theorem [1, p. 109]. \(\square\)

Theorem 1 (given the finding of Lemma 5) and Theorem 2 attracting Theorem 14.C [4, p. 28] and Theorem 3 [3] do a fair conclusion:

**Corollary 2.** The following three properties are equivalent:

\[ L_2(T, v_\ast^\ast, R) \subset \chi_L L_2(T, v_\ast^\ast, R) \]

\[ \iff L_2(T, v_\ast^\ast, R) = \chi_L L_2(T, v_\ast^\ast, R) \iff \Omega_e \cup \Omega^\ast = \emptyset \]

and if \(T_0 = \emptyset\) (mod \(\mu\)), then any signified property turns the beam \(N \cup \{(x_\ast, u_\ast^\ast, u_\ast^\ast(x_\ast))\}\) into the set of dynamic processes with the differential realization (1). \(\square\)

**Remark 2.** Corollary 2 allows to call Theorem 2 as “direct theorem” about elementary algebraic extension of differential realization while hypothesis: \(T_0 = \emptyset\) (mod \(\mu\)), \(N \cup \{(x_\ast, u_\ast^\ast, u_\ast^\ast(x_\ast))\}\) has a realization (1) \(\Rightarrow L_2(T, v_\ast^\ast, R) \subset L_2(T, v_\ast^\ast, R)\) in general case isn’t confirmed that the following example illustrates.

**Example 1.** Let \(X = Y = R, T = [-1, 1] \times \mathbb{R}^n\) (\(u^\ast(\cdot) = 0\) and \(N = \{n \in \mathbb{R} : n \neq 0\}\)).

\[ \{(x_\ast, u_\ast^\ast, u_\ast^\ast(x_\ast)) : (x_\ast, u_\ast^\ast, u_\ast^\ast(x_\ast)) \in \mathbb{R}^n, x_\ast + u_\ast^\ast \in \mathbb{R} \} \}

It is obvious that \(T_0 = \emptyset\) (mod \(\mu\)) and the beam \(N \cup \{(x_\ast, u_\ast^\ast, u_\ast^\ast(x_\ast))\}\) has a realization (1); we note that \(T_0 \neq \emptyset\). Then \(L_2(T, v_\ast^\ast, R) = L_2(T, v_\ast, R)\) and \(L_2(T, v_\ast^\ast, R) = L_2(T, v_\ast^\ast, R)\), \(v_\ast = \int_0^t f(t) dt\), because \(x_\ast + u_\ast^\ast(t) = (0, t, 0)\).

Next statement shows that the construction similar to Example 1 can’t be realized in the functional class \(AC(T, X) \times \{0\} \times \{0\} \subset \Pi, i.e.\, for\, free\, trajectories\, (C-solutions)\, it\, can\, be\, said\, that\, for\, N \subset AC(T, X) \times \{0\} \times \{0\}\) Corollary 3 in a known sense is opposite to Corollary 2 (see above Remark 2).

**Corollary 3.** If \(N \subset AC(T, X) \times \{0\} \times \{0\}, \) Card \(N < \infty\) and \(N \cup \{(x_\ast, 0, 0)\}\) be a set of trajectories with the realization (1) with \(u = 0, u^\ast = 0\), then the following relations are true:

\[ T_0 = \emptyset \]

\[ L_2(T, v_\ast^\ast, R) = L_2(T, v_\ast, R) \]

\[ \Omega_e \cup \Omega^\ast = \Omega_e \cup \Omega^\ast. \]

**Proof.** It is easy to see that \(T_0 = \emptyset\), because otherwise there exists a period of time \(t \in T\), which \(x_\ast(t) = \Sigma \alpha_n x_n(t)\), where all \(\alpha_n\) except the finite number are zero, \(x_{n_0} \) – the first component of the triple \((x_\ast, 0, 0)\) \(\in \Omega_e\). Consequently, the trajectory \(x_\ast(\cdot)\) has a representation \(\Sigma \alpha_n x_n(\cdot)\) by the uniqueness of solution, extending at time \(t\) through the point \(x_\ast(\cdot)\), for the differential system (1) with \((A, 0, 0)\)-model, corresponding to a set of dynamic processes \(N \cup \{(x_\ast, 0, 0)\}\); that is contrary to its earlier condition \((x_\ast, 0, 0) \notin E\).

Further, because of the continuity of the trajectory \(x_\ast(\cdot)\) and the compactness of the interval \(T\), there exist such real constants \(c_1, c_2 > 0\), that equalities are true

\[ \inf \{\|x_\ast(\cdot)|_E, \|x_\ast(\cdot)|_{T} \} = c_1, \sup \{\|x_\ast(\cdot)|_{T} \} = c_2. \]

Similarly (including \(T_0 = \emptyset\), Card \(N < \infty\)), for some \(c_3, c_4 \geq 0\) will be

\[ \inf \{\|x_\ast(\cdot)|_{T} \} = c_3, \sup \{\|x_\ast(\cdot)|_{T} \} = c_4. \]

Consequently, the classes of real-valued functions summable with square on \(T\) on measures \(v_\ast = \int_0^T \mu(t) dt\) and \(v_\ast = \int_0^T \mu(t) dt\), or in other words \(L_2(T, v_\ast, R) = L_2(T, v_\ast^\ast, R)\), and hence (see Theorem 2) \(\Omega_e \cup \Omega^\ast = \emptyset\) \(\square\)

We look at Theorem 2 under foreshortening of unmanaged trajectories of a differential system (1), we can see that the analyst of output condition \(L_2(T, v_\ast, R) = L_2(T, v_\ast^\ast, R)\) in the proof of Corollary 3 enables us to strengthen this theorem to the characteristic feature of elementary algebraic extension of differential realization of a finite beam of unmanaged implementation processes \(N \subset AC(T, X) \times \{0\} \times \{0\}\).

**Theorem 3.** In the family of free \(K\)-solutions the problem of singleton expansion of differential realization of the finite beam of trajectories is solvable if and only if \(T_0 = \emptyset\).

This work was supported by the Program “Leading Scientific Schools” (project no. NSh-5007.2014.09).