Exact Solutions to a Generalized Benjamin-Bona-Mahony Equation

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Abstract

Mathematical techniques based on auxiliary equations and the symbolic computation system Maple are employed to investigate a generalized Benjamin-Bona-Mahony differential equation. The Jacobian elliptic function solution, the soliton solutions and the triangle function solutions to the equation are constructed under various circumstances.

Keywords: Benjamin-Bona-Mahony equation, Jacobian elliptic function solution, Solitons, Triangle function solutions

1. Introduction

Many mathematical techniques have been employed to find traveling wave solutions of nonlinear partial differential equations. Rosenau and Hyman[1] used the pseudo spectral methods in space and a variable order, variable time-step Adams-Basford-Moulton method in time to study a family of nonlinear KdV equations, and obtained a class of solitary waves with compact support, which were called compactons. Wadati[2][3] developed the trace method to investigate the exact traveling wave solutions for a modified Kortweg-de Vries equation. The tanh method developed by Malfliet et al.[4][5] is a reliable algebraic technique to obtain exact solutions of many nonlinear equations. Fan and Zhang[6][7] extended tanh method and investigated the generalized $mKdV$ equation and the generalized ZK equation. This extended method was a powerful tool to seek exact solutions of nonlinear equations. By decomposing the time and space variables of nonlinear partial differential equations into two integrable ordinary differential equation, Ma and Wu[8] have found some exact solutions of $KdV, mKdV$ and KPP equations.

Benjamin, Bona and Mahony[9] established the model

$$u_t + au_x - bu_{xxt} + k(u^2)_x = 0, \tag{1}$$

which was called BBM equation. It is used as an alternative to the KdV equation which describes unidirectional propagation of weakly long dispersive waves[10]. As a model that characterizes long waves in nonlinear dispersive media, the BBM equation, like KdV equation, was formally derived to describe an approximation for surface water waves in a uniform channel. The equation covers not only the surface waves of long wavelength in liquids, but also hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, and acoustic gravity waves in compressible fluids. Many researchers are attracted by the wide applicability of the BBM equation[10][11].

Using the tanh method and the sine-cosine method, Wazwaz[12] obtained compactons, solitons, solitary patterns and periodic solutions for the following generalized form of Eq.(1)

$$u_t + au_x - bu_{xxt} + k(u^m)_x = 0, \tag{2}$$

where $a \neq 0, b \neq 0, k \neq 0$ and $m > 1$ are constants.

In the present work, making use of two different kinds of auxiliary equations, we will focus on deriving the exact traveling wave solutions including Jacobian elliptic function solution, solitons and triangle function solutions for Eq.(2). Our results includes those presented in Wazwaz's paper[12] as a special case.

2. Exact traveling wave solutions to the Eq.(2)

Firstly, we illustrate the main approach used in this work.

The transformation $u(x,t) = u(\xi)$ $(\xi = \mu(x - ct))$ turns a given nonlinear equation

$$P(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt} \cdots) = 0 \tag{3}$$

into the following nonlinear ordinary different equation

$$Q(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \cdots) = 0. \tag{4}$$

We seek for the solutions of Eq.(4) in the form

$$u(\xi)=\sum_{i=0}^{N} g_i z^i(\xi), \tag{5}$$

where $g_i (i = 0, 1, 2, \cdots, N)$ are constants which will be determined later. The parameter N is a positive integer and can be determined by balancing the highest order derivative terms and the highest power nonlinear terms in Eq.(4). The highest degree can be calculated by

$$\begin{cases} O[\frac{\partial^p u}{\partial \xi^p}] = N + p, \quad p = 0, 1, 2 \cdots \\ O[u^q \frac{\partial^p u}{\partial \xi^p}] = qN + p, \quad q, p = 0, 1, 2, \cdots . \end{cases} \tag{6}$$

2.1. The first auxiliary equation for solving Eq.(2)

To find the traveling solutions for Eq.(2), we know that the wave variable $\xi = \mu(x - ct)$ turns Eq.(2) into the ordinary differential equation

$$(a-c)u' + bc\mu^2 u''' + k(u^m)' = 0. \tag{7}$$

Integrating Eq.(7) once and ignoring the integral constant give rise to

$$(a-c)u + bc\mu^2 u'' + ku^m = 0. \tag{8}$$

Setting $u^{m-1}(\xi) = v(\xi)$ yields

$$(a-c)v^2 + \frac{bc\mu^2(2-m)}{(m-1)^2}v'^2 + \frac{bc\mu^2}{m-1}v''v + kv^3 = 0. \tag{9}$$

From (6) and (9), we can assume that $v(\xi)$ takes the form

$$v(\xi) = g_0 + g_1 z + g_2 z^2. \tag{10}$$

Supposing $z(\xi) = sn(\xi)$, that is

$$v(\xi) = g_0 + g_1 sn(\xi) + g_2 sn^2(\xi), \tag{11}$$

where g_0, g_1 and g_2 are constants to be determined later. Function $sn(\xi) = sn(\xi, r)$ is a Jacobian elliptic function and $r (0 < r < 1)$ is the modus of the function.

It follows from (11) that

$$\begin{cases} v' = g_1 cn(\xi)dn(\xi) + 2g_2 sn(\xi)cn(\xi)dn(\xi), \\ v'' = 2g_2 cn^2(\xi)dn^2(\xi) - sn(\xi)\times \\ \qquad (g_1 + 2g_2 sn(\xi))(dn^2(\xi) + r^2 cn^2(\xi)). \end{cases} \tag{12}$$

Substituting Eq.(11) and (12) into Eq.(9) and equating each coefficients of $sn^i (0 \le i \le 6)$ to be zero in the resulting equation, we get the following algebraic equations

$$\begin{aligned} &\frac{6bc\mu^2 g_2^2 r^2}{m-1} + kg_2^3 + \frac{8bc\mu^2 g_2^2 r^2}{(m-1)^2} \\ &-\frac{4bc\mu^2 m g_2^2 r^2}{(m-1)^2} = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} &\frac{8bc\mu^2 g_1 r^2 g_2}{(m-1)^2} - \frac{4bc\mu^2 m g_1 r^2 g_2}{(m-1)^2} \\ &+\frac{8bc\mu^2 g_1 r^2 g_2}{m-1} + 3kg_1 g_2^2 = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} &-\frac{bc\mu^2 m g_1^2 r^2}{(m-1)^2} - \frac{4bc\mu^2 g_2^2 r^2}{(m-1)^2} - \frac{4bc\mu^2 g_2^2}{(m-1)^2} \\ &+\frac{2bc\mu^2 g_1^2 r^2}{m-1} + \frac{6bc\mu^2 g_2 r^2 g_0}{m-1} + (a-c+3kg_0)g_2^2 \\ &+3kg_1^2 g_2 + \frac{2bc\mu^2 g_1^2 r^2}{(m-1)^2} = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} &-\frac{3bc\mu^2 g_2 g_1 r^2}{(m-1)^2} - \frac{3bc\mu^2 g_2 g_1}{(m-1)^2} - \frac{bc\mu^2 m g_2 g_1 r^2}{(m-1)^2} \\ &-\frac{bc\mu^2 m g_2 g_1}{(m-1)^2} + 6kg_0 g_1 g_2 + 2(a-c)g_1 g_2 \\ &+\frac{2bc\mu^2 g_1 r^2 g_0}{m-1} + kg_1^3 = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} &(a-c)g_1^2 - \frac{4bc\mu^2 m g_2^2}{(m-1)^2} - \frac{4bc\mu^2 g_2 g_0}{m-1} \\ &+\frac{2bc\mu^2 g_2^2}{m-1} + \frac{8bc\mu^2 g_2^2}{(m-1)^2} + 3kg_0 g_1^2 \\ &+2(a-c)g_0 g_2 - \frac{4bc\mu^2 g_2 r^2 g_0}{m-1} + 3kg_0^2 g_2 \\ &-\frac{bc\mu^2 g_1^2 (r^2+1)}{(m-1)^2} = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &(2a-2c+3kg_0)g_0 g_1 + \frac{(6-2m)bc\mu^2 g_2 g_1}{(m-1)^2} \\ &-\frac{bc\mu^2 g_1 (r^2+1) g_0}{m-1} = 0 \end{aligned} \tag{18}$$

$$\begin{aligned} &kg_0^3 + \frac{2bc\mu^2 g_2 g_0}{m-1} + (a-c)g_0^2 \\ &-\frac{bc\mu^2 m g_1^2}{(m-1)^2} + \frac{2bc\mu^2 g_1^2}{(m-1)^2} = 0. \end{aligned} \tag{19}$$

Solving Eqs.(13) to (19) with the Maple, we get

$$c = \frac{a}{b\mu^2 r^2 + b\mu^2 + 1}, m = 3, g_0 = 0, \tag{20}$$

$$g_1 = 0, g_2 = -\frac{2ab\mu^2 r^2}{k(b\mu^2 r^2 + b\mu^2 + 1)}, \tag{21}$$

or

$$r = 1, \quad m \neq 3, \quad g_1 = 0, \tag{22}$$

$$c = -\frac{(m^2-2m+1)a}{4b\mu^2 - 1 + 2m - m^2}, \tag{23}$$

$$g_0 = -\frac{2(m+1)ab\mu^2}{k(4b\mu^2 - 1 + 2m - m^2)}, \tag{24}$$

$$g_2 = \frac{2(m+1)ab\mu^2}{k(4b\mu^2 - 1 + 2m - m^2)}. \tag{25}$$

Substituting Eq.(20) and (21) into Eq.(11) and using the transformations $u^{m-1}(\xi) = v(\xi)$ admit the exact solution of Eq.(2) to have the form

$$\begin{aligned} u(x,t) &= \{-\frac{2ab\mu^2r^2}{k(b\mu^2r^2+b\mu^2+1)}sn^2 \\ &\quad (\mu(x - \frac{a}{b\mu^2r^2+b\mu^2+1}t))\}^{\frac{1}{2}}. \end{aligned} \tag{26}$$

From (22), (23), (24) and (25), the exact solution is expressed by

$$\begin{aligned} u(x,t) = \{&-\frac{2(m+1)ab\mu^2}{k(4b\mu^2-1+2m-m^2)}(1 - tanh^2 \\ &[\mu(x + \frac{(m^2-2m+1)a}{4b\mu^2-1+2m-m^2}t)])\}^{\frac{1}{m-1}}. \end{aligned} \tag{27}$$

2.2. The second auxiliary equation for solving Eq.(2)

We assume that $z(\xi)$ of Eq.(10) satisfies the following auxiliary equation

$$(\frac{dz}{d\xi})^2 = a_1z^6 + a_2z^4 + a_3z^2. \tag{28}$$

Substituting Eq.(10) and (28) into Eq.(9) and setting the coefficients of each order of z to be zero, we get a set of algebraic equations

$$\begin{aligned} &\frac{8bc\mu^2g_2^2a_1}{m-1} - \frac{4bc\mu^2mg_2^2a_1}{(m-1)^2} \\ &+\frac{8bc\mu^2g_2^2a_1}{(m-1)^2} = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} &\frac{-4bc\mu^2mg_1g_2a_1}{(m-1)^2} + \frac{8bc\mu^2g_1g_2a_1}{(m-1)^2} \\ &+\frac{11bc\mu^2g_1g_2a_1}{m-1} = 0, \end{aligned} \tag{30}$$

$$\begin{aligned} &3kg_1g_2^2 + \frac{3bc\mu^2g_1a_1g_0}{m-1} + \frac{8bc\mu^2g_1g_2a_2}{m-1} \\ &+\frac{(8-4m)bc\mu^2g_1g_2a_2}{(m-1)^2} = 0, \end{aligned} \tag{31}$$

$$\begin{aligned} &\frac{kg_2^3}{bc\mu^2} + \frac{2mg_2^2a_2}{(m-1)^2} + \frac{8g_2a_1g_0}{m-1} + \frac{2g_2^2a_2}{(m-1)^2} \\ &-\frac{g_1^2a_1}{(m-1)^2} + \frac{2mg_1^2a_1}{(m-1)^2} = 0, \end{aligned} \tag{32}$$

$$\begin{aligned} &(3kg_0 - c + a)g_2^2 - \frac{4bc\mu^2mg_2^2a_3}{(m-1)^2} \\ &+\frac{6bc\mu^2g_2a_2g_0}{m-1} + 3kg_1^2g_2 + \frac{2bc\mu^2g_1^2a_2}{m-1} \\ &-\frac{bc\mu^2mg_1^2a_2}{(m-1)^2} + \frac{4bc\mu^2g_2^2a_3}{(m-1)^2} = 0, \end{aligned} \tag{33}$$

$$\begin{aligned} &\frac{2(a-c+3kg_0)g_1g_2 + kg_1^3}{bc\mu^2} + \frac{3g_1g_2a_3}{(m-1)^2} \\ &+\frac{mg_1g_2a_3}{(m-1)^2} + \frac{2g_1a_2g_0}{m-1} = 0, \end{aligned} \tag{34}$$

$$\begin{aligned} &(3kg_0 - 2c + 2a)g_0g_2 + \frac{4bc\mu^2g_2a_3g_0}{m-1} \\ &+(a-c+3kg_0)g_1^2 + \frac{bc\mu^2g_1^2a_3}{m-1} \\ &-\frac{bc\mu^2mg_1^2a_3}{(m-1)^2} + \frac{2bc\mu^2g_1^2a_3}{(m-1)^2} = 0, \end{aligned} \tag{35}$$

$$\begin{aligned} &2(a-c)g_0g_1 + 3kg_0^2g_1 \\ &+\frac{bc\mu^2g_1a_3g_0}{m-1} = 0, \end{aligned} \tag{36}$$

$$-cg_0^2 + ag_0^2 + kg_0^3 = 0. \tag{37}$$

Solving Eq.(29) to (37) by using the Maple, we obtain

$$g_0 = 0, g_1 = 0, g_2 = \frac{-2bc\mu^2(m+1)}{k(m-1)^2}a_2, \tag{38}$$

$$a_1 = 0, \quad a_3 = -\frac{(a-c)(m-1)^2}{4bc\mu^2}, \tag{39}$$

where a_2 is a nonzero constant. The equation $(\frac{dz}{d\xi})^2 = a_2z^4 - \frac{(a-c)(m-1)^2}{4bc\mu^2}z^2$ admits the solutions in the case where $\phi = \frac{(a-c)(m-1)^2}{4bc}$ and $\frac{a-c}{bc} > 0$

$$z(\xi) = \{\frac{\phi}{\mu^2 a_2}\sec^2(\sqrt{\phi}(x-ct))\}^{\frac{1}{2}}, \qquad (40)$$

$$z(\xi) = \{\frac{\phi}{\mu^2 a_2}\csc^2(\sqrt{\phi}(x-ct))\}^{\frac{1}{2}}, \qquad (41)$$

and in the case $\frac{a-c}{bc} < 0$

$$z(\xi) = \{\frac{\phi}{\mu^2 a_2} sech^2(\sqrt{-\phi}(x-ct))\}^{\frac{1}{2}}, \qquad (42)$$

$$z(\xi) = \{\frac{\phi}{\mu^2 a_2} csch^2(\sqrt{-\phi}(x-ct))\}^{\frac{1}{2}}, \qquad (43)$$

$$z(\xi) = 4\{\frac{a_3 \exp(\pm\sqrt{-2\phi}(x-ct))}{\exp(\pm\sqrt{-2\phi}(x-ct) - 4a_2)}\}^{\frac{1}{2}}. \qquad (44)$$

From Eq.(38) and (39), the solutions of $z(\xi)$ and the transformations $u^{m-1}(\xi) = v(\xi)$, we get the following results

$$u(x,t) = \{\frac{(c-a)(m+1)}{2k}\sec^2(\sqrt{\phi}(x-ct))\}^{\frac{1}{m-1}}, \qquad (45)$$

$$u(x,t) = \{\frac{(c-a)(m+1)}{2k}\csc^2(\sqrt{\phi}(x-ct))\}^{\frac{1}{m-1}}, \qquad (46)$$

where $\frac{a-c}{bc} > 0$.

$$u(x,t) = \{\frac{(c-a)(m+1)}{2k} sech^2(\sqrt{-\phi}(x-ct))\}^{\frac{1}{m-1}}, \qquad (47)$$

$$u(x,t) = \{\frac{(c-a)(m+1)}{2k} csch^2(\sqrt{-\phi}(x-ct))\}^{\frac{1}{m-1}}, \qquad (48)$$

$$u(x,t) = \{\frac{8(a-c)(m+1)a_2 \exp(\pm\sqrt{-2\phi}(x-ct))}{k(\exp(\pm\sqrt{-2\phi}(x-ct)) - 4a_2)}\}^{\frac{1}{m-1}}, \qquad (49)$$

where $\frac{a-c}{bc} < 0$ and a_2 is a nonzero constant.

Solution formulas (45)-(48) are in full agreement with the solutions presented in Wazwaz[12].

3. Conclusions

In this paper, by using the ansatz method with the help of two forms of auxiliary equations and the Maple, we have obtained some exact solutions to a generalized Benjamin-Bona-Mahony equation. It is worthwhile mention that the ansatz method can also be applied to many other evolution equations, which is our future work.

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