Discrete Time Risk Model with Discounted Rate*

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Abstract - In this paper, we consider a discrete time risk model with discount rate. By using the recursive method, recursion formulas for distribution of the largest surplus before ruin, ruin probabilities in finite time and ultimate ruin probability are derived. Moreover, recursion formulas for the ruin lasting time are also derived.

Index Terms - Discount rate, Discrete time risk model, Finite time ruin probability, Ultimate ruin probability, Distribution of the largest surplus before ruin

1. Introduction

Ruin theory is a hot research topic of the actuarial mathematics, it is mainly because that financial company make ruin probability and it's related distribution as main indexes of investigating risk measurement, so ruin theory has strong actual guiding effects. In recent years, people are paying more and more attention to the study of ruin theory's risk model. Traditionally, most of the conclusions of the risk theory are all about continuous time, in real application, the discrete time model is more easily applied. Bowers et all discussed discrete time risk model, the model of the premiums received within the unit time is a constant, and the claims of each period regarded as an independent and identically distributed random variables. Sun and Gu studied a class of discrete time constant interest rate risk model and derived the expressions of the distribution of surplus immediately before ruin and distribution of the time in the red which describe severity of ruin, using mainly update distribution method. In this paper, we consider a discrete time risk model with discounted rate. By using the recursive method, the recursion formulas for distribution of the largest surplus before ruin, ruin probabilities in finite time and ultimate ruin probability, as well as the recursion formulas for the ruin lasting time are derived. What we must point out is that we adopt each period premiums and claims as random variable in this paper.

We consider the premium income and claims expending random variable for discrete time risk model, in which the accumulated surplus of the insurance company at time n discounted to the initial time expressed by the following equation.

$$U_n^{\delta}(u) = u + \sum_{i=1}^{n} X_i e^{-\delta(i-1)} - \sum_{i=1}^{n} Y_i e^{-\delta i} \tag{1}$$

where $U_0 = u$ is the initial surplus of an insurance company, δ is constant discount rate, and X_{i}, Y_{i} are independent and identically distributed nonnegative random variables, X_{i} denotes the amount of premiums during the nth period, and is received at the beginning of the nth period. Y_{i} is the amount of claims during the nth period, and is paid at the end of the nth period.

We assume X_{i} and Y_{i} expectations are limited and set

$$Z_{\mathrm{i}} = Y_{\mathrm{i}} - e^{\delta} X_i$$

Then Z_{i} is also a net loss of independent and identically distributed random variable sequence. So the surplus process can be written in the following form

$$U_n^{\delta}(u) = u - S_{\mathrm{n}} \tag{2}$$

Where

$$S_{\mathrm{n}} = \sum_{i=1}^{n} \frac{Z_i}{\mathrm{e}^{\mathrm{i}\delta}} \quad (S_0 = 0)$$

In order to guarantee the normal operation of insurance company, it must add certain risk load $E(Z_{\mathrm{i}}) < 0$.

Set respectively $F_X(\mathrm{x})$ and $F_Y(\mathrm{y})$ to be the distribution functions of X_{i} and Y_{i} then the distribution function of Z_{i} is

$$G(\mathrm{u}) = P\{Z \le u\} = \int_0^{\infty} P\{Y - \mathrm{e}^{\delta} X \le u \,|\, X = x\} dF_X(x)$$

$$= \int_0^{\infty} F_Y((e^{\delta} x + u) dF_X(x) \tag{3}$$

It can be seen from (3), the distribution of net loss determine jointly by the distribution of premium and the distribution of claim.

Let $T(u) = \inf_n \{n : n > 0, U_n^{\delta} < 0\} = \inf_n \{n : n > 0, S_n > u\}$

be the time of ruin, obviously T is stopping time.

2. The Distribution of the Largest Surplus before Ruin

Understand the distribution of the largest surplus of insurance company before ruin and timely invest to the insurance funds, which is necessary to increase the solvency of the insurance company.

The distribution function of the largest surplus before ruin with the initial reserve u is defined as

$$H(\mathrm{u}, \mathrm{x}) = P\left\{\sup_{0 \le k < T} U_k^{\delta} \le x, \quad T < \infty \,|\, U_0 = u\right\} \tag{4}$$

* Supported by the National Natural Science Foundation of China (11361058)

Obviously, if $x < u$, then $H(u,x)=0$, here we only discuss the case $x \ge u$, from (4)

$$H(u,x)=\sum_{n=1}^{+\infty} P\{\sup_{1\le k<T} U_k^{\delta} < x, T=n\}$$

$$=\sum_{n=1}^{+\infty} P\{0 \le U_1^{\delta} < x,\ldots,0 \le U_{n-1}^{\delta} < x, U_n^{\delta} < 0\}$$

$$=\sum_{n=1}^{+\infty} P\{u-x \le S_1 < u,\ldots,u-x \le S_{n-1} < u, S_n > u\}$$

$$\overset{\nabla}{=}\sum_{n=1}^{+\infty} h_n(u,x) \tag{5}$$

Where $h_n(u,x)$ is the distribution of the largest surplus at the ruin time n. by the definition we have

$$h_1(u,x)=P\{S_1>u\}=P\{Z_1>ue^{\delta}\}=1-G(ue^{\delta})$$

$$h_2(u,x)=P\{u-x \le S_1 < u, S_2 > u\}$$

$$=P\{(u-x)e^{\delta} \le Z_1 < ue^{\delta}, Z_1+\frac{Z_2}{e^{\delta}} > ue^{\delta}\}$$

$$=\int_{(u-x)e^{\delta}}^{ue^{\delta}} P\{S_1 > ue^{\delta}-s\}dG(s)$$

$$=\int_{(u-x)e^{\delta}}^{ue^{\delta}} h_1(ue^{\delta}-s,x)dG(s)$$

$$h_3(u,x)=P\{u-x \le S_1 < u, u-x \le S_2 < u, S_3 > u\}$$

$$=P\{(u-x)e^{\delta} \le Z_1 < ue^{\delta}, (u-x)e^{\delta}-Z_1$$

$$\le \frac{Z_2}{e^{\delta}} < ue^{\delta}-Z_1, Z_1+\frac{Z_2}{e^{\delta}}+\frac{Z_3}{e^{2\delta}} > ue^{\delta}\}$$

$$=\int_{(u-x)e^{\delta}}^{ue^{\delta}} P\{(u-x)e^{\delta}-s \le S_1 < ue^{\delta}-s, S_2 > ue^{\delta}-s\}dG(s)$$

$$=\int_{(u-x)e^{\delta}}^{ue^{\delta}} h_2(ue^{\delta}-s, xe^{\delta})dG(s)$$

By using the induction method on $n \ge 2$, we have

$$h_n(u,x)=\int_{(u-x)e^{\delta}}^{ue^{\delta}} h_{n-1}(ue^{\delta}-s, xe^{\delta})dG(s) \tag{6}$$

From (4) it is clear that the series converges. From (5) and (6)

$$H(u,x)=\sum_{n=1}^{\infty} h_n(u,x)=h_1(u,x)+\sum_{n=2}^{\infty} h_n(u,x)$$

$$=h_1(u,x)+\sum_{n=2}^{\infty}\int_{(u-x)e^{\delta}}^{ue^{\delta}} h_{n-1}(ue^{\delta}-s, xe^{\delta})dG(s)$$

$$=1-G(ue^{\delta})+\int_{(u-x)e^{\delta}}^{ue^{\delta}} H(ue^{\delta}-s, xe^{\delta})dG(s) \tag{7}$$

In summary, the distribution of the largest surplus of insurance company before ruin satisfies the integral equation (7).

3. Ruin Probability

Ruin probability is what the insurance company most concern, it is also the most concern of the key research risk model. In this section, by using the recursion method, we can find an integral equation satisfying finite time ruin probability and ultimate ruin probability.

3.1 Finite time ruin probability.

In the theory of ruin, more research is concerning with ultimate ruin probability due to that the finite time ruin probability is usually not easy to get, here we discuss on this issue bellow.

For $n=1,2,3,\cdots$, let

$$\phi_n(u)=P\{T \le n\}$$

Then $\varphi_n(u)$ denotes the probability of ruin before or at time n with initial reserve u ,thus the probability of non-ruin within time n with initial reserve u .

$$\Phi_n(u)=1-\varphi_n(u)=P\{T>n\}$$

We obtain from the above definition

$$\Phi_n(u)=P\{T>n\}=P\{U_1^{\delta} \ge 0,\ldots,U_n^{\delta} \ge 0\}=P\{S_1 \le u,\ldots,S_n \le u\}$$

Using the recursive method, we get the following recursion formulas

$$\Phi_1(u)=P\{T>1\}=P\{S_1 \le u\}=P\{Z_1 \le ue^{\delta}\}=G(ue^{\delta})$$

$$\Phi_2(u)=P\{T>2\}=P\{S_1 \le u, S_2 \le u\}=P\{Z_1 \le ue^{\delta}, Z_1+\frac{Z_2}{e^{\delta}} \le ue^{\delta}\}$$

$$=\int_{-\infty}^{ue^{\delta}} P\{S_1 \le ue^{\delta}-s\}dG(s)=\int_{-\infty}^{ue^{\delta}} \Phi_1(ue^{\delta}-s)dG(s)$$

$$\Phi_3(u)=P\{T>3\}=P\{S_1 \le u, S_2 \le u, S_3 \le u\}$$

$$=P\{Z_1 \le ue^{\delta}, Z_1+\frac{Z_2}{e^{\delta}} \le ue^{\delta}, Z_1+\frac{Z_2}{e^{\delta}}+\frac{Z_3}{e^{2\delta}} \le ue^{\delta}\}$$

$$=\int_{-\infty}^{ue^{\delta}} P\{S_1 \le ue^{\delta}-s, S_2 \le ue^{\delta}-s\}dG(s)$$

$$=\int_{-\infty}^{ue^{\delta}} \Phi_2(ue^{\delta}-s)dG(s)$$

Then, recursively, for any $n \ge 2$

$$\Phi_n(u)=\int_{-\infty}^{ue^{\delta}} \Phi_{n-1}(ue^{\delta}-s)dG(s)$$

Thus, the corresponding ruin probabilities are that

$$\varphi_1(u)=1-\Phi_1(u)=1-G(ue^{\delta})$$

$$\varphi_2(u)=1-\Phi_2(u)=1-\int_{-\infty}^{ue^{\delta}} \Phi_1(ue^{\delta}-s)dG(s)$$

$$=1-\int_{-\infty}^{ue^{\delta}} (1-\phi_1(ue^{\delta}-s))dG(s)$$

$$=1-G(ue^{\delta})+\int_{-\infty}^{ue^{\delta}} \phi_1(ue^{\delta}-s))dG(s)$$

Then, recursively, for any

$$\varphi_{\mathrm{n}}(\mathrm{u})=1-G(\mathrm{ue}^{\delta})+\int_{-\infty}^{ue^{\delta}}\phi_{\mathrm{n-1}}(\mathrm{ue}^{\delta}\text{-s})\,)dG(s)$$

3.2 Ultimate ruin probability.

Ultimate ruin probability describes the probability of ultimate ruin of insurance company

$$\Psi(u)=P\{T<\infty\}$$

From the above definition we obtain

$$\Psi(\mathrm{u})=\sum_{n=1}^{\infty}P\{T=n\}=\sum_{n=1}^{\infty}P\{U_1^{\delta}\geq 0,\cdots U_{n-1}^{\delta}\geq 0,U_n^{\delta}<0\}$$

$$\overset{\nabla}{=}\sum_{n=1}^{\infty}Q_n(u) \tag{8}$$

Where $Q_{\mathrm{n}}(u)$ represents the probability of ruin at moment n.

Taking advantage of results of the next section, we have

$$Q_1(u)=P\{U_1^{\delta}<0\}=P\{S_1>u\}=P\{Z_1>ue^{\delta}\}=1-G(ue^{\delta})$$

$$Q_2(u)=P\{U_1^{\delta}\geq 0,U_2^{\delta}<0\}=P\{S_1\leq u,S_2>u\}$$

$$=P\{Z_1\leq ue^{\delta},Z_1+\frac{Z_2}{e^{\delta}}>ue^{\delta}\}=\int_{-\infty}^{ue^{\delta}}P\{S_1>ue^{\delta}-s\}dG(s)$$

$$=\int_{-\infty}^{ue^{\delta}}Q_1(\mathrm{ue}^{\delta}\text{-s})dG(s)$$

Recursively, for any $\mathrm{n}\geq 2$

$$Q_{\mathrm{n}}(\mathrm{u})=\int_{-\infty}^{ue^{\delta}}Q_{\mathrm{n-1}}(\mathrm{ue}^{\delta}\text{-s})\,)dG(s)$$

From (8) we obtain

$$\Psi(\mathrm{u})=\sum_{n=1}^{+\infty}Q_n(u)=Q_1(\mathrm{u})+\sum_{n=2}^{+\infty}Q_n(u)$$

$$=Q_1(\mathrm{u})+\sum_{n=2}^{+\infty}\int_{-\infty}^{ue^{\delta}}Q_{\mathrm{n-1}}(\mathrm{ue}^{\delta}\text{-s})\,)dG(s)$$

$$=1\text{-}G(\mathrm{ue}^{\delta})+\int_{-\infty}^{\mathrm{ue}^{\delta}}\Psi(\mathrm{u}e^{\delta}-s)dG(s) \tag{9}$$

To sum up, the probability of ultimate ruin of the insurance company satisfies the integral equation (9).

4. The Ruin Lasting Time

After the insurance company ruins, what circumstances cause the financial affairs to be worse? How long does the predicament last? Which not only concern the fates of the insurers, but also effect the profits of the insured. In this section, we shall discuss the probability character of the ruin lasting time.

We define the time when the surplus crosses the level zero for the first time after ruining

$$\tau(u)=\inf_{n}\left\{n:n>T,U_n^{\delta}>0\right\} \tag{10}$$

Thus, the ruin lasting time is that

$$\overline{T}(u)=\begin{cases}\tau(\mathrm{u})\text{-}T(\mathrm{u}), & T(\mathrm{u})<\infty\\ 0, & T(\mathrm{u})=\infty\end{cases} \tag{11}$$

Then $\overline{T}(u)=1$, the probability of the ruin time lasting one period is that

$$\nu_1(u)=P\{\overline{T}(u)=1\}=P\{\tau(u)=T(u)+1\}$$

$$=\sum_{k=1}^{\infty}P\{\tau(u)=k+1|T(u)=k\}P\{T(u)=k\}$$

$$=\sum_{k=1}^{\infty}P\{U_1^{\delta}\geq 0,\ U_2^{\delta}\geq 0,\cdots,U_{k-1}^{\delta}\geq 0,U_k^{\delta}<0,U_{k+1}^{\delta}\geq 0\}Q_k(u)$$

$$\overset{\nabla}{=}\sum_{k=1}^{+\infty}M_{\mathrm{k}}^{(1)}Q_k(u) \tag{12}$$

Where $Q_{\mathrm{k}}(u)$ has been obtained in the preceding section. Denote

$$A_1(u)=\int_{ue^{\delta}}^{\infty}G((ue^{\delta}-s)e^{\delta})dG(s)$$

will be derived in the following

$$M_1^{(1)}(\mathrm{u})=P\{U_1^{\delta}<0,U_2^{\delta}\geq 0\}=P\{S_1>u,S_2\leq u\}$$

$$=P\{Z_1>ue^{\delta},Z_1+\frac{Z_2}{e^{\delta}}\leq ue^{\delta}\}$$

$$=\int_{ue^{\delta}}^{\infty}p\{Z_2\leq(ue^{\delta}-s)e^{\delta}\}dG(s)$$

$$=\int_{ue^{\delta}}^{\infty}G((ue^{\delta}-s)e^{\delta})dG(s)\overset{\nabla}{=}A_1(\mathrm{u})$$

$$M_2^{(1)}(\mathrm{u})=P\{U_1^{\delta}\geq 0,\ \ U_2^{\delta}<0,U_3^{\delta}\geq 0\}$$

$$=P\{S_1\leq u,S_2>u,S_3\leq u\}$$

$$=P\{Z_1\leq ue^{\delta},Z_1+\frac{Z_2}{e^{\delta}}>ue^{\delta},Z_1+\frac{Z_2}{e^{\delta}}+\frac{Z_3}{e^{2\delta}}\leq ue^{\delta}\}$$

$$=\int_{-\infty}^{ue^{\delta}}P(S_1>ue^{\delta}-s,\ \ S_2\leq ue^{\delta}-s)dG(s)$$

$$=\int_{-\infty}^{ue^{\delta}}M_1^{(1)}(\mathrm{ue}^{\delta}\text{-s})dG(s)$$

By induction on $\mathrm{k}\geq 2$ we have

$$M_{\mathrm{k}}^{(1)}(\mathrm{u})=\int_{-\infty}^{ue^{\delta}}M_{\mathrm{k-1}}^{(1)}(\mathrm{ue}^{\delta}\text{-s})dG(s) \tag{13}$$

which is the probability of ruining at time k and lasting one

period. Hence, the probability of the ruin time lasting one period is that

$$\nu_1(u) = P\{\overline{T}(u) = 1\} = \sum_{k=1}^{+\infty} M_{\mathrm{k}}^{(1)}(\mathrm{u}) Q_k(u) \tag{14}$$

When $\overline{T}(u) = 2$, similarly

$$\nu_2(u) = P\{\overline{T}(u) = 2\} = P\{\tau(u) = T(u) + 2\}$$

$$= \sum_{k=1}^{\infty} P\{\tau(u) = k + 2 \,|\, T(u) = k\} P\{T(u) = k\}$$

$$= \sum_{k=1}^{\infty} P\{U_1^\delta \ge 0, \ldots, U_{k-1}^\delta \ge 0, U_k^\delta < 0, U_{k+1}^\delta < 0, U_{k+2}^\delta \ge u\} Q_k(u)$$

$$\overset{\nabla}{=} \sum_{k=1}^{+\infty} M_{\mathrm{k}}^{(2)}(\mathrm{u}) Q_k(u) \tag{15}$$

Denote

$$A_2(\mathrm{u}) = \int_{ue^\delta}^{\infty} A_1(ue^\delta - s) dG(s)$$

From (15) we have

$$M_1^{(2)}(\mathrm{u}) = P\{U_1^\delta < 0, U_2^\delta < 0, U_3^\delta \ge 0\}$$

$$= P\{S_1 > u, S_2 > u, S_3 \le u\}$$

$$= P\{Z_1 > ue^\delta, Z_1 + \frac{Z_2}{e^\delta} > ue^\delta, Z_1 + \frac{Z_2}{e^\delta} + \frac{Z_3}{e^{2\delta}} \le ue^\delta\}$$

$$= \int_{ue^\delta}^{\infty} p\{S_1 > ue^\delta - s, S_2 \le ue^\delta - s\} dG(s)$$

$$= \int_{ue^\delta}^{\infty} A_1(ue^\delta - s) dG(s) \overset{\nabla}{=} A_2(u)$$

$$M_2^{(2)}(\mathrm{u}) = P\{U_1 \ge 0, \ U_{,2}^\delta < 0, U_3^\delta < 0, U_4^\delta \ge 0\}$$

$$= P\{S_1 \le \mathrm{u}, \ S_2 > u, S_3 > u, S_4 \le u\}$$

$$= P\{Z_1 \le ue^\delta, Z_1 + \frac{Z_2}{e^\delta} > ue^\delta,$$

$$Z_1 + \frac{Z_2}{e^\delta} + \frac{Z_3}{e^{2\delta}} > ue^\delta, Z_1 + \frac{Z_2}{e^\delta} + \frac{Z_3}{e^{2\delta}} + \frac{Z_4}{e^{3\delta}} \le u\}$$

$$= \int_{-\infty}^{ue^\delta} P(S_1 > ue^\delta - s, \ S_2 > ue^\delta - s, S_3 \le ue^\delta - s) dG(s)$$

$$= \int_{-\infty}^{ue^\delta} M_1^{(2)}(\mathrm{ue}^\delta\text{-s}) dG(s)$$

By induction on $\mathrm{k} \ge 2$ we have

$$M_{\mathrm{k}}^{(2)}(\mathrm{u}) = \int_{-\infty}^{ue^\delta} M_{\mathrm{k-1}}^{(2)}(\mathrm{ue}^\delta - \mathrm{s}) dG(s)$$

This yields the probability of the ruin time lasting two period is that

$$\nu_2(u) = P\{\overline{T}(u) = 2\} = \sum_{k=1}^{+\infty} M_{\mathrm{k}}^{(2)}(\mathrm{u}) Q_k(u) \tag{16}$$

And so on, denote

$$A_{\mathrm{n}}(\mathrm{u}) = \int_{ue^\delta}^{\infty} A_{\mathrm{n-1}}(ue^\delta - s) dG(s)$$

When $\overline{T}(u) = \mathrm{n}$ as the same above

$$M_1^{(\mathrm{n})}(\mathrm{u}) = A_{\mathrm{n}}(\mathrm{u}) \tag{17}$$

$$M_{\mathrm{k}}^{(\mathrm{n})}(\mathrm{u}) = \int_{-\infty}^{ue^\delta} M_{\mathrm{k-1}}^{(\mathrm{n})}(\mathrm{ue}^\delta\text{-s}) dG(s) \tag{18}$$

Thus, the probability of the ruin time lasting n period is that

$$\nu_{\mathrm{n}}(u) = P\{\overline{T}(u) = \mathrm{n}\} = \sum_{k=1}^{+\infty} M_{\mathrm{k}}^{(\mathrm{n})}(\mathrm{u}) Q_k(u) \tag{19}$$

5. Conclusions

It can be seen that from the integral equations derived from the model we discussed in this paper satisfy various bankruptcy indicators, as long as the distribution function of the random variable Z_k is known, we can use the recursion formula to obtain further bankruptcy described the severity of insurance company go bankrupt.

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