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Rota-Baxter operators on pre-Lie algebras

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Abstract

Rota-Baxter operators or relations were introduced to solve certain analytic and combinatorial problems and then applied to many fields in mathematics and mathematical physics. In this paper, we commence to study the Rota-Baxter operators of weight zero on pre-Lie algebras. Such operators satisfy (the operator form of) the classical Yang-Baxter equation on the sub-adjacent Lie algebras of the pre-Lie algebras. We not only study the invertible Rota-Baxter operators on pre-Lie algebras, but also give some interesting construction of Rota-Baxter operators. Furthermore, we give all Rota-Baxter operators on 2-dimensional complex pre-Lie algebras and some examples in higher dimensions.

1 Introduction

Rota-Baxter operator or its corresponding Rota-Baxter relation was introduced by G. Baxter to solve an analytic problem [12]. Later it was intensively studied in many fields in mathematics [3, 17, 30, 39, 40, 41, 42, 43]. In particular, G.-C. Rota [40, 41, 42] showed its importance in combinatorics. In fact, Rota-Baxter relation may be regarded as one possible generalization of the standard shuffle relation [30, 40].

The Rota-Baxter relation is also closely related to many topics in mathematical physics. For example, the Rota-Baxter operator on an associative algebra satisfies the famous (operator form of) classical Yang-Baxter equation on the Lie algebra which is the commutator of the associative algebra [2, 22, 23]. Thus it plays an important role in the integrable systems [13, 44]. Recently, Rota-Baxter relation was found to be very important in quantum field theory [21, 24].

Up to now, most of study on Rota-Baxter operators was on the associative algebras. It was mentioned in [24] that the Rota-Baxter operator and relation can be extended to be on the case of Lie algebras or pre-Lie algebras. In the case of Lie algebras, when the weight $\lambda = 0$, the Rota-Baxter relation is just the operator form of classical Yang-Baxter equation and when the weight $\lambda = 1$, it corresponds to the operator form of the modified classical Yang-Baxter equation [24].

However, there was very few study about the Rota-Baxter relation on pre-Lie algebras. Pre-Lie algebras (also named left-symmetric algebras) are a class of non-associative algebras which first have been studied in the theory of affine manifolds, affine structures on Lie groups and convex homogeneous cones [4, 33, 38, 47] and then appeared in many fields in mathematics and mathematical physics, such as complex and symplectic structures on Lie groups and Lie algebras [1, 20, 45], integrable systems [14, 46], classical and quantum Yang-Baxter equations [25, 26, 29, 35, 36], Poisson brackets and infinite-dimensional Lie algebras [11, 28, 49], quantum field theory [21], operads [18] and so on. Therefore, it should be quite interesting and necessary to study the Rota-Baxter operators on pre-Lie algebras independently.

In this paper, we commence to study the Rota-Baxter relation in the pre-Lie algebraic version, that is, the Rota-Baxter operators on pre-Lie algebras. We mainly consider the Rota-Baxter operators of weight zero. Like the case of associative algebras, a Rota-Baxter operator of weight zero on a pre-Lie algebra also satisfies the classical Yang-Baxter equation on its sub-adjacent Lie algebra. Moreover, it can induce a series of pre-Lie algebras (the double construction). The paper is organized as follows. In section 2, we give some fundamental results on both Rota-Baxter relations and pre-Lie algebras. In section 3, we study the invertible Rota-Baxter operators. In section 4, we provide some approach for constructing Rota-Baxter operators on certain pre-Lie algebras. In section 5, we give all Rota-Baxter operators on 2-dimensional complex pre-Lie algebras. In section 6, some examples in higher dimensions are given. In section 7, we give some further discussion and questions.

Throughout this paper, without special saying, all algebras are of finite dimension and over the complex field $\mathbf{C}$. And $\langle \mid \rangle$ stands a Lie or pre-Lie algebra with a basis and nonzero products at each side of $|$.

2 Preliminaries and some basic results

Definition 2.1 Let A be a vector space over a filed $\mathbf{F}$ with a bilinear product $(x, y) \to xy$. A is called a pre-Lie algebra (or left-symmetric algebra), if for any $x, y, z \in A$, the associator

$$(x, y, z) = (xy)z - x(yz) \tag{2.1}$$

is symmetric in x, y, that is

$$(x, y, z) = (y, x, z), \text{ or equivalently } (xy)z - x(yz) = (yx)z - y(xz). \tag{2.2}$$

For a pre-Lie algebra A, the commutator [38]

$$[x, y] = xy - yx, \tag{2.3}$$

defines a Lie algebra $\mathcal{G} = \mathcal{G}(A)$, which is called the sub-adjacent Lie algebra of A. For any $x, y \in A$, let L_x and R_x denote the left and right multiplication operator respectively, that is, $L_x(y) = xy$ and $R_x(y) = yx$. Then the left-symmetry (2.2) is just

$$[L_x, L_y] = L_{[x,y]}, \quad \forall x, y \in A, \tag{2.4}$$

which means that $L : \mathcal{G}(A) \to gl(\mathcal{G}(A))$ with $x \to L_x$ gives a (regular) representation of the Lie algebra $\mathcal{G}(A)$.

Some subclasses of pre-Lie algebras are very important:

Definition 2.2 Let A be a pre-Lie algebra.

(1) If A has no ideals except itself and zero, then A is called simple. A is called semisimple if A is the direct sum of simple pre-Lie algebras [15, 16].

(2) If for every $x \in A$, R_x is nilpotent, then A is said to be transitive or complete. The transitivity corresponds to the completeness of the affine manifolds in geometry [33, 38]. Moreover, the sub-adjacent Lie algebra of a transitive pre-Lie algebra is solvable.

(3) If for any $x, y \in A$, $R_xR_y = R_yR_x$, then A is called a Novikov algebra. It was introduced in connection with the Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus [11, 27, 28, 48, 49].

(4) If for any $x, y, z \in A$, the associator (x, y, z) is right-symmetric, that is, $(x, y, z) = (x, z, y)$, then A is said to be bi-symmetric. It is just the assosymmetric ring in the study of near associative algebras [6, 34].

Definition 2.3 Let $\mathcal{A}$ be an (associative or non-associative) algebra over a field $\mathbf{F}$. A linear operator $R : \mathcal{A} \to \mathcal{A}$ is called a Rota-Baxter operator of weight $\lambda \in \mathbf{F}$ on $\mathcal{A}$ if R satisfies the following (Rota-Baxter) relation:

$$R(x)R(y) + \lambda R(xy) = R(R(x)y + xR(y)), \forall x, y \in \mathcal{A}. \tag{2.5}$$

For any $\lambda \neq 0$, $R \to \lambda^{-1}R$ can reduce the Rota-Baxter operator R of weight λ to be of weight $\lambda = 1$.

The Lie algebraic version of equation (2.5) with $\lambda = 0$ is just the well-known classical Yang-Baxter equation which plays an important role in intergrable systems [13, 19, 44].

Definition 2.4 Let $\mathcal{G}$ be a Lie algebra. A linear operator $R : \mathcal{G} \to \mathcal{G}$ is called a solution of the (operator form) of classical Yang-Baxter equation (CYBE) on $\mathcal{G}$ if R satisfies the following relation:

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]), \forall x, y \in \mathcal{G}. \tag{2.6}$$

We denote the set of such solutions of CYBE on $\mathcal{G}$ by CYB($\mathcal{G}$).

Remark 2.5 The original form of CYBE is given as follows. Let $\mathcal{G}$ be a Lie algebra and $r \in \mathcal{G} \otimes \mathcal{G}$. r is called a solution of classical Yang-Baxter equation (CYBE) on $\mathcal{G}$ if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \ \text{ in } U(\mathcal{G}), \tag{2.7}$$

where $U(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$ and for $r = \sum\limits_i a_i \otimes b_i$,

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1; \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i; \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i. \tag{2.8}$$

It is known that equation (2.6) is equivalent to equation (2.7) when r is skew-symmetric and there is a nondegenerate symmetric invariant bilinear form on $\mathcal{G}$ [44].

In the following, all Rota-Baxter operators are assumed to be of weight $\lambda = 0$. We mainly study such Rota-Baxter operators on pre-Lie algebras, that is,

$$R(x)R(y) = R(R(x)y + xR(y)), \forall x, y \in A, \tag{2.9}$$

where A is a pre-Lie algebra. We denote the set of Rota-Baxter operators on a pre-Lie algebra A by $\mathrm{RB}(A)$. Obviously, we have the following two conclusions.

Proposition 2.6 Let R be a Rota-Baxter operator on a pre-Lie algebra A. Then R is a solution of CYBE on its sub-adjacent Lie algebra $\mathcal{G}(A)$, that is, $\mathrm{RB}(A) \subset \mathrm{CYB}(\mathcal{G}(A))$.

Proposition 2.7 Let $(A, \cdot)$ be a pre-Lie algebra. Let A' denote the algebra defined by a product $(x, y) \to x \circ y$ on A which satisfies $x \circ y = y \cdot x$ for any $x, y \in A$. If $(A', \circ)$ is still a pre-Lie algebra, then $\mathrm{RB}(A) = \mathrm{RB}(A')$.

Remark 2.8 It is easy to know that the above $(A', \circ)$ is a pre-Lie algebra if and only if $(A, \cdot)$ is bi-symmetric.

3 Invertible Rota-Baxter operators on pre-Lie algebras

Recall that a derivation of a pre-Lie algebra A is a linear operator $D : A \to A$ satisfying

$$D(xy) = D(x)y + xD(y),\ \forall x,\ y \in A. \tag{3.1}$$

Proposition 3.1 Let A be a pre-Lie algebra and $R : A \to A$ be a linear isomorphism. Then R is a Rota-Baxter operator on A if and only if R^{-1} is a derivation on A.

Proof For any $x, y \in A$, R is a Rota-Baxter operator on A if and only if $R(x)R(y) = R(R(x)y + xR(y))$, which is equivalent to

$$R^{-1}(uv) = uR^{-1}(v) + R^{-1}(u)v,$$

where $u = R(x), v = R(y)$. Therefore the conclusion follows. ■

Theorem 3.2 Let $(A, \cdot)$ be a pre-Lie algebra. If there exists an invertible derivation D, then for any $x \in A$, L_x is nilpotent.

Proof Let $A = \oplus_{\alpha\in\Omega} A_\alpha$ be the decomposition of A into the direct sum of the weight subspaces of D associated to the weight α, where $A_\alpha = \{x \in A | (D - \alpha \mathrm{Id})^m x = 0,\ \text{some}\ m \in \mathbf{N}\}$, Id is the identity transformation and Ω is the set of the weights of D. Obviously Ω is a finite set and $0 \notin \Omega$ since D is invertible. Because D is a derivation on A, for any $\alpha, \beta \in \Omega$, it is easy to know that $A_\alpha \cdot A_\beta \subset A_{\alpha+\beta}$ when $\alpha + \beta \in \Omega$ and $A_\alpha \cdot A_\beta = 0$ when $\alpha + \beta \notin \Omega$. For any $x \in A$, suppose $x \in A_\alpha, \alpha \neq 0$. Therefore there exits a large $N \in \mathbf{N}$ such that $N\alpha + \beta \notin \Omega$ for any $\beta \in \Omega$. Hence for any $y \in A$, assume $y \in A_\beta$, we have

$$L_x^N(y) = x \cdot (x \cdot (\cdots (x \cdot y) \cdots)) \in A_\alpha \cdot (A_\alpha \cdot (\cdots \cdot (A_\alpha \cdot A_\beta) \cdots)) = 0.$$

Hence L_x is nilpotent. ■

Lemma 3.3 [33, Theorem 2.1 and 2.2] Let A be a pre-Lie algebra. The following two conditions are equivalent.

(1) For any $x \in A$, L_x is nilpotent;

(2) A is transitive and the sub-adjacent Lie algebra $\mathcal{G}(A)$ is nilpotent.

Corollary 3.4 Let A be a pre-Lie algebra. If there exists an invertible Rota-Baxter operator on A, then for every $x \in A$, L_x is nilpotent. Moreover, A is transitive and the sub-adjacent Lie algebra $\mathcal{G}(A)$ is nilpotent.

Remark 3.5 Obviously, any derivation D of a pre-Lie algebra A is also a derivation of its sub-adjacent Lie algebra $\mathcal{G}(A)$, that is, D satisfies

$$[D(x), D(y)] = [D(x), y] + [x, D(y)],\ \ \forall x, y \in A. \tag{3.2}$$

Therefore, the above conclusion (that is, both R_x and $\mathrm{ad}x = L_x - R_x$ are nilpotent) can be obtained directly from the proof in Theorem 3.2 by replacing L_x by R_x and $\mathrm{ad}x$ respectively. On the other hand, it also partly coincides with a well-known conclusion in Lie algebra [31]: any Lie algebra with an invertible derivation is nilpotent.

The converse of the above conclusion is not true in general. In fact, there exists a transitive pre-Lie algebra on a nilpotent Lie algebra without any invertible derivation (see Example 3.11).

Furthermore, there is a simple construction of invertible derivations or Rota-Baxter operators. Let $(A, \cdot)$ be a pre-Lie algebra. A is called graded if $A = \oplus_{\lambda\in\Gamma} A_\lambda$ as a direct sum of vector spaces, $A_\lambda \neq \{0\}$ and $A_\alpha \cdot A_\beta \subset A_{\alpha+\beta}$. Γ is called a gradation set.

Proposition 3.6 Let A be a graded pre-Lie algebra graded by Γ. If $0 \notin \Gamma$, then A has an invertible derivation.

Proof Let $A = \oplus_{\lambda\in\Gamma} A_\lambda$. Set $D : A \to A$ given by $D(x_\alpha) = \alpha x_\alpha$ for any $x_\alpha \in A_\alpha$. Then it is easy to know that D is an derivation of A. D is invertible since $0 \notin \Gamma$. ■

On the other hand, we have the following conclusion.

Theorem 3.7 Let $\mathcal{G}$ be a Lie algebra. If D is an invertible derivation of $\mathcal{G}$ (hence $\mathcal{G}$ is nilpotent), then there exists a compatible pre-Lie algebra on $\mathcal{G}$ given by

$$x * y = D^{-1}([x, D(y)]), \quad \forall x, y \in \mathcal{G}. \tag{3.3}$$

D is also an invertible derivation of the above pre-Lie algebra. Hence D^{-1} is a Rota-Baxter operator on this pre-Lie algebra.

Proof The fact that the product (3.3) defines a pre-Lie algebra can be checked directly or by a conclusion in [16]. Let $x, y \in \mathcal{G}$. Then we have

$$\begin{aligned} D(x) * y + x * D(y) &= D^{-1}([D(x), D(y)]) + D^{-1}([x, D^2(y)]) \\ &= D^{-1}(D([x, D(y)])) = [x, D(y)] = D(x * y). \end{aligned}$$

Hence D is an invertible derivation of the pre-Lie algebra $(\mathcal{G}, *)$. ■

Corollary 3.8 Let $\mathcal{G}$ be a nilpotent Lie algebra with an invertible derivation D. Then the pre-Lie algebra defined by equation (3.3) is transitive.

Example 3.9 According to [16], there are exactly two transitive pre-Lie algebras in dimension 2 whose sub-adjacent Lie algebras are nilpotent: the trivial pre-Lie algebra (all products are zero) (C4) and (C5) $=< e_1, e_2 | e_1 e_1 = e_2 >$. From [9], both of them have the invertible derivations. In fact, they are graded by a set without zero. For example, for the pre-Lie algebra (C5), the linear map $R : (C5) \to (C5)$ given by $R(e_1) = 2e_1, R(e_2) = e_2$ is a Rota-Baxter operator on (C5) (also see section 5). ■

Example 3.10 In [33], a classification of 3-dimensional transitive pre-Lie algebras whose sub-adjacent Lie algebras are nilpotent was given as follows (also see Proposition 6.1 and Remark 6.2 in section 6).

Trivial algebra (T1); (T2)$=< e_1, e_2, e_3 | e_3 e_3 = e_2 >$;
(T3)$=< e_1, e_2, e_3 | e_2 e_2 = e_3 e_3 = e_1 >$;
(T4)$=< e_1, e_2, e_3 | e_2 e_3 = e_3 e_2 = e_1, e_3 e_3 = e_2 >$;
(T5)$=< e_1, e_2, e_3 | e_2 e_3 = e_1, e_3 e_2 = -e_1 >$;
$(T6)_l =< e_1, e_2, e_3 | e_2 e_2 = e_2 e_3 = e_1, e_3 e_2 = -e_1, e_3 e_3 = l e_1 >$.
$(T7)_l =< e_1, e_2, e_3 | e_2 e_3 = e_1, e_3 e_2 = l e_1, e_3 e_3 = e_2 >, l \neq 1$;
(T8)$=< e_1, e_2, e_3 | e_3 e_2 = e_1, e_3 e_3 = e_2 >$.

It is easy to know that all of them are Novikov algebras and then from [10], all of them have the invertible derivations. In fact, all of them are graded by a set without zero. In particular, the algebras of type $(\mathrm{T7})_l (l \neq 1)$ and (T8) are not associative. ■

Example 3.11 There exists a 4-dimensional transitive pre-Lie algebra on a nilpotent Lie algebra without an invertible derivation [33]. For example, let A be a pre-Lie algebra given by

$$< e_1, e_2, e_3, e_4 | e_2e_3 = e_1, e_3e_3 = e_2, e_4e_1 = e_2, e_1e_4 = e_2, e_2e_4 = -e_3, e_4e_4 = e_1 > .$$

Then A is transitive and $\mathcal{G}(A)$ is nilpotent. It is easy to know that $D : A \to A$ is a derivation if only if D satisfies

$$D(e_1) = D(e_2) = 0, D(e_3) = ae_1, D(e_4) = -ae_3, a \in \mathbf{F}.$$

Obviously, D cannot be invertible. ■

4 Construction of Rota-Baxter operators on some pre-Lie algebras

(1) Pre-Lie algebras from direct sums and tensor products

The following proposition is obvious:

Proposition 4.1 Let $A = A_1 \oplus A_2$ be a direct sum of two ideals of pre-Lie algebras. Then for any Rota-Baxter operators R_i on A_i $(i = 1, 2)$, the linear map $R : A \to A$ given by $R(x_1, x_2) = (R_1(x_1), R_2(x_2))$ (for any $x_1 \in A_1, x_2 \in A_2$) defines a Rota-Baxter operator on A.

Remark 4.2 It is not true that every Rota-Baxter operator on $A = A_1 \oplus A_2$ can be obtained from the above way. Some counterexamples can be found in sections 5 and 6.

In general, it is hard to define a pre-Lie algebra structure on a tensor space of two pre-Lie algebras. However, we have the following conclusion.

Proposition 4.3 Let $(A, \cdot)$ be a pre-Lie algebra and $(A', \circ)$ be a commutative associative algebra. Then there is a pre-Lie algebra structure on the vector space $A \otimes A'$ given by

$$(x \otimes x') * (y \otimes y') = (x \cdot y) \otimes (x' \circ y'), \quad \forall x, y \in A, x', y' \in A'. \tag{4.1}$$

Moreover, if R_1 is a Rota-Baxter operator on A and R_2 is a Rota-Baxter operator on A', then for any $x \in A, x' \in A'$, $R(x \otimes x') = R(x) \otimes R(x')$ defines a Rota-Baxter operator on $(A \otimes A', *)$.

Proof The conclusion can be obtained by a direct computation. ■

(2) Pre-Lie algebras from linear functions

In [5], the pre-Lie algebras satisfying the following conditions are considered: for any two vectors x, y in a pre-Lie algebra A, the product $x \cdot y$ is still in the subspace spanned by x, y.

Proposition 4.4 [5] Let A be a vector space in dimension $n \geq 2$. Let $f, g : A \to \mathbf{C}$ be two linear functions. Then the product

$$x * y = f(y)x + g(x)y, \quad \forall x, y \in A \tag{4.2}$$

defines a left-symmetric algebra if and only if $f = 0$ or $g = 0$. Furthermore, we have

(a) If $f=0, g\neq 0$, then there exists a basis $\{e_1,\cdots,e_n\}$ in A such that $L_{e_1}=\mathrm{Id}, L_{e_i}=0, i=2,\cdots,n$.

(b) If $g=0, f\neq 0$, then there exists a basis $\{e_1,\cdots,e_n\}$ in A such that $R_{e_1}=\mathrm{Id}, R_{e_i}=0, i=2,\cdots,n$.

(c) If $f=g=0$, then A is a trivial algebra.

Moreover, all the above algebras are associative.

Proposition 4.5 Let A be a non-trivial pre-Lie algebra given by (4.2). Then $R: A\to A$ is a Rota-Baxter operator if and only if $R^2=0$.

Proof Let A be the pre-Lie algebra in the case (a) in Proposition 4.4. Let $R(e_i)=\sum_{j=1}^n r_{ij}e_j$. Then by the Rota-Baxter relation (2.9), we only need to check the following equations (other equations hold naturally):

$$R(e_1)R(e_i)=R(e_1R(e_i)+R(e_1)e_i),\ \ \forall\ \ i=1,\cdots,n.$$

For any i, the left hand side is $r_{11}R(e_i)$ and the right hand side is $R^2(e_i)+r_{11}R(e_i)$. Therefore R is Rota-Baxter operator if and only if $R^2=0$. With a similar discussion or by Proposition 2.7, the conclusion still holds for the pre-Lie algebra in the case (b) in Proposition 4.4. ■

(3) Pre-Lie algebras from Lie algebras

Lemma 4.6 [29] Let $\mathcal{G}$ be a Lie algebra and R be a linear transformation on $\mathcal{G}$ satisfying the operator form of CYBE (2.6). Then on $\mathcal{G}$ a new product

$$x*y=[R(x),y],\forall x,y\in\mathcal{G} \tag{4.3}$$

defines a pre-Lie algebra.

Proposition 4.7 Let $\mathcal{G}$ be a Lie algebra and R be a linear transformation on $\mathcal{G}$ satisfying the operator form of CYBE (2.6). Then R is a Rota-Baxter operator on the pre-Lie algebra given by equation (4.3).

Proof For any $x,y\in A$, we know that

$$\begin{aligned}R(x)*R(y)=[R^2(x),R(y)] &= R([R^2(x),y]+[R(x),R(y)])\\ &= R(R(x)*y+x*R(y)).\end{aligned}$$

Therefore R is a Rota-Baxter operator on $(\mathcal{G},*)$. ■

Corollary 4.8 Let $(A,\cdot)$ be a pre-Lie algebra and R be a Rota-Baxter operator. Then the product given by

$$x*y=[R(x),y]_\cdot=R(x)\cdot y-y\cdot R(x),\ \ x,\ y\in A, \tag{4.4}$$

defines a new pre-Lie algebra $(A,*)$. Moreover R is still a Rota-Baxter operator on $(A,*)$.

Proof The first half of the conclusion follows directly from Proposition 2.6 and Lemma 4.6. The second half of the conclusion follows from Proposition 4.7. ■

We call the above pre-Lie algebra $(A,*)$ the (1st) double of $(A,\cdot)$ associated to the Rota-Baxter R. Moreover, for any pre-Lie algebra $(A,\cdot)$ with a Rota-Baxter operator R, we can define a series of pre-Lie algebras $(A,*_i)$ as follows. $(A,*_0)=(A,\cdot)$ and the product on $(A,*_i)$ $(i\geq 1)$ is given by

$$x*_i y=[R(x),y]_{i-1}=R(x)*_{i-1}y-y*_{i-1}R(x),\ \ \forall x,y\in A. \tag{4.5}$$

$(A, *_i)$ is called the ith double of $(A, \cdot)$. It is the (1st) double of $(A, *_{i-1})$ associated to R.

Proposition 4.9 Let $(A, \cdot)$ be a pre-Lie algebra and R be a Rota-Baxter operator. Then for any $i \geq 0$, we have

$$\begin{aligned} x *_{i+1} (y) &= \sum_{k=0}^{i} C_i^k [R^{i+1-k}(x), R^k(y)]. \\ &= \sum_{k=0}^{i} C_i^k (R^{i+1-k}(x) \cdot R^k(y) - R^k(y) \cdot R^{i+1-k}(x)), \quad \forall x, y \in A; \end{aligned} \tag{4.6}$$

$$\begin{aligned} [x, y]_i &= x *_i y - y *_i x = \sum_{k=0}^{i} C_i^k [R^{i-k}(x), R^k(y)]. \\ &= \sum_{k=0}^{i} C_i^k (R^{i-k}(x) \cdot R^k(y) - R^k(y) \cdot R^{i-k}(x)), \quad \forall x, y \in A; \end{aligned} \tag{4.7}$$

Proof It can be obtained by induction on i. ■

Corollary 4.10 Let $(A, \cdot)$ be a pre-Lie algebra and R be a Rota-Baxter operator. If R is nilpotent, then there exists N such that $(A, *_n)$ is trivial for $n > N$.

Proof Set $R^m = 0$. Then for any $n \geq 2m - 1$ and $k \leq n$, either $k \geq m$ or $n - k \geq m$. Hence by equation (4.6), $x *_n y = 0$ for any $x, y \in A$. ■

Obviously, the converse of the above conclusion does not hold in general. For example, the 1st double of any commutative associative algebra associated to any Rota-Baxter operator is trivial. From Proposition 4.5, it is easy to get the following conclusion.

Corollary 4.11 Let A be a pre-Lie algebra given by (4.2). Then the 3rd double of A associated to any Rota-Baxter operator is trivial.

On the other hand, the idempotent Rota-Baxter R (that is, $R^2 = R$) is quite useful in quantum field theory [24]. The following conclusion is obvious.

Corollary 4.12 Let $(A, \cdot)$ be a pre-Lie algebra and R be a Rota-Baxter operator. If R is idempotent, then we have

$$x *_{i+1} y = [R(x), y]_{\cdot} + (2^i - 1)[R(x), R(y)]_{\cdot}, \quad \forall x, y \in A, \ i \geq 0. \tag{4.8}$$

(4) Novikov algebras from commutative associative algebras

There is a kind of realization theory of Novikov algebras in [7, 8]. Let A be a commutative associative algebra with the product $(, \cdot ,)$ and D be its derivation. Then the new product

$$x *_a y = x \cdot Dy + a \cdot x \cdot y, \tag{4.9}$$

makes $(A, *_a)$ become a Novikov algebra for $a = 0$ by S. Gel'fand [28], for $a \in \mathbf{F}$ by Filipov [27] and for a fixed element $a \in A$ by Xu [48]. In [7, 8], we show that the algebra $(A, *) = (A, *_0)$ given by S. Gel'fand is transitive, and the other two kinds of Novikov algebras given by Filipov and Xu are the special deformations of the former. Moreover, in [7, 8] a deformation theory of Novikov algebras is constructed and we prove that the Novikov algebras in dimension ≤ 3 can be realized as the algebras defined by S. Gel'fand and their compatible linear deformations. We conjecture that this conclusion can be extended to higher dimensions.

Proposition 4.13 Let $(A, \cdot)$ be a commutative associative algebra and D be its derivation. Let R be a Rota-Baxter operator on $(A, \cdot)$. If $RD = DR$, then R is a Rota-Baxter operator on $(A, *_a)$ for $a = 0$ and $a \in \mathbf{F}$. If $DR = RD$ and $RL^{\cdot}_a = L^{\cdot}_a R$, then R is a Rota-Baxter operator on $(A, *_a)$ for $a \in A$.

Proof Let $a \in A$, $a \neq 0$ and $DR = RD$, $RL^{\cdot}_a = L^{\cdot}_a R$. Then for any $x, y \in A$, we have

$$\begin{aligned} R(x) *_a R(y) &= R(x) \cdot DR(y) + a \cdot R(x) \cdot R(y) \\ &= R(x) \cdot RD(y) + a \cdot R(R(x) \cdot y + x \cdot R(y)) \\ &= R(R(x) \cdot D(y) + x \cdot RD(y)) + R(a \cdot R(x) \cdot y + a \cdot x \cdot R(y)) \\ &= R(R(x) *_a y + x *_a R(y)). \end{aligned}$$

Hence R is a Rota-Baxter operator of $(A, *_a)$. Similarly, the conclusion also holds for the cases $a = 0$ and $a \in \mathbf{F}$. ■

Example 4.14 From [9], we know that the Novikov algebra $(\mathrm{NIV})_1 =< e_1, e_2 | e_1 e_2 = e_1, e_2 e_2 = e_1 + e_2 >$ is obtained from $(\mathrm{C2}) =< e_1, e_2 | e_1 e_2 = e_2 e_1 = e_1, e_2 e_2 = e_2 >$ with a derivation D given by $D(e_1) = -e_1, D(e_2) = 0$ and $a = e_1 + e_2$. From section 5, we know that $\mathrm{RB}((\mathrm{NVI})_1) = \mathrm{RB}((\mathrm{C2})) = \{\begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix} | r_{21} \in \mathbf{C}\}$. ■

(5) Complex structures and phase spaces (parakähler structures) on Lie algebras

Let $\mathcal{G}$ be a Lie algebra and $\rho : \mathcal{G} \to gl(V)$ be its representation. It is known that there is a natural Lie algebra structure on $\mathcal{G} \oplus V$ as a direct sum of vector spaces given as follows.

$$[x_1 + v_1, x_2 + v_2] = [x_1, x_2] + \rho(x_1)v_2 - \rho(x_2)v_1, \ \forall x_1, x_2 \in \mathcal{G}, v_1, v_2 \in V. \tag{4.10}$$

We denote this Lie algebra by $\mathcal{G} \ltimes_\rho V$.

Lemma 4.15 [1, 35] Let $(A, \cdot)$ be a pre-Lie algebra. Let $L : \mathcal{G}(A) \to gl(A)$ be the regular representation of its sub-adjacent Lie algebra and $L^* : \mathcal{G}(A) \to gl(A^*)$ be its dual representation.

(1) There exists a compatible pre-Lie algebra structure on $\mathcal{G}(A) \ltimes_L \mathcal{G}(A)$ given by

$$(x, y) * (x', y') = (x \cdot x', x \cdot y'), \ \forall x, y, x', y' \in A. \tag{4.11}$$

(2) There exist two compatible pre-Lie algebra structures on $\mathcal{G}(A) \ltimes_{L^*} \mathcal{G}(A)^*$ given as follows.

$$(x, a^*) \circ_1 (y, b^*) = (x \cdot y, L^*(x)b^*), \ \forall x, y \in A, \ a^*, b^* \in A^*; \tag{4.12}$$

$$(x, a^*) \circ_2 (y, b^*) = (x \cdot y, \ \mathrm{ad}^*(x)b^* + (\mathrm{ad}^* - L^*)(y)a^*), \ \forall x, y \in A, \ a^*, b^* \in A^*, \tag{4.13}$$

where ad^* is the dual representation of the adjoint representation of $\mathcal{G}(A)$.

On the other hand, the pre-Lie algebras play a crucial role in some structures on Lie algebras related to geometry, like the complex structures and parakähler structures.

Definition 4.16 Let $\mathcal{G}$ be a real Lie algebra. A complex structure on $\mathcal{G}$ is a linear endomorphism $J : \mathcal{G} \to \mathcal{G}$ satisfying $J^2 = -\mathbf{1}$ and the integrable condition:

$$J[x, y] = [Jx, y] + [x, Jy] + J[Jx, Jy], \ \forall x, y \in \mathcal{G}. \tag{4.14}$$

Recall that a skew-symmetric 2-cocycle ω of a Lie algebra $\mathcal{G}$ satisfies

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0, \ \forall \ x, y, z \in \mathcal{G}. \tag{4.15}$$

Definition 4.17 Let $\mathcal{G}$ be a Lie algebra. If there is a Lie algebra structure in the direct sum (as vector spaces) of $\mathcal{G}$ and its dual space $\mathcal{G}^* = \mathrm{Hom}(\mathcal{G}, \mathbf{F})$ such that $\mathcal{G}$ and $\mathcal{G}^*$ are Lie subalgebras and the natural skew-symmetric bilinear (symplectic) form on $\mathcal{G} \oplus \mathcal{G}^*$

$$\omega((x, a^*), (y, b^*)) =< a^*, y > - < x, b^* >, \ \ \forall x, y \in \mathcal{G}, a^*, b^* \in \mathcal{G}^*, \tag{4.16}$$

is a 2-cocycle ($<,>$ is the ordinary pair between the vector space $\mathcal{G}$ and its dual space $\mathcal{G}^*$), then it is called a phase space of the Lie algebra $\mathcal{G}$ [35]. Such a structure is also called a parakähler structure [32].

In geometry, a parakähler manifold is a symplectic manifold with a pair of transversal Lagrangian foliations [37]. A parakähler Lie algebra $\mathcal{G}$ is just the Lie algebra of a Lie group G with a G-invariant parakähler structure [32]. Furthermore, from [1] and [35], we know

Proposition 4.18 Let A be a pre-Lie algebra. Then there is a complex structure J on $\mathcal{G}(A) \ltimes_L \mathcal{G}(A)$ given by $J(x, y) = (-y, x)$ for any $x, y \in A$ and $\mathcal{G}(A) \ltimes_{L^*} \mathcal{G}(A)^*$ is a phase space of the Lie algebra of $\mathcal{G}(A)$ with the symplectic form ω given by (4.16).

Proposition 4.19 Let $(A, \cdot)$ be a pre-Lie algebra and R be its Rota-Baxter operator.

(1) The map $\mathcal{R} : A \oplus A \to A \oplus A$ given by $\mathcal{R}(x, y) = (R(x), R(y))$ for any $x, y \in A$ is a Rota-Baxter operator on the pre-Lie algebra given by (4.11). Moreover, $\mathcal{R}J = J\mathcal{R}$, where J is the complex structure given in Proposition 4.18.

(2) Let $R^* : A^* \to A^*$ be a operator defined by $< R^*(a^*), x >=< a^*, R(x) >$, where $a^* \in A^*$, $x \in A$. Then the map $\mathcal{R} : A \oplus A^* \to A \oplus A^*$ given by $\mathcal{R}(x, a^*) = (R(x), -R^*(a^*))$ for any $x \in A$ and $a^* \in A^*$ is a Rota-Baxter operator on both pre-Lie algebras given by (4.12) and (4.13). Moreover, $\omega(\mathcal{R}(u), v) + \omega(u, \mathcal{R}(v)) = 0$ for any $u, v \in \mathcal{G}(A) \ltimes_{L^*} \mathcal{G}(A)^*$, where ω is given by (4.16).

Proof (1) Let $x, y, x', y' \in A$. Then

$$\begin{aligned}
\mathcal{R}(x, y) * \mathcal{R}(x', y') &= (R(x), R(y)) * (R(x'), R(y')) = (R(x) \cdot R(x'), R(x) \cdot R(y')) \\
&= (R(R(x) \cdot x' + x \cdot R(x')), R(R(x) \cdot y' + x \cdot R(y'))) \\
&= \mathcal{R}((R(x) \cdot x', R(x) \cdot y') + (x \cdot R(x'), x \cdot R(y'))) \\
&= \mathcal{R}((R(x), R(y)) * (x', y') + (x, y) * (R(x'), R(y'))) \\
&= \mathcal{R}(\mathcal{R}(x, y) * (x', y') + (x, y) * \mathcal{R}(x', y')).
\end{aligned}$$

Hence $\mathcal{R}$ is a Rota-Baxter operator on the pre-Lie algebra given by (4.11). Furthermore, we have

$$\mathcal{R}J(x, y) = \mathcal{R}(-y, x) = (-R(y), R(x)) = J(R(x), R(y)) = J\mathcal{R}(x, y).$$

(2) Let $x, y \in A, a^*, b^* \in A^*$. Then

$$\begin{aligned}
\mathcal{R}(x, a^*) \circ_1 \mathcal{R}(y, b^*) &= (R(x), -R^*(a^*)) \circ_1 (R(y), -R^*(b^*)) \\
&= (R(x) \cdot R(y), -L^*(R(x))R^*(b^*)); \\
\mathcal{R}(\mathcal{R}(x, a^*) \circ_1 (y, b^*) &+ (x, a^*) \circ_1 \mathcal{R}(y, b^*)) \\
&= \mathcal{R}((R(x) \cdot y, L^*(R(x))b^*) + (x \cdot R(y), -L^*(x)R^*(b^*)) \\
&= (R(R(x) \cdot y + x \cdot R(y)), R^*(-L^*(R(x))b^* + L^*(x)R^*(b^*))).
\end{aligned}$$

Since for any $z \in A$, we have

$$< -L^*(R(x))R^*(b^*), z >=< R^*(b^*), R(x) \cdot z >=< b^*, R(R(x) \cdot z) >;$$
$$< R^*(-L^*(R(x))b^*), z >= - < L^*(R(x))b^*, R(z) >=< b^*, R(x) \cdot R(z) >;$$
$$< R^*(L^*(x)R^*(b^*)), z >=< L^*(x)R^*(b^*), R(z) >=< b^*, -R(x \cdot R(z)) > .$$

Because R is a Rota-Baxter operator of $(A, \cdot)$, $\mathcal{R}$ is a Rota-Baxter operator on the pre-Lie algebra given by (4.12). Similarly, $\mathcal{R}$ is also a Rota-Baxter operator on the pre-Lie algebra given by (4.13). Furthermore, we have

$$\begin{aligned}
&\omega(\mathcal{R}(x, a^*), (y, b^*)) + \omega((x, a^*), \mathcal{R}(y, b^*)) \\
&= \omega((R(x), -R^*(a^*)), (y, b^*)) + \omega((x, a*), (R(y), -R^*(b^*))) \\
&=< -R^*(a^*), y > - < R(x), b^* > + < a^*, R(y) > + < x, R^*(b^*) >= 0.
\end{aligned}$$

■

Remark 4.20 It is easy to see that in the above cases, $\mathcal{R}(x, y) = (R(x), 0)$ or $\mathcal{R}(x, a^*) = (R(x), 0)$ also defines a Rota-Baxter operator on the corresponding pre-Lie algebras respectively. However, these operators do not satisfy the additional conditions as given in the above proposition: in case (1), $J\mathcal{R} = \mathcal{R}J$; in case (2), $\omega(\mathcal{R}(u), v) + \omega(u, \mathcal{R}(v)) = 0$. In fact, these conditions are very interesting in complex and symplectic geometry [1, 45].

5 The Rota-Baxter operators on 2-dimensional complex pre-Lie algebras

Let $\{e_1, \cdots, e_n\}$ be a basis of a pre-Lie algebra $(A, \cdot)$ and $e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k$. Then any Rota-Baxter operator R on A can be presented by a matrix (r_{ij}), where $R(e_i) = \sum\limits_{j=1}^n r_{ij} e_j$. Moreover, r_{ij} satisfy the following (quadratic) equations:

$$\sum_{k,l,m} [c_{kl}^m r_{ik} r_{jl} - c_{kj}^l r_{ik} r_{lm} - c_{il}^k r_{jl} r_{km}] = 0,\ i, j = 1, \cdots, n. \tag{5.1}$$

It is known that there exist exactly two Lie algebras in dimension 2: Abelian Lie algebra $\mathcal{C}$ and non-abelian Lie algebra given by $\mathcal{N} =< e_1, e_2 | [e_1, e_2] = e_1 >$. It is easy to know (we also list the corresponding pre-Lie algebras given by (4.3); we use the notations as in the following Proposition 5.1)

$$\begin{aligned}
\mathrm{CYB}(\mathcal{C}) &= \{\begin{pmatrix} r_{11} & r_{22} \\ r_{21} & r_{22} \end{pmatrix} | r_{ij} \in \mathbf{C}\} \Longrightarrow \text{(C4)} \\
\mathrm{CYB}(\mathcal{N}) &= \{\begin{pmatrix} r_{11} & r_{12} \\ -\frac{r_{12}^2}{r_{11}} & -r_{11} \end{pmatrix} | r_{11} \neq 0, r_{12} \neq 0\} \\
&\Longrightarrow \begin{cases} e_1 * e_1 = -r_{12} e_1 \\ e_2 * e_2 = -\frac{r_{11}^2}{r_{12}} e_1 \\ e_1 * e_2 = e_2 * e_1 = r_{11} e_1 \end{cases} \cong \text{(C3)} \\
&\bigcup \{\begin{pmatrix} 0 & 0 \\ r_{21} & r_{22} \end{pmatrix} | r_{22} \neq 0\} \Longrightarrow \begin{cases} e_2 * e_1 = -r_{22} e_1 \\ e_2 * e_2 = r_{21} e_1 \end{cases} \cong (\mathrm{NII})_0
\end{aligned}$$

$$\bigcup \quad \{\begin{pmatrix} 0 & r_{12} \\ 0 & 0 \end{pmatrix} | r_{12} \neq 0\} \Longrightarrow e_1 * e_1 = -r_{12}e_1 \cong \text{(C3)}$$

$$\bigcup \quad \{\begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix} |\} \Longrightarrow e_2 * e_2 = r_{21}e_1 \cong \text{(C4)}, \text{(C5)}$$

By a direct computation, we have

Proposition 5.1 The Rota-Baxter operators on 2-dimensional pre-Lie algebras are given in the following table (without special saying, any parameter belongs to the complex field **C**).

Pre-Lie algebra A	Rota-Baxter operators RB(A)
(C1) $e_1e_1 = e_1, e_2e_2 = e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
(C2) $e_2e_2 = e_2, e_1e_2 = e_2e_1 = e_1$	$\begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix}$
(C3) $e_1e_1 = e_1$	$\begin{pmatrix} 0 & r_{12} \\ 0 & r_{22} \end{pmatrix}$
(C4) $e_ie_j = 0$	$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$
(C5) $e_1e_1 = e_2$	$\begin{pmatrix} 0 & r_{12} \\ 0 & r_{22} \end{pmatrix} \bigcup \begin{pmatrix} 2r_{22} & r_{12} \\ 0 & r_{22} \end{pmatrix} (r_{22} \neq 0)$
(NI)$e_2e_1 = -e_1, e_2e_2 = -e_2$	$\{R \mid R^2 = 0\} = \left\{ \begin{array}{l} \begin{pmatrix} r_{11} & r_{12} \\ -\frac{r_{11}^2}{r_{12}} & -r_{11} \end{pmatrix} (r_{11} \neq 0, r_{12} \neq 0) \\ \bigcup \begin{pmatrix} 0 & r_{12} \\ 0 & 0 \end{pmatrix} (r_{12} \neq 0) \bigcup \begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix} \end{array} \right\}$
(NII)$_0$ $e_2e_1 = -e_1$	$\begin{pmatrix} 0 & 0 \\ 0 & r_{22} \end{pmatrix} (r_{22} \neq 0) \bigcup \begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix}$
(NII)$_{-1}$ $e_2e_1 = -e_1, e_2e_2 = e_1 - e_2$	$\begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix}$
(NII)$_k$ $e_2e_1 = -e_1, e_2e_2 = ke_2$ $(k \neq 0, -1)$	$\begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix}$
(NIII) $e_1e_2 = e_1, e_2e_2 = e_2$	$\{R \mid R^2 = 0\} = \left\{ \begin{array}{l} \begin{pmatrix} r_{11} & r_{12} \\ -\frac{r_{11}^2}{r_{12}} & -r_{11} \end{pmatrix} (r_{11} \neq 0, r_{12} \neq 0) \\ \bigcup \begin{pmatrix} 0 & r_{12} \\ 0 & 0 \end{pmatrix} (r_{12} \neq 0) \bigcup \begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix} \end{array} \right\}$
(NIV)$_k$ $e_1e_2 = ke_1, e_2e_1 = (k-1)e_2$ $e_2e_2 = e_1 + ke_2$	$\begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix}$
(NV) $e_1e_1 = e_2, e_2e_1 = -e_1, e_2e_2 = -2e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Corollary 5.2 Besides the type (NII)$_0$, any 2-dimensional pre-Lie algebra's 2nd double (associated to any Rota-Baxter operator) is trivial.

Proof According to the above discussion, besides the type (NII)$_0$, any 2-dimensional pre-Lie algebra's double is commutative. Hence its 2-nd double is trivial. ■

Example 5.3 $R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a Rota-Baxter operator on $(\text{NII})_0 =< e_1, e_2 | e_2e_1 = -e_1 >$. Then for any $i \geq 1$, the ith double of $(\text{NII})_0$ associated to R is isomorphic to $(\text{NII})_0$ (they have the same structural constants). ■

There are exactly two (complex) semisimple pre-Lie algebras in dimension 2 [16]: (C1) and (NV), and (NV) is simple.

Corollary 5.4 There does not exist a nonzero Rota-Baxter operator on any 2-dimensional complex semisimple pre-Lie algebra.

6 The Rota-Baxter operators on some pre-Lie algebras in higher dimensions

The classification of 3-dimensional complex pre-Lie algebras is very complicated. In this section, we mainly give the Rota-Baxter operators on the following three kinds of 3-dimensional pre-Lie algebras (some of them are overlapped) which are important and interesting:

(1) 3-dimensional associative algebras;

(2) Pre-Lie algebras on 3-dimensional Heisenberg Lie algebra $\mathcal{H} =< e_1, e_2, e_3 | [e_1, e_2] = e_3 >$;

(3) 3-dimensional semisimple pre-Lie algebras.

By a direct computation, we have

Proposition 6.1 The Rota-Baxter operators on the above three kinds of pre-Lie algebras (with their complete classification) are given in the following table (without special saying, any parameter belongs to the complex field $\mathbf{C}$).

Pre-Lie algebra A	Type	Rota-Baxter operators RB(A)
(C1) $e_ie_j = 0$	Commutative Associative	$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$
(C2) $e_3e_3 = e_1$	Commutative Associative	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 2r_{11} \end{pmatrix} (r_{11} \neq 0)$
(C3) $\begin{cases} e_2e_2 = e_1 \\ e_3e_3 = e_1 \end{cases}$	Commutative Associative	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & \pm\sqrt{-1}r_{22} \\ r_{31} & \mp\sqrt{-1}r_{33} & r_{33} \end{pmatrix}$ $\bigcup \begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22}^{\pm} & r_{23} \\ r_{31} & r_{23} & r_{33}^{\pm} \text{ (or } r_{33}^{\mp}) \end{pmatrix} (r_{11} \neq 0)$ $(r_{22}^{\pm} = r_{11} \pm (r_{11}^2 - r_{23}^2)^{\frac{1}{2}}, r_{33}^{\pm} = r_{11} \pm (r_{11}^2 - r_{23}^2)^{\frac{1}{2}})$
(C4) $\begin{cases} e_2e_3 = e_3e_2 = e_1 \\ e_3e_3 = e_2 \end{cases}$	Commutative Associative	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix} (r_{11} \neq 0)$ $\bigcup \begin{pmatrix} \frac{2}{3}r_{22} & 0 & 0 \\ \frac{2}{3}r_{32} & r_{22} & 0 \\ r_{31} & r_{32} & 2r_{22} \end{pmatrix} (r_{22} \neq 0)$
(C5)-(S1) $\begin{cases} e_1e_1 = e_1 \\ e_2e_2 = e_2 \\ e_3e_3 = e_3 \end{cases}$	Commutative Associative Semisimple	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
(C6) $\begin{cases} e_2e_2 = e_2 \\ e_3e_3 = e_3 \end{cases}$	Commutative Associative	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix}$
(C7) $\begin{cases} e_1e_3 = e_3e_1 = e_1 \\ e_2e_2 = e_2 \\ e_3e_3 = e_3 \end{cases}$	Commutative Associative	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix}$
(C8) $e_3e_3 = e_3$	Commutative Associative	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$
(C9) $\begin{cases} e_1e_3 = e_3e_1 = e_1 \\ e_3e_3 = e_3 \end{cases}$	Commutative Associative	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & 0 \end{pmatrix} (r_{12} \neq 0)$

Pre-Lie algebra A	Type	Rota-Baxter operators RB(A)
(C10) $\begin{cases} e_1e_3 = e_3e_1 = e_1 \\ e_2e_3 = e_3e_2 = e_2 \\ e_3e_3 = e_3 \end{cases}$	Commutative Associative	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} (r_{32} \neq 0)$ $\bigcup \begin{pmatrix} r_{11} & r_{12} & 0 \\ \frac{-r_{11}^2}{r_{12}} & -r_{11} & 0 \\ \frac{r_{32}r_{11}}{r_{12}} & r_{32} & 0 \end{pmatrix} (r_{12} \neq 0)$
(C11) $\begin{cases} e_1e_1 = e_2 \\ e_3e_3 = e_3 \end{cases}$	Commutative Associative	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 2r_{11} & 0 \\ 0 & r_{32} & 0 \end{pmatrix} (r_{11} \neq 0)$
(C12) $\begin{cases} e_1e_1 = e_2 \\ e_1e_3 = e_3e_1 = e_1 \\ e_2e_3 = e_3e_2 = e_2 \\ e_3e_3 = e_3 \end{cases}$	Commutative Associative	$\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 2r_{12} & r_{32} & 0 \end{pmatrix} (r_{12} \neq 0)$
(A1-H1) $\begin{cases} e_1e_2 = \frac{1}{2}e_3 \\ e_2e_1 = -\frac{1}{2}e_3 \end{cases}$	Associative on $\mathcal{H}$	$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ 0 & 0 & \frac{r_{11}r_{22}-r_{12}r_{21}}{r_{11}+r_{22}} \end{pmatrix} (r_{11}+r_{22} \neq 0)$ $\bigcup \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ -\frac{r_{11}^2}{r_{12}} & -r_{11} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} (r_{12} \neq 0)$ $\bigcup \begin{pmatrix} 0 & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$
(A2-H2) $e_2e_1 = -e_3$	Associative on $\mathcal{H}$	$\begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 0 \end{pmatrix} (r_{12} \neq 0)$ $\bigcup \begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix} (r_{21} \neq 0) \bigcup \begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$ $\bigcup \begin{pmatrix} r_{11} & 0 & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & \frac{r_{11}r_{22}}{r_{11}+r_{22}} \end{pmatrix} (r_{11}+r_{22} \neq 0)$
(A3-H3)$_\lambda$ $\begin{cases} e_1e_1 = e_3 \\ e_1e_2 = e_3 \\ e_2e_2 = \lambda e_3 \end{cases}$ $\lambda \neq 0$	Associative on $\mathcal{H}$	$\begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} (r_{33} \neq 0)$ $\bigcup \begin{pmatrix} 2r_{33} & 0 & r_{13} \\ 0 & 2r_{33} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} (r_{33} \neq 0)$ $\bigcup \begin{pmatrix} r_{11}^{\pm} & r_{12} & r_{13} \\ r_{21} & r_{22}^{\mp} & r_{23} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_{11}^{\pm} = \frac{-r_{12}\pm(1-4\lambda)^{\frac{1}{2}}r_{12}}{2} \\ r_{22}^{\pm} = \frac{-r_{21}\pm(1-4\lambda)^{\frac{1}{2}}r_{21}}{2\lambda} \end{pmatrix}$ $\bigcup \begin{pmatrix} r_{11}^{\pm} & r_{12} & r_{13} \\ -\lambda r_{12} & r_{22}^{\pm} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$ $\begin{pmatrix} r_{12} \neq 0, r_{33} \neq 0 \\ r_{11}^{\pm} = \frac{(-r_{12}+2r_{33})\pm(r_{12}^2-4\lambda r_{12}^2+4r_{33}^2)^{\frac{1}{2}}}{2} \\ r_{22}^{\pm} = \frac{(r_{12}+2r_{33})\pm(r_{12}^2-4\lambda r_{12}^2+4r_{33}^2)^{\frac{1}{2}}}{2} \end{pmatrix}$
(A4) $\begin{cases} e_3e_2 = e_2 \\ e_3e_3 = e_3 \end{cases}$	Associative	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix} (r_{21} \neq 0)$ $\bigcup \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & -\frac{r_{22}^2}{r_{23}} & -r_{22} \end{pmatrix} (r_{23} \neq 0)$

Pre-Lie algebra A	Type	Rota-Baxter operators RB(A)
(A5) $\begin{cases} e_2e_3=-e_2 \\ e_3e_3=-e_3 \end{cases}$	Associative	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix} (r_{21}\neq 0)$ $\bigcup \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & -\frac{r_{22}^2}{r_{23}} & -r_{22} \end{pmatrix} (r_{23}\neq 0)$
(A6) $\begin{cases} e_1e_1=e_1 \\ e_3e_2=e_2 \\ e_3e_3=e_3 \end{cases}$	Associative	$\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & 0 & 0 \\ r_{23} & r_{22} & r_{23} \\ -r_{22} & -\frac{r_{22}^2}{r_{23}} & -r_{22} \end{pmatrix} (r_{23}\neq 0)$ $\bigcup \begin{pmatrix} 0 & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & -\frac{r_{22}^2}{r_{23}} & -r_{22} \end{pmatrix}$
(A7) $\begin{cases} e_1e_1=e_1 \\ e_2e_3=-e_2 \\ e_3e_3=-e_3 \end{cases}$	Associative	$\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & 0 & 0 \\ -r_{23} & r_{22} & r_{23} \\ r_{22} & -\frac{r_{22}^2}{r_{23}} & -r_{22} \end{pmatrix} (r_{23}\neq 0)$ $\bigcup \begin{pmatrix} 0 & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & -\frac{r_{22}^2}{r_{23}} & -r_{22} \end{pmatrix} (r_{23}\neq 0)$
(A8) $\begin{cases} e_1e_3=e_1 \\ e_3e_1=e_1 \\ e_3e_2=e_2 \\ e_3e_3=e_3 \end{cases}$	Associative	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & 0 & 0 \\ 0 & r_{22} & r_{23} \\ r_{31} & -\frac{r_{22}^2}{r_{23}} & -r_{22} \end{pmatrix} (r_{23}\neq 0)$ $\bigcup \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix} (r_{21}\neq 0) \bigcup \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix} (r_{12}\neq 0)$
(A9) $\begin{cases} e_1e_1=e_1 \\ e_1e_2=e_2 \\ e_1e_3=e_3 \\ e_2e_1=e_2 \\ e_3e_1=e_3 \\ e_3e_2=e_2 \\ e_3e_3=e_3 \end{cases}$	Associative	$\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & 0 & 0 \\ -r_{23} & r_{22} & r_{23} \\ r_{22} & -\frac{r_{22}^2}{r_{23}} & -r_{22} \end{pmatrix} (r_{23}\neq 0)$ $\bigcup \begin{pmatrix} 0 & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & -\frac{r_{22}^2}{r_{23}} & -r_{22} \end{pmatrix} (r_{23}\neq 0)$
(A10) $\begin{cases} e_3e_1=e_1 \\ e_3e_2=e_2 \\ e_3e_3=e_3 \end{cases}$	Associative	$\{R \mid R^2=0\}$
(A11) $\begin{cases} e_1e_3=-e_1 \\ e_2e_3=-e_2 \\ e_3e_3=-e_3 \end{cases}$	Associative	$\{R \mid R^2=0\}$
(A12) $\begin{cases} e_2e_3=e_2 \\ e_3e_1=e_1 \\ e_3e_3=e_3 \end{cases}$	Associative	$\bigcup \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix} (r_{21}\neq 0)$ $\bigcup \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix} (r_{12}\neq 0)$ $\bigcup \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & r_{23} \\ -\frac{r_{21}r_{22}}{r_{23}} & -\frac{r_{22}^2}{r_{23}} & -r_{22} \end{pmatrix} (r_{23}\neq 0)$ $\bigcup \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & 0 & 0 \\ -\frac{r_{11}^2}{r_{13}} & -\frac{r_{11}r_{12}}{r_{13}} & -r_{11} \end{pmatrix} (r_{13}\neq 0)$

Pre-Lie algebra A	Type	Rota-Baxter operators RB(A)
(S2) $\begin{cases} e_1e_2 = e_2e_1 = e_3 \\ e_3e_1 = e_1 \\ e_3e_2 = \lambda e_2 \\ e_3e_3 = (\lambda+1)e_3 \end{cases}$ $\lambda \neq 0, \lvert\lambda\rvert \leq 1$	Simple	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (r_{12} \neq 0);$ when $\lambda = 1$ $\bigcup \begin{pmatrix} r_{11} & -\frac{r_{13}^2}{2r_{11}} & r_{13} \\ \frac{2r_{11}^3}{r_{13}^2} & -r_{11} & \frac{2r_{11}^2}{r_{13}} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_{11} \neq 0, \\ r_{13} \neq 0 \end{pmatrix};$ when $\lambda = -1$ $\bigcup \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r_{33} \end{pmatrix} (r_{33} \neq 0)$
(S3) $\begin{cases} e_1e_2 = e_3 \\ e_2e_1 = e_3 \\ e_2e_2 = e_1 \\ e_3e_1 = e_1 \\ e_3e_2 = \frac{1}{2}e_2 \\ e_3e_3 = \frac{3}{2}e_3 \end{cases}$	Simple	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
(S4) $\begin{cases} e_1e_1 = e_1 \\ e_2e_2 = e_3 \\ e_3e_2 = e_2 \\ e_3e_3 = 2e_3 \end{cases}$	Semisimple	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
(H4) $\begin{cases} e_1e_1 = e_1 \\ e_1e_2 = e_2 + e_3 \\ e_2e_1 = e_2 \\ e_3e_1 = e_1e_3 = e_3 \end{cases}$	on $\mathcal{H}$	$\begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (r_{12} \neq 0)$
(H5) $\begin{cases} e_1e_1 = e_1 \\ e_1e_2 = e_2 + e_3 \\ e_2e_1 = e_2 \\ e_2e_2 = e_3 \\ e_3e_1 = e_1e_3 = e_3 \end{cases}$	on $\mathcal{H}$	$\begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & 2r_{23} & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix} (r_{23} \neq 0)$
(H6) $\begin{cases} e_1e_1 = e_1 \\ e_1e_2 = e_3 \\ e_2e_2 = e_3 \end{cases}$	on $\mathcal{H}$	$\begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$
(H7) $\begin{cases} e_1e_1 = e_1 \\ e_1e_2 = e_3 \end{cases}$	on $\mathcal{H}$	$\begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} (r_{33} \neq 0)$
(H8) $\begin{cases} e_2e_1 = -e_3 \\ e_2e_2 = e_1 \end{cases}$	on $\mathcal{H}$	$\begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} (r_{33} \neq 0)$ $\bigcup \begin{pmatrix} r_{11} & 0 & -\frac{1}{3}r_{21} \\ r_{21} & 2r_{11} & r_{23} \\ 0 & 0 & \frac{2}{3}r_{11} \end{pmatrix} (r_{11} \neq 0)$
(H9) $\begin{cases} e_1e_2 = e_3 \\ e_2e_2 = e_1 \end{cases}$	on $\mathcal{H}$	$\begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} (r_{33} \neq 0)$ $\bigcup \begin{pmatrix} r_{11} & 0 & \frac{1}{3}r_{21} \\ r_{21} & 2r_{11} & r_{23} \\ 0 & 0 & \frac{2}{3}r_{11} \end{pmatrix} (r_{11} \neq 0)$
(H10)$_\lambda$ $\begin{cases} e_1e_2 = \frac{\lambda}{\lambda-1}e_3 \\ e_2e_1 = \frac{1}{\lambda-1}e_3 \\ e_2e_2 = \lambda e_1 \end{cases}$ $\lambda \neq 0, 1$	on $\mathcal{H}$	$\begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ 0 & 0 & 0 \end{pmatrix} \bigcup \begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} (r_{33} \neq 0)$ $\bigcup \begin{pmatrix} r_{11} & 0 & \frac{\lambda+1}{3\lambda(\lambda-1)}r_{21} \\ r_{21} & 2r_{11} & r_{23} \\ 0 & 0 & \frac{2}{3}r_{11} \end{pmatrix} (r_{11} \neq 0)$

Remark 6.2 Among the above results, the conclusion on the invertible Rota-Baxter operators is consistent with Example 3.10. In fact, we have the following (notations) correspondence:

(T1) $\leftrightarrow$ (C1); (T2) $\leftrightarrow$ (C2); (T3) $\leftrightarrow$ (C3); (T4) $\leftrightarrow$ (C4); (T5) $\leftrightarrow$ (A1-H1);

(T8) $\leftrightarrow$ (H8); $(\text{T6})_l \leftrightarrow$ (A2-H2) and $(\text{A3-H3})_\lambda$ $(\lambda \neq 0)$;

$(\text{T7})_l$ $(l \neq 1) \leftrightarrow$ (H9) and $(\text{H10})_\lambda$ $(\lambda \neq 0, 1)$. ■

With the notations as above, we have the following conclusions by a direct computation.

Corollary 6.3 Let A be one of the following pre-Lie algebras: (C1)-(C12), (H6), (S4). Then its 1st double (associated to any Rota-Baxter operator R) is trivial (that is, the Rota-Baxter operator maps A into the center of $\mathcal{G}(A)$).

Corollary 6.4 Let A be one of the following pre-Lie algebras: (A6), (A7), (A8), (A9), (A12), (S2), (S3), (H4), (H5). Then its 2nd double (associated to any Rota-Baxter operator R) is trivial.

Corollary 6.5 Let A be one of the following pre-Lie algebras: (A10), (A11). Then its 3rd double (associated to any Rota-Baxter operator R) is trivial.

Example 6.6 Let A be one of the following pre-Lie algebras with a fixed Rota-Baxter operator R. Then for any $i \geq 1$, the ith double of A associated to the corresponding R is isomorphic to A (even they have the same structural constants).

(A1-H1): $R = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$; (A2-H2): $R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$;

$(\text{A3-H3})_\lambda : R = \begin{pmatrix} 1 & -1 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$.

■

Example 6.7 Let A be one of the following pre-Lie algebras with a fixed Rota-Baxter operator R. Then for any $i \geq 1$, the ith double of A associated to the corresponding R is isomorphic to (A1-H1).

(A4) or (A5): $R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; (H7): $R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$;

(H8) or (H9) or $(\text{H10})_\lambda$ $(\lambda \neq 0, 1)$: $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. ■

At the end of this section, we construct some Rota-Baxter operators on a class of simple pre-Lie algebras given as follows [16]:

$$I_n = < e_1, \cdots, e_n | e_n e_n = 2e_n, e_n e_j = e_j, e_j e_j = e_n, j \leq n-1 > . \tag{6.1}$$

Proposition 6.8 When $n \geq 3$, there exists a non-zero Rota-Baxter operator R on I_n defined by

$$R(e_1) = e_n + \frac{1}{\sqrt{2-n}} \sum_{i=2}^{n-1} e_i, \;\; R(e_i) = 0, \; 2 \leq i \leq n. \tag{6.2}$$

Proof It is easy to prove that $R(e_1)R(e_1) = R(R(e_1)e_1 + e_1 R(e_1)) = R(e_1)$. The other equations hold naturally. So R is a Rota-Baxter operator on I_n. ■

Notice that the simple pre-Lie algebra (S2) with $\lambda = 1$ in Proposition 6.1 is isomorphic to I_3 by a linear transformation $e_1 \to \frac{\sqrt{2}}{2}(e_1+e_2)$, $e_2 \to \frac{\sqrt{-2}}{2}(e_1-e_2)$, $e_3 \to e_3$ and $R = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 \end{pmatrix}$ corresponding to the Rota-Baxter operator given by (6.2). Moreover, the 1st double of I_n associated to R is given by

$$(I_n, *_1) =< e_1, \cdots, e_n | e_1 * e_n = -\frac{1}{\sqrt{2-n}} \sum_{i=2}^{n-1} e_i, e_1 * e_j = e_j, 1 \le j \le n-1 >, \tag{6.3}$$

which includes an ideal (whose all products are zero) spanned by $e_2, \cdots, e_n$. The (1st) double of $(I_n, *_1)$ which is the second double of I_n associated to R is trivial. Furthermore, similarly, for any $2 \le k \le n-1$, R_k given by

$$R_k(e_k) = e_n + \frac{1}{\sqrt{2-n}} \sum_{j \ne k}^{n-1} e_j, \quad R_k(e_i) = 0, \ i \ne k, \tag{6.4}$$

is also a Rota-Baxter operator on I_n.

7 Further discussion and some questions

From the discussion in the previous sections, we have obtained some properties of Rota-Baxter operators on pre-Lie algebras. We would like to give the following discussion.

(1) As we did in section 5 and 6, we can find all Rota-Baxter operators on a pre-Lie algebra A through solving the quadratic equations (5.1) on r_{ij}. However, RB(A) will relay on the choice of a basis of A and its corresponding structural constants. So there are two natural questions arising here:

(a) Whether we can give a meaningful "classification rules" so that the classification of Rota-Baxter operators can be more "interesting"?

(b) It is hard and less practicable to extend what we have done in section 5 and 6 to other cases since the Rota-Baxter relation involves the nonlinear quadratic equations. Whether we can find a "linearization" process like we did for the classification of pre-Lie algebras?

(2) We have obtained some examples (Example 5.2, 6.6 and 6.7) that for a pre-Lie algebra A and its Rota-Baxter operator R, the double series $(A, *_i)$ will be stable, that is, there exists a N such that $(A, *_n)$ is isomorphic to $(A, *_N)$ which is not trivial for $n > N$. Obviously, R cannot be nilpotent due to Corollary 4.10. However, even R is semisimple, the double $(A, *_i)$ associated to R may still be trivial. Therefore, it seems interesting to consider whether and when the double series $(A, *_i)$ will be stable for a general pre-Lie algebra.

(3) We have known that the CYBE on a Lie algebra can be re-written (although the equivalence need some additional conditions) as an algebraic equation in the tensor form as (2.7). For an associative algebra A, there is another approach for the tensor form of Rota-Baxter relation [2] as follows. Let $r = \sum_i a_i \otimes b_i \in A \otimes A$ and we use the same

notations as in (2.8), where $U(\mathcal{G})$ is replaced by $A \oplus \mathbf{F}$. Then if r satisfies the associative analog of the classical Yang-Baxter equation

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0, \tag{7.1}$$

then $R : A \to A$ given by

$$R(x) = \sum_i a_i x b_i, \quad \forall x \in A, \tag{7.2}$$

defines a Rota-Baxter operator on A. However, in the case of Lie algebras, it depends on the skew-symmetry of Lie brackets, and in the case of associative algebras, it depends on the associativity. So it seems that both of the above approaches cannot be extended to pre-Lie algebras. We would like to ask whether there is a tensor form of Rota-Baxter relation on pre-Lie algebras.

(4) The counterexample in Example 3.11 is not graded by a non-zero set and its kernel ideal $N(A) = \{x | L_x = 0\} = \{0\}$. The kernel ideal of a pre-Lie algebra corresponds to a one-parameter subgroup of translation in affine geometry [38]. Whether the existence of invertible Rota-Baxter operators is related to these two conditions?

(5) There are close relations between Rota-Baxter operators and certain geometric structures related to pre-Lie algebras, like complex and parakähler structures in section 4. It is quite interesting to give a further study and consider the possible application in both geometry and physics.

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