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Construction of q-discrete two-dimensional Toda lattice equation with self-consistent sources

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Abstract

The q-discrete two-dimensional Toda lattice equation with self-consistent sources is presented through the source generalization procedure. In addition, the Gramm-type determinant solutions of the system are obtained. Besides, a bilinear Bäcklund transformation (BT) for the system is given.

1 Introduction

Soliton equations with self-consistent sources (SESCSs) have important physical applications, e.g., the areas of hydrodynamics, plasma physics, solid-state physics, among others [13]-[17]. There are several ways to study SECSs, such as the inverse scattering method, the Daboux transformation, and the bilinear method (see Refs [18]-[22]). Recently, a valid algebraic method called 'source generalization procedure' [7] has been proposed to construct and solve SESCSs, starting from Hirota's bilinear form of the original equations without sources. We describe the process of the source generalization procedure as follows:

1. to express N-soliton solutions of a soliton equation without sources in the form of determinant or pfaffian with some arbitrary constants, say $c_{i,j}$.

2. to introduce the corresponding determinant or pfaffian using arbitrary functions of an independent variable, e.g. $c_{i,j}(t)$.

3. to seek coupled bilinear equations whose solutions are the above generalized determinants or pfaffians. The coupled bilinear equations that we derive are the equations with self-consistent sources.

The procedure has been successfully applied to the KP equation, the 2D Toda lattice equation, the BKP-type equation, the discrete KP equation and so on[7, 23, 24]. We will take the KP equation as an example to explain the process of source generalization procedure in the next section. Meanwhile, the so-called q-discrete integrable systems have attracted some attention (see [8]–[2]) recently. In Ref. [8], the authors proposed the q-discrete version of the two-dimensional Toda lattice equation and gave its determinantal solution. The q-discrete two-dimensional Toda lattice equation is given by

$$\begin{aligned}&\delta_{q_1^\alpha,x}\delta_{q_2^\beta,y}\tau_n(x,y)\cdot\tau_n(x,y)-\delta_{q_1^\alpha,x}\tau_n(x,y)\cdot\delta_{q_2^\beta,y}\tau_n(x,y)\\&\qquad=\tau_{n+1}(x,q_2^\beta y)\tau_{n-1}(q_1^\alpha x,y)-\tau_n(q_1^\alpha x,q_2^\beta y)\tau_n(x,y),\end{aligned}\tag{1.1}$$

where the q-difference operators $\delta_{q_1^\alpha,x}$ and $\delta_{q_2^\beta,y}$ are defined by

$$\delta_{q_1^\alpha,x}F(x)=\frac{F(x)-F(q_1^\alpha x)}{(1-q_1)x},\quad \delta_{q_2^\beta,y}F(x)=\frac{F(y)-F(q_2^\beta y)}{(1-q_2)y}.$$

Then equation (1.1) can be written in the equivalent form

$$\begin{aligned}&(1+a(x)b(y))\tau_n(q_1^\alpha x,q_2^\beta y)\tau_n(x,y)-\tau_n(q_1^\alpha x,y)\tau_n(x,q_2^\beta y)\\&\qquad=a(x)b(y)\tau_{n+1}(x,q_2^\beta y)\tau_{n-1}(q_1^\alpha x,y),\end{aligned}\tag{1.2}$$

where $a(x)=(1-q_1)x$ and $b(y)=(1-q_2)y$. Equation (1.2) is in quadratic form which is similar to the discrete KP equation (or Hirota-Miwa equation)[20, 3]. However, we can find an obvious distinction between the equation (1.2) and the discrete KP equation. That is, the latter can be expressed as the following form using Hirota's bilinear difference operators,

$$[z_1e^{\frac{1}{2}(-D_{k_1}+D_{k_2}+D_{k_3})}+z_2e^{\frac{1}{2}(D_{k_1}-D_{k_2}+D_{k_3})}+z_3e^{\frac{1}{2}(-D_{k_1}-D_{k_2}+D_{k_3})}]f\cdot f=0,\tag{1.3}$$

where k_1, k_2, k_3 are discrete variables, and z_1, z_2, z_3 are constants satisfying $z_1+z_2+z_3=0$. Here Hirota's bilinear difference operator is defined by

$$\exp(\delta D_z)f_z\cdot g_z=f_{z+\delta}g_{z-\delta}.$$

but as for the equation (1.2), we can not express it as the form

$$F(D_x,D_y,D_n)\tau_n\cdot\tau_n=0,$$

where F denotes the combinations of Hirota's difference operators D_x, D_y and D_n. Therefore it would be interesting to study the q-discrete two-dimensional Toda lattice equation with self-consistent sources. The purpose of this paper is to construct and solve the q-discrete two-dimensional Toda lattice equation with self-consistent sources utilizing the source generalization procedure.

2 Source generalization procedure to the KP equation

In this section, we will take the KP equation as an example to briefly review the process of source generalization procedure. The KP equation is a $2+1$-dimensional nonlinear partial differential equation which can be transformed into the bilinear equation [5]

$$(D_x^4-4D_xD_t+3D_y^2)\tau\cdot\tau=0,\tag{2.1}$$

where D is the Hirota's differential operator

$$D_x^l D_t^n f \cdot g = (\partial_x - \partial_{x'})^l (\partial_t - \partial_{t'})^n f(x,t) g(x',t')|_{x=x',t=t'}.$$

Firstly, we give the Grammian determinant solution the equation (2.1):

$$\tau = \det(\beta_{ij} + \int^x f_i \tilde{f}_j \mathrm{d}x)_{1\leqslant i,j\leqslant N}, \quad \beta_{ij} = \text{constant},$$

with each functions f_i and $\tilde{f}_j$ satisfying

$$\frac{\partial f_i}{\partial x_n} = \frac{\partial^n f_i}{\partial x^n}, \quad \frac{\partial \tilde{f}_i}{\partial x_n} = (-1)^{n-1} \frac{\partial^n \tilde{f}_i}{\partial x^n}, \quad (x_1 = x, \quad x_2 = y, \quad x_3 = t). \tag{2.2}$$

Secondly we generalize τ to the form

$$f = \det(a_{ij})_{1\leqslant i,j\leqslant N} = \mathrm{pf}(1, 2, \cdots, N, N^*, \cdots, 1^*), \tag{2.3}$$

where pfaffian elements are defined by

$$a_{ij} = \mathrm{pf}(i, j^*) = \beta_{ij}(t) + \int^x f_i \tilde{f}_j \mathrm{d}x, \quad \mathrm{pf}(i,j) = \mathrm{pf}(i^*, j^*) = 0, \quad i, j = 1, 2, \cdots, N,$$

with $\beta_{ij}(t)$ satisfying

$$\beta_{ij}(t) = \begin{cases} \beta_i(t), & i = j \text{ and } 1 \leqslant i \leqslant K \leqslant N, \quad K, N \in Z^+, \\ \beta_{ij}, & \text{otherwise.} \end{cases}$$

Following the source generalization procedure, we introduce other new functions g_j and h_j expressed by

$$g_j = 2\sqrt{2\dot{\beta}_j(t)}\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*), \quad j = 1, 2, \cdots, K, \tag{2.4}$$

$$h_j = 2\sqrt{2\dot{\beta}_j(t)}\mathrm{pf}(d_0, 1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, 1^*), \quad j = 1, 2, \cdots, K \tag{2.5}$$

where the dot denotes the derivative of the function $\beta_j(t)$ with respect to t, and new pfaffian entries are defined as

$$\mathrm{pf}(d_m^*, i) = \frac{\partial^m}{\partial x^m} f_i, \quad \mathrm{pf}(d_n, j^*) = \frac{\partial^n}{\partial x^n} \tilde{f}_j,$$

$$\mathrm{pf}(d_m^*, d_n) = \mathrm{pf}(d_m^*, d_n^*) = \mathrm{pf}(d_m, d_n) = \mathrm{pf}(d_m^*, j^*) = \mathrm{pf}(d_m, i) = 0.$$

Then we can show that the f, g_j and h_j so defined satisfy the bilinear equations:

$$(D_x^4 - 4D_x D_t + 3D_y^2) f \cdot f = \sum_{j=1}^{K} g_j h_j, \tag{2.6}$$

$$(D_y + D_x^2) f \cdot g_j = 0, \quad j = 1, 2, \cdots, K, \tag{2.7}$$

$$(D_y + D_x^2) h_j \cdot f = 0, \quad j = 1, 2, \cdots, K. \tag{2.8}$$

Equations (2.6)-(2.8) are just the KP ESCS in the bilinear form. They can be proved through pfaffian identities, and here we omit the process of proof. In Ref. [24], we have also given the nonlinear forms of the bilinear system (2.6)-(2.8). These results are consistent with the results in Ref. [25].

3 Gramm-type determinant solution of equation (1.2)

In [8], the Casorati determinant solution of equation (1.2) has been given. Now we will give the Gramm-type determinant solution of this equation. The Gramm-type determinant solution can be expressed in the form

$$\begin{aligned}\tau_n(x,y) &= |M_0|_{N\times N} = |m_{ij}^{(n)}(x,y)| \\ &= |c_{ij} + \sum_{s=0}^{\infty} \varphi_i^{(s+n)}(x,y)\bar{\varphi}_j^{(-s-n-1)}(x,y)|_{1\leqslant i,j\leqslant N},\end{aligned} \tag{3.1}$$

where each c_{ij} is an arbitrary constant, and functions $\varphi_i^{(m)}(x,y)$ and $\bar{\varphi}_j^{(-m)}(x,y)$ satisfy the relations

$$\begin{aligned}&\delta_{q_1^\alpha,x}\varphi_i^{(m)}(x,y) = -\varphi_i^{(m+1)}(x,y), \quad \delta_{q_2^\beta,y}\varphi_i^{(m)}(x,y) = \varphi_i^{(m-1)}(x,y), \\ &\delta_{q_1^\alpha,x}\bar{\varphi}_j^{(-m)}(x,y) = \bar{\varphi}_j^{(-m+1)}(q_1^\alpha x,y), \quad \delta_{q_2^\beta,y}\bar{\varphi}_j^{(-m)}(x,y) = -\bar{\varphi}_j^{(-m-1)}(x,q_2^\beta y).\end{aligned} \tag{3.2}$$

In the following, we prove that $\tau_n(x,y)$ actually satisfy equation (1.2). In fact, we get the following relations from (3.1) and (3.2):

$$\delta_{q_1^\alpha,x}m_{ij}^{(n)}(x,y) = \varphi_i^{(n)}(x,y)\bar{\varphi}_j^{(-n)}(q_1^\alpha x,y), \tag{3.3a}$$

$$\delta_{q_2^\beta,y}m_{ij}^{(n)}(x,y) = \varphi_i^{(n-1)}(x,y)\bar{\varphi}_j^{(-n-1)}(x,q_2^\beta y), \tag{3.3b}$$

$$m_{ij}^{(n+1)}(x,y) - m_{ij}^{(n)}(x,y) = -\varphi_i^{(n)}(x,y)\bar{\varphi}_j^{(-n-1)}(x,y), \tag{3.3c}$$

$$m_{ij}^{(n-1)}(x,y) - m_{ij}^{(n)}(x,y) = \varphi_i^{(n-1)}(x,y)\bar{\varphi}_j^{(-n)}(x,y). \tag{3.3d}$$

Then we have the difference formulas

$$\tau_n(q_1^\alpha x,y) = a(x)\begin{vmatrix} M_0 & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & a(x)^{-1}\end{vmatrix},$$

$$\tau_{n-1}(q_1^\alpha x,y) = -\begin{vmatrix} M_0 & \phi^{(n-1)}(x,y)^T \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & -1\end{vmatrix},$$

$$\tau_n(x,q_2^\beta y) = b(y)\begin{vmatrix} M_0 & \phi^{(n-1)}(x,y)^T \\ \bar{\phi}^{(-n-1)}(x,q_2^\beta y) & b(y)^{-1}\end{vmatrix},$$

$$[1+\frac{1}{a(x)b(y)}]\tau_n(q_1^\alpha x,q_2^\beta y) = \begin{vmatrix} M_0 & \phi^{(n)}(x,y)^T & \phi^{(n-1)}(x,y)^T \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & a(x)^{-1} & -1 \\ \bar{\phi}^{(-n-1)}(x,q_2^\beta y) & 1 & b(y)^{-1}\end{vmatrix},$$

where $\phi^{(n)}(x,y)$ and $\bar{\phi}^{(-n)}(x,y)$ denote the following $1\times N$ matrices, respectively:

$$(\varphi_1^{(n)}(x,y),\varphi_2^{(n)}(x,y),\cdots,\varphi_N^{(n)}(x,y)), \quad (\bar{\varphi}_1^{(-n)}(x,y),\bar{\varphi}_2^{(-n)}(x,y),\cdots,\bar{\varphi}_N^{(-n)}(x,y)).$$

Substituting the above results into equation (1.2) yields the determinant identity

$$
0 = |M_0| \begin{vmatrix} M_0 & \phi^{(n)}(x,y)^T & \phi^{(n-1)}(x,y)^T \\ \bar{\phi}^{(-n)}(q_1^\alpha x, y) & a(x)^{-1} & -1 \\ \bar{\phi}^{(-n-1)}(x, q_2^\beta y) & 1 & b(y)^{-1} \end{vmatrix}
$$

$$
- \begin{vmatrix} M_0 & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n)}(q_1^\alpha x, y) & a(x)^{-1} \end{vmatrix} \begin{vmatrix} M_0 & \phi^{(n-1)}(x,y)^T \\ \bar{\phi}^{(-n-1)}(x, q_2^\beta y) & b(y)^{-1} \end{vmatrix}
$$

$$
+ \begin{vmatrix} M_0 & \phi^{(n-1)}(x,y)^T \\ \bar{\phi}^{(-n)}(q_1^\alpha x, y) & -1 \end{vmatrix} \begin{vmatrix} M_0 & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n-1)}(x, q_2^\beta y) & 1 \end{vmatrix},
$$

which is just a Jacobi identity of determinants [5]. This indicates that $\tau_n(x,y)$ in (3.1) is a Gramm-type determinant solution of equation (1.2).

4 Construction of q-discrete two-dimensional Toda equation with self-consistent sources

In this section, we will apply the source generalization procedure to the q-discrete two-dimensional Toda equation. We first change the function $\tau_n(x,y)$ in (3.1) to form

$$
\begin{aligned}
f_n(x,y) &= |M|_{N\times N} = |C_{ij} + \bar{m}_{ij}^{(n)}(x,y)| \\
&= |C_{ij} + \sum_{s=0}^{\infty} \varphi_i^{(s+n)}(x,y)\bar{\varphi}_j^{(-s-n-1)}(x,y)|_{1\leqslant i,j\leqslant N},
\end{aligned} \tag{4.1}
$$

where each parameter C_{ij} satisfy

$$
C_{ij} = \begin{cases} c_j(y), & i = 1,2,\cdots,N \text{ and } 1 \leqslant j \leqslant K \leqslant N,\ K \in Z^+, \\ c_{ij}, & \text{otherwise.} \end{cases}
$$

Here we have assumed that each function $c_j(y)$ satisfies $\triangle c_j(y) = \triangle c_j(q_2^\beta y)$, where $\triangle c_j(y) = c_j(q_2^\beta y) - c_j(y)$. According to the determinant identity in [5], $f_n(x,y)$ can also be expressed as

$$
f_n(x,y) = \begin{vmatrix} M_1 & -1 \\ C(y) & 1 \end{vmatrix},
$$

where the $N \times N$ matrix $M_1 = (\bar{m}_{ij}^{(n)}(x,y))_{1\leqslant i,j\leqslant N}$, and $C(y)$ is a $1 \times N$ matrix defined by

$$
(c_1(y), \cdots, c_K(y), 0, \cdots, 0).
$$

Then we have the formulas

$$f_n(q_1^\alpha x, y) = a(x)\begin{vmatrix} M_1 & -1 & \phi^{(n)}(x,y)^T \\ C(y) & 1 & 0 \\ \bar{\phi}^{(-n)}(q_1^\alpha x, y) & 0 & a(x)^{-1} \end{vmatrix}, \tag{4.2a}$$

$$f_n(x, q_2^\beta y) = b(y)\begin{vmatrix} M_1 & -1 & \phi^{(n-1)}(x,y)^T \\ C(q_2^\beta y) & 1 & 0 \\ \bar{\phi}^{(-n-1)}(x, q_2^\beta y) & 0 & b(y)^{-1} \end{vmatrix}, \tag{4.2b}$$

$$f_{n+1}(x, q_2^\beta y) = \begin{vmatrix} M_1 & -1 & \phi^{(n)}(x,y)^T \\ C(q_2^\beta y) & 1 & 0 \\ \bar{\phi}^{(-n-1)}(x, q_2^\beta y) & 0 & 1 \end{vmatrix}, \tag{4.2c}$$

$$[1+\frac{1}{a(x)b(y)}]f_n(q_1^\alpha x, q_2^\beta y) = \begin{vmatrix} M_1 & -1 & \phi^{(n)}(x,y)^T & \phi^{(n-1)}(x,y)^T \\ C(q_2^\beta y) & 1 & 0 & 0 \\ \bar{\phi}^{(-n)}(q_1^\alpha x, y) & 0 & a(x)^{-1} & -1 \\ \bar{\phi}^{(-n-1)}(x, q_2^\beta y) & 0 & 1 & b(y)^{-1} \end{vmatrix}. \tag{4.2d}$$

Following the source generalization procedure, we introduce new functions defined by

$$g_{j,n}(x,y) = a(x)\sqrt{\triangle c_j(y)}\begin{vmatrix} M & -1 \\ \bar{\phi}^{(-n)}(x,y) & 0 \end{vmatrix}, \quad j = 1, 2, \cdots, K, \tag{4.3a}$$

$$h_{j,n}(x,y) = (-1)^{j+N}\sqrt{\triangle c_j(y)}\begin{vmatrix} \hat{M}_j & \phi^{(n)}(x,y)^T \end{vmatrix}, \quad j = 1, 2, \cdots, K, \tag{4.3b}$$

where $\hat{M}_j$ denotes the $N \times (N-1)$ matrix eliminating the $j-$th column from the matrix M. We can show that functions in (4.1) and (4.3) satisfy the bilinear equations

$$\begin{aligned} &(1+a(x)b(y))f_n(q_1^\alpha x, q_2^\beta y)f_n(x,y) - f_n(q_1^\alpha x, y)f_n(x, q_2^\beta y) \\ &\qquad = a(x)b(y)f_{n+1}(x, q_2^\beta y)f_{n-1}(q_1^\alpha x, y) + \sum_{j=1}^{K} g_{j,n}(q_1^\alpha x, y)h_{j,n}(x, q_2^\beta y), \end{aligned} \tag{4.4}$$

$$f_n(q_1^\alpha x, y)g_{j,n+1}(x,y) - a(x)f_{n+1}(x,y)g_{j,n}(q_1^\alpha x, y) - f_n(x,y)g_{j,n+1}(q_1^\alpha x, y) = 0, \tag{4.5}$$

$$h_{j,n}(q_1^\alpha x, y)f_{n+1}(x,y) - a(x)h_{j,n+1}(x,y)f_n(q_1^\alpha x, y) - h_{j,n}(x,y)f_{n+1}(q_1^\alpha x, y) = 0. \tag{4.6}$$

Now we verify that the functions $f_n(x,y)$, $g_{j,n}(x,y)$, and $h_{j,n}(x,y)$ so defined are really solutions of equations (4.4)-(4.6). In fact, if we set

$$\bar{g}_{j,n}(x,y) = \begin{vmatrix} M & -1 \\ \bar{\phi}^{(-n)}(x,y) & 0 \end{vmatrix}, \quad \bar{h}_{j,n}(x,y) = \begin{vmatrix} \hat{M}_j & \phi^{(n)}(x,y)^T \end{vmatrix},$$

we have the following difference formulas:

$$f_{n+1}(x,y)=\begin{vmatrix} M & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n-1)}(x,y) & 1 \end{vmatrix}, \tag{4.7a}$$

$$f_n(q_1^\alpha x,y)=a(x)\begin{vmatrix} M & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & a(x)^{-1} \end{vmatrix}, \tag{4.7b}$$

$$f_{n+1}(q_1^\alpha x,y)=a^2(x)\begin{vmatrix} M & \phi^{(n+1)}(x,y)^T & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n-1)}(x,y) & 0 & 1 \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & -a(x)^{-2} & a(x)^{-1} \end{vmatrix}, \tag{4.7c}$$

$$\bar{g}_{j,n+1}(x,y)=\begin{vmatrix} M & -1 \\ \bar{\phi}^{(-n-1)}(x,y) & 0 \end{vmatrix},\quad \bar{g}_{j,n}(q_1^\alpha x,y)=\begin{vmatrix} M & -1 \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & 0 \end{vmatrix}, \tag{4.8a}$$

$$\bar{g}_{j,n+1}(q_1^\alpha x,y)=-a(x)\begin{vmatrix} M & -1 & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & 0 & a(x)^{-1} \\ \bar{\phi}^{(-n-1)}(x,y) & 0 & 1 \end{vmatrix}, \tag{4.8b}$$

$$\bar{h}_{j,n+1}(x,y)=\begin{vmatrix} \hat{M}_j & \phi^{(n+1)}(x,y)^T & \phi^{(n)}(x,y)^T \\ \hat{\bar{\phi}}_j^{(-n-1)}(x,y) & 0 & 1 \end{vmatrix}, \tag{4.9a}$$

$$\bar{h}_{j,n}(q_1^\alpha x,y)=a^2(x)\begin{vmatrix} \hat{M}_j & \phi^{(n+1)}(x,y)^T & \phi^{(n)}(x,y)^T \\ \hat{\bar{\phi}}_j^{(-n)}(q_1^\alpha x,y) & -a(x)^{-2} & a(x)^{-1} \end{vmatrix}, \tag{4.9b}$$

where $\hat{\bar{\phi}}_j^{(-n)}(x,y)$ is a $1\times(N-1)$ matrix eliminating the $j-$th column from $\bar{\phi}^{(-n)}(x,y)$. Substituting (4.2)-(4.3) into equation (4.4) yields the determinant identity

$$\begin{aligned}
0=a(x)b(y)[&\begin{vmatrix} M_1 & -1 \\ C(y) & 1 \end{vmatrix}\begin{vmatrix} M_1 & -1 & \phi^{(n)}(x,y)^T & \phi^{(n-1)}(x,y)^T \\ C(q_2^\beta y) & 1 & 0 & 0 \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & 0 & a(x)^{-1} & -1 \\ \bar{\phi}^{(-n-1)}(x,q_2^\beta y) & 0 & 1 & b(y)^{-1} \end{vmatrix} \\
&-\begin{vmatrix} M_1 & -1 & \phi^{(n)}(x,y)^T \\ C(y) & 1 & 0 \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & 0 & a(x)^{-1} \end{vmatrix}\begin{vmatrix} M_1 & -1 & \phi^{(n-1)}(x,y)^T \\ C(q_2^\beta y) & 1 & 0 \\ \bar{\phi}^{(-n-1)}(x,q_2^\beta y) & 0 & b(y)^{-1} \end{vmatrix} \\
&+\begin{vmatrix} M_1 & -1 & \phi^{(n-1)}(x,y)^T \\ C(y) & 1 & 0 \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & 0 & -1 \end{vmatrix}\begin{vmatrix} M_1 & -1 & \phi^{(n)}(x,y)^T \\ C(q_2^\beta y) & 1 & 0 \\ \bar{\phi}^{(-n-1)}(x,q_2^\beta y) & 0 & 1 \end{vmatrix} \\
&+\begin{vmatrix} M_1 & -1 \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & 0 \end{vmatrix}\begin{vmatrix} M_1 & -1 & \phi^{(n)}(x,y)^T & \phi^{(n-1)}(x,y)^T \\ C(q_2^\beta y) & 1 & 0 & 0 \\ \bar{\phi}^{(-n-1)}(x,q_2^\beta y) & 0 & 1 & b(y)^{-1} \\ C(y) & 1 & 0 & 0 \end{vmatrix}].
\end{aligned}$$

The determinant identity above can be proved through the Laplace expansion of a $2(N+2)\times 2(N+2)$ matrix that is equal to zero. Hence equation (4.4) holds. Similarly, substitution of (4.7)-(4.8) into equation (4.5) gives the Jacobi determinant identity

$$
\begin{aligned}
a(x)[|M| & \begin{vmatrix} M & -1 & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & 0 & a(x)^{-1} \\ \bar{\phi}^{(-n-1)}(x,y) & 0 & 1 \end{vmatrix} \\
& - \begin{vmatrix} M & -1 \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & 0 \end{vmatrix} \begin{vmatrix} M & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n-1)}(x,y) & 1 \end{vmatrix} \\
& + \begin{vmatrix} M & -1 \\ \bar{\phi}^{(-n-1)}(x,y) & 0 \end{vmatrix} \begin{vmatrix} M & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & a(x)^{-1} \end{vmatrix}] = 0.
\end{aligned}
$$

Substituting (4.7) and (4.9) into equation (4.6), we get the determinant identity

$$
\begin{aligned}
a^2(x)[& \begin{vmatrix} M & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n-1)}(x,y) & 1 \end{vmatrix} \begin{vmatrix} \hat{M}_j & \phi^{(n+1)}(x,y)^T & \phi^{(n)}(x,y)^T \\ \hat{\bar{\phi}}_j^{(-n)}(q_1^\alpha x,y) & -a(x)^{-2} & a(x)^{-1} \end{vmatrix} \\
& - \begin{vmatrix} M & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & a(x)^{-1} \end{vmatrix} \begin{vmatrix} \hat{M}_j & \phi^{(n+1)}(x,y)^T & \phi^{(n)}(x,y)^T \\ \hat{\bar{\phi}}_j^{(-n-1)}(x,y) & 0 & 1 \end{vmatrix} \\
& + \begin{vmatrix} \hat{M}_j & \phi^{(n)}(x,y)^T \end{vmatrix} \begin{vmatrix} M & \phi^{(n+1)}(x,y)^T & \phi^{(n)}(x,y)^T \\ \bar{\phi}^{(-n-1)}(x,y) & 0 & 1 \\ \bar{\phi}^{(-n)}(q_1^\alpha x,y) & -a(x)^{-2} & a(x)^{-1} \end{vmatrix}] = 0,
\end{aligned}
$$

which can also be verified through the Laplace expansion of a $2(N+2)\times 2(N+2)$ matrix that is equal to zero. Therefore functions in (4.1) and (4.3) are Gramm-type determinant solutions of equations (4.4)-(4.6), and equations (4.4)-(4.6) construct the q-discrete two-dimensional Toda lattice equation with self-consistent sources.

5 Bilinear Bäcklund transformation of equations (4.4)-(4.6)

In this section, we will give the bilinear Bäcklund transformation for equations (4.4)-(4.6).

Proposition 1. *The system (4.4)-(4.6) has the bilinear Bäcklund transformation*

$$\lambda f_n(q_1^\alpha x,y)f'_{n+1}(x,y) - af_{n+1}(x,y)f'_n(q_1^\alpha x,y) - \mu f_n(x,y)f'_{n+1}(q_1^\alpha x,y) = 0, \tag{5.1}$$

$$\begin{aligned}&\lambda g_{j,n}(q_1^\alpha x,y)g'_{j,n+1}(x,y) - ag_{j,n+1}(x,y)g'_{j,n}(q_1^\alpha x,y)\\ &-\mu g_{j,n}(x,y)g'_{j,n+1}(q_1^\alpha x,y) = 0,\end{aligned} \tag{5.2}$$

$$\begin{aligned}&\lambda h_{j,n}(q_1^\alpha x,y)h'_{j,n+1}(x,y) - ah_{j,n+1}(x,y)h'_{j,n}(q_1^\alpha x,y)\\ &-\mu h_{j,n}(x,y)h'_{j,n+1}(q_1^\alpha x,y) = 0,\end{aligned} \tag{5.3}$$

$$\mu f'_{n+1}(x,y)g_{j,n}(x,y) - f'_n(x,y)g_{j,n+1}(x,y) + \theta_j f_n(x,y)g'_{j,n+1}(x,y) = 0, \tag{5.4}$$

$$\mu f_n(x,y)h'_{j,n+1}(x,y) - f_{n+1}(x,y)h'_{j,n}(x,y) + \theta_j f'_{n+1}(x,y)h_{j,n}(x,y) = 0, \tag{5.5}$$

$$\mu(1+ab)f_n(x,y)f'_{n+1}(q_1^\alpha x, q_2^\beta y) - f_n(q_1^\alpha x, y)f'_{n+1}(x, q_2^\beta y)$$
$$+\nu f_{n+1}(x, q_2^\beta y)f'_n(q_1^\alpha x, y) = -\sum_{j=1}^{K} \frac{\mu}{\theta_j} g_{j,n}(q_1^\alpha x, y)h'_{j,n+1}(x, q_2^\beta y), \tag{5.6}$$

where $a = a(x)$, $b = b(y)$, and λ, μ, ν and θ_j are arbitrary constants.

Proof. Let $f_n(x,y)$, $g_{j,n}(x,y)$ and $h_{j,n}(x,y)$ be solutions of equations (4.4)-(4.6). What we need to prove is that $f'_n(x,y)$, $g'_{j,n}(x,y)$ and $h'_{j,n}(x,y)$ in (5.1)-(5.6) are also solutions of (4.4)-(4.6). In fact, according to the relations (5.1)-(5.6), we have

$$\begin{aligned}
P = {} & [(1+ab)f_n(q_1^\alpha x, q_2^\beta y)f_n(x,y) - abf_{n+1}(x, q_2^\beta y)f_{n-1}(q_1^\alpha x, y) \\
& -f_n(q_1^\alpha x, y)f_n(x, q_2^\beta y) - \sum_{j=1}^{K} g_{j,n}(q_1^\alpha x, y)h_{j,n}(x, q_2^\beta y)]f'_{n+1}(x, q_2^\beta y)f'_{n-1}(q_1^\alpha x, y) \\
& -f_{n+1}(x, q_2^\beta y)f_{n-1}(q_1^\alpha x, y)[(1+ab)f'_n(q_1^\alpha x, q_2^\beta y)f'_n(x,y) - f'_n(q_1^\alpha x, y)f'_n(x, q_2^\beta y) \\
& -abf'_{n+1}(x, q_2^\beta y)f'_{n-1}(q_1^\alpha x, y) - \sum_{j=1}^{K} g'_{j,n}(q_1^\alpha x, y)h'_{j,n}(x, q_2^\beta y)] \\
= {} & \mu(1+ab)[f_n(x,y)f'_{n-1}(q_1^\alpha x, y)f_n(x, q_2^\beta y)f'_{n+1}(q_1^\alpha x, q_2^\beta y) \\
& -f_{n+1}(x, q_2^\beta y)f'_n(q_1^\alpha x, q_2^\beta y)f_{n-1}(x,y)f'_n(q_1^\alpha x, y)] \\
& -f_n(q_1^\alpha x, y)f_n(x, q_2^\beta y)f'_{n+1}(x, q_2^\beta y)f'_{n-1}(q_1^\alpha x, y) \\
& +f_{n+1}(x, q_2^\beta y)f_{n-1}(q_1^\alpha x, y)f'_n(q_1^\alpha x, y)f'_n(x, q_2^\beta y) \\
& +\sum_{j=1}^{K}[f_n(x, q_2^\beta y)f'_{n-1}(q_1^\alpha x, y)g_{j,n}(q_1^\alpha x, y)h'_{j,n+1}(x, q_2^\beta y) \\
& \qquad -f_{n+1}(x, q_2^\beta y)f'_n(q_1^\alpha x, y)g_{j,n-1}(q_1^\alpha x, y)h'_{j,n}(x, q_2^\beta y)] \\
= {} & f_n(x, q_2^\beta y)f'_{n-1}(q_1^\alpha x, y)[\mu(1+ab)f_n(x,y)f'_{n+1}(q_1^\alpha x, q_2^\beta y) \\
& -f_n(q_1^\alpha x, y)f'_{n+1}(x, q_2^\beta y) + \sum_{j=1}^{K} g_{j,n}(q_1^\alpha x, y)h'_{j,n+1}(x, q_2^\beta y)] \\
& -f_{n+1}(x, q_2^\beta y)f'_n(q_1^\alpha x, y)[\mu(1+ab)f_{n-1}(x,y)f'_n(q_1^\alpha x, q_2^\beta y) \\
& -f_{n-1}(q_1^\alpha x, y)f'_n(x, q_2^\beta y) + \sum_{j=1}^{K} g_{j,n-1}(q_1^\alpha x, y)h'_{j,n}(x, q_2^\beta y)] \\
= {} & \nu[f_n(x, q_2^\beta y)f'_{n-1}(q_1^\alpha x, y)f_{n+1}(x, q_2^\beta y)f'_n(q_1^\alpha x, y) \\
& \quad -f_{n+1}(x, q_2^\beta y)f'_n(q_1^\alpha x, y)f_n(x, q_2^\beta y)f'_{n-1}(q_1^\alpha x, y)] = 0.
\end{aligned}$$

The above result indicates that $f'_n(x,y)$, $g'_{j,n}(x,y)$ and $h'_{j,n}(x,y)$ satisfy equation (4.4). Much in the same way, equations (4.5)-(4.6) can also be satisfied for $f'_n(x,y)$, $g'_{j,n}(x,y)$ and $h'_{j,n}(x,y)$. Therefore we have completed the proof of the proposition. ■

6 Conclusion and discussion

In this paper, we apply the source generalization procedure to the q-discrete two-dimensional Toda lattice equation. As a result, the q-discrete two-dimensional Toda lattice equation with self-consistent sources is presented. Besides, the bilinear Bäcklund transformation for the q-discrete Toda equation with self-consistent sources is given. When the system has K ($K \geqslant 1$) pairs of sources, we can obtain the N-soliton ($N \geqslant K$) solutions of the q-discrete two-dimensional Toda equation with self-consistent sources. When we set each function $c_j(y)$ to be constant, the sources become zero, and the q-discrete two-dimensional Toda equation with self-consistent sources is reduced to the original q-discrete two-dimensional Toda latice equation. Accordingly, the N-soliton solutions of the q-discrete two-dimensional Toda equation with self-consistent sources are reduced to the solution of the q-discrete two-dimensional Toda equation. Hence the q-discrete Toda equation with self-consistent sources can be viewed as a kind of coupled generalization of the q-discrete two-dimensional Toda lattice equation.

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