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Approximate Lie symmetries of the Navier-Stokes equations

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Abstract

In the framework of the theory of approximate transformation groups proposed by Baikov, Gaziziv and Ibragimov [1], the first-order approximate symmetry operator is calculated for the Navier-Stokes equations. The symmetries of the coupled system obtained by expanding the dependent variables of the Navier-Stokes equations in the perturbation series with respect to a small parameter (viscosity) are used to derive approximate symmetries in the sense by Baikov *et al.*

1 Introduction

Symmetries of differential equations are pivotal to a profound understanding of the physics of the underlying the problems under investigations. Symmetry group analysis of differential equations on the basis of Lie (Lie-Bäcklund) groups unify a wide variety of *ad hoc* methods to analyze and exactly solve differential equations. Detailed description of the methods and their applications to various problems of mathematical physics, fluid dynamics and others may be found e.g. in [8], [9].

In the context of Symmetry Group Methods an approach to derive certain turbulent scaling laws arising in the statistical theory of turbulence was given in [10]. In particular, it unifies a large set of scaling laws for the mean velocity of stationary parallel turbulent shear flows. The approach is derived from the Reynolds averaged Navier-Stokes equations, the fluctuations equations, and the velocity product equations, which are the dyad product of the velocity fluctuations with the equations for the velocity fluctuations. From the knowledge of the symmetries a broad variety of invariant solutions (scaling laws) were derived. Since the symmetries of fluid motion are admitted by all statistical quantities of turbulent flow, the necessary conditions on turbulent models such that they "capture" the proper physics (the symmetries and their corresponding invariant solutions) were presented in [11]. For the plane case the results include the logarithmic law of the wall, the algebraic law, the viscous sublayer, the linear region in the center of a Couette flow and in the center of a rotating channel flow, and a new exponential mean velocity profile that is found in the mid-wake region of high Reynolds number flat-plate boundary layers. Therefore, it was shown that in the case of $Re \to \infty$, respectively $\nu \to 0$, the knowledge of symmetries make it possibly to derive a family of turbulent scaling laws (invariant solutions). Still, we have to note that these invariant solutions we were derived using the symmetries of the Euler equations only. From a physical point of view as the viscosity tends to zero turbulence becomes highly intermittent, and vorticity is concentrated on sets of a small measure.

Reconsidering the derivation of the different scaling laws in [11] we note that the use of symmetries of the Navier-Stokes equations do not enable us to introduce the Reynolds number dependence into scaling laws explicitly. In fact, viscosity is symmetry breaking of one scaling symmetry and as a consequence the entire scaling law theory will brake down. The crucial point for understanding of Reynolds number dependence is that viscosity is primarily significant for small scale turbulence at the order of the Kolmogorov length scale and, if wall bounded flows are considered, in the inner region (viscous sublayer) of a turbulent motion. The so-called outer region of this motion is mainly determined by the Euler equations. According to Kolmogorov's sub-range theory there is a region in correlation space obeying the limits $\eta_K \ll r \ll l_t$ where viscosity is negligible and large-scale influence are also asymptotically small. l_t is the integral length-scale and $\eta_K = (\nu^3/\varepsilon)^{1/4}$ is the Kolmogorov length scale with ε is the energy of dissipation. The eddies of size η_K have a negligible amount of energy but provide the necessary dissipation for the energy balance equation. In contrast the energy containing large scale eddies of size l_t determine the mean velocity, the Reynolds stress tensor and similar variables. It is this distinction and the corresponding difference in symmetries which is the basis for the understanding of the invariant solutions of turbulent flows [11]. The disentangling of these regions leads to a singular asymptotic expansion of the multi-point correlation equations in correlation space r [12].

Barenblatt and Chorin in a series of papers [13]–[15] investigate of the influence of the intermittency phenomenon on certain scaling laws presented by the von Kármán-Prandtl universal logarithmic law of the wall (in the intermediate region of wall-bounded turbulence), and the Kolmogorov-Obukhov scaling for the local structure of turbulence. It was shown that when the viscosity is small the universal logarithmic law for the intermediate region of wall-bounded shear flow must be replaced by a power law. The concept of the so-called incomplete similarity and intermediate asymptotics was used to make a correction of the classical scaling laws when the Reynolds number is finite but large. The analysis extended the classical form of dependency between the velocity gradient and the spatial coordinate y, the shear stress at the wall τ, the pipe diameter d, the kinematic viscosity ν and density ρ without using the Navier-Stokes equations directly.

Our aim is to find symmetries which to leading order correspond to the Euler equations but to higher order allows for the Reynolds number dependence of a turbulent motion. As the first step we apply the theory of approximate symmetries developed by Fushchich and Shtelen [5], Euler *et al.* [3], [4], Ibragimov, Bykov and Gazizov [1], [2] for studying differential equations with a small parameter and consider the Navier-Stokes equations as a perturbation of the Euler equations. In this note we calculate the so-called approximate Lie symmetry tangent vector field to the manifold defined by the Navier-Stokes equations which is motivated by their application to the theory of turbulence. In particular, we show that the Lie symmetries of the Euler equations are inherited by the Navier-Stokes equations in the form of approximate symmetries.

We also mention the paper [16] where the comparison of the above-mentioned approaches are given in details. Moreover, therein the authors give another method for the construction of approximate symmetries which is consistent with perturbation theory. Higher symmetries including a small parameters are studied in [17] to construct approximate solutions of perturbed equations. The reccurent relations for an expansion of symmetry by a small parameter are found for evolution equations with one spatial variable. An illustrative example for the Burgers equation is given and the use of approximate conservation laws for perturbed equations are discussed.

2 Approximate Lie symmetry of the Navier-Stokes equations

Let us consider the Navier- Stokes equation

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \nu\Delta\vec{u}, \qquad \text{div } \vec{u} = 0 \tag{2.1}$$

and the perturbation series for $\vec{u}$ and p in this small viscosity ν limit

$$\begin{aligned} u^\alpha(\vec{x},t) &= u_0^\alpha(\vec{x},t) + \sum_{i=1}^{s} \nu^i u_i^\alpha(\vec{x},t) + o(\nu^s), \quad s = 1,\ldots,n \\ p(\vec{x},t) &= p_0(\vec{x},t) + \sum_{i=1}^{s} \nu^i p_i(\vec{x},t) + o(\nu^s). \end{aligned} \tag{2.2}$$

Inserting these series into the Navier-Stokes equations, we obtain the s-order approximate system for the Navier-Stokes equations in the following denoted by *coupled system.* In a first step we compute the exact Lie symmetry for the coupled system. This symmetry is called s-order approximate symmetries of the Navier-Stokes equations. The infinitesimal operator for this system of any order approximation can be written in the following form

$$X_s = \xi^0\frac{\partial}{\partial t} + \xi^i\frac{\partial}{\partial x^i} + \eta^{0,\alpha}\frac{\partial}{\partial u_0^\alpha} + \eta^{l,\alpha}\frac{\partial}{\partial u_l^\alpha} + \zeta^0\frac{\partial}{\partial p_0} + \zeta^l\frac{\partial}{\partial p_l}. \tag{2.3}$$

Remark 1. We note that symmetries of a coupled system obtained in the framework of the reduction of differential equations with a small parameter using the expansion of the depending variables asymptotically in terms of a small parameter (such as the one in (2.2)) have been considered by Fushchich, Shtelen and Euler *et al.* in [3]–[5] as approximate symmetries of the corresponding differential equations with the small parameter.

In the present section we are primarily interested in the calculation of the exact symmetries of the coupled system for the Navier-Stokes equations. However, that this is in view of finding the first-order approximate symmetry to be derived in the next subsection. We expand a solution $(\vec{u}, p)$ in a perturbation series according to (2.2) to obtain the coupled system of equations for $(\vec{u}_0, \vec{u}_1)$, and (p_0, p_1)

$$\begin{aligned} &\vec{u}_{0t} + (\vec{u}_0\cdot\nabla)\vec{u}_0 + \nabla p_0 = 0, \\ &\vec{u}_{1t} + (\vec{u}_0\cdot\nabla)\vec{u}_1 + (\vec{u}_1\cdot\nabla)\vec{u}_0 + \nabla p_1 = \Delta\vec{u}_0 \\ &\text{div } \vec{u}_0 = 0, \quad \text{div } \vec{u}_1 = 0. \end{aligned} \tag{2.4}$$

The corresponding infinitesimal operator is due to (2.3), here denoted by X_1, and the corresponding prolongation of this operator $\tilde{X}_1$ can be written in the form

$$\tilde{X}_1 = X_1 + \eta_m^{0,\alpha}\frac{\partial}{\partial u_{0,m}^\alpha} + \eta_m^{1,\alpha}\frac{\partial}{\partial u_{1,m}^\alpha} + \zeta_m^0\frac{\partial}{\partial p_{0,m}} + \zeta_m^1\frac{\partial}{\partial p_{1,m}} + \eta_{mn}^{0,\alpha}\frac{\partial}{\partial u_{0,mn}^\alpha} + \eta_{mn}^{1,\alpha}\frac{\partial}{\partial u_{1,mn}^\alpha}, \quad (2.5)$$

where $\eta_m^{i\alpha} = D_m(\eta^{i\alpha}) - u_{i,j}^\alpha D_m(\xi^j)$, $\zeta_m^i = D_m(\zeta^i) - p_{i,j}D_m(\xi^j)$, $\eta_{mn}^{i\alpha} = D_n(\eta_m^{i\alpha}) - u_{i,mr}^\alpha D_n(\xi^r)$, $i = 0, 1$. Here D_m denotes the total derivative operator

$$D_m = \frac{\partial}{\partial x^m} + u_{i,m}^\alpha\frac{\partial}{\partial u_i^\alpha} + u_{i,mn}^\beta\frac{\partial}{\partial u_{i,n}^\beta} + p_{i,m}\frac{\partial}{\partial p_i} + p_{i,mn}\frac{\partial}{\partial p_{i,n}} + \cdots .$$

Applying the operator $\tilde{X}_1$ to the coupled system (2.4), we find that the smooth coefficients ξ^0, ξ^1, $\eta^{0,i}$, $\eta^{1,i}$, ζ^0 and ζ^1 are given by

$$\begin{aligned}
\xi^0 &= (2a_0 + b_0)t + c_0,\\
\xi^i &= (a_0 + b_0)x^i + a_{ij}x^j + h_j(t),\\
\eta^{0,i} &= -a_0u_0^i + a_{ij}u_0^j + h_j'(t),\\
\eta^{1,i} &= -a_0u_1^i + a_{ij}u_1^j - b_0u_1^i,\\
\zeta^0 &= -2a_0p_0 - x^ih_i''(t) + g^0(t),\\
\zeta^1 &= -2a_0p_1 + g^1(t) - b_0p_1,
\end{aligned} \quad (2.6)$$

where a_0, b_0, c_0 are arbitrary constants, the numbers a_{ij} are connected by the relationships $a_{ii} = 0$, $a_{ij} + a_{ji} = 0$ for $i \neq j$ and h_j, g^0, g^1 are arbitrary smooth functions of the variable t. We note that the functions ξ^0, ξ^i, $\eta^{0,i}$, ζ^0 coincide with the coefficient functions of the infinitesimal operator for the Euler equations.

To adopt the symmetry operator (2.3), (2.6) for their application in turbulence, we need to rewrite (approximately) this operator (or the Lie symmetry vector field) in the original variables $(\partial/\partial t, \partial/\partial x^i, \partial/\partial u^\alpha, \partial/\partial p)$. For this aim we use the concept of Approximate Group Transformations by Ibragimov, Baikov and Gazizov [1], [2] wherein the infinitesimal operator is expanded in a perturbation series with the small parameter ν.

3 Approximate Lie symmetry vector field defined by the Navier-Stokes equations

Following central idea in paper [6], we consider a family G of invertible transformations

$$\begin{aligned}
&\bar{x}^i \approx \omega^i(t, \vec{x}, \vec{u}, p, a; \nu) \equiv \omega_0^i(t, \vec{x}, \vec{u}, p, a) + \nu\omega_1^i(t, \vec{x}, \vec{u}, p, a) + o(\nu),\\
&\bar{t} \approx \lambda(t, \vec{x}, \vec{u}, p, a; \nu) \equiv \lambda_0(t, \vec{x}, \vec{u}, p, a) + \nu\lambda_1(t, \vec{x}, \vec{u}, p, a) + o(\nu),\\
&\bar{u}^\alpha \approx \tau^\alpha(t, \vec{x}, \vec{u}, p, a; \nu) \equiv \tau_0^\alpha(t, \vec{x}, \vec{u}, p, a) + \nu\tau_1^\alpha(t, \vec{x}, \vec{u}, p, a) + o(\nu),\\
&\bar{p} \approx \mu(t, \vec{x}, \vec{u}, p, a; \nu) \equiv \mu_0(t, \vec{x}, \vec{u}, p, a) + \nu\mu_1(t, \vec{x}, \vec{u}, p, a) + o(\nu),\\
&\bar{\nu} \approx \nu\theta(a; \nu) \equiv \nu\theta_0(a) + o(\nu),
\end{aligned}$$

where $f \approx g$ means that $f - g = o(\nu)$, $|o(\nu)| \leq C\nu^2$. According to the theory of approximate Lie symmetries (see, for example [6]), the first-order approximate infinitesimal operator can be written in the form

$$\begin{aligned}
X_1^{appr} &= [\xi^0_{(0)}(t,\vec{x},\vec{u},p) + \nu\xi^0_{(1)}(t,\vec{x},\vec{u},p)]\frac{\partial}{\partial t} \\
&+ [\xi^i_{(0)}(t,\vec{x},\vec{u},p) + \nu\xi^i_{(1)}(t,\vec{x},\vec{u},p)]\frac{\partial}{\partial x^i} \\
&+ [\eta^\alpha_{(0)}(t,\vec{x},\vec{u},p) + \nu\eta^\alpha_{(1)}(t,\vec{x},\vec{u},p)]\frac{\partial}{\partial u^\alpha} \\
&+ [\zeta_{(0)}(t,\vec{x},\vec{u},p) + \nu\zeta_{(1)}(t,\vec{x},\vec{u},p)]\frac{\partial}{\partial p} + \nu\kappa\frac{\partial}{\partial \nu},
\end{aligned}$$

where

$$\begin{aligned}
\xi^0_{(l)} &= \left.\frac{\partial\lambda_{(l)}}{\partial a}\right|_{a=0}, \quad \xi^i_{(l)} = \left.\frac{\partial\omega^i_{(l)}}{\partial a}\right|_{a=0}, \\
\eta^\alpha_{(l)} &= \left.\frac{\partial\tau^\alpha_{(l)}}{\partial a}\right|_{a=0}, \quad \zeta_{(l)} = \left.\frac{\partial\mu_{(l)}}{\partial a}\right|_{a=0}, \quad \kappa = \left.\frac{\partial\theta_0}{\partial a}\right|_{a=0}.
\end{aligned}$$

An algorithm for the direct calculation of the coefficients of X_1^{appr} can be taken from [6]. The following assertion establishes a relationship between the operators X_1 and X_1^{appr}:

The operator X_1 of an exact symmetry of the coupled system (2.4) can be rewritten in the form of an approximate infinitesimal operator X_1^{appr} if and only if it has the form

$$\begin{aligned}
X_1 &= \xi^0_{(0)}\frac{\partial}{\partial t} + \xi^i_{(0)}\frac{\partial}{\partial x^i} + \eta^\alpha_{(0)}\frac{\partial}{\partial u^\alpha_0} + \zeta_{(0)}\frac{\partial}{\partial p_0} \\
&+ [\eta^\alpha_{(1)} - \xi^j_{(1)}u^\alpha_{0,j} + \frac{\partial\eta^\alpha_{(0)}}{\partial u^\beta_0}u^\beta_1 - \frac{\partial\xi^j_{(0)}}{\partial u^\beta_0}u^\alpha_{0,j}u^\beta_1 - \kappa u^\alpha_1]\frac{\partial}{\partial u^\alpha_1} \\
&+ [\zeta_{(1)} - \xi^j_{(1)}p_{0,j} + \frac{\partial\zeta_0}{\partial p_0}p_1 - \frac{\partial\xi^j_{(0)}}{\partial p_0}p_{0,j}p_1 - \kappa p_1]\frac{\partial}{\partial p_1}
\end{aligned} \tag{3.1}$$

which is obtained by substituting (2.2) into the operator X_1^{appr} and expanding the coefficient functions into Taylor series (for details see Theorem 1 [6]). The nice form of the coefficient functions obtained in (2.6) of the operator X_1 enables us to apply (3.1) for calculating X_1^{appr}. Indeed, comparing (2.6) and (3.1) we obtain that

$$\xi^i_{(1)} \equiv 0, \quad \eta^\alpha_{(1)} \equiv 0, \quad \zeta^{(1)} = g^1(t), \quad \kappa = b_0. \tag{3.2}$$

taking into account that (see (2.6))

$$\frac{\partial\xi^j_{(0)}}{\partial u^\beta_0} = 0, \quad \frac{\partial\xi^j_{(0)}}{\partial p_0} = 0, \quad \frac{\partial\eta^\alpha_{(0)}}{\partial u^\beta_0} = a_{\alpha\beta} \quad (\beta \neq \alpha), \quad \frac{\partial\eta^\alpha_{(0)}}{\partial u^\alpha_0} = -a_0, \quad \frac{\partial\zeta_0}{\partial p_0} = -2a_0.$$

As a result, we obtain that the operator X_1 is transformed to

$$X_1^{appr} = \xi^0\frac{\partial}{\partial t} + \xi^i\frac{\partial}{\partial x^i} + \eta^{0,\alpha}\frac{\partial}{\partial u^\alpha} + [\zeta^0 + \nu g^1(t)]\frac{\partial}{\partial p} + \nu b_0\frac{\partial}{\partial \nu}. \tag{3.3}$$

Moreover, this operator is admitted by the Navier-Stokes equations in the sense of the first-order approximation of the theory of approximate transformation groups and the operator X_0 (unperturbed term of X_1^{appr})

$$X_0 = \xi^0 \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x^i} + \eta^{0,\alpha} \frac{\partial}{\partial u^\alpha} + \zeta^0 \frac{\partial}{\partial p}$$

coincides with the infinitesimal operator for the Euler equations. Therefore we showed that the operator X_0 is inherited [7] by the Navier-Stokes equations in the form of approximate symmetry (3.3).

Remark 2. In general, symmetry operators obtained in the framework as developed by Fushchich, Shtelen [5] and Euler *et al.* [3], [4] and approximate symmetry operators in the sense by Baikov *et al.* (based on the theory of approximate transformation groups [2]) are not equivalent to each other. Some corresponding examples are given by Gazizov in [6]. Also we note that an infinitesimal operator admitted by an unperturbed equation cannot always be extended in the form of approximate symmetry operator of the perturbed equation under consideration, see [2].

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References

[1] Baikov V A, Gazizov R K and Ibragimov N H, Approximate symmetries of equations with a small parameter, *Mat. USSR Sb.* **64** (1989), 427–441.

[2] Baikov V A, Gazizov R K and Ibragimov N H, Approximate transformation groups and deformations of symmetry Lie algebras, Chapter 2, in CRC Handbook of Lie groups analysis of differential equations, Vol. 3, Editor: Ibragimov N H, CRC Press, Boca Raton, Florida, 1996.

[3] Euler N, Shulga M and Steeb W, Approximate symmetries and approximate solutions for a multi-dimensional Landau-Ginzburg equation, *J. Phys. A.* **25** (1992), L1095–L1103.

[4] Euler M, Euler N and Kohler A, On the construction of approximate solutions for a multidimensional nonlinear heat equation *J. Phys. A.* **27** (1994), 2083–2092.

[5] Fushchich W and Shtelen W, On approximate symmetries and approximate solutions of the non-linear wave equation with a small parameter, *J. Phys. A.* **22** (1989), L887–L890.

[6] Gazizov R K, Symmetries of differential equations with a small parameters: a comparison of two approaches, *Modern Group Anal.* **7** (1997), 107–113.

[7] Ibragimov N H, CRC Handbook of Lie groups analysis of differential equations, Vol. 2, CRC Press, 1995.

[8] Ovsyannikov L V, Group methods of differential equations, Academic Press, New York, 1982.

[9] Olver P J, Applications of Lie groups to differential equations, Graduate Texts in Mathematics, Springer-Verlag, 1986.

[10] Oberlack M, Symmetries, invariance and scaling-laws in inhomogeneous turbulent shear flows, *Flow Turbul. Combust.* **62** (1999), 111–135.

[11] Oberlack M, Unified approach for symmetries in plane parallel turbulent shear flows, *J. Fluid Mech.* **427** (2001) 299–328.

[12] Oberlack M and Busse F H, Theories of turbulence, Springer, Wienn, New York, 2002.

[13] Barenblatt G and Chorin A, New perspective in turbulence, scaling laws, asymptotics, and intermittency, *SIAM Rev.* **40(2)** (1998) 265–291.

[14] Barenblatt G, Scaling laws for turbulent wall-bounded shear flow at very large Reynolds number, *J. Eng. Math.* **36(4)** (1999) 361–384.

[15] Barenblatt G and Chorin A, Scaling laws for turbulent at very large Reynolds number, *Proc. Est. Acad. Sci., Phys. Math.* **48** (1999) 199–205.

[16] Pakdemirli M, Yürüsoy M and Dolarci T, Comparison of approximate symmetries methods for differential equations, *Acta Appl. Math.* **80** (2004) 243–271.

[17] Samokhin A, Symmetries of equations containing a small parameter, The international conference on secondary calculus and cohomological physics, Moscow, Russia, August 24–31, Moscow: Diffiety Institute of the Russian Academy of Natural Sciences, 9 p., electronic only, 1997.