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A note on Bernoulli polynomials and solitons

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Abstract

The dependence on time of the moments of the one-soliton KdV solutions is given by Bernoulli polynomials. Namely, we prove the formula

$$\int_{\mathbb{R}} x^n \operatorname{sech}^2(x-t)\, dx = 2\,\pi^n\,(-i)^n\, B_n(\frac{1}{2}+\frac{t}{\pi}i)\ ,$$

expressing the moments of the one-soliton function $\operatorname{sech}^2(x-t)$ in terms of the Bernoulli polynomials $B_n(x)$. We also provide an alternative short proof to the Grosset-Veselov formula connecting the one-soliton to the Bernoulli numbers

$$\int_{\mathbb{R}} \left(D^{m-1}\operatorname{sech}^2 x\right)^2\, dx = (-1)^{m-1}\, 2^{2m+1}\, B_{2m}\ ,$$

$(D = d/dx)$ published recently in this journal.

1 Introduction

The Bernoulli polynomials $B_n(t)$ are defined by the generating function

$$\frac{z\,e^{zt}}{e^z-1} = \sum_{n=0}^{\infty} B_n(t)\,\frac{z^n}{n!}\ , \tag{1.1}$$

$(|z| < 2\pi)$ and the Bernoulli numbers are their values at zero, $B_n = B_n(0)$. In two recent papers Fairlie and Veselov [2] and Grosset and Veselov [4] revealed an interesting connection between Bernoulli polynomials and the theory of the Korteweg-deVries equation

$$u_t - 6\,u\,u_x + u_{xxx} = 0\ , \tag{1.2}$$

and in particular, with its remarkable single soliton solution [5]

$$u(x,t) = -2\operatorname{sech}^2(x-4\,t)\ . \tag{1.3}$$

In their letter [4] Grosset and Veselov established the formula

$$B_{2m} = \frac{(-1)^{m-1}}{2^{2m+1}} \int_{\mathbb{R}} \left(D^{m-1}\operatorname{sech}^2 x\right)^2\, dx\ , \tag{1.4}$$

which connects the Bernoulli numbers and the derivatives of the single soliton. In section 3 we give a short proof of this formula based on Fourier theory. First of all, we point out another interesting connection between the single soliton and the Bernoulli polynomials.

2 Main result

Writing $\mathrm{sech}(x) = 1/\cosh(x)$, we have the following proposition.

Proposition 1. *For all t and $n = 0, 1, 2, \ldots$,*

$$\int_{\mathbb{R}} \frac{x^n}{\cosh^2(x-t)}\, dx = 2\,(-i\,\pi)^n\, B_n(\frac{1}{2} + \frac{t}{\pi} i) \ . \tag{2.1}$$

Remark 1. The Bernoulli polynomials have the addition property [7, p. 4]

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x)\, y^{n-k} \tag{2.2}$$

and also the property

$$B_k(\frac{1}{2}) = (2^{1-k} - 1)\, B_k \ . \tag{2.3}$$

Therefore, we can put (2.1) in the form

$$\int_{\mathbb{R}} \frac{x^n}{\cosh^2(x-t)}\, dx = 2 \sum_{k=0}^{n} \binom{n}{k} (-\pi\, i)^k\, (2^{1-k} - 1)\, t^{n-k}\, B_k \ , \tag{2.4}$$

where on the right hand side all terms with odd indices are zeros, as $B_k = 0$ when $k > 1$ is odd, and for $k = 1$ the factor $2^{1-k} - 1$ is zero.

Remark 2. If

$$f(x) = \sum_{n=0}^{m} a_n\, x^n \tag{2.5}$$

is a polynomial, we can multiply equation (2.1) by a_n and sum for $n = 0, 1, \ldots, m$ to obtain the superposition formula

$$\int_{\mathbb{R}} \frac{f(x)}{\cosh^2(x-t)}\, dx = 2 \sum_{n=0}^{m} a_n\, (-i\,\pi)^n\, B_n(\frac{1}{2} + \frac{t}{\pi}\, i) \ . \tag{2.6}$$

Proof of the Proposition. We use the Fourier transform formula [3, 3.982.1, p. 505] or [6, 1.7.2, p. 33]

$$\int_{\mathbb{R}} \frac{e^{ixy}}{\cosh^2 x}\, dx = \frac{\pi\, y}{\sinh \frac{\pi\, y}{2}} \ , \tag{2.7}$$

and change the variable $x \to x - t$ to obtain

$$\int_{\mathbb{R}} \frac{e^{ixy}}{\cosh^2(x-t)}\, dx = \frac{\pi\, y\, e^{ity}}{\sinh \frac{\pi\, y}{2}} = \frac{2\,\pi\, y\, e^{\pi y (\frac{1}{2} + \frac{i\,t}{\pi})}}{e^{\pi y} - 1} \ . \tag{2.8}$$

This is explicitly the Fourier transform of the single soliton solution. Next, we expand the right hand side of (2.8) in a power series on the powers of $\pi\, y$ for $|y| < 2$. In view of (1.1) this provides the representation

$$\int_{\mathbb{R}} \frac{e^{ixy}}{\cosh^2(x-t)}\, dx = 2\sum_{n=0}^{\infty} B_n\Big(\frac{1}{2}+\frac{t}{\pi}\, i\Big)\, \frac{\pi^n\, y^n}{n!}\ . \tag{2.9}$$

Expanding now in power series the exponential function inside the integral, changing the order of summation and integration, and comparing the coefficients for y^n we arrive at (2.1). The proof is complete. ■

3 The Grosset-Veselov Formula

We show here that Grosset-Veselov's formula (1.4) is equivalent to the representation

$$B_{2m} = \frac{(-1)^{m-1}}{\pi^{2\,m}} \int_0^{+\infty} \frac{x^{2m}}{\sinh^2 x}\, dx\ , \tag{3.1}$$

via Fourier transform theory. Formula (3.1) is known and can be found in [1, 1.13 (27)] or [3, 3.527.2, p. 352]. To understand the nature of (3.1) better, we give a short derivation in the Appendix.

Proof. We use the notation

$$F(t) = \int_{\mathbb{R}} f(x)\, e^{-2\pi ixt}\, dx \tag{3.2}$$

for the Fourier transform and write (2.7) as

$$\int_{\mathbb{R}} \frac{e^{-2\pi ixt}}{\cosh^2 x}\, dx = \frac{2\,\pi^2\, t}{\sinh(\pi^2\, t)}\ . \tag{3.3}$$

According to the derivative property we find

$$\int_{\mathbb{R}} \left(D^{m-1}\frac{1}{\cosh^2 x}\right) e^{-2\pi ixt}\, dx = (2\,\pi\, i\, t)^{m-1}\frac{2\,\pi^2\, t}{\sinh(\pi^2\, t)} = \frac{-\pi\, i\,(2\,\pi\, i\, t)^m}{\sinh(\pi^2\, t)}\ . \tag{3.4}$$

Applying now Plancherel's theorem, i.e.

$$\int_{\mathbb{R}} |F(t)|^2\, dt = \int_{\mathbb{R}} |f(x)|^2\, dx\ , \tag{3.5}$$

we obtain

$$\int_{\mathbb{R}} \left(D^{m-1}\frac{1}{\cosh^2 x}\right)^2 dx = \int_{\mathbb{R}} \left|\frac{\pi\, i\,(2\,\pi\, i\, t)^m}{\sinh(\pi^2\, t)}\right|^2 dt = 2^{2m+1}\,\pi^{2m+2}\int_0^{+\infty} \frac{t^{2m}}{\sinh^2(\pi^2\, t)}\, dt\ . \tag{3.6}$$

Therefore, according to (3.1),

$$\int_{\mathbb{R}} \left(D^{m-1}\frac{1}{\cosh^2 x}\right)^2 dx = (-1)^{m-1}\, 2^{2m+1}\, B_{2m}\ , \tag{3.7}$$

which is Grosset-Veselov's formula. ■

We want to point out that this proof was independently suggested by Professor A. Staruszkiewicz (see Note added in Proofs at the end of [4]). For the convenience of the reader it is appropriate to have it recorded here together with the derivation of (3.1) below.

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Appendix

A

We deduce (3.1) from the well-known Fourier sine transform formula [3, 3.911.2, p. 481] or [6, 2.3.12, p. 125]

$$\int_0^{+\infty} \frac{\sin(t\,y)}{e^t - 1}\, dt = \frac{\pi}{2}\, \coth(\pi\, y) - \frac{1}{2\,y}\ . \tag{A.1}$$

After integration by parts the left hand side takes the form

$$\frac{1}{y} \int_0^{+\infty} \frac{(1-\cos(t\,y))\, e^t}{(e^t-1)^2}\, dt = \frac{1}{2\,y} \int_0^{+\infty} \frac{1-\cos(2\,x\,y)}{\sinh^2 x}\, dx \quad \text{(setting } \ t = 2\,x) \ . \tag{A.2}$$

Equation (A.1) can now be written as

$$\int_0^{+\infty} \frac{1-\cos(2\,x\,y)}{\sinh^2 x}\, dx = \pi\, y - 1 + \frac{2\,\pi\, y}{e^{2\pi y} - 1}\ . \tag{A.3}$$

Expanding both sides on powers of y by using the two series

$$1 - \cos(2\,x\,y) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}\, 2^{2n}\, x^{2n}\, y^{2n}}{(2\,n)!}\ ; \tag{A.4}$$

$$\pi\, y - 1 + \frac{2\,\pi\, y}{e^{2\pi y} - 1} = \sum_{n=1}^{+\infty} B_{2n} \frac{(2\,\pi)^{2n}\, y^{2n}}{(2\,n)!}\ , \tag{A.5}$$

and comparing coefficients we arrive at (3.1).

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