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On $\mathfrak{sl}(2)$-relative cohomology of the Lie algebra of vector fields and differential operators

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Abstract

Let Vect($\mathbb{R}$) be the Lie algebra of smooth vector fields on $\mathbb{R}$. The space of symbols Pol($T^*\mathbb{R}$) admits a non-trivial deformation (given by differential operators on weighted densities) as a Vect($\mathbb{R}$)-module that becomes trivial once the action is restricted to $\mathfrak{sl}(2) \subset \mathrm{Vect}(\mathbb{R})$. The deformations of Pol($T^*\mathbb{R}$), which become trivial once the action is restricted to $\mathfrak{sl}(2)$ and such that the Vect($\mathbb{R}$)-action on them is expressed in terms of differential operators, are classified by the elements of the weight basis of $\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu})$, where $\mathrm{H}^i_{\mathrm{diff}}$ denotes the differential cohomology (i.e., we consider only cochains that are given by differential operators) and where $\mathcal{D}_{\lambda,\mu} = \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ is the space of differential operators acting on weighted densities. The main result of this paper is computation of this cohomology. In addition to relative cohomology, we exhibit 2-cocycles spanning $\mathrm{H}^2(\mathfrak{g}; \mathcal{D}_{\lambda,\mu})$ for $\mathfrak{g} = \mathrm{Vect}(\mathbb{R})$ and $\mathfrak{sl}(2)$.

1 Introduction

Notations. Let Vect($\mathbb{R}$) be the Lie algebra of smooth vector fields on $\mathbb{R}$. Let $\mathcal{F}_\lambda$ be the space of weighted densities of degree λ on $\mathbb{R}$, i.e., the space of sections of the line bundle $(T^*\mathbb{R})^{\otimes\lambda}$, so its elements can be represented as $\phi(x)dx^\lambda$, where $\phi(x)$ is a function and dx^λ is a formal (for a time being) symbol. This space coincides with the space of vector fields, functions and differential forms for $\lambda = -1$, 0 and 1, respectively. The Lie algebra Vect($\mathbb{R}$) acts on $\mathcal{F}_\lambda$ by the Lie derivative: we set

$$L^\lambda_X(\phi\, dx^\lambda) = (X(\phi) + \lambda\, \phi\, \mathrm{div} X)\, dx^\lambda \text{ for any } X \in \mathrm{Vect}(\mathbb{R}) \text{ and } \phi\, dx^\lambda \in \mathcal{F}_\lambda. \quad (1.1)$$

We denote by $\mathcal{D}_{\lambda,\nu}$ the space of linear differential operators that act on the spaces of weighted densities:

$$A : \mathcal{F}_\lambda \to \mathcal{F}_\mu. \quad (1.2)$$

The Lie algebra Vect($\mathbb{R}$) acts on $\mathcal{D}_{\lambda,\nu}$ as follows. For any $X \in \mathrm{Vect}(\mathbb{R})$, we set (here L^λ_X is the action (1.1)):

$$L^{\lambda,\mu}_X(A) = L^\mu_X \circ A - A \circ L^\lambda_X. \quad (1.3)$$

Motivations. This work has its genesis in the study of the Vect($\mathbb{R}$)-module $\mathcal{D}_{\lambda,\mu}$. Duval, Lecomte and Ovsienko showed [6, 13] that this space cannot be isomorphic, as a Vect($\mathbb{R}$)-module, to the corresponding space of symbols of these operators but is its deformation in the sense of Richardson-Neijenhuis [15]. As is well known, deformation theory of modules is closely related to the Lie algebra cohomology [15]. More precisely, given a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V; the *infinitesimal* deformations of the $\mathfrak{g}$-module structure on V, i.e., deformations that are linear in the parameter of deformation, are described by the elements (up to proportionality) of $\mathrm{H}^1(\mathfrak{g}; \mathrm{End}(V))$. The obstructions to extension of any infinitesimal deformation to a formal one are similarly described by $\mathrm{H}^2(\mathfrak{g}; \mathrm{End}(V))$. Computation of H^1 in our situation (with $\mathfrak{g} = \mathrm{Vect}(\mathbb{R})$ and $\mathcal{D}_{\lambda,\mu}$ instead of $\mathrm{End}(V)$) was carried out by Feigin and Fuchs [7]. Ovsienko and I computed the corresponding $\mathfrak{sl}(2)$-relative cohomology (see [5]). Gordan's classification of bilinear differential operators on weighted densities [11] played a central role in our computation. Later, a generalization to multi-dimensional manifolds has been carried out by Lecomte and Ovsienko in [13]; for further results, see [4]. Note that the $\mathfrak{sl}(2)$-relative cohomology measures infinitesimal deformations that become trivial once the action is restricted to $\mathfrak{sl}(2)$. This is actually the case for the space of differential operators since, as $\mathfrak{sl}(2)$-module, it is isomorphic to the space of symbols for generic λ and μ (cf. [9]). Let $\mathrm{H}^i_{\mathrm{diff}}$ be the differential cohomology (i.e., we consider only cochains that are given by differential operators). Recently I realized that a description of (here $\mathrm{Vect}_{\mathrm{P}}(\mathbb{R})$ is the Lie algebra of polynomial vector fields)

$$\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{P}}(\mathbb{R}); \mathcal{D}_{\lambda,\mu}) \tag{1.4}$$

can be deduced from the work by Feigin and Fuchs [7]. Feigin-Fuchs gave details of computation of $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{P}}(\mathbb{R}); \mathcal{D}_{\lambda,\mu})$ but not of higher cohomology and no explicit 2-cocycles were provided. The $\mathfrak{sl}(2)$-relative cohomology cannot, however, be deduced from their computation. Several authors (see, e.g., [14, 19]) have also studied $\mathrm{H}^i(\mathrm{Vect}(\mathbb{R}); \mathcal{A})$ for an arbitrary Vect($\mathbb{R}$)-module $\mathcal{A}$. But it is not easy to get a description of the cohomology (1.4) nor the $\mathfrak{sl}(2)$-relative cohomology from their results. Our main result is computation of the $\mathfrak{sl}(2)$-relative cohomology and explicit expressions of 2-cocycles that span (1.4). This work is the first step towards the study of formal deformations of symbols.

For investigation of all deformations of symbols in case of $\mathbb{R}^n$ for $n > 1$, see [1]. The authors use the Neijenhuis-Richardson product to prove the existence of cocycles but do not compute any cohomology. The cohomology similar to (1.4) with $\mathbb{R}^n$ instead of $\mathbb{R}$ is still out of reach for $n > 1$.

2 Basic definitions

Consider the standard (local) action of SL(2) on $\mathbb{R}$ by linear-fractional transformations. Although the action is local, it generates global vector fields

$$\frac{d}{dx}, \quad x\frac{d}{dx}, \quad x^2\frac{d}{dx},$$

that form a Lie subalgebra of Vect($\mathbb{R}$) isomorphic to the Lie algebra $\mathfrak{sl}(2)$ (cf. [16]). This realization of $\mathfrak{sl}(2)$ is understood throughout this paper.

2.1 The Gelfand-Fuchs cocycle

We need to introduce the following cocycle (of Gelfand-Fuchs):

$$\omega(X,Y) = \begin{vmatrix} f' & g'' \\ f' & g'' \end{vmatrix} dx \quad \text{for } X = f\frac{d}{dx}, Y = g\frac{d}{dx}. \tag{2.1}$$

Here ω is a cohomology class in $\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}), \mathcal{F}_1)$. Related is the element of $\mathrm{H}^2(\mathrm{Vect}(S^1))$, the 2-cocycle on $\mathrm{Vect}(S^1)$ given by the formula (see [10]):

$$\int_{S^1} \omega(X,Y).$$

This 2-cocycle generates the central extension of $\mathrm{Vect}(S^1)$ called the *Virasoro* algebra.

3 The $\mathfrak{sl}(2)$-relative cohomology of $\mathrm{Vect}(\mathbb{R})$ acting on $\mathcal{D}_{\lambda,\mu}$

The following steps to compute the relative cohomology has intensively been used in [3, 4, 5, 13]. First, we classify $\mathfrak{sl}(2)$-invariant differential operators, then we isolate among them those that are 2-cocycles. To do that, we need the following Lemma.

Lemma 1. *Any 2-cocycle vanishing on the Lie subalgebra* $\mathfrak{sl}(2)$ *of* $\mathrm{Vect}(\mathbb{R})$ *is* $\mathfrak{sl}(2)$-*invariant.*

Proof. The 2-cocycle condition reads as follows:

$$c([X,Y], Z, \phi\, dx^\lambda) - L_X^{\lambda,\mu}\, c(Y, Z, \phi\, dx^\lambda) + \circlearrowleft (X,Y,Z) = 0$$

for every $X, Y, Z \in \mathrm{Vect}(\mathbb{R})$ and $\phi\, dx^\lambda \in \mathcal{F}_\lambda$, where $\circlearrowleft (X,Y,Z)$ denotes the summands obtained from the two written ones by the cyclic permutation of the symbols X, Y, Z. Now, if $X \in \mathfrak{sl}(2)$, then the equation above becomes

$$c([X,Y], Z, \phi\, dx^\lambda) - c([X,Z], Y, \phi\, dx^\lambda) = L_X^{\lambda,\mu}\, c(Y, Z, \phi\, dx^\lambda).$$

This condition is nothing but the invariance property. ■

3.1 $\mathfrak{sl}(2)$-invariant differential operators

As our 2-cocycles vanish on $\mathfrak{sl}(2)$, we will investigate $\mathfrak{sl}(2)$-invariant bilinear differential operators that vanish on $\mathfrak{sl}(2)$.

Proposition 1. *The space of skew-symmetric bilinear differential operators* $\mathrm{Vect}(\mathbb{R}) \wedge \mathrm{Vect}(\mathbb{R}) \to \mathcal{D}_{\lambda,\mu}$, *which are* $\mathfrak{sl}(2)$-*invariant and vanish on* $\mathfrak{sl}(2)$, *is as follows:*

1. *It is* $\frac{1}{2}(k-3)$-*dimensional if* $\mu - \lambda = k$ *and* k *is odd.*
2. *It is* $\frac{1}{2}(k-4)$-*dimensional if* $\mu - \lambda = k$ *and* k *is even.*
3. *It is* 0-*dimensional, otherwise.*

Proof. The generic form of any such a differential operator is (here $X = f\frac{d}{dx}, Y = g\frac{d}{dx} \in$ Vect($\mathbb{R}$) and $\phi\, dx^\lambda \in \mathcal{F}_\lambda$):

$$c(X, Y, \phi\, dx^\lambda) = \sum_{i+j+l\leq k} c_{i,j}\, f^{(i)}\, g^{(j)}\, \phi^{(l)} dx^\mu,$$

where $c_{i,j} = -c_{j,i}$ and $f^{(i)}$ stands for $\frac{d^i f}{dx^i}$.

The invariance property with respect to the vector field $X = x\frac{d}{dx}$ with arbitrary Y and Z implies that $c'_{i,j} = 0$ and $\mu = \lambda + i + j + l$. Therefore $c_{i,j}$ are constants. Now, the invariance property with respect $X = x^2\frac{d}{dx}$ with arbitrary Y and Z is equivalent to the system (where $2 < \beta < \gamma < k$):

$$(\beta+1)(\beta-2)\, c_{\beta+1,\gamma} - (\gamma+1)(\gamma-2)\, c_{\gamma+1,\beta} + (k+2-\beta-\gamma)(k+1-\beta-\gamma+2\lambda)\, c_{\beta,\gamma} = 0. \quad (3.1)$$

For $\beta = 3$, the equation (3.1) implies that all the constants $c_{t,3}$ can be determined *uniquely* in terms of $c_{4,3}$ and $c_{4,s}$. More precisely,

$$c_{\gamma+1,3} = \frac{4\, c_{4,\gamma} + (k-1-\gamma)\,(k-2-\gamma+2\,\lambda)\ c_{3,\gamma}}{(\gamma+1)(\gamma-2)}.$$

For $\beta = 4$ and $\gamma = 5$, and from the system (3.1), we have

$$c_{6,4} = \frac{1}{12}\,(k-7)\,(k-8+2\,\lambda\,)\ c_{4,5}.$$

Thus the constant $c_{6,4}$ is determined. But for $\beta = 4$ and $\gamma > 5$, the system (3.1) implies that

$$c_{5,\gamma} = \frac{1}{10}(\gamma+1)(\gamma-2)\, c_{\gamma+1,4} - \frac{1}{10}(k-\gamma-2)(k-\gamma-3+2\,\lambda)\, c_{4,\gamma}.$$

Therefore all $c_{5,\gamma}$ can be determined for any $\gamma \geq 6$.

By continuing this procedure we see that $c_{6,\gamma}, c_{7,\gamma}, \ldots$ can be determined as well as $c_{4,\gamma}$ for γ even. Finally, we have proved that the space of $\mathfrak{sl}(2)$-invariant operators is as follows:

(i) for k even, it is generated by $c_{4,3}, c_{4,5}, c_{4,7}, \ldots, c_{4,k-3}$. The space of solution is $\frac{1}{2}(k-4)$-dimensional.

(ii) for k odd, it is generated by $c_{4,3}, c_{4,5}, c_{4,7}, \ldots, c_{4,k-2}$. The space of solution is $\frac{1}{2}(k-3)$-dimensional. ∎

3.2 The $\mathfrak{sl}(2)$-relative cohomology of Vect($\mathbb{R}$)

Theorem 1. *We have*

$$\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu}) = \begin{cases} \mathbb{R} & \text{if } \begin{cases} (\lambda,\mu) = (0,5), (-2,3), (-4,1), (-\frac{5}{2}, \frac{7}{2}) \text{ or} \\ \qquad \left(-\frac{5}{2} \pm \frac{\sqrt{19}}{2}, \frac{7}{2} \pm \frac{\sqrt{19}}{2}\right), \\ \mu - \lambda = 7, 8, 9, 10, 11 \text{ for all } \lambda \neq \frac{1-k}{2}, \\ \mu - \lambda = k = 12, 13, 14 \text{ and } \lambda = \frac{1-k}{2} \pm \frac{\sqrt{12k-23}}{2}, \\ \mu - \lambda = k = 15 \text{ and } \lambda = \frac{1-k}{2}, \end{cases} \\ \mathbb{R}^2 & \text{if } \ \mu - \lambda = k = 7, \ldots, 14 \text{ for } \lambda = \frac{1-k}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1. $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu})$ has been computed in [5].

4 Proof of Theorem 1

Every 2-cocycle on Vect($\mathbb{R}$) retains the following general form (here $X = f\frac{d}{dx}, Y = g\frac{d}{dx} \in$ Vect($\mathbb{R}$) and $\phi\, dx^\lambda \in \mathcal{F}_\lambda$):

$$c(X, Y, \phi\, dx^\lambda) = \sum_{i+j+l\leq k} c_{i,j}\, f^{(i)}\, g^{(j)}\, \phi^{(l)} dx^\mu, \qquad (4.1)$$

where $c_{i,j} = -c_{j,i}$. Since this 2-cocycle vanishes on $\mathfrak{sl}(2)$, Lemma 1 implies that this 2-cocycle is $\mathfrak{sl}(2)$-invariant. Therefore all $c_{i,j}$ are zero and $i+j+l = \mu-\lambda$. The last statement means that the 2-cocycle (4.1) is homogenous. Besides, we have $c_{0,j} = c_{1,j} = c_{2,j} = 0$.

Before starting with the proof proper, we explain our strategy. This method has already been used in [3]. First, we investigate operators that belong to $Z^2(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu})$. The 2-cocycle condition imposes conditions on the constants $c_{i,j}$: we get a linear system for $c_{i,j}$. Second, taking into account these conditions, we eliminate all constants underlying coboundaries. Gluing these bits of information together we deduce that $\dim \mathrm{H}^2$ is equal to the number of independent constants $c_{i,j}$ remaining in the expression of the 2-cocycle (4.1).

Proposition 2. ([11]) *There exist $\mathfrak{sl}(2)$-invariant bilinear differential operators $J_k^{\tau,\lambda}$: $\mathcal{F}_\tau \otimes \mathcal{F}_\lambda \to \mathcal{F}_{\tau+\lambda+k}$ given by:*

$$J_k^{\tau,\lambda}(\varphi\, dx^\tau, \phi\, dx^\lambda) = \sum_{i+j=k} \gamma_{i,j}\, \varphi^{(i)}\, \phi^{(j)}\, dx^{\tau+\lambda+k}, \qquad (4.2)$$

where the constants $\gamma_{i,j}$ satisfy

$$(i+1)(i+2\tau)\, \gamma_{i+1,j} + (j+1)(j+2\lambda)\, \gamma_{i,j+1} = 0. \qquad (4.3)$$

Remark 2. The operators (4.2) are called *transvectants*. Amazingly, they appear in many contexts, especially in the computation of cohomology (cf. [3, 5]). We refer to [18] for their history.

Now we will study properties of the coboundaries. Let B : Vect($\mathbb{R}$) $\to \mathcal{D}_{\lambda,\mu}$ be an operator defined by (for any $X = f\frac{d}{dx} \in$ Vect($\mathbb{R}$) and $\phi\, dx^\lambda \in \mathcal{F}_\lambda$):

$$B(X, \phi\, dx^\lambda) = \sum_{i+j=k+1} \gamma_{i,j}\, f^{(i)}\, \phi^{(j)}\, dx^{\lambda+k}.$$

Proposition 3. *Every coboundary $\delta(B) \in B^2(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu})$ possesses the following properties. The operator B coincides (up to a nonzero factor) with the transvectant $J_{k+1}^{-1,\lambda}$, where $\gamma_{0,k+1} = \gamma_{1,k} = \gamma_{2,k-1} = 0$. In addition (here $X = f\frac{d}{dx} \in$ Vect($\mathbb{R}$) and $\phi\, dx^\lambda \in \mathcal{F}_\lambda$),*

$$\delta(B)(X, Y, \phi\, dx^\lambda) = \sum_{i+j+l=k+2} \beta_{i,j}\, f^{(i)}\, g^{(j)}\, \phi^{(l)}\, dx^{\lambda+k}, \qquad (4.4)$$

where

$$\beta_{0,j} = \beta_{1,j} = \beta_{2,j} = 0,$$

and

$$\begin{aligned}\beta_{3,4} &= -\tfrac{1}{24}\tbinom{k-2}{3}\left(k^2+4(\lambda-1)\lambda+k(4\lambda-5)\right)(k-1+2\lambda)\gamma_{3,k-2}\\ \beta_{4,5} &= -\tfrac{1}{480}\tbinom{k-2}{5}(k-3+2\lambda)(k^3+4(\lambda-1)\lambda(2\lambda-19)+3k^2(2\lambda-7)+2k(49+6(\lambda-7)\lambda))\\ &\quad\times(k-1+2\lambda)\gamma_{3,k-3}.\end{aligned}$$

Proof. From the very definition of coboundaries, we have (for any $X, Y \in \mathrm{Vect}(\mathbb{R})$ and $\phi\, dx^\lambda \in \mathcal{F}_\lambda$):

$$\delta(B)(X,Y,\phi\, dx^\lambda) = B([X,Y],\phi\, dx^\lambda) - L_X\, B(Y,\phi\, dx^\lambda) + L_Y\, B(X,\phi\, dx^\lambda).$$

The coboundary above vanishes on the Lie algebra $\mathfrak{sl}(2)$. It means that if $X \in \mathfrak{sl}(2)$, we have

$$B([X,Y],\phi\, dx^\lambda) = L_X\, B(Y,\phi\, dx^\lambda) - L_Y\, B(X,\phi\, dx^\lambda).$$

Hence, the operator B is $\mathfrak{sl}(2)$-invariant; therefore it coincides with the transvectants. The conditions $\gamma_{0,k+1} = \gamma_{1,k} = \gamma_{2,k-1} = 0$ come from the fact that the operator B vanishes on $\mathfrak{sl}(2)$. Now, the conditions $\beta_{0,j} = \beta_{1,j} = \beta_{2,j} = 0$ are consequences of $\mathfrak{sl}(2)$-invariance, while the values of $\beta_{3,4}$ and $\beta_{4,5}$ follow by a direct computation. ■

4.1 The case where $\mu - \lambda = 5$

In this case, the 2-cocycle has the form

$$c(X,Y,\phi\, dx^\lambda) = \begin{vmatrix} f^{(3)} & g^{(3)} \\ f^{(4)} & g^{(4)} \end{vmatrix} \phi\, dx^{\lambda+5} \quad \text{for } X = f\frac{d}{dx}, Y = g\frac{d}{dx}. \tag{4.5}$$

The 2-cocycle condition is always satisfied. On the other hand, the coboundary (4.4) takes the form

$$\frac{1}{3}\lambda\;(2+\lambda)\;(4+\lambda)\;\gamma_{3,k-2}\;\left(g^{(3)}\,f^{(4)} - f^{(3)}\,g^{(4)}\right)\;\phi\, dx^{\lambda+5}.$$

This coboundary coincides with the 2-cocycle (4.5) except for $\lambda = 0, -2$ or -4. Therefore the cohomology in Theorem 1 is trivial except for $\lambda = 0, -2$ or -4.

4.2 The case where $\mu - \lambda = 6$

The 2-cocycle has the form

$$c(X,Y,\phi\, dx^\lambda) = \left(\begin{vmatrix} f^{(3)} & g^{(3)} \\ f^{(4)} & g^{(4)} \end{vmatrix}\phi' - \frac{\lambda}{5}\begin{vmatrix} f^{(3)} & g^{(3)} \\ f^{(5)} & g^{(5)} \end{vmatrix}\phi\right) dx^{\lambda+6} \quad \text{for } X = f\frac{d}{dx}, Y = g\frac{d}{dx}.$$

On the other hand, the coboundary (4.4) takes the form

$$\begin{aligned}&\tfrac{1}{3}\,(5+2\,\lambda)\;(3+2\,\lambda\;(5+\lambda))\;\gamma_{3,k-2}\;\left(g^{(3)}\,f^{(4)} - f^{(3)}\,g^{(4)}\right)\phi'\, dx^{\lambda+6}\\ &-\tfrac{1}{15}\lambda\;(5+2\,\lambda)\;(3+2\,\lambda\;(5+\lambda))\;\gamma_{3,k-2}\;\left(g^{(3)}\,f^{(5)} - f^{(3)}\,g^{(5)}\right)\phi\, dx^{\lambda+6}.\end{aligned}$$

This coboundary coincides with our 2-cocycle except when $\lambda = -\frac{5}{2}$ or λ is a solution to $3 + 2\lambda(5+\lambda) = 0$.

4.3 The case where $\mu - \lambda \geq 7$

In this case, the 2-cocycle condition is equivalent to the system (where $2 \leq \alpha < \beta < \gamma$):

$$\begin{aligned}
&\left(\binom{\alpha+\beta-1}{\alpha} - \binom{\alpha+\beta-1}{\alpha-1}\right) c_{\alpha+\beta-1,\gamma} - \left(\binom{\alpha+\gamma-1}{\alpha} - \binom{\alpha+\gamma-1}{\alpha-1}\right) c_{\alpha+\gamma-1,\beta} \\
&+ \left(\binom{\beta+\gamma-1}{\beta} - \binom{\beta+\gamma-1}{\beta-1}\right) c_{\beta+\gamma-1,\alpha} + \left(\binom{k+2-\beta-\gamma}{\alpha} + \lambda\binom{k+2-\beta-\gamma}{\alpha-1}\right) c_{\beta,\gamma} \\
&- \left(\binom{k+2-\alpha-\gamma}{\beta} + \lambda\binom{k+2-\alpha-\gamma}{\beta-1}\right) c_{\alpha,\gamma} + \left(\binom{k+2-\alpha-\beta}{\gamma} + \lambda\binom{k+2-\alpha-\beta}{\gamma-1}\right) c_{\alpha,\beta} = 0.
\end{aligned} \tag{4.6}$$

This system can be deduced by a simple computation. Of course, such a system has at least one solution in which the solutions $c_{i,j}$ are just the coefficients $\beta_{i,j}$ of the coboundaries (4.4).

4.3.1 The case where $\mu - \lambda = 7, 8, 9, 10, 11$

Let us show that the solutions to the system (4.6) are expressed in terms of $c_{3,4}$ and $c_{4,5}$.

In the case $\alpha = 2$, the system (4.6) has been studied in Section 3.1; its study corresponds to the investigation of $\mathfrak{sl}(2)$-invariant differential operators. We have seen that all the constants $c_{i,j}$ can be expressed in terms of $c_{3,4}, c_{5,4}, c_{7,4}, c_{9,4}, \ldots$.

For $k = 7$. According to Proposition 1, the space of solutions is generated by $c_{3,4}$ and $c_{4,5}$. Note that the coefficients $c_{4,i}$, where $i \geq 6$, are zero. The following coefficients can be deduced from the system (4.6):

$$c_{3,5} = \tfrac{1}{10}(5-k)(k-6+2\lambda)\, c_{3,4}, \quad c_{3,6} = \tfrac{1}{18}((6-k)(k-7+2\lambda)\, c_{3,5} - 4\, c_{4,5}). \tag{4.7}$$

For $k = 8$. According to Proposition 1, the space of solutions is generated by $c_{3,4}$ and $c_{4,5}$. Moreover, the coefficients $c_{4,i}$, where $i \geq 7$, are zero. The solutions to (4.6) are given by (4.7) together with

$$c_{3,7} = \tfrac{1}{28}((7-k)(k+2(\lambda-4))\, c_{3,6} - 4\, c_{4,6}), \quad c_{4,6} = \tfrac{1}{18}(k-7)(k-8+2\lambda)\, c_{4,5}. \tag{4.8}$$

Now for $k = 9, 10$ and 11 we have to deal with the system (4.6) for $\alpha = 3$:

$$\begin{aligned}
&\left(\binom{\beta+2}{3} - \binom{\beta+2}{2}\right) c_{\beta+2,\gamma} - \left(\binom{\gamma+2}{3} - \binom{\gamma+2}{2}\right) c_{\gamma+2,\beta} + \left(\binom{\gamma+\beta-1}{\beta} - \binom{\gamma+\beta-1}{\beta-1}\right) c_{\gamma+\beta-1,3} \\
&+ \left(\binom{k+2-\beta-\gamma}{3} + \lambda\binom{k+2-\beta-\gamma}{2}\right) c_{\beta,\gamma} - \left(\binom{k-1-\gamma}{\beta} + \lambda\binom{k-1-\gamma}{\beta-1}\right) c_{3,\gamma} \\
&+ \left(\binom{k-1-\beta}{\gamma} + \lambda\binom{k-1-\beta}{2}\right) c_{3,\beta} = 0.
\end{aligned}$$

For $\beta = 4$ and $\gamma = 5$, the coefficient $c_{4,7}$ is given by

$$\begin{aligned}
c_{4,7} = \tfrac{1}{105840}\binom{k-7}{2} \quad &\Big(\binom{k-5}{2}(2\lambda+k-3)(-288+k(194+k(k-27))+268\lambda+6(k-18)k\lambda \\
&+12(k-9)\lambda^2+8\lambda^3)\, c_{3,4} - 80 c_{4,5}(279+2k^2+\lambda(8\lambda-113)+k(8\lambda-49))\Big).
\end{aligned} \tag{4.9}$$

We continue like this until we determine all the constants $c_{4,k-3}$ for k even and $c_{4,k-2}$ for k is odd. Therefore the system (4.6) admits solutions generated by $c_{3,4}$ and $c_{4,5}$. Let us give explicitly these solutions.

For $k = 9$**.** The coefficients are given by (4.7), (4.8), (4.9) together with

$$c_{3,8} = \tfrac{1}{40}((8-k)(k-9+2\lambda)\,c_{3,7} - 4\,c_{4,7}), \quad c_{5,6} = \tfrac{1}{45}\binom{k-8}{2}\binom{k+2\lambda-7}{2}c_{3,4} - \tfrac{14}{5}c_{4,7}. \quad (4.10)$$

For $k = 10$**.** The coefficients are given by (4.7), (4.8), (4.9), (4.10) together with

$$c_{3,9} = \tfrac{1}{54}((9-k)(k+2\lambda-10)\,c_{3,8} - 4\,c_{4,8}), \quad c_{5,7} = \tfrac{1}{10}(9-k)(k-10+2\lambda)\,c_{4,7} - 4\,c_{4,8}, \quad (4.11)$$

and

$$\begin{aligned} c_{4,8} &= \tfrac{1}{20160}(9-k)(k+2(\lambda-5))\times \\ &\quad ((k-8)(k-7)(k+2(\lambda-4))(k-9+2\lambda)\,c_{3,4} + 10008\,c_{4,7})\,. \end{aligned} \quad (4.12)$$

For $k = 11$. The coefficients are given by (4.7), (4.8), (4.9), (4.10), (4.11), (4.12) together with

$$c_{3,10} = \tfrac{1}{70}((10-k)(k-11+2\lambda)\,c_{3,9} - 4\,c_{4,9}), \quad c_{5,8} = \tfrac{1}{10}(10-k)(k-11+2\lambda)\,c_{4,8} - \tfrac{27}{5}\,c_{4,9},$$

and

$$\begin{aligned} c_{6,7} &= \tfrac{1}{45360}\,((k-10)\,(k-9)\,(k+2\,(\lambda-5))\,(k+2\,\lambda-11)\times \\ &\quad ((k-8)\,(k-7)\,(k+2\,\lambda-8)\,(k+2\,\lambda-9)\,c_{3,4} + 756\,c_{7,4})) + 12\,c_{4,9}. \end{aligned} \quad (4.13)$$

The explicit value of $c_{4,9}$ is too long; hereafter we omit such expressions obtained with the help of *Mathematica*.

We have just proved that the coefficients of every 2-cocycle is expressed in terms of the two constants $c_{3,4}$ and $c_{4,5}$. But this general formula may contain coboundaries. We explain how the coboundaries can be removed. Consider any coboundary given as in (4.4). We discuss the following cases:

1) $\lambda = \frac{1-k}{2}$. Then the constant $\beta_{3,4}$ and $\beta_{4,5}$ vanish simultaneously. Hence the constants $c_{4,5}$ and $c_{3,4}$ cannot be eliminated by adding the coboundary (4.4). It follows that the coefficients of the 2-cocycle are generated by $c_{3,4}$ and $c_{4,5}$. Therefore the cohomology is two-dimensional. The 2-cocycles are given explicitly by the constants (4.7), (4.8), (4.9), (4.10), (4.11), (4.12) by taking $c_{3,4} = 1$ and $c_{4,5} = 0$ then by taking $c_{3,4} = 0$ and $c_{4,5} = 1$.

2) $\lambda = \frac{1}{2}(1 - k \pm \sqrt{1+3k})$. Then the constant $c_{4,5}$ can be eliminated by adding the coboundary (4.4). On the other hand, the constant $c_{3,4}$ cannot be eliminated because $\beta_{3,4} = 0$. It follows that the coefficients of the 2-cocycle are generated by $c_{3,4}$. Therefore the cohomology is one-dimensional. The 2-cocycle is given explicitly by the constants (4.7), (4.8), (4.9), (4.10), (4.11), (4.12) upon taking $c_{3,4} = 1$ and $c_{4,5} = 0$.

3) $\lambda = \frac{3-k}{2}$. First, we observe that there is no common solutions for λ in 2) and 3) except for $\lambda = 1$ and $k = 1$; or $\lambda = -1$ and $k = 1$. But these cases are not taken into consideration because $k \geq 7$. The constant $c_{3,4}$ can be eliminated by adding the coboundary (4.4). On the other hand, the constant $c_{4,5}$ cannot be eliminated because $\beta_{4,5} = 0$. It follows that the coefficients of the 2-cocycle are generated by $c_{4,5}$. Therefore the cohomology is one-dimensional. The 2-cocycle is given by the constants (4.7), (4.8), (4.9), (4.10), (4.11), (4.12) upon taking $c_{3,4} = 0$ and $c_{4,5} = 1$.

4) λ is a solution to the equation

$$k^3 + 4(\lambda-1)\lambda(2\lambda-19) + 3k^2(2\lambda-7) + 2k(49+6(\lambda-7)\lambda) = 0.$$

In this case, $c_{3,4}$ can be eliminated by adding the coboundary (4.4). On the other hand, the constant $c_{4,5}$ cannot be eliminated as $\beta_{4,5} = 0$. It follows that the coefficients of the 2-cocycle are generated by $c_{4,5}$. Therefore the cohomology is one-dimensional. The coefficients of the 2-cocycle are given by constants as above upon taking $c_{3,4} = 0$ and $c_{4,5} = 1$.

5) λ is not like 1)–4). In this case, whatever the weight λ is, one of the constant $c_{3,4}$ or $c_{4,5}$ can be eliminated by adding the coboundary. It follows that the cohomology is one-dimensional. The coefficients of the 2-cocycle are given by the constants as above upon taking, for instance, $c_{3,4} = 1$ and $c_{4,5} = 0$.

4.3.2 The case where $\mu - \lambda = 12, 13, 14$

Let us prove that the system (4.6) has solutions that can be expressed in terms of one parameter if λ is generic, and in terms of two parameters for particular values of λ. But we have already seen in the previous section that all the solutions can be expressed in terms of $c_{3,4}$ and $c_{3,5}$. As $k \geq 12$, we are required to study (4.6) for $\alpha = 4$. For $\alpha = 4, \beta = 5$ and $\gamma = 6$, the system has one more equation

$$\begin{aligned}
&\binom{k-7}{5} (2\lambda + k - 1) \left(k^2 + 2\, k\, (2\,\lambda - 7) + 4\, (6 + (-1 + \lambda)\, \lambda)\right) \times \\
&[(k-5)\, k\, ((k-14)\, (k-7)\, (k-6)\, (k-3)\, c_{3,4} - 400\, c_{4,5}) \\
&+4\, ((k-6)\, (k-5)\, (-57 + k\, (131 + 2\, (-18 + k)\, k))\, c_{3,4} - 400\, (k-1)\, c_{4,5})\, \lambda \\
&+4\, ((k-6)\, (k-5)\, (101 + 6\, (k-12)\, k)\, c_{3,4} - 400\, c_{4,5})\, \lambda^2 \\
&+32\, (k-6)^2\, (k-5)\, c_{3,4}\, \lambda^3 + 16\, (k-6)\, (k-5)\, c_{3,4}\, \lambda^4\Big] = 0.
\end{aligned} \tag{4.14}$$

We have three cases:

1) If $\lambda \neq \frac{1}{2}(1-k)$ or $\frac{1}{2}(1 - k \pm \sqrt{12k - 23})$, then from Eq. (4.14) the constant $c_{4,5}$ can be expressed in terms of $c_{3,4}$. Here we have two subcases:

1.1) If $\lambda = \frac{1}{2}(1 - k \pm \sqrt{1 + 3k})$, then Eq. (4.14) implies that $c_{3,4} = 0$. The constant $c_{4,5}$ can be eliminated by adding the coboundary (4.4) for a suitable $\gamma_{3,k-2}$. Therefore the cohomology is zero.

1.2) If $\lambda \neq \frac{1}{2}(1 - k \pm \sqrt{1 + 3k})$, then Eq. (4.14) implies that $c_{4,5}$ can be determined in terms of $c_{3,4}$. We omit here the explicit expression because it is too long.

The constant $c_{3,4}$ can be eliminated upon adding the coboundary (4.4) for a suitable $\gamma_{3,k-2}$. Therefore the cohomology is zero.

2) If $\lambda = \frac{1}{2}(1 - k \pm \sqrt{12k - 23})$, then the system (4.6) has solutions that still depend on $c_{3,4}$ and $c_{4,5}$. Now, the coboundary (4.4) can be added in order to eliminate the constant

$c_{3,4}$. The constants are as follows:

$$
\begin{array}{lll lll}
c_{3,11} & = & \frac{1}{88}((11-k)(k+2\lambda-12)\,c_{3,10}-4\,c_{4,10}), & c_{3,4} & = & 0,\\
c_{3,13} & = & \frac{1}{130}((13-k)(k+2\lambda-14)\,c_{3,12}-4\,c_{4,12}), & c_{5,9} & = & \frac{1}{10}(k-11)(k+2\lambda-12)\,c_{9,4}\\
 & & & & & +7\,c_{10,4},\\
c_{5,10} & = & \frac{1}{10}(k-12)(k+2\lambda-13)\,c_{10,4}+\frac{44}{5}\,c_{11,4}, & c_{5,11} & = & \frac{1}{10}(k-13)(k+2\lambda-14)\,c_{11,4}\\
 & & & & & +\frac{54}{5}\,c_{12,4},\\
c_{6,8} & = & \frac{1}{18}(63\,c_{9,5}-(k-12)(k-11+2\,\lambda)\,c_{5,8}), & c_{6,9} & = & \frac{1}{18}((-k+13)(k-12+2\,\lambda)\,c_{5,9})\\
 & & & & & +\frac{40}{9}\,c_{10,5},\\
c_{6,10} & = & \frac{1}{18}(99\,c_{11,5}-(k-14)(k-13+2\,\lambda)\,c_{5,10}), & c_{7,8} & = & \frac{1}{14}(35\,c_{10,6}-(k-13)\,c_{6,9}),\\
c_{7,9} & = & \frac{1}{28}(54\,c_{9,6}-(k-12)(k-11+2\,\lambda)\,c_{6,8}), & c_{11,4} & = & 0,\\
c_{3,12} & = & \frac{1}{108}((12-k)(k+2\lambda-13)\,c_{3,11}-4\,c_{4,11}), & c_{4,5} & = & 1.
\end{array}
$$

Here we omit the expressions of $c_{10,4}$ and $c_{12,4}$ as they are too long.

3) If $\lambda=\frac{1}{2}(1-k)$, then the cohomology is two-dimensional.

4.3.3 The case where $\mu-\lambda\geq 15$

Let us prove that the system (4.6) has solutions that depend on one parameter for all λ. We have seen in the previous section that the solutions to the system (4.6) depend on one parameter if λ is generic and on two parameters if $\lambda=\frac{1}{2}(1-k)$ or $\frac{1}{2}(1-k\pm\sqrt{12k-23})$. But here $k\geq 15$; we have to study (4.6) for $\alpha=5$. For $\alpha=5, \beta=6$ and $\gamma=7$, the system (4.6) has one more equation

$$
\begin{aligned}
&\left(\tbinom{10}{5}-\tbinom{10}{4}\right)\,c_{10,7}-\left(\tbinom{11}{5}-\tbinom{11}{4}\right)\,c_{11,6}+\left(\tbinom{12}{6}-\tbinom{12}{5}\right)\,c_{12,5}+\left(\tbinom{k-11}{5}+\lambda\tbinom{k-11}{4}\right)\,c_{6,7}\\
&-\left(\tbinom{k-10}{6}+\lambda\tbinom{k-10}{5}\right)\,c_{5,7}+\left(\tbinom{k-9}{7}+\lambda\tbinom{k-9}{6}\right)\,c_{5,6}=0.
\end{aligned}
\tag{4.15}
$$

1) For $\lambda=\frac{1}{2}(1-k)$, Eq. (4.15) implies that the constant $c_{4,5}$ is expressed in terms of $c_{3,4}$. Once more we omits its explicit expression. If $k=15$, then $c_{3,4}$ generates the system and consequently the cohomology is one-dimensional since $\beta_{3,4}=\beta_{4,5}=0$. If $k>15$, then the system (4.6) adds another condition that implies $c_{3,4}=0$. Therefore the cohomology is zero.

2) For $\lambda=\frac{1}{2}(1-k\pm\sqrt{12k-23})$, we proceed as before. The cohomology is zero.

■

Remark 3. The study of $\mathfrak{sl}(2)$-invariant differential operators over polynomial vector fields on $\mathbb{R}$, $\mathrm{Vect}_{\mathrm{P}}(\mathbb{R})$, or over smooth vector fields on the circle, $\mathrm{Vect}(\mathbb{S}^1)$, (in the case of $\mathbb{S}^1$ we express such operators in an affine coordinate) is identical with the study of $\mathfrak{sl}(2)$-invariant differential operators over $\mathrm{Vect}(\mathbb{R})$. Therefore, Theorem 1 remains true whether for $\mathrm{Vect}(\mathbb{S}^1)$ or $\mathrm{Vect}_{\mathrm{P}}(\mathbb{R})$ since its proof is based on the classification of $\mathfrak{sl}(2)$-invariant differential operators.

5 Explicit 2-cocycles for $\mathrm{Vect}(\mathbb{R})$ and $\mathfrak{sl}(2)$

The following cohomology was computed by Lecomte [12]:

$$\mathrm{H}^2(\mathfrak{sl}(2);\mathcal{D}_{\lambda,\mu}) = \begin{cases} \mathbb{R} & \text{if } (\lambda,\mu)=(\frac{1-k}{2},\frac{1+k}{2}), \text{ and } k\in\mathbb{N}\backslash\{0\}, \\ 0 & \text{otherwise.}\end{cases}$$

The 2-cocycle that spans this cohomology is given by (here ω is the Gelfand-Fuchs cocycle (2.1)):

$$\Omega(X,Y,\phi\,dx^\lambda) = \omega(X,Y)\,\phi^{(k-1)}\,dx^{\frac{1+k}{2}}.$$

The following cohomology can be deduced from the work of Feigin-Fuchs [7] (where $\mathrm{Vect_P}(\mathbb{R})$ is the Lie algebra of polynomial vector fields) :

$$\mathrm{H}^2(\mathrm{Vect_P}(\mathbb{R});\mathcal{D}_{\lambda,\mu}) = \begin{cases} \mathbb{R} & \text{if } \begin{cases} (\mu,\lambda)=(1,0), \\ \mu-\lambda = 2,3,4 \text{ for all } \lambda, \\ \mu-\lambda=7,8,9,10,11 \text{ for all } \lambda, \\ \mu-\lambda=k=12,13,14 \text{ but } \lambda \text{ is either } \frac{1-k}{2}, \\ \text{or } \frac{1-k}{2}\pm\frac{\sqrt{12k-23}}{2}, \end{cases} \\ \mathbb{R}^2 & \text{if } \begin{cases} (\lambda,\mu)=(0,5) \text{ or } (-4,1), \\ (\lambda,\mu)=\left(-\frac{5}{2}\pm\frac{\sqrt{19}}{2},\frac{7}{2}\pm\frac{\sqrt{19}}{2}\right), \end{cases} \\ 0 & \text{otherwise.}\end{cases} \tag{5.1}$$

The 2-cocycles spanning (5.1) for $k=1,2,3,4,5$ and 6 are as follows (here $X=f\frac{d}{dx}$, $Y=g\frac{d}{dx}$):

(i) For $(\lambda,\mu)=(0,1)$, the 2-cocycle is given by

$$\Omega_1(X,Y,\phi\,dx^\lambda) = \omega(X,Y)\,\phi\,dx^\lambda. \tag{5.2}$$

(ii) For $\mu-\lambda=2$, the 2-cocycle is given by

$$\Omega_2(X,Y) = c_{1,2}\,\omega(X,Y)\,\frac{d}{dx} + c_{1,3}\begin{vmatrix} f' & g' \\ f''' & g''' \end{vmatrix}, \tag{5.3}$$

where $c_{1,2}=1$ and $c_{1,3}=0$ for $\lambda=-\frac{1}{2}$; whereas $c_{1,2}=0$ and $c_{1,3}=1$ for $\lambda\neq-\frac{1}{2}$.

(iii) For $\mu-\lambda=3$, the 2-cocycle is given by

$$\Omega_3(X,Y) = c_{1,2}\,\omega(X,Y)\,\frac{d^2}{dx^2} + c_{1,3}\begin{vmatrix} f' & g' \\ f''' & g''' \end{vmatrix}\frac{d}{dx} + \frac{\lambda}{2}(c_{1,2}-c_{1,3})\begin{vmatrix} f' & g' \\ f^{(4)} & g^{(4)} \end{vmatrix}, \tag{5.4}$$

where $c_{1,2}=1$ and $c_{1,3}=0$ for $\lambda=-1$; whereas $c_{1,2}=0$ and $c_{1,3}=1$ for $\lambda\neq-1$.

(iv) For $\mu-\lambda=4$, the 2-cocycle is given by

$$\begin{aligned}\Omega_4(X,Y) &= c_{1,2}\,\omega(X,Y)\,\tfrac{d^3}{dx^3} + \tfrac{1}{2}((1+2\lambda)c_{1,3}-(1+3\lambda)c_{1,2})\begin{vmatrix} f' & g' \\ f^{(4)} & g^{(4)} \end{vmatrix}\tfrac{d}{dx} \\ &+c_{1,3}\begin{vmatrix} f' & g' \\ f''' & g''' \end{vmatrix}\tfrac{d^2}{dx^2} + \tfrac{\lambda}{10}((1-3\lambda)c_{1,2}+(1+2\lambda)c_{1,3})\begin{vmatrix} f' & g' \\ f^{(5)} & g^{(5)} \end{vmatrix},\end{aligned} \tag{5.5}$$

where $c_{1,3} = 0$ and $c_{1,2} = 1$ for $\lambda = -\frac{3}{2}$; whereas $c_{1,3} = 1$ and $c_{1,2} = 0$ for $\lambda \neq -\frac{3}{2}$.

(v) For $\mu - \lambda = 5$, the two 2-cocycles are given by (where α and β are constants):

$$\begin{aligned} \Omega_5(X,Y) &= 3\alpha(1+\lambda)(1+2\lambda)\,\omega(X,Y)\,\tfrac{d^4}{dx^4} + 2\alpha(1+3\lambda+6\lambda^2)\begin{vmatrix} f' & g' \\ f^{(3)} & g^{(3)} \end{vmatrix}\tfrac{d^3}{dx^3} \\ &+3\alpha(1+\lambda)(1+4\lambda)\begin{vmatrix} f' & g' \\ f^{(4)} & g^{(4)} \end{vmatrix}\tfrac{d^2}{dx^2} - \tfrac{1}{5}\alpha\lambda(1+9\lambda)\begin{vmatrix} f' & g' \\ f^{(6)} & g^{(6)} \end{vmatrix} \\ &+\beta\begin{vmatrix} f''' & g''' \\ f^{(4)} & g^{(4)} \end{vmatrix}. \end{aligned} \tag{5.6}$$

(vi) For $\mu - \lambda = 6$, the two 2-cocycles are given by (where α and β are constants):

$$\begin{aligned} \Omega_6(X,Y) &= \alpha(4+3\lambda(5+2\lambda))\,\omega(X,Y)\,\tfrac{d^5}{dx^5} + 5\alpha(2+\lambda(4+3\lambda))\begin{vmatrix} f' & g' \\ f^{(3)} & g^{(3)} \end{vmatrix}\tfrac{d^4}{dx^4} \\ &+5\alpha(\lambda(3+4\lambda)-2)\begin{vmatrix} f' & g' \\ f^{(4)} & g^{(4)} \end{vmatrix}\tfrac{d^3}{dx^3} + 5\alpha(2+\lambda(4+3\lambda))\begin{vmatrix} f' & g' \\ f^{(5)} & g^{(5)} \end{vmatrix}\tfrac{d^2}{dx^2} \\ &+\beta\begin{vmatrix} f^{(3)} & g^{(3)} \\ f^{(4)} & g^{(4)} \end{vmatrix}\tfrac{d}{dx} + \alpha(4+15\lambda+6\lambda^2)\begin{vmatrix} f' & g' \\ f^{(6)} & g^{(6)} \end{vmatrix}\tfrac{d}{dx} - \tfrac{\lambda}{5}\beta\begin{vmatrix} f^{(3)} & g^{(3)} \\ f^{(5)} & g^{(5)} \end{vmatrix}. \end{aligned} \tag{5.7}$$

In order to complete the list of 2-cocycles spanning (5.1) we need the following two Lemmas.

Lemma 2. *Every 2-cocycle in* $\mathrm{H}^2(\mathrm{Vect_P}(\mathbb{R}); \mathcal{D}_{\lambda,\mu})$ *can be reduced to a 2-cocycle vanishing on* $\mathfrak{sl}(2)$, *except those given in* (5.2) – (5.7).

Proof. Consider a general form of a 2-cocycle (where $X = f\frac{d}{dx}, Y = g\frac{d}{dx} \in \mathrm{Vect_P}(\mathbb{R})$ and $\phi\, dx^\lambda \in \mathcal{F}_\lambda$):

$$c(X,Y,\phi\, dx^\lambda) = \sum_{i+j+l=k+2} c_{i,j}\, f^{(i)}\, g^{(j)}\, \phi^{(l)} dx^{\lambda+k}. \tag{5.8}$$

We will eliminate coboundaries in order to turn the 2-cocycle above into a 2-cocycle vanishing on $\mathfrak{sl}(2)$. Consider a general expression of a coboundary

$$\begin{aligned} \delta B(X,Y,\phi\, dx^\lambda) &= -\beta_0\, fg'\,\phi^{(k+1)} - \beta_0(\tbinom{k+1}{\alpha} + \lambda\tbinom{k+1}{\alpha-1})\, fg^{(\alpha)}\,\phi^{(k+2-\alpha)} \\ &\quad -\textstyle\sum_{\alpha\geq 2}\beta_1(\tbinom{k}{\alpha} + \lambda\tbinom{k}{\alpha-1})f'g^{(\alpha)}\,\phi^{(k+1-\alpha)} + \text{higher order terms} \\ &\quad -(f \leftrightarrow g). \end{aligned}$$

Immediately we see that the constant $c_{0,1}$ can be eliminated upon putting $c_{0,1} = -\beta_0$. On the other hand, the 2-cocycle condition implies that $c_{\gamma,0} = -c_{0,1}\,(\binom{k+1}{\gamma} + \lambda\,\binom{k+1}{\gamma-1})$.

1) For $k = 1$, the 2-cocycle takes the form

$$\Omega_1(X,Y,\phi) = c_{1,2}\,\omega(X,Y)\,\phi.$$

On the other hand, the coboundary takes the form

$$\delta B(X,Y,\phi) = \lambda\,\alpha_1\,\omega(X,Y)\,\phi,$$

where α_1 is a constant. The 2-cocycle is trivial except for $\lambda = 0$.

2) For $k=2,3,4,5,6$, we proceed as before.

Suppose now that $k>6$. We will deal with the coefficients $c_{1,\gamma}$. The 2-cocycle condition implies that the component of $f'\, g^\beta\, h^\gamma\, \phi^{k+2-\beta-\gamma}$, which should be zero, is equal to

$$\begin{array}{l} c_{\beta+\gamma-1,1}\left(\binom{\beta+\gamma-1}{\beta}-\binom{\beta+\gamma-1}{\beta-1}\right)-c_{1,\gamma}\left(\binom{k+1-\gamma}{\beta}+\lambda\binom{k+1-\gamma}{\beta-1}\right)\\ +c_{1,\beta}\left(\binom{k+1-\beta}{\gamma}+\lambda\binom{k+1-\beta}{\gamma-1}\right)=0. \end{array} \tag{5.9}$$

We have two cases:

i) For $\lambda=\frac{1-k}{2}$. In this case, the coefficient of $f'g''\,\phi^{k-1}$ is zero in the expression of the coboundary. But $c_{1,3}$ can be eliminated upon putting $c_{1,3}=\frac{1}{6}\,k(k-1)\,(k-2+3\lambda)\,\beta_1$. By putting $\beta=2$, we can see from (5.9) that all $c_{t,1}$ can be expressed in terms of $c_{1,2}$. They are given by the induction formula:

$$c_{1,i}=\tfrac{2}{i-3}\left(-c_{1,i-1}\left(\binom{k+2-i}{2}+\lambda\binom{k+2-i}{1}\right)+c_{1,2}(\binom{k-1}{i-1}+\lambda\binom{k-1}{i-2})\right)\quad \text{for } i>3. \tag{5.10}$$

However, for $\beta=3$ and $\gamma=4$ the system (5.9) becomes

$$\binom{k-1}{4}(1+k)(1+3k)\,c_{1,2}=0.$$

As $k>4$, the equation above admits a solution only for $c_{1,2}=0$. Thus, all $c_{1,\gamma}$ are zero.

ii) If $\lambda\neq\frac{1-k}{2}$, then the constant $c_{1,2}$ can be eliminated and we proceed as before.

Now we deal with the coefficients $c_{2,s}$. These coefficients can be eliminated upon taking

$$\begin{array}{rcl} \beta_{s+1,k-s-1} &=& \frac{1}{(s+1)(s-2)}\,(c_{2,s}+(k-s)(k-s-1+2\,\lambda)\,\beta_{s,k-s})\\ && +\frac{1}{(s+1)(s-2)}\left(-2\,\left(\binom{k-2}{s}+\lambda\binom{k-2}{s-1}\right)\,\beta_{2,k-2}\right). \end{array}$$

Finally, the remaining 2-cocycle vanishes on $\mathfrak{sl}(2)$. ■

Lemma 3. *Every coboundary $\delta(B)\in B^2(\mathrm{Vect}(\mathbb{R});\mathcal{D}_{\lambda,\mu})$ vanishing on $\mathfrak{sl}(2)$ possesses the following properties. The operator B coincides (up to a nonzero factor) with the transvectant $J^{-1,\lambda}_{k+1}$, where $\gamma_{0,k+1}=\gamma_{1,k}=0$. In addition (here $X=f\frac{d}{dx}, Y=g\frac{d}{dx}\in\mathrm{Vect}(\mathbb{R})$ and $\phi\,dx^\lambda\in\mathcal{F}_\lambda$)*

$$\delta(B)(X,Y,\phi\,dx^\lambda)=\sum_{i+j+l=k+2}\beta_{i,j}\,f^{(i)}\,g^{(j)}\,\phi^{(l)}\,dx^{\lambda+k}, \tag{5.11}$$

where

$$\beta_{0,j}=\beta_{1,j}=\beta_{2,j}=0,$$

and

$$\begin{array}{rcl} \beta_{3,4} &=& \frac{1}{24}\binom{k-2}{3}\left(k^2+4(\lambda-1)\lambda+k(4\lambda-5)\right)((k-1)(k-2+3\lambda)\gamma_{2,k-1}-(k-1+2\lambda)\gamma_{3,k-2})\\ \beta_{4,5} &=& -\frac{1}{480}\binom{k-2}{5}(k-3+2\lambda)(k^3+4(\lambda-1)\lambda(2\lambda-19)+3k^2(2\lambda-7)+2k(49+6(\lambda-7)\lambda))\\ && \times((k-1)(k-2+3\lambda)\gamma_{2,k-1}-(k-1+2\lambda)\gamma_{3,k-3}). \end{array}$$

Proof. Similar to Proposition 3. ■

Now we will explain how we can deduce the explicit expressions of the 2-cocycles that span $\mathrm{H}^2(\mathrm{Vect_P}(\mathbb{R});\mathcal{D}_{\lambda,\mu})$ by using the results of Sec. 4.3. To save space, we give details of the computation only for $\mu-\lambda=7,8,9,10,11$. The other cases, namely $\mu-\lambda=12,13,14$, can be deduced by the same way. We start with any 2-cocycle $c\in Z^2(\mathrm{Vect_P}(\mathbb{R});\mathcal{D}_{\lambda,\mu})$ vanishing on $\mathfrak{sl}(2)$. This is actually possible, thanks to Lemma 2. The 2-cocycle condition of c has already been studied Sec. 4.3.1. The 2-cocycle c is generated by the two constants $c_{3,4}$ and $c_{4,5}$. We have the following cases:

1) $\lambda=\frac{1-k}{2}$. By Lemma 3, one of the constants $c_{3,4}$ or $c_{4,5}$ can be eliminated by adding a coboundary with an appropriate value of $\gamma_{2,k-2}$. We obtain, therefore, a unique 2-cocycle that is non-trivial in $\mathrm{H}^2(\mathrm{Vect_P}(\mathbb{R});\mathcal{D}_{\lambda,\mu})$.

2) $\lambda=\frac{2-k}{3}$. By Lemma 3, one of the constants $c_{3,4}$ or $c_{4,5}$ can be eliminated by adding a coboundary with an appropriate value of $\gamma_{3,k-3}$. We obtain, therefore, a unique 2-cocycle that is non-trivial in $\mathrm{H}^2(\mathrm{Vect_P}(\mathbb{R});\mathcal{D}_{\lambda,\mu})$.

3) λ is a solution to the equation

$$k^2+4(\lambda-1)\lambda+k(4\lambda-5)=0.$$

Then $\beta_{3,4}=0$. Therefore the constant $c_{4,5}$ can be eliminated with an appropriate value of $\gamma_{2,k-1}$. We obtain, therefore, a unique 2-cocycle that is non-trivial in $\mathrm{H}^2(\mathrm{Vect_P}(\mathbb{R});\mathcal{D}_{\lambda,\mu})$.

4) λ is a solution to the equation

$$(k-3+2\lambda)(k^3+4(\lambda-1)\lambda(2\lambda-19)+3k^2(2\lambda-7)+2k(49+6(\lambda-7)\lambda))=0.$$

Then $\beta_{4,5}=0$. Therefore the constant $c_{3,4}$ can be eliminated with an appropriate value of $\gamma_{2,k-1}$. We obtain, therefore, a unique 2-cocycle that is non-trivial in $\mathrm{H}^2(\mathrm{Vect_P}(\mathbb{R});\mathcal{D}_{\lambda,\mu})$.

5) If λ is not as in 1)–4). Whatever the value of λ is the constant $c_{3,4}$ can be eliminated with an appropriate value of $\gamma_{2,k-1}$. We obtain, therefore, a unique 2-cocycle that is non-trivial in $\mathrm{H}^2(\mathrm{Vect_P}(\mathbb{R});\mathcal{D}_{\lambda,\mu})$.

5.1 Further remarks

It would be interesting to study the cohomology arising in the deformation of symbols at the group level, $\mathrm{Diff}(\mathbb{R})$. We do not know whether our 2-cocycles introduced here can be integrated to the group. Nevertheless, the 2-cocycle (5.2) can be integrated to a 2-cocycle $A\in\mathrm{H}^2(\mathrm{Diff}(\mathbb{R});\mathcal{D}_{\lambda,\lambda+1})$ (here $F,G\in\mathrm{Diff}(\mathbb{R})$ and $\phi\,dx^\lambda\in\mathcal{F}_\lambda$):

$$A(F,G,\phi\,dx^\lambda):=\log(F\circ G)'\,\frac{G''}{G'}\,\phi\,dx^{\lambda+1}.$$

This 2-cocycle is just the multiplication operator by the well-know Bott-Thurston cocycle [2]. Let $S(f):=\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2$ be the Schwarz derivative. Then the 2-cocycle (4.5) can be integrated to $B\in\mathrm{H}^2(\mathrm{Diff}(\mathbb{R}),\mathrm{PSL}(2,\mathbb{R});\mathcal{D}_{\lambda,\lambda+5})$:

$$B(F,G,\phi\,dx^\lambda):=\begin{vmatrix} G^*S(F) & S(F)\\ G^*S(F)' & S(F)'\end{vmatrix}\,\phi\,dx^{\lambda+5}.$$

This 2-cocycle is also the multiplication operator by a 2-cocycle introduced by Ovsienko-Roger [17].

It would also be interesting to study the cohomology arising in the context of deformation of the space of symbols on multi-dimensional manifolds.

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