Journal of Nonlinear Mathematical Physics Volume 14, Number 1 (2007), 82–94 Article

A note on the relationship between rational and trigonometric solutions of the WDVV equations

Andrew RILEY [a] *and Ian A B STRACHAN* [b]

[a] *Department of Mathematics, University of Hull, Hull HU6 7RX, U.K.*
E-mail: a.riley@maths.hull.ac.uk

[b] *Department of Mathematics, University of Glasgow, Glasgow G12 8QQ, U.K.*
E-mail: i.strachan@maths.gla.ac.uk

Received May 5, 2006; Accepted in Revised Form August 22, 2006

Abstract

Legendre transformations provide a natural symmetry on the space of solutions to the WDVV equations, and more specifically, between different Frobenius manifolds. In this paper a twisted Legendre transformation is constructed between solutions which define the corresponding dual Frobenius manifolds. As an application it is shown that certain trigonometric and rational solutions of the WDVV equations are related by such a twisted Legendre transform.

1 Introduction

The Witten-Dijkgraaf-Verlinde-Verlinde (or WDVV) equations of associativity

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\gamma \partial t^\delta} - \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\gamma \partial t^\alpha} = 0\,, \quad \alpha\,,\beta\,,\gamma\,,\delta = 1\,\ldots\,,n$$

have been much studied from a variety of different points of view, amongst them, topological quantum field theories, Seiberg-Witten theory, singularity theory and integrable systems. Here η is a flat metric and the coordinates $\{t^\alpha\}$ are the corresponding flat coordinates, i.e. the components of the metric in this coordinate system are constants[1] and the solution F is known as the prepotential. In what follows it will be convenient to denote this metric as $<,>$.

Geometrically, a solution defines a multiplication $\circ : TM \times TM \rightarrow TM$ of vector fields, i.e.

$$\begin{aligned} \partial_{t^\alpha} \circ \partial_{t^\beta} &= \left(\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\sigma} \eta^{\sigma\gamma} \right) \partial_{t^\gamma}\,, \\ &:= c^\gamma_{\alpha\beta}(t)\,\partial_{t^\gamma}\,, \end{aligned}$$

[1]These are, of course, only defined up to linear transformations.

where the metric η is used to raise and lower indices. The total symmetry of the third derivatives implies that

$$< X \circ Y, Z > = < X, Y \circ Z >$$

for all vector fields X , Y , Z . In applications there are often additional structures which tie together the metric and the multiplication. For example, the existence of a unity vector field e implies that

$$\begin{aligned} < X, Y > &= < X \circ e, Y > , \\ &= < e, X \circ Y > . \end{aligned}$$

If, in addition, the unity field is covariantly constant, one may, without loss of generality, fix

$$e = \frac{\partial}{\partial t^1} ,$$

so that

$$\eta_{ij} = c_{1ij} .$$

This then fixes the t^1-dependence of the prepotential F in terms of the components of the metric η .

Symmetries map solutions of equations to other solutions of the same equations, and the WDVV equations have a particularly rich symmetry structure [1]. More specifically, a symmetry of the WDVV equations consists of transformations

$$\begin{aligned} t^\alpha &\longmapsto \hat{t}^\alpha , \\ \eta_{\alpha\beta} &\longmapsto \hat{\eta}_{\alpha\beta} , \\ F &\longmapsto \hat{F} \end{aligned}$$

that preserve the equations. These were studied in [1] where, in particular, Legendre-type transformations were defined. These transformations (or symmetries), denoted by S_κ ($\kappa = 1 , \ldots , n$) in [1], act as follows:

$$\begin{aligned} \hat{t}_\alpha &= \partial_{t^\alpha} \partial_{t^\kappa} F(t) , \qquad \text{(N.B.} \quad \hat{t}_\alpha = \eta_{\alpha\beta} t^\beta) \\ \frac{\partial^2 \hat{F}}{\partial \hat{t}^\alpha \partial \hat{t}^\beta} &= \frac{\partial^2 F}{\partial t^\alpha \partial t^\beta} , \\ \hat{\eta}_{\alpha\beta} &= \eta_{\alpha\beta} . \end{aligned}$$

Note that (here we denote ∂_κ to be the vector field, which, in the $\{t^\alpha\}$-coordinates, is the coordinate vector field ∂_{t^κ})

$$\partial_{t^\alpha} = \partial_\kappa \circ \partial_{\hat{t}^\alpha} .$$

It follows [1] that the new metric $<,>_\kappa$ is related to the original metric by

$$< a, b >_\kappa = < \partial_\kappa^2, a \circ b > ,$$

where $\partial_\kappa^2 = \partial_\kappa \circ \partial_\kappa$, and that the variables $\{\hat{t}^\alpha\}$ are flat coordinates for this new metric.

While such Legendre transformations may be applied to any solution of the WDVV equations, in this paper we will concentrate on solutions that define a Frobenius manifold. Such solutions have, by definition, the quasihomogeneity property

$$\mathcal{L}_E F = d_F \, F + \text{quadratic terms}$$

where d_F is a constant and the Euler vector field is linear in the flat-coordinates,

$$E = \sum_\alpha d_\alpha t^\alpha \partial_{t^\alpha} + \sum_{\alpha | d_\alpha = 0} r^\alpha \partial_{t^\alpha} \, .$$

For the precise definition of a Frobenius manifold see [1] .

Example 1. (see [1] example B1)

$$\left\{ \begin{array}{lcl} F & = & \frac{1}{2}(t^1)^2 t^2 + e^{t^2} \\ E & = & t^1 \partial_{t^1} + 2\partial_{t^2} \end{array} \right\} \xrightarrow{S_2} \left\{ \begin{array}{lcl} \hat{F} & = & \frac{1}{2}(\hat{t}^2)^2 \hat{t}^1 + \frac{1}{2}((\hat{t}^1)^2(\log \hat{t}^1 - \frac{3}{2}) \\ \hat{E} & = & \hat{t}^2 \partial_{\hat{t}^2} + 2\hat{t}^1 \partial_{\hat{t}^1} \end{array} \right\} .$$

More specifically:

$$\begin{array}{lcl} t^1 & = & \hat{t}^2 \, , \\ t^2 & = & \log \hat{t}^1 \, , \end{array}$$

and from the defining relation $\hat{F}_{\hat{i}\hat{j}} = F_{ij}$ one obtains

$$\begin{array}{lclcl} \hat{F}_{\hat{1}\hat{1}} & = & t^2 & = & \log \hat{t}^1 \, , \\ \hat{F}_{\hat{1}\hat{2}} & = & t^1 & = & \hat{t}^2 \, , \\ \hat{F}_{\hat{2}\hat{2}} & = & e^{t^2} & = & \hat{t}^1 \, . \end{array}$$

Integrating yields the above prepotential $\hat{F}$.

Note that the transformation $E \rightarrow \hat{E}$ is just a coordinate transformation, so technically it does not require the hat (geometrically nothing has changed) and so this hat may be dropped. Similar remarks hold for the multiplication $\circ$ - this just transforms as a $(2,1)$-tensor under the above transformation and so does not require a hat.

Recently, it was shown by Dubrovin [2] that one may construct a so-called dual solution to the WDVV equations starting from a given Frobenius manifold (and hence a specific solution of the WDVV equations). This is constructed in two stages, from a new multiplication and a new metric.

From the specific Frobenius manifold M one may define a dual multiplication $\star$: $T\overset{\star}{M} \rightarrow T\overset{\star}{M}$ by

$$X \star Y = E^{-1} \circ X \circ Y \, , \qquad \forall \, X \, , Y \in T\overset{\star}{M} \, .$$

Here E^{-1} is the vector field defined by the equation $E^{-1} \circ E = e$ and $\overset{\star}{M} = M \backslash \Sigma$, where Σ is the (discriminant) submanifold where E^{-1} is undefined and e being the unity vector field on the manifold. This new multiplication is clearly commutative and associative with the Euler field playing the role of the unity field for the new multiplication.

On such Frobenius manifolds there exists a second metric, denoted by $(\,,\,)$ (alternatively, by g) and known as the intersection form, which is defined by the formula

$$(\omega_1, \omega_2) = i_E(\omega_1 \circ \omega_2) \tag{1.1}$$

where $\omega_1\,,\omega_2 \in T^*M$ and the metric η has been used to extend the multiplication from the tangent bundle to the cotangent bundle. This metric is also flat and hence there exists another distinguished coordinate system $\{p^i\,, i = 1\,, \ldots\,, n\}$ in which the components of the intersection form are constant. It follows [1] that these two metrics are related by the formula

$$(E \circ u, v) = < u, v >$$

or

$$(u, v) = < E^{-1} \circ u, v >\,, \qquad \forall\, u\,, v \in \overset{\star}{TM}\,. \tag{1.2}$$

These new structures are compatible

$$(X \star Y, Z) = (X, Y \star Z)\,.$$

From this and various other properties inherited from the original Frobenius structure, one may derive a dual prepotential $F^\star$ satisfying the WDVV equations in the flat coordinates of the intersection form. Explicitly (here G_{ij} are the components of the intersection form in its flat coordinate system):

Theorem 1. *There exists a function $F^\star(p)$ such that*[2]

$$\frac{\partial^3 F^\star(p)}{\partial p^i \partial p^j \partial p^k} = G_{ia} G_{jb} \frac{\partial t^\gamma}{\partial p^k} \frac{\partial p^a}{\partial t^\alpha} \frac{\partial p^b}{\partial t^\beta} c_\gamma^{\alpha\beta}(t)$$

and which satisfies the WDVV-equations in the $\{p^i\}$ coordinates.

For the simple two-dimensional examples of Frobenius manifolds given above, it is straightforward to calculate the flat-coordinates for the two intersection forms and hence to calculate the new multiplication and prepotentials $F^\star$ and $\hat{F}^\star$:

Example 2. From each of the two sets of solutions $\{F, E\}$ and $\{\hat{F}, \hat{E}\}$ in example 1 one may calculate the corresponding dual-prepotential.

- From the data $\{F, E\}$ in example 1, one may calculate the intersection form (1.1) yielding

$$g_{ij} = \begin{pmatrix} 2e^{t^2} & t^1 \\ t^1 & 2 \end{pmatrix}.$$

[2] In such formulae, Greek indices are raised and lowered using the metric η and Latin indices using the metric G.

By construction this metric is flat, though clearly the coordinates $\{t^i\}$ are not the flat coordinates. The flat coordinates $\{z^i\}$ are defined by the equations

$$\begin{aligned} t^1 &= -\left(e^{z^1} + e^{z^2}\right), \\ t^2 &= z^1 + z^2 . \end{aligned}$$

and in these coordinates $g = 2\, dz^1\, dz^2$. The vector field E^{-1} may easily be calculated and using this one finds that the third derivatives of the dual prepotential are:

$$\begin{aligned} F^\star{}_{111} &= -F^\star{}_{112} = \frac{e^{z^1}}{e^{z^2} - e^{z^1}}, \\ F^\star{}_{222} &= -F^\star{}_{122} = \frac{e^{z^2}}{e^{z^1} - e^{z^2}}, \end{aligned}$$

The integration of these equation involve the trilogarithm function. The definition of the general polylogarithm functions is, for positive integers N ,

$$Li_N(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^N} .$$

The series converges for $|z| < 1$, but the functions may be analytically continued elsewhere as well as extended to non-integer values of N via integral representations of the function. Immediate consequences of the definitions are:

$$\begin{aligned} Li_0(z) &= \frac{z}{1-z}, \\ \frac{d}{dz} Li_n(z) &= \frac{1}{z} Li_{n-1}(z) . \end{aligned}$$

With this the resulting dual-prepotential is

$$F^\star = \frac{1}{4} z^1 z^2 (z^1 + z^2) - \frac{1}{12}\left((z^1)^3 + (z^2)^3\right) + \frac{1}{2}\left\{ Li_3(e^{z^1 - z^2}) + Li_3(e^{z^2 - z^1})\right\} \quad (1.3)$$

- From the data $\{\hat{F}, \hat{E}\}$ in example 1, one may calculate the intersection form (1.1) yielding

$$\hat{g}_{ij} = \begin{pmatrix} 2\hat{t}^1 & \hat{t}^2 \\ \hat{t}^2 & 2 \end{pmatrix} .$$

By construction this metric is flat, though clearly the coordinates $\{\hat{t}^i\}$ are not the flat coordinates. The flat coordinates $\{\hat{z}^i\}$ are defined by the equations

$$\begin{aligned} \hat{t}^1 &= \hat{z}^1 \hat{z}^2 , \\ \hat{t}^2 &= \hat{z}^1 + \hat{z}^2 . \end{aligned}$$

and in these coordinates $\hat{g} = 2d\hat{z}^1\, d\hat{z}^2$. The vector field $\hat{E}^{-1}$ may easily be calculated and using this one finds that the third derivatives of the dual prepotential are:

$$\begin{aligned} \hat{F}^\star_{111} &= \frac{1}{\hat{z}^1} + \frac{1}{\hat{z}^2 - \hat{z}^1}\,, \\ \hat{F}^\star_{112} &= \frac{1}{\hat{z}^1 - \hat{z}^2} \qquad\qquad = \; -F^\star{}_{122}\,, \\ \hat{F}^\star_{222} &= \frac{1}{\hat{z}^2} + \frac{1}{\hat{z}^1 - \hat{z}^2}\,, \end{aligned}$$

The resulting dual-prepotential is

$$\hat{F}^\star = \frac{1}{4}\Big\{(\hat{z}^1)^2\log(\hat{z}^1)^2 + (\hat{z}^2)^2\log(\hat{z}^2)^2 - (\hat{z}^1-\hat{z}^2)^2\log(\hat{z}^1-\hat{z}^2)^2\Big\}\,.$$

Note that with a further change $z^1 = w^2 - iw^1\,, z^2 = -w^2 - iw^1$, the dual prepotential (1.3) takes the form

$$F^\star = \frac{1}{2}\sum_{k=\pm1} Li_3(e^{2kw^2}) + 2i\left[\frac{1}{6}(w^1)^3 + \frac{1}{2}w^1(w^2)^2\right]$$

and so falls into the class of solutions of the WDVV equations studied in [5] .

Schematically we have the following structure:

$$\begin{array}{ccc} F & \xrightarrow{S_\kappa} & \hat{F} \\ \downarrow & & \downarrow \\ F^\star & & \hat{F}^\star \end{array}$$

So given a prepotential F one may calculate a dual-prepotential $F^\star$. Alternatively one may construct, via a Legendre transformation, a new prepotential $\hat{F}$ and its corresponding dual-prepotential $\hat{F}^\star$. The aim of this note is two-fold: firstly to construct a transformation

$$F^\star \xrightarrow{\hat{S}_\kappa} \hat{F}^\star\,.$$

This will turn out to be a twisted-Legendre transformation, where the twist is provided by the Euler vector field. Secondly, the above example will be generalised to arbitrary dimension. This provides a transformation between certain rational and trigonometric solutions of the WDVV equations.

2 Twisted Legendre Transformations

We summarize the various structure in the following diagram:

$$\begin{array}{ccc} \{<a,b>\,,\circ\,,E\} & \xrightarrow{S_\kappa} & \{<a,b>_\kappa := <\partial_\kappa\circ\partial_\kappa, a\circ b>\,,\circ\,,E\} \\ \downarrow & & \downarrow \\ \left\{\begin{array}{ccc} (a,b) & := & <E^{-1}\circ a,\, b> \\ a\star b & := & E^{-1}\circ a\circ b \end{array}\right\} & & \left\{\begin{array}{ccc} (a,b)_\kappa & := & <E^{-1}\circ a,\, b>_\kappa \\ a\star b & := & E^{-1}\circ a\circ b \end{array}\right\} \end{array}$$

Proposition 1. *There exists a vector field $\hat{\partial}_\kappa$ generating a twisted Legendre transformation*

$$F^\star \xrightarrow{\hat{S}_\kappa} \hat{F}^\star$$

such that

$$(a,b)_\kappa = (\hat{\partial}_\kappa \star \hat{\partial}_\kappa, a \star b)\,.$$

Explicitly,

$$\hat{\partial}_\kappa = E \circ \partial_\kappa\,.$$

Proof. The proof is straightforward:

$$\begin{aligned}
(a,b)_\kappa &= \ < E^{-1} \circ a, b >_\kappa\,,\\
&= \ < \partial_\kappa \circ \partial_\kappa, E^{-1} \circ a \circ b >\,,\\
&= (E \circ \partial_\kappa \circ \partial_\kappa, a \star b)\,,\\
&= ((E \circ \partial_\kappa) \star (E \circ \partial_\kappa), a \star b)\ ,\\
&= (\hat{\partial}_\kappa \star \hat{\partial}_\kappa, a \star b)\,,
\end{aligned}$$

on defining $\hat{\partial}_\kappa = E \circ \partial_\kappa$.

■

It also follows from this result that if $\partial_{\hat{p}^a}$ are flat vector fields for the metric $(,)_\kappa$ then $\hat{\partial}_\kappa \star \partial_{\hat{p}^a}$ are flat vector fields for the metric $(,)$.

3 Hurwitz spaces and Frobenius structures

The original example of the prepotential F in example 1 falls into a wider class of Frobenius manifolds constructed on Hurwitz spaces. Hurwitz spaces are moduli spaces of pairs $(\mathcal{C}, \lambda)$, where $\mathcal{C}$ is a Riemann surface of genus g and λ is a meromorphic function on $\mathcal{C}$ of degree N. It was shown in [1] that such spaces may be endowed with the structure of a Frobenius manifold. The $g = 0$ case is particularly simple - meromorphic functions from the Riemann sphere to itself are just given by rational functions. It is into this category of Frobenius manifolds that the examples constructed above fall.

More specifically, the Hurwitz space $H_{g,N}(k_1, \ldots, k_l)$ is the space of equivalence classes $[\lambda : \mathcal{C} \to \mathbb{P}^1]$ of N-fold branched covers[3] with:

- M simple ramification points $P_1, \ldots, P_M \in \mathcal{L}$ with distinct *finite* images $l_1, \ldots, l_M \in \mathbb{C} \subset \mathbb{P}^1$;
- the preimage $\lambda^{-1}(\infty)$ consists of l points: $\lambda^{-1}(\infty) = \{\infty_1, \ldots, \infty_l\}$, and the ramification index of the map p at the point ∞_j is k_j $(1 \le k_j \le N)$.

[3]Dubrovin uses the different notation $H_{g,k_1-1,\ldots,k_l-1}$.

The Riemann-Hurwitz formula implies that the dimension of this space is $M = 2g + l + N - 2$. One has also the equality $k_1 + \cdots + k_l = N$. For $g > 0$ one has to introduce a covering space, but this is unnecessary in the $g = 0$ case that will be considered here.

In this construction there is a certain ambiguity; one has to choose a so-called primary differential (also known as a primitive form). Different choices produce different solutions to the WDVV equations, but such solutions are related by Legendre transformation S_κ. The Hurwitz data $\{\lambda, \omega\}$ from which one constructs a solution $F_{\{\lambda,\omega\}}$ consists of the map λ (also known as the superpotential) and a particular primary differential ω. Thus, again schematically, one has:

$$F_{\{\lambda,\omega\}} \stackrel{S_\kappa}{\longleftrightarrow} \hat{F}_{\{\lambda,\hat{\omega}\}}$$

(note the map λ does not change, though it might undergo a coordinate transformation). The metrics $<,>$, $(,)$ and multiplications $\circ$, $\star$ are determined by calculating certain residues at the critial points of the map λ.

Theorem 2.

$$\begin{aligned} <\partial', \partial''> &= -\sum \operatorname*{res}_{d\lambda=0} \frac{\partial'(\lambda(p)dp)\partial''(\lambda(p)dp)}{d\lambda(p)}, \\ <\partial' \circ \partial'', \partial'''> &= -\sum \operatorname*{res}_{d\lambda=0} \frac{\partial'(\lambda(p)dp)\partial''(\lambda(p)dp)\partial'''(\lambda(p)dp)}{d\lambda(p)}, \\ (\partial', \partial'') &= -\sum \operatorname*{res}_{d\lambda=0} \frac{\partial'(\log\lambda(p)dp)\partial''(\log\lambda(p)dp)}{d\log\lambda(p)}, \\ (\partial' \star \partial'', \partial''') &= -\sum \operatorname*{res}_{d\lambda=0} \frac{\partial'(\log\lambda(p)dp)\partial''(\log\lambda(p)dp)\partial'''(\log\lambda(p)dp)}{d\log\lambda(p)}. \end{aligned}$$

The first three formulae appeared in [1] while the last follows immediately from the results in [2]. With these, and the basic result that

$$\sum \operatorname*{res}_{v\in\mathcal{L}} \omega = 0 \tag{3.1}$$

for any meromorphic differential ω on a compact Riemann surface $\mathcal{L}$, the calculations reduce to the calculation of residues. More specifically, the locations of the zeros of $d\lambda = 0$ are only known implicitly while the zeros of λ are known explicitly. Thus using this one finds

$$\sum \operatorname*{res}_{d\lambda=0} \omega = -\sum \operatorname*{res}_{\lambda=0} \omega - \sum \operatorname*{res}_{\text{special points}} \omega$$

where the 'special points' are points like zero or infinity, where the residues may be easily calculated. Thus the residues, and hence the various tensors, in Theorem 2 may be calculated very simply. In the appendix we illustrate this by calculating the dual-prepotential for the simplest case of a Hurwitz space, namely[4] $H_{0,N+1}(N+1)$. The proofs of Propositions 2 and 3 below follow this simple example very closely: conceptually they are identical,

[4]This space is also isomorphic to the orbit space $\mathbb{C}^N/W(A_N)$ corresponding to Coxeter group A_N. This interpretation will not be used here, but it does provide a starting point for the extension of these ideas to other, non-Hurwitz, Frobenius manifolds.

just notationally more complicated. Proofs of these propositions will therefore not be given here, but will appear in [6].

Rather than applying these methods to the full $g = 0$ Hurwitz space we consider the space $H_{0,k+m}(k,m)$. This coincides with the space[5] of trigonometric polynomials of bidegree (k,m) [3]. Here the primary differential is dp and

$$\lambda(p) = e^{ikp} + a_1 e^{i(k-1)p} + \ldots + a_k + \ldots a_{k+m} e^{-imp} \,.$$

As in the above example, the coordinates $\{a_i\}$ are not flat coordinates for the metric $<,>$ but these may be constructed via various residue formulae. In particular, it turns out that the flat coordinates for the intersection form are just the zeros of λ [3] .

Example 3. The prepotential F and $F^\star$ in examples 1 and 2 may be derived from the Hurwitz data

$$\left\{ \lambda(p) = e^p + t^1 + e^{t^2}\, e^{-p} \,, dp \right\}$$

(here $\{t^1, t^2\}$ are the flat coordinates for the metric $<,>$). The dual coordinates $\{z^1, z^2\}$ are the zeros of the superpotential

$$\lambda(p) = e^{-p}(e^p - e^{z^1})(e^p - e^{z^2}) \,.$$

Thus in general one may write

$$\lambda(p) = e^{-imp} \prod_{\alpha=1}^{k+m} \left(e^{ip} - e^{iz^\alpha} \right)$$

and it follows from [3] that the $\{z^\alpha\}$ are the flat coordinates of the intersection form. Using the above residue formulae one may construct the dual structure functions for the multiplication $\star$ and then integrate to find the dual prepotential $F^\star$ [6] .

Proposition 2. *The dual prepotential on the space of trigonometric polynomials of bidegree* (k,m) *is:*

$$F^\star = A \sum_{i=1}^{k+m} (z^i)^3 + B \sum_{i=1}^{k+m} (z^i)^2 \sum_{j \neq j} z^j + C \sum_{\substack{i,j,k \\ distinct}} z^i z^j z^k + \frac{1}{2} \sum_{i \neq j} Li_3 \left[e^{i(z^i - z^j)} \right]$$

where

$$\begin{aligned} A &= \frac{i}{12} \left[(m-2)(m-1) - mk \right] \,, \\ B &= \frac{i}{4m} \left(2 - m \right) \,, \\ C &= \frac{i}{m} \end{aligned}$$

[5]This space is also isomorphic to the orbit space $\mathbb{C}^{k+m}/\widetilde{W}^{(k)}(A_{k+m-1})$ corresponding to a certain extended affine Weyl group [3].

An alternative way to construct the Frobenius structure on the Hurwitz spaces $H_{0,k+m}(k,m)$ is to use the superpotential (this is not quite true if $k_1 = 1$ but a similar form still holds in this case)

$$\lambda(\hat{p}) = \hat{p}^k + \sum_{r=0}^{k-2} a_r \hat{p}^r - \sum_{\alpha=1}^{m} \frac{c_\alpha}{(\hat{p}-\hat{z}^0)^\alpha}\,.$$

Geometrically this just corresponds to a change in primary differential and hence the prepotentials are related via a Legendre transform.

Example 4. The prepotentials $\hat{F}$ and $\hat{F}^\star$ in examples 1 and 2 may be derived from the Hurwitz data

$$\left\{ \lambda(\hat{p}) = \hat{p} + \frac{\hat{t}^1}{\hat{p}-\hat{t}^2}\,, d\hat{p} \right\}.$$

The dual coordinates $\{\hat{z}^1, \hat{z}^2\}$ are just the zeros of the superpotential

$$\lambda(\hat{p}) = \frac{(\hat{p}-\hat{z}^1)(\hat{p}-\hat{z}^2)}{(\hat{p}-(\hat{z}^1+\hat{z}^2))}\,.$$

On changing the primary differential:

$$dp = \partial_{\hat{t}^1}[\lambda(\hat{p})d\hat{p}]$$

so $e^p = \hat{p} - \hat{t}^2$ the Hurwitz data transforms to

$$\left\{ \lambda(p) = e^p + t^1 + e^{t^2} e^{-p}\,, dp \right\}$$

and this generates the prepotentials F and $F^\star$.

As in the previous case, by factorizing $\lambda(\hat{p})$ one may derive the dual prepotential very simply. Explicitly one may write

$$\lambda(\hat{p}) = \left. \frac{\prod_{i=1}^{k+m}\left(\hat{p}-\hat{z}^i\right)}{\left(\hat{p}-\hat{z}^0\right)^m} \right|_{m\hat{z}^0=\sum_{i=1}^{k+m}\hat{z}^i}.$$

By calculating various residues and integrating to find the dual prepotential one finds:

Proposition 3. *The dual prepotential on the Hurwitz space $H_{0,k+m}(k,m)$ is:*

$$\hat{F}^\star = \left. \frac{1}{4} \sum_{i,j=0\,,i\neq j}^{k+m} \alpha_i\alpha_j \left(\hat{z}^i-\hat{z}^j\right)^2 \log\left(\hat{z}^i-\hat{z}^j\right)^2 \right|_{m\hat{z}^0=\sum_{i=1}^{k+m}\hat{z}^i},$$

where $\alpha_0 = -1$ and $\alpha_{i>0} = +1$.

This part of the calculation may be generalised very simply to an arbitrary $g=0$ Hurwitz space, and more generally to the induced structures on the discriminants of such Hurwitz spaces [4, 6]. Since a change in primary differential induces a Legendre transformation between the corresponding solutions of the WDVV equations, applying proposition 1 yields the following result:

Theorem 3. *The prepotentials $\hat{F}$ and $\hat{F}^\star$ constructed in propositions 2 and 3 are connected via a twisted Legendre transformation.*

This provides a map between certain rational solutions and trigonometric solutions to the WDVV equations.

4 Comments

It would be interesting to extend these results to other classes of Frobenius manifold; all the Hurwitz space examples considered above have an underlying A_n-structure but the construction will work for arbitrary an Weyl group. Such proofs would be more algebraic in style, using properties of the roots systems (or, conjecturally, of $\vee$-systems [8] via their geometrical interpretation in terms of induced structures on discriminants [4]). Such 'dual' solutions are solutions of the WDVV equations even though there is no 'undual'-prepotential (the metric on the discriminant induced by $<,>$ has curvature, though Frobenius-type structures remain [7]). It would also be of interest to apply Legendre transformations (both twisted and non-twisted) directly to such solutions.

Also, given the close similarities between Calogero-Moser type operators and the flatness of the (dual) Dubrovin connection, it would be of interest to see if the notion of a (twisted) Legendre transformation holds for such operators, and in particular between rational and trigonometric operators.

Acknowledgments. *Andrew Riley would like to thank the EPSRC for a research studentship.*

Appendix

In this appendix we calculate the dual prepotential on the Hurwitz space $H_{0,N+1}(N+1)$. In this case the superpotential is polynomial and using the freedom $p \to ap + b$ one may set the sum of the roots to be zero and the leading coefficient to be one. Thus:

$$\begin{aligned} \lambda(p) &= \left(p + \sum_{i=1}^{N} z^i\right) \prod_{r=1}^{N} (p - z^i)\,, \\ &= \prod_{i=0}^{N} (p - z^i)\,, \qquad\qquad \text{where } z^0 = -\sum_{i=1}^{N} z^i \end{aligned}$$

(to agree with the notation in the main body of the paper, the variables p and z should be hatted. Here we drop these hats for notational convenience).

Lemma 1. *The coordinates $\{z^i \,:\, i = 1\,, \ldots\,, N\}$ are flat coordinates for the intersection form.*

Proof. Note that since the sum of the roots of the superpotential is zero, the roots are not independent, so care has to be taken when taking derivatives. Since

$$\frac{\partial \log \lambda(p)}{\partial z^i} = \frac{1}{p - z^0} - \frac{1}{p - z^i}\,, \qquad i = 1\,, \ldots\,, N$$

it follows from the third formula in Theorem 2 that

$$(\partial_{z^i}, \partial_{z^j}) = -\sum \operatorname*{res}_{d\lambda=0} \left[\left(\frac{1}{p - z^0} - \frac{1}{p - z^i}\right) \left(\frac{1}{p - z^0} - \frac{1}{p - z^j}\right) \frac{\lambda}{\lambda'}\, dp \right].$$

One now uses the result (3.1) to express this in terms of residues at $p = z^0\,, z^i$ and $p = z^j$, there being to pole at infinity. The analysis splits depending on whether i and j are distinct or not.

Case I: $i \neq j$

$$\begin{aligned}(\partial_{z^i}, \partial_{z^j}) &= \operatorname*{res}_{p=z^0}\left[\frac{1}{(p-z^0)^2}\,\frac{\lambda}{\lambda'}\,dp\right],\\ &= 1\end{aligned}$$

since the simple poles cancel with the zeros in the numerator.
Case II: $i = j$

$$\begin{aligned}(\partial_{z^i}, \partial_{z^i}) &= \operatorname*{res}_{p=z^0}\left[\frac{1}{(p-z^0)^2}\,\frac{\lambda}{\lambda'}\,dp\right] + \operatorname*{res}_{p=z^i}\left[\frac{1}{(p-z^i)^2}\,\frac{\lambda}{\lambda'}\,dp\right],\\ &= 2\,.\end{aligned}$$

Hence $(\partial_{z^i}, \partial_{z^j}) = 1 + \delta_{ij}$. This shows that the coordinates $\{z^i \,:\, i = 1, \ldots, N\}$ are flat coordinates for the intersection form. Thus

$$\begin{aligned} g &= \sum_{i,j=1}^{N} (1+\delta_{ij})\, dz^i\, dz^j\,,\\ &= \sum_{i=0}^{N} \left(dz^i\right)^2\Bigg|_{z^0=-\sum_{r=1}^{N} z^i}\,. \end{aligned}$$

■

The dual-prepotential $F^\star$ may be calculated similarly, using the fourth formula in Theorem 2.

Proposition 4. *The dual-prepotential is given by*

$$F^\star = \frac{1}{4}\sum_{i,j=0,i\neq j}^{N} (z^i - z^j)^2 \log(z^i - z^j)^2\,.$$

Proof. Let

$$\overset{\star}{c}_{ijk} = (\partial_{z^i} \star \partial_{z^j}, \partial_{z^k})\,.$$

Thus, using the fourth formula in Theorem 2:

$$\overset{\star}{c}_{ijk} = -\sum \operatorname*{res}_{d\lambda=0}\left[\left(\frac{1}{p-z^0} - \frac{1}{p-z^i}\right)\left(\frac{1}{p-z^0} - \frac{1}{p-z^j}\right)\left(\frac{1}{p-z^0} - \frac{1}{p-z^k}\right)\frac{\lambda}{\lambda'}\,dp\right].$$

Again, the analysis splits depending on the values of i , j and k . Here we present the details of the case i , j , k all distinct, the remaining cases being similar.

$$\begin{aligned}\overset{\star}{c}_{ijk} &= -\sum \operatorname*{res}_{d\lambda=0}\left[\left(\frac{1}{p-z^0} - \frac{1}{p-z^i}\right)^3 \frac{\lambda}{\lambda'}\,dp\right],\\ &= \operatorname*{res}_{p=z^0}\left[\left(\frac{1}{(p-z^0)^3} - \frac{1}{(p-z^i)^2}\left\{\frac{1}{p-z^i} + \frac{1}{p-z^j} + \frac{1}{p-z^k}\right\}\right)\frac{\lambda}{\lambda'}\,dp\right],\\ &= -\frac{1}{2}\,\frac{\lambda''}{\lambda'}\bigg|_{p=z^0} - \left\{\frac{1}{z^0-z^i} + \frac{1}{z^0-z^j} + \frac{1}{z^0-z^k}\right\},\\ &= \sum_{r\neq i,j,k} \frac{1}{z^r - z^0} + 2\left\{\frac{1}{z^i-z^0} + \frac{1}{z^j-z^0} + \frac{1}{z^k-z^0}\right\}.\end{aligned}$$

Again, the residue result (3.1) has been used. Similar results may be derived for $\overset{\star}{c}_{iij}$ and $\overset{\star}{c}_{iii}$. Since

$$\overset{\star}{c}_{ijk} = \frac{\partial^3 F^\star}{\partial z^i \partial z^j \partial z^k}\,, \qquad\qquad i,j,k = 1\,,\ldots\,,N\,,$$

one may integrate to obtain the dual-prepotential $F^\star$. ■

As remarked in section 3, the proofs of Propositions 2 and 3 follow the calculations in this appendix very closely (the only real difference being a change of variable, as used in [3], in proposition 2): conceptually they are identical, notationally they are more complicated.

References

[1] Dubrovin B, Geometry of 2D topological field theories, in Integrable Systems and Quantum Groups, Editors Francaviglia M and Greco S, *Springer Lect. notes Math.* **1620** (1996), 120–348.

[2] Dubrovin B, On almost duality for Frobenius manifolds, in Geometry, topology, and mathematical physics, American Mathematical Society Transl. Ser. 2 **212**, Providence, RI, 2004, 75–132.

[3] Dubrovin B and Zhang Y, Extended affine Weyl groups and Frobenius manifolds, *Compos. Math.* **111** (1998), 167–219.

[4] Feigin M V and Veselov A P, Logarithmic Frobenius structures and Coxeter discriminants, *Adv. Math.*, to appear; `arXiv:math-ph/0512095`.

[5] Martini R and Hoevenaars L K, Trigonometric Solutions of the WDVV Equations from Root Systems, *Lett. Math. Phys.* **65** (2003), 15–18.

[6] Riley A, University of Hull PhD thesis, 2007, in preparation.

[7] Strachan I A B, Frobenius manifolds: natural submanifolds and induced bi-Hamiltonian structures, Differ. Geom. Appl. **20** (2004), 67–99.

[8] Veselov A P, Deformations of root systems and new solutions to generalised WDVV equations, *Phys. Lett.* **A 261** (1999), 297–302.