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An invariant p-adic q-integral associated with q-Euler numbers and polynomials

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Abstract

The purpose of this paper is to consider q-Euler numbers and polynomials which are q-extensions of ordinary Euler numbers and polynomials by the computations of the p-adic q-integrals due to T. Kim, cf. [1, 3, 6, 12], and to derive the "complete sums for q-Euler polynomials" which are evaluated by using multivariate p-adic q-integrals. These sums help us to study the relationships between p-adic q-integrals and non-archimedean combinatorial analysis.

1 Introduction

Let p be a fixed odd prime, and let $\mathbb{C}_p$ denote the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. For d a fixed positive integer with $(p,d)=1$, let

$$X = X_d = \varprojlim_{N} \mathbb{Z}/dp^N, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup\nolimits_{(a,p)=1}^{0<a<dp} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$, (cf. [1], [2], [14]).

The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. Let q be variously considered as an indeterminate a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we always assume $|q| < 1$. If $q \in \mathbb{C}_p$, we always assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. Throughout this paper, we use the following notation :

$$[x]_q = [x : q] = \frac{1-q^x}{1-q}.$$

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p-$ and denote this property by $f \in UD(\mathbb{Z}_p)-$ if the difference quotients

$$F_f(x,y) = \frac{f(x)-f(y)}{x-y}$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$, [1, 11, 12]. For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \le j < p^N} q^j f(j) = \sum_{0 \le j < p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p), \text{ cf. [2, 4]},$$

representing a q-analogue of Riemann sums for f.

The integral of f on $\mathbb{Z}_p$ will be defined as limit $(n \to \infty)$ of these sums, when it exists. An invariant p-adic q-integral of a function $f \in \mathrm{UD}(\mathbb{Z}_p)$ on $\mathbb{Z}_p$ is defined by

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{0 \le j < p^N} f(j) q^j.$$

Note that if $f_n \to f$ in $UD(Z_p)$; then

$$\int_{\mathbb{Z}_p} f_n(x) d\mu_q(x) \to \int_{\mathbb{Z}_p} f(x) d\mu_q(x).$$

It was well known that the ordinary Euler numbers are defined by

$$F(t) = \frac{2}{e^t+1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

where we use the technique method notation by replacing E^m by E_m $(m \ge 0)$, symbolically, cf.[2, 3, 6, 12]. In this paper, we consider q-Euler numbers and polynomials which are q-extensions of ordinary Euler numbers and polynomials by the computations of the p-adic q-integrals, and derive the" complete sums for q-Euler polynomials" which are evaluated by using multivariate p-adic q-integrals. These sums help us to study the relationships between p-adic q-integrals and non-archimedean combinatorial analysis.

2 q-Euler and Genocchi numbers associated with p-adic q-integral

The Euler polynomials are defined by means of the following generating function: $\frac{2}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$. Note that $E_n(0) = E_n$. From these Euler polynomials, we can evaluate the value of the following alternating sums of powers of consecutive integers [1, 2, 3, 11]:

$$-1^m + 2^m - 3^m + \cdots + (-1)^{m-1}(n-1)^m = \frac{1}{2}\left((-1)^{n+1}E_m(n) - E_m\right). \tag{2.1}$$

In a fermionic sense, we now consider the following p-adic q-integrals:

$$\int_{X_f} [x]_q^k d\mu_{-q}(x) = \int_{\mathbb{Z}_p} [x]_q^k d\mu_{-q}(x) = E_{k,q} \quad \text{for} \quad k, f \in \mathbb{N}. \tag{2.2}$$

From the computation of this p-adic q-integral, we derive the following Eq.(3):

$$E_{k,q} = [2]_q \left(\frac{1}{1-q}\right)^k \sum_{l=0}^{k} \binom{k}{l} (-1)^l \frac{1}{1+q^{l+1}}, \tag{2.3}$$

where $\binom{k}{i}$ is the binomial coefficient. Note that $\lim_{q\to 1} E_{k,q} = E_k$. Hence, $E_{k,q}$ is a q-extension of Euler numbers which are called q-Euler numbers. Let $F_q(t) = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}$ be the generating function of these q-Euler numbers. Then we easily see that [6, 8, 9, 10]

$$F_q(t) = e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{[2]_q}{[2]_{q^{j+1}}} \left(\frac{1}{q-1}\right)^j \frac{t^j}{j!} = [2]_q \sum_{l=0}^{\infty} (-q)^l e^{[l]_q t}. \tag{2.4}$$

By using an invariant p-adic q-integral on $\mathbb{Z}_p$, we can also consider a q-extension of ordinary Euler polynomials which are called q-Euler polynomials[3,8,12]. For $x \in \mathbb{Z}_p$, we define q-Euler polynomials as follows:

$$\int_{\mathbb{Z}_p} [x+y]_q^k d\mu_{-q}(y) = E_{k,q}(x). \tag{2.5}$$

By (5), we easily see that

$$E_{k,q}(x) = \sum_{n=0}^{k} \binom{k}{n} [x]_q^{k-n} q^{nx} E_{n,q}.$$

In Eq.(5), it is easy to see that

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y) = [2]_q \left(\frac{1}{1-q}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{xk} \frac{1}{1+q^{k+1}}.$$

By using the definition of Eq.(5), we will give the distribution of q-Euler polynomials. From the definition of a p-adic q-integral , we derive the below formula:

$$\int_{X_m} [x+y]_q^n d\mu_{-q}(y) = \frac{[m]_q^m}{[m]_{-q}} \sum_{a=0}^{m-1} (-1)^a q^a \int_{\mathbb{Z}_p} [\frac{a+x}{m} + y]_{q^m}^n d\mu_{-q^m}(y), \quad \text{if } m \text{ is odd.}$$

Thus, if m is an odd integer, then we have

$$E_{n,q}(x) = \frac{[m]_q^n}{[m]_{-q}} \sum_{a=0}^{m-1} (-1)^a q^a E_{n,q^m}(\frac{a+x}{m}).$$

From the definition of the q-Euler polynomials, we note that

$$F_q(x,t) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]_q t}.$$

As is well know, the Genocchi numbers are also defined by

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}.$$

Thus, we easily see that $G_n = \sum_{l=0}^{n-1} \binom{n}{l} 2^l B_l$, where B_l are ordinary Bernoulli numbers. We now define a q-extension of Genocchi number which are called q-Genocchi numbers as follows:

$$F_q^*(t) = [2]_q t \sum_{l=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}, \text{ see [8].} \tag{2.6}$$

From Eq. (2.6), we can derive the following, see Refs. [8, 12]

$$G_{n,q} = n \left(\frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{[2]_{q^{l+1}}}, \text{ when } m \text{ is odd .} \tag{2.7}$$

From Eq. (2.6), we can also recover the defining relation for the definition of q-Genocchi polynomials as follows:

$$F_q^*(x,t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}, \quad \text{when } n \text{ is odd, (see [8]).} \tag{2.8}$$

Let $a_1, a_2, \cdots, a_k$ be positive integers . For $w \in \mathbb{Z}_p$, we define multiple Daehee q-Euler polynomials by using the invariant p-adic q-integrals as follows, cf. [7, 8, 12]:

$$E_n^{(k)}(w, q|a_1, a_2, \cdots, a_k) = \int_{\mathbb{Z}_p^k} [w + \sum_{j=1}^{k} a_j x_j]^n d\mu_{-q}(x), \tag{2.9}$$

and

$$E_n^{(k)}(q|a_1, \cdots, a_k) = \int_{\mathbb{Z}_p^k} [\sum_{j=1}^{k} a_j x_j]^n d\mu_{-q}(x),$$

where

$$\int_{\mathbb{Z}_p^k} f(x) d\mu_{-q}(x) = \underbrace{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} f(x) d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r).$$

From Eq. (2.9), we can derive the following theorem:

Theorem 1. *Let $a_1, a_2, \cdots, a_k$ be positive integers. Then we have*

$$E_n^{(k)}(w, q|a_1, \cdots, a_k) = \frac{[2]_q^k}{(1-q)^n} \sum_{r=0}^{n} \binom{n}{r} (-q^w)^r \prod_{j=1}^{k} \left(\frac{1}{[2]_{q^{1+ra_j}}} \right). \tag{2.10}$$

Given elements $\alpha_1, \cdots, \alpha_m \in \mathbb{C}_p$ and positive integers $N_1, \cdots, N_m, n$, it is easy to see that [1, 6]

$$[N_1(x_1 + \alpha_1) + \cdots + N_m(x_m + \alpha_m)]^n \tag{2.11}$$

$$= \sum_{\substack{i_1, \cdots, i_m \geq 0 \\ i_1 + \cdots + i_m = n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \cdots \sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}} \tag{2.12}$$

$$\times \binom{n}{i_1, \cdots, i_m} \binom{n-i_1}{k_1} \binom{n-i_1-i_2}{k_2} \cdots \binom{n-i_1-i_2-\cdots-i_{m-1}}{k_{m-1}} \tag{2.13}$$

$$\times (q-1)^{k_1+\cdots+k_{m-1}} [N_1]^{i_1+k_1} \cdots [N_{m-1}]^{i_{m-1}+k_{m-1}} [N_m]^{i_m} \tag{2.14}$$

$$\times [x_1 + \alpha_1 : q^{N_1}]^{k_1+i_1} \cdots [x_{m-1} + \alpha_{m-1} : q^{N_{m-1}}]^{k_{m-1}+i_{m-1}} [x_m + \alpha_m : q^{N_m}]^{i_m}, \tag{2.15}$$

Hence, we have

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{m \text{ times}} [N_1(x_1 + \alpha_1) + \cdots + N_m(x_m + \alpha_m)]^n d\mu_{-q^{N_1}}(x_1) \cdots d\mu_{-q^{N_m}}(x_m) \tag{2.16}$$

$$= \sum_{\substack{i_1, \cdots, i_m \geq 0 \\ i_1 + \cdots + i_m = n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \cdots \sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}} \tag{2.17}$$

$$\times \binom{n}{i_1, \cdots, i_m} \binom{n-i_1}{k_1} \binom{n-i_1-i_2}{k_2} \cdots \binom{n-i_1-i_2-\cdots-i_{m-1}}{k_{m-1}} \tag{2.18}$$

$$\times (q-1)^{k_1+\cdots+k_{m-1}} [N_1]^{i_1+k_1} \cdots [N_{m-1}]^{i_{m-1}+k_{m-1}} [N_m]^{i_m} \tag{2.19}$$

$$\times E_{k_1+i_1}(\alpha_1, q^{N_1}) \cdots E_{k_{m-1}+i_{m-1}}(\alpha_{m-1}, q^{N_{m-1}}) E_{i_m}(\alpha_m, q^{N_m}). \tag{2.20}$$

From (2.9), (2.10), (2.11) and (2.16), we can derive the following theorem:

Theorem 2. *(Complete sum for multiple Daehee q-Euler polynomials)*
Given elements $\alpha_1, \cdots, \alpha_m \in \mathbb{C}_p$ and positive integers $N_1, \cdots, N_m, n$,

$$\sum_{\substack{i_1, \cdots, i_m \geq 0 \\ i_1 + \cdots + i_m = n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \cdots \sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}}$$

$$\times \binom{n}{i_1, \cdots, i_m} \binom{n-i_1}{k_1} \binom{n-i_1-i_2}{k_2} \cdots \binom{n-i_1-i_2-\cdots-i_{m-1}}{k_{m-1}}$$

$$\times (q-1)^{k_1+\cdots+k_{m-1}} [N_1]^{i_1+k_1} \cdots [N_{m-1}]^{i_{m-1}+k_{m-1}} [N_m]^{i_m}$$

$$\times E_{k_1+i_1}(\alpha_1, q^{N_1}) \cdots E_{k_{m-1}+i_{m-1}}(\alpha_{m-1}, q^{N_{m-1}}) E_{i_m}(\alpha_m, q^{N_m})$$

$$= E_n^{(m)}(N_1\alpha_1 + \cdots + N_m\alpha_m, q|N_1, \cdots, N_m).$$

3 Further Remarks and Observations

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let $\Gamma(s)$ be the ordinary gamma function given by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$, $s \in \mathbb{C}$. From (8) and complex integration, we can derive the following formula:

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} F_q^*(x,-t)dt = [2]_q \sum_{n=0}^\infty \frac{(-1)^{n+1} q^{n+x}}{[n+x]_q}, \quad \text{for } s \in \mathbb{C}. \tag{3.1}$$

For $s \in \mathbb{C}$, we define the (Hurwitz's type) q-Genocchi zeta function as follows [3, 12]:

$$\zeta_{q,G}(s,x) = [2]_q \sum_{n=0}^\infty \frac{(-1)^{n+1} q^{x+n}}{[n+x]_q^s}, \text{ where } x \in \mathbb{R} \text{ with } 0 < x < 1. \tag{3.2}$$

By (2.8), (3.1) and (3.2), we can see that

$$\zeta_{q,G}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} F_q^*(x,-t)dt = \sum_{n=0}^\infty \frac{G_{n,q}(x)}{n!} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{n+s-2} dt \right). \tag{3.3}$$

By using Laurent series in Eq. (3.3), we easily see that (see Refs. [3, 12, 13])

$$\zeta_{q,G}(1-n,x) = \frac{(-1)^{n-1}}{n} G_{n,q}(x), \ n \in \mathbb{N} .$$

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