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Symmetry reduction and exact solutions of the Navier-Stokes equations. I

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Abstract

Ansatzes for the Navier-Stokes field are described. These ansatzes reduce the Navier-Stokes equations to system of differential equations in three, two, and one independent variables. The large sets of exact solutions of the Navier-Stokes equations are constructed.

1 Introduction

The Navier-Stokes equations (NSEs)

$$\begin{aligned} &\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \triangle\vec{u} + \vec{\nabla}p = \vec{0}, \\ &\mathrm{div}\vec{u} = 0 \end{aligned} \tag{1.1}$$

which describe the motion of an incompressible viscous fluid are the basic equations of modern hydrodynamics. In (1.1) and below $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity field of a fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian, the kinematic coefficient of viscosity and fluid density are set equal to unity. Repead indices denote summation whereby we consider the indices a, b to take on values in $\{1,2,3\}$ and the indices i, j to take on values in $\{1,2\}$.

The problem of finding exact solutions of non-linear equations (1.1) is an important but rather complicated one. There are some ways to solve it. Considerable progress in this field can be achieved by means of making use of a symmetry approach. Equations (1.1) have non-trivial symmetry properties. It was known long ago [37, 2] that they are invariant under the eleven-parametric extended Galilei group. Let us denote it by $G_1(1,3)$. This group includes the Galilei group and scale transformations. The Lie algebra $AG_1(1,3)$ of $G_1(1,3)$

is generated by the operators

$$P_0, \quad J_{ab}, \quad D, \quad P_a, \quad G_a,$$

where

$$P_0 = \partial_t, \quad D = 2t\partial_t + x_a\partial_a - u^a\partial_{u^a} - 2p\partial_p,$$

$$J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a}, \quad a \neq b,$$

$$G_a = t\partial_a + \partial_{u^a}, \quad P_a = \partial_a.$$

Relatively recently it was found by means of the Lie method [8, 5, 26] that the maximal Lie invariance algebra (MIA) of the NSEs (1.1) is the infinite-dimensional algebra $A(NS)$ with the basis elements

$$\partial_t, \quad D, \quad J_{ab}, \quad R(\vec{m}), \quad Z(\chi), \tag{1.2}$$

where

$$R(\vec{m}) = R(\vec{m}(t)) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p, \tag{1.3}$$

$$Z(\chi) = Z(\chi(t)) = \chi(t)\partial_p, \tag{1.4}$$

$m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (degree of their smoothness is discussed in Note A.1).

The algebra $AG_1(1,3)$ is a subalgebra of $A(NS)$. Indeed, setting $m^a = \delta_{ab}$, where b is fixed, we obtain $R(\vec{m}) = \partial_b$, and if $m^a = \delta_{ab}t$ then $R(\vec{m}) = G_b$. Here δ_{ab} is the Kronecker symbol ($\delta_{ab} = 1$ if $a = b$, $\delta_{ab} = 0$ if $a \neq b$).

Operators (1.2) generate the following invariance transformations of system (1.1):

∂_t: $\tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t+\varepsilon,\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t+\varepsilon,\vec{x})$
(translations with respect to t),

J_{ab}: $\tilde{\vec{u}}(t,\vec{x}) = B\vec{u}(t,B^T\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t,B^T\vec{x})$
(space rotations),

D: $\tilde{\vec{u}}(t,\vec{x}) = e^{\varepsilon}\vec{u}(e^{2\varepsilon}t,e^{\varepsilon}\vec{x}), \quad \tilde{p}(t,\vec{x}) = e^{2\varepsilon}p(e^{2\varepsilon}t,e^{\varepsilon}\vec{x})$
(scale transformations), (1.5)

$R(\vec{m})$: $\tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t,\vec{x}-\vec{m}(t)) + \vec{m}_t(t)$,
$\tilde{p}(t,\vec{x}) = p(t,\vec{x}-\vec{m}(t)) - \vec{m}_{tt}\cdot\vec{x} - \frac{1}{2}\vec{m}\cdot\vec{m}_{tt}$
(these transformations include the space translations and the Galilei transformations),

$Z(\chi)$: $\tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t,\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t,\vec{x}) + \chi(t).$

Here $\varepsilon \in \mathbb{R}$, $B = \{\beta_{ab}\} \in O(3)$, i.e. $BB^T = \{\delta_{ab}\}$, B^T is the transposed matrix.

Besides continuous transformations (1.5) the NSEs admit discrete transformations of the form

$$\begin{aligned} &\tilde{t} = t, \quad \tilde{x}_a = x_a,\ a \neq b, \quad \tilde{x}_b = -x_b, \\ &\tilde{p} = p, \quad \tilde{u}^a = u^a,\ a \neq b, \quad \tilde{u}^b = -u^b, \end{aligned} \tag{1.6}$$

where b is fixed. Invariance under transformations (1.5) and (1.6) means that $(\vec{\tilde{u}}, \tilde{p})$ is a solution of (1.1) if $(\vec{u}, p)$ is a solution of (1.1).

A complete review of exact solutions found for the NSEs before 1963 is contained in [1]. We should like also to mark more modern reviews [16, 7, 36] despite their subjects slightly differ from subjects of our investigations. To find exact solutions of (1.1), symmetry approach in explicit form was used in [2, 31, 32, 6, 20, 21, 4, 17, 15, 12, 10, 11, 30]. This article is a continuation and a extention of our works [15, 12, 10, 11, 30]. In it we make symmetry reduction of the NSEs to systems of PDEs in three and two independent variables and to systems of ODEs, using subalgebraic structure of $A(NS)$. We investigate symmetry properties of the reduced systems of PDEs and construct exact solutions of the reduced systems of ODEs when it is possible. As a result, large classes of exact solutions of the NSEs are obtained.

The reduction problem for the NSEs is to describe ansatzes of the form [9]:

$$u^a = f^{ab}(t,\vec{x})v^b(\omega) + g^a(t,\vec{x}), \quad p = f^0(t,\vec{x})q(\omega) + g^0(t,\vec{x}) \tag{1.7}$$

that reduce system (1.1) in four independent variables to systems of differential equations in the functions v^a and q depending on the variables $\omega = \{\omega_n\}$ ($n = \overline{1,N}$), where N takes on a fixed value from the set $\{1,2,3\}$. In formulas (1.7) f^{ab}, g^a, f^0, g^0, and ω_n are smooth functions to be described. In such a general formulation the reduction problem is too complex to solve. But using Lie symmetry, some ansatzes (1.7) reducing the NSEs can be obtained. According to the Lie method, first a complete set of $A(NS)$-inequivalent subalgebras of dimension $M = 4 - N$ is to be constructed. For $N = 3$, $N = 2$, and $N = 1$ such sets are given in Subsections A.2, A.3, and A.4, correspondingly. Knowing subalgebraic structure of $A(NS)$, one can find explicit forms for the functions f^{ab}, g^a, f^0, g^0, and ω_n and obtain reduced systems in the functions v^k and q. This is made in Sec. 2 ($N = 3$), Sec. 3 ($N = 2$) and Sec. 4 ($N = 1$). Moreover, in Subsec. 2.3 symmetry properties of all reduced systems of PDEs in three independent variables are investigated, and in Subsec. 4.3 exact solutions of the reduced systems of ODEs are constructed. Symmetry properties and exact solutions of some reduced systems of PDEs in two independent variables are discussed in Sections 5 and 6. In Sec. 7 we make symmetry reduction of a some reduced system of PDEs in three independent variables.

In conclusion of the section, for convenience, we give some abbreviations, notations, and default rules used in this article.

Abbreviations:

the NSEs: the Navier-Stokes equations

the MIA: the maximal Lie invariance algebra (of either a some equation or a some system of equations)

a ODE: a ordinary differential equation

a PDE: a partial differential equation

Notations:

$C^\infty((t_0,t_1),\mathbb{R})$**:** the set of infinite-differentiable functions from (t_0,t_1) into $\mathbb{R}$, where $-\infty \le t_0 < t_1 \le +\infty$

$C^\infty((t_0,t_1),\mathbb{R}^3)$**:** the set of infinite-differentiable vector-functions from (t_0,t_1) into $\mathbb{R}^3$, where $-\infty \le t_0 < t_1 \le +\infty$

$\partial_t = \partial/\partial_t$, $\partial_a = \partial/\partial_{x_a}$, $\partial_{u^a} = \partial/\partial_{u^a}$, ...

Default rules:

Repead indices denote summation whereby we consider the indices a, b to take on values in $\{1,2,3\}$ and the indices i, j to take on values in $\{1,2\}$.

All theorems on the MIAs of PDEs are proved by means of the standard Lie algorithm.

Subscripts of functions denote differentiation.

2 Reduction of the Navier-Stokes equations to systems of PDEs in three independent variables

2.1 Ansatzes of codimension one

In this subsection we give ansatzes that reduce the NSEs to systems of PDEs in three independent variables. The ansatzes are constructed with the subalgebraic analysis of $A(NS)$ (see Subsec.A.2) by means of the method discribed in Sec.B .

1. $$\begin{aligned} u^1 &= |t|^{-1/2}(v^1\cos\tau - v^2\sin\tau) + \tfrac{1}{2}x_1t^{-1} - \varkappa x_2t^{-1},\\ u^2 &= |t|^{-1/2}(v^1\sin\tau + v^2\cos\tau) + \tfrac{1}{2}x_2t^{-1} + \varkappa x_1t^{-1},\\ u^3 &= |t|^{-1/2}v^3 + \tfrac{1}{2}x_3t^{-1},\\ p &= |t|^{-1}q + \tfrac{1}{2}\varkappa^2t^{-2}r^2 + \tfrac{1}{8}t^{-2}x_ax_a, \end{aligned} \tag{2.1}$$

where

$$y_1 = |t|^{-1/2}(x_1\cos\tau + x_2\sin\tau),\quad y_2 = |t|^{-1/2}(-x_1\sin\tau + x_2\cos\tau),$$

$y_3 = |t|^{-1/2}x_3, \quad \varkappa \geq 0, \quad \tau = \varkappa \ln|t|.$

Here and below $v^a = v^a(y_1, y_2, y_3), \quad q = q(y_1, y_2, y_3), \quad r = (x_1^2 + x_2^2)^{1/2}.$

$$\begin{aligned} 2. \quad u^1 &= v^1 \cos \varkappa t - v^2 \sin \varkappa t - \varkappa x_2, \\ u^2 &= v^1 \sin \varkappa t + v^2 \cos \varkappa t + \varkappa x_1, \\ u^3 &= v^3, \\ p &= q + \tfrac{1}{2}\varkappa^2 r^2, \end{aligned} \tag{2.2}$$

where

$y_1 = x_1 \cos \varkappa t + x_2 \sin \varkappa t, \quad y_2 = -x_1 \sin \varkappa t + x_2 \cos \varkappa t,$

$y_3 = x_3, \quad \varkappa \in \{0; 1\}.$

$$\begin{aligned} 3. \quad u^1 &= x_1 r^{-1} v^1 - x_2 r^{-1} v^2 + x_1 r^{-2}, \\ u^2 &= x_2 r^{-1} v^1 + x_1 r^{-1} v^2 + x_2 r^{-2}, \\ u^3 &= v^3 + \eta(t) r^{-1} v^2 + \eta_t(t) \arctan x_2/x_1, \\ p &= q - \tfrac{1}{2}\eta_{tt}(t)(\eta(t))^{-1} x_3^2 - \tfrac{1}{2} r^{-2} + \chi(t) \arctan x_2/x_1, \end{aligned} \tag{2.3}$$

where

$y_1 = t, \quad y_2 = r, \quad y_3 = x_3 - \eta(t) \arctan x_2/x_1, \quad \eta, \chi \in C^\infty((t_0, t_1), \mathbb{R}).$

Note 2.1 *The expression for the pressure p from ansatz (2.3) is indeterminate in the points $t \in (t_0, t_1)$ where $\eta(t) = 0$. If there are such points t, we will consider ansatz (2.3) on the intervals (t_0^n, t_1^n) that are contained in the interval (t_0, t_1) and that satisfy one of the conditions:*

a) $\eta(t) \neq 0 \quad \forall t \in (t_0^n, t_1^n)$;

b) $\eta(t) = 0 \quad \forall t \in (t_0^n, t_1^n)$.

In the last case we consider $\eta_{tt}/\eta := 0$.

$$\begin{aligned} 4. \quad \vec{u} &= v^i \vec{n}^i + (\vec{m} \cdot \vec{m})^{-1} v^3 \vec{m} + (\vec{m} \cdot \vec{m})^{-1} (\vec{m} \cdot \vec{x}) \vec{m}_t - y_i \vec{n}^i_t, \\ p &= q - \tfrac{3}{2}(\vec{m} \cdot \vec{m})^{-1}((\vec{m}_t \cdot \vec{n}^i) y_i)^2 - (\vec{m} \cdot \vec{m})^{-1}(\vec{m}_{tt} \cdot \vec{x})(\vec{m} \cdot \vec{x}) + \\ &\quad + \tfrac{1}{2}(\vec{m}_{tt} \cdot \vec{m})(\vec{m} \cdot \vec{m})^{-2}(\vec{m} \cdot \vec{x})^2, \end{aligned} \tag{2.4}$$

where

$y_i = \vec{n}^i \cdot \vec{x}, \quad y_3 = t, \quad \vec{m}, \vec{n}^i \in C^\infty((t_0, t_1), \mathbb{R}^3).$

$$\vec{n}^i \cdot \vec{m} = \vec{n}^1 \cdot \vec{n}^2 = \vec{n}^1_t \cdot \vec{n}^2 = 0, \quad |\vec{n}^i| = 1. \tag{2.5}$$

Note 2.2 *There exist vector-functions $\vec{n}^i$ which satisfy conditions (2.5). They can be constructed in the following way: let us fix the vector-functions $\vec{k}^i = \vec{k}^i(t)$ such that $\vec{k}^i \cdot \vec{m} = \vec{k}^1 \cdot \vec{k}^2 = 0$, $|\vec{k}^i| = 1$, and set*

$$\begin{aligned} \vec{n}^1 &= \vec{k}^1 \cos\psi(t) - \vec{k}^2 \sin\psi(t), \\ \vec{n}^2 &= \vec{k}^1 \sin\psi(t) + \vec{k}^2 \cos\psi(t). \end{aligned} \tag{2.6}$$

Then $\vec{n}^1_t \cdot \vec{n}^2 = \vec{k}^1_t \cdot \vec{k}^2 - \psi_t = 0$ *if* $\psi = \int(\vec{k}^1_t \cdot \vec{k}^2)dt$.

2.2 Reduced systems

1–2. Substituting ansatzes (2.1) and (2.2) into the NSEs (1.1), we obtain reduced systems of PDEs with the same general form

$$\begin{aligned} & v^a v^1_a - v^1_{aa} + q_1 + \gamma_1 v^2 = 0, \\ & v^a v^2_a - v^2_{aa} + q_2 - \gamma_1 v^1 = 0, \\ & v^a v^3_a - v^3_{aa} + q_3 = 0, \\ & v^a_a = \gamma_2. \end{aligned} \tag{2.7}$$

Hereafter subscripts 1, 2, and 3 of functions denote differentiation with respect to y_1, y_2, and y_3, accordingly. The constants γ_i take the values

1. $\gamma_1 = -2\varkappa$, $\gamma_2 = -\frac{3}{2}$ if $t > 0$, $\gamma_1 = 2\varkappa$, $\gamma_2 = \frac{3}{2}$ if $t < 0$.
2. $\gamma_1 = -2\varkappa$, $\gamma_2 = 0$.

For ansatzes (2.3) and (2.4) the reduced equations have the form

3. $$\begin{aligned} & v^1_1 + v^1 v^1_2 + v^3 v^1_3 - y_2^{-1} v^2 v^2 - (v^1_{22} + (1 + \eta^2 y_2^{-2}) v^1_{33}) - \\ & -2\eta y_2^{-2} v^2_3 + q_2 = 0, \\ & v^2_1 + v^1 v^2_2 + v^3 v^2_3 + y_2^{-1} v^1 v^2 - (v^2_{22} + (1 + \eta^2 y_2^{-2}) v^2_{33}) + \\ & +2\eta y_2^{-2} v^1_3 + 2 y_2^{-2} v^2 - \eta y_2^{-1} q_3 + \chi y_2^{-1} = 0, \\ & v^3_1 + v^1 v^3_2 + v^3 v^3_3 - (v^3_{22} + (1 + \eta^2 y_2^{-2}) v^3_{33}) - 2\eta^2 y_2^{-3} v^1_3 + \\ & +2\eta_1 y_2^{-1} v^2 + 2\eta y_2^{-1} (y_2^{-1} v^2)_2 + (1 + \eta^2 y_2^{-2}) q_3 - \\ & -\eta_{11} \eta^{-1} y_3 - \chi \eta y_2^{-2} = 0, \\ & y_2^{-1} v^1 + v^1_2 + v^3_3 = 0. \end{aligned} \tag{2.8}$$

4. $$\begin{aligned} & v^i_3 + v^j v^i_j - v^i_{jj} + q_i + \rho^i(y_3) v^3 = 0, \\ & v^3_3 + v^j v^3_j - v^3_{jj} = 0, \\ & v^i_i + \rho^3(y_3) = 0, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} &\rho^i = \rho^i(y_3) = 2(\vec{m}\cdot\vec{m})^{-1}(\vec{m}_t\cdot\vec{n}^i), \\ &\rho^3 = \rho^3(y_3) = (\vec{m}\cdot\vec{m})^{-1}(\vec{m}_t\cdot\vec{m}). \end{aligned} \tag{2.10}$$

2.3 Symmetry of reduced systems

Let us study symmetry properties of systems (2.7), (2.8), and (2.9). All results of this subsection are obtained by means of the standard Lie algorithm [28, 27]. First, let us consider system (2.7).

Theorem 2.1 *The MIA of system (2.7) is the algebra*

a) $<\partial_a, \partial_q, J^1_{12}>$ *if* $\gamma_1 \neq 0$;

b) $<\partial_a, \partial_q, J^1_{ab}>$ *if* $\gamma_1 = 0,\ \gamma_2 \neq 0$;

c) $<\partial_a, \partial_q, J^1_{ab}, D^1_1>$ *if* $\gamma_1 = \gamma_2 = 0$.

Here $J^1_{ab} = y_a\partial_b - y_b\partial_a + v^a\partial_{v^b} - v^b\partial_{v^a}$,

$D^1_1 = y_a\partial_a - v^a\partial_{v^a} - 2q\partial_q$.

Note 2.3 *All Lie symmetry operators of (2.7) are induced by operators from* $A(NS)$*: The operators* J^1_{ab} *and* D^1_1 *are induced by* J_{ab} *and* D*. The operators* $c_a\partial_a$ $(c_a = \mathrm{const})$ *and* ∂_q *are induced by either*

$$R(|t|^{1/2}(c_1\cos\tau - c_2\sin\tau, c_1\sin\tau + c_2\cos\tau, c_3)), \quad Z(|t|^{-1}),$$

where $\tau = \varkappa\ln|t|$*, for ansats (2.1) or*

$$R(c_1\cos\varkappa t - c_2\sin\varkappa t, c_1\sin\varkappa t + c_2\cos\varkappa t, c_3), \quad Z(1)$$

for ansatz (2.2), respectively. Therefore, Lie reductions of system (2.7) give only solutions that can be obtained by reducing the NSEs with two- and three-dimensional subalgebras of $A(NS)$*.*

Let us continue to system (2.8). We denote A^{max} as the MIA of (2.8). Studying symmetry properties of (2.8), one has to consider the following cases:

A. $\eta, \chi \equiv 0$. Then

$$A^{max} = <\partial^1, D^1_2, R_1(\psi(y_1)), Z^1(\lambda(y_1))>,$$

where $D^1_2 = 2y_1\partial_1 + y_2\partial_2 + y_3\partial_3 - v^a\partial_{v^a} - 2q\partial_q$,

$R_1(\psi(y_1)) = \psi\partial_3 + \psi_1\partial_{v^3} - \psi_{11}y_3\partial_q, \quad Z^1(\lambda(y_1)) = \lambda(y_1)\partial_q.$

Here and below $\psi = \psi(y_1)$ and $\lambda = \lambda(y_1)$ are arbitrary smooth functions of $y_1 = t$.

B. $\eta \equiv 0,\ \chi \not\equiv 0$. In this case an extension of A^{max} exists for $\chi = (C_1y_1 + C_2)^{-1}$, where $C_1, C_2 = \mathrm{const}$. Let $C_1 \neq 0$. We can make C_2

vanish by means of equivalence transformation (A.6), i.e., $\chi = Cy_1^{-1}$, where $C = \text{const}$. Then

$$A^{max} =< D_2^1, R_1(\psi(y_1)), Z^1(\lambda(y_1)) > .$$

If $C_1 = 0$, $\chi = C = \text{const}$ and

$$A^{max} =< \partial_1, R_1(\psi(y_1)), Z^1(\lambda(y_1)) > .$$

For other values of χ, i.e., when $\chi_{11}\chi \neq \chi_1\chi_1$,

$$A^{max} =< R_1(\psi(y_1)), Z^1(\lambda(y_1)) > .$$

C. $\eta \neq 0$. By means of equivalence transformation (A.6) we make $\chi = 0$. In this case an extension of A^{max} exists for $\eta = \pm|C_1y_1 + C_2|^{1/2}$, where $C_1, C_2 = \text{const}$. Let $C_1 \neq 0$. We can make C_2 vanish by means of equivalence transformation (A.6), i.e., $\eta = C|y_1|^{1/2}$, where $C = \text{const}$. Then

$$A^{max} =< D_2^1, R_2(|y_1|^{1/2}), R_2(|y_1|^{1/2}\ln|y_1|), Z^1(\lambda(y_1)) >,$$

where $R_2(\psi(y_1)) = \psi\partial_3 + \psi_1\partial_{v^3}$. If $C_1 = 0$, i.e., $\eta = C = \text{const}$,

$$A^{max} =< \partial^1, \partial_3, y_1\partial_3 + \partial_{v^3} Z^1(\lambda(y_1)) > .$$

For other values of η, i.e., when $(\eta^2)_{11} \neq 0$,

$$A^{max} =< R_2(\eta(y_1)), R_2(\eta(y_1)\textstyle\int(\eta(y_1))^{-2}dy_1), Z^1(\lambda(y_1)) > .$$

Note 2.4 *In all cases considered above the Lie symmetry operators of (2.8) are induced by operators from $A(NS)$: The operators ∂_1, D_2^1, and $Z^1(\lambda(y_1))$ are induced by ∂_t, D, and $Z(\lambda(t))$, respectively. The operator $R(0,0,\psi(t))$ induces the operator $R_1(\psi(y_1))$ for $\eta \equiv 0$ and the operator $R_2(\psi(y_1))$ (if $\psi_{11}\eta - \psi\eta_{11} = 0$) for $\eta \neq 0$. Therefore, the Lie reduction of system (2.8) gives only solutions that can be obtained by reducing the NSEs with two- and three-dimentional subalgebras of $A(NS)$.*

When $\eta = \chi = 0$, system (2.8) describes axially symmetric motion of a fluid and can be transformed into a system of two equations for a stream function Ψ^1 and a function Ψ^2 that are determined by

$$\Psi^1_3 = y_2v^1, \quad \Psi^1_2 = -y_2v^3, \quad \Psi^2 = y_2v^2.$$

The transformed system was studied by L.V. Kapitanskiy [20, 21].

Consider system (2.9). Let us introduce the notations

$$t = y_3, \quad \rho = \rho(t) = \textstyle\int \rho^3(t)dt,$$

$$R_3(\psi^1(t), \psi^2(t)) = \psi^i\partial_{y_i} + \psi^i_t\partial_{v^i} - \psi^i_{tt}y_i\partial_q,$$

$$Z^1(\lambda(t)) = \lambda(t)\partial_q, \quad S = \partial_{v^3} - \rho^i(t)y_i\partial_q,$$

$$E(\chi(t)) = 2\chi\partial_t + \chi_t y_i\partial_{y_i} + (\chi_{tt}y_i - \chi_t v^i)\partial_{v^i} - (2\chi_t q + \tfrac{1}{2}\chi_{ttt}y_jy_j)\partial_q,$$

$$J^1_{12} = y_1\partial_2 - y_2\partial_1 + v^1\partial_{v^2} - v^2\partial_{v^1}.$$

Theorem 2.2 *The MIA of(2.9) is the algebra*

1) $< R_3(\psi^1(t),\ \psi^2(t)),\ Z^1(\lambda(t)),\ S,\ E(\chi^1(t)),\ E(\chi^2(t)),\ v^3\partial_{v^3},\ J^1_{12} >$, *where* $\chi^1 = e^{-\rho(t)}\int e^{\rho(t)}dt$ *and* $\chi^2 = e^{-\rho(t)}$, *if* $\rho^i = 0$;

2) $< R_3(\psi^1(t), \psi^2(t)), Z^1(\lambda(t)), S, E(\chi(t))+2a_1v^3\partial_{v^3}+2a_2J^1_{12} >$, *where* a_1, a_2, *and* a_3 *are fixed constants,* $\chi = e^{-\rho(t)}\left(\int e^{\rho(t)}dt + a_3\right)$, *if*

$$\rho^1 = e^{\frac{3}{2}\rho}\hat{\rho}^{-\frac{3}{2}-a_1}\Big(C_1\cos(a_2\ln\hat{\rho}) - C_2\sin(a_2\ln\hat{\rho})\Big),$$

$$\rho^2 = e^{\frac{3}{2}\rho}\hat{\rho}^{-\frac{3}{2}-a_1}\Big(C_1\sin(a_2\ln\hat{\rho}) + C_2\cos(a_2\ln\hat{\rho})\Big)$$

with $\hat{\rho} = \hat{\rho}(t) = |\int e^{\rho(t)}dt + a_3|$, $C_1, C_2 = \mathrm{const}$, $(C_1, C_2) \neq (0,0)$;

3) $< R_3(\psi^1(t), \psi^2(t)), Z^1(\lambda(t)), S, E(\chi(t))+2a_1v^3\partial_{v^3}+2a_2J^1_{12} >$, *where* a_1 *and* a_2 *are fixed constants,* $\chi = e^{-\rho(t)}$, *if*

$$\rho^1 = e^{\frac{3}{2}\rho - a_1\hat{\rho}}\Big(C_1\cos(a_2\hat{\rho}) - C_2\sin(a_2\hat{\rho})\Big),$$

$$\rho^2 = e^{\frac{3}{2}\rho - a_1\hat{\rho}}\Big(C_1\sin(a_2\hat{\rho}) + C_2\cos(a_2\hat{\rho})\Big)$$

with $\hat{\rho} = \hat{\rho}(t) = \int e^{\rho(t)}dt$, $C_1, C_2 = \mathrm{const}$, $(C_1, C_2) \neq (0,0)$;

4) $< R_3(\psi^1(t),\ \psi^2(t)),\ Z^1(\lambda(t)),\ S >$ *in all other cases.*

Here $\psi^i = \psi^i(t)$, $\lambda = \lambda(t)$ *are arbitrary smooth function of* $t = y_3$.

Note 2.5 *If functions* ρ^b *are determined by (2.10), then* $e^{\rho(t)} = C|\vec{m}(t)|$, *where* $C = \mathrm{const}$, *and the condition* $\rho^i = 0$ *implies that* $\vec{m} = |\vec{m}(t)|\vec{e}$, *where* $\vec{e} = \mathrm{const}$ *and* $|\vec{e}| = 1$.

Note 2.6 *The vector-functions* $\vec{n}^i$ *from Note 2.2 are determined up to the transformation*

$$\vec{n}^1 = \vec{n}^1\cos\delta - \vec{n}^2\sin\delta, \quad \vec{n}^2 = \vec{n}^1\sin\delta + \vec{n}^2\cos\delta,$$

where $\delta = \mathrm{const}$. *Therefore,* δ *can be chosen such that* $C_2 = 0$ *(then* $C_1 \neq 0$*).*

Note 2.7 *The operators* $R_3(\psi^1, \psi^2)+\alpha S$ *and* $Z^1(\lambda)$ *are induced by* $R(\vec{l})+Z(\chi)$ *and* $Z(\lambda)$, *respectively. Here* $\vec{l} = \psi^i\vec{n}^i + \psi^3\vec{m}$, $\psi^3_t(\vec{m}\cdot\vec{m}) + 2\psi^i(\vec{n}^i_t\cdot\vec{m}) = \alpha$,

$$\chi - \tfrac{3}{2}(\vec{m}\cdot\vec{m})^{-1}((\vec{m}_t\cdot\vec{n}^i)\psi^i)^2 - \tfrac{1}{2}(\vec{m}_{tt}\cdot\vec{n}^i)\psi^3\psi^i + \tfrac{1}{2}(\vec{l}_{tt}\cdot\vec{n}^i)\psi^i = 0.$$

If $\vec{m} = |\vec{m}|\vec{e}$, *where* $\vec{e} = \mathrm{const}$ *and* $|\vec{e}| = 1$, *the operator* J^1_{12} *is induced by* $e^1J_{23} + e^2J_{31} + e^3J_{12}$.

For

$$\vec{m} = \beta_3 e^{\sigma t}(\beta_2 \cos\tau, \beta_2 \sin\tau, \beta_1)^T$$

with $\tau = \varkappa t + \delta$ *and* $\beta_a = \mathrm{const}$, *where* $\beta_1^2 + \beta_2^2 = 1$, *the operator* $\partial_t + \varkappa J_{12}$ *induces the operator* $\partial_{y_3} - \beta_1 \varkappa J^1_{12} + \sigma v^3 \partial_{v^3}$ *if the following vector-functions* $\vec{n}^i$ *are chosen:*

$$\vec{n}^1 = \vec{k}^1 \cos\beta_1\tau + \vec{k}^2 \sin\beta_1\tau, \quad \vec{n}^2 = -\vec{k}^1 \sin\beta_1\tau + \vec{k}^2 \cos\beta_1\tau, \tag{2.11}$$

where $\vec{k}^1 = (-\sin\tau, \cos\tau, 0)^T$ *and* $\vec{k}^2 = (\beta_1\cos\tau, \beta_1\sin\tau, -\beta_2)^T$.

For

$$\vec{m} = \beta_3 |t + \beta_4|^{\sigma+1/2}(\beta_2 \cos\tau, \beta_2 \sin\tau, \beta_1)^T$$

with $\tau = \varkappa \ln|t + \beta_4| + \delta$ *and* $\beta_a, \beta_4 = \mathrm{const}$, *where* $\beta_1^2 + \beta_2^2 = 1$, *the operator* $D + 2\beta_4\partial_t + 2\varkappa J_{12}$ *induces the operator*

$$D^1_3 + 2\beta_4\partial_{y_3} - 2\beta_1\varkappa J^1_{12} + 2\sigma v^3 \partial_{v^3},$$

where $D^1_3 = y_i\partial_{y_i} + 2y_3\partial_{y_3} - v^i\partial_{v^i} - 2q\partial_q$, *if the vector-functions* $\vec{n}^i$ *are chosen in form (2.11). In all other cases the basis elements of the MIA of (2.9) are not induced by operators from* $A(NS)$.

Note 2.8 *The invariance algebras of systems of form (2.9) with different parameter-functions* $\rho^3 = \rho^3(t)$ *and* $\tilde{\rho}^3 = \tilde{\rho}^3(t)$ *are similar . It suggests that there exists a local transformation of variables which make* ρ^3 *vanish. So, let us transform variables in the following way:*

$$\tilde{y}_i = y_i e^{\frac{1}{2}\rho(t)}, \quad \tilde{y}_3 = \int e^{\rho(t)} dt,$$

$$\tilde{v}^i = \Big(v^i + \tfrac{1}{2} y_i \rho^3(t)\Big) e^{-\frac{1}{2}\rho(t)}, \quad \tilde{v}^3 = v^3, \tag{2.12}$$

$$\tilde{q} = q e^{-\rho(t)} + \tfrac{1}{8} y_i y_i \Big((\rho^3(t)^2) - 2\rho^3_t(t)\Big) e^{-\rho(t)}.$$

As a result, we obtain the system

$$\tilde{v}^i_3 + \tilde{v}^j \tilde{v}^i_j - \tilde{v}^i_{jj} + \tilde{q}_i + \tilde{\rho}^i(\tilde{y}_3)\tilde{v}^3 = 0,$$

$$\tilde{v}^3_3 + \tilde{v}^j v^3_j - \tilde{v}^3_{jj} = 0,$$

$$\tilde{v}^i_i = 0$$

for the functions $\tilde{v}^a = \tilde{v}^a(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ *and* $\tilde{q} = \tilde{q}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$. *Here subscripts 1, 2, and 3 denote differentiation with respect to* $\tilde{y}_1$, $\tilde{y}_2$, *and* $\tilde{y}_3$, *accordingly. Also* $\tilde{\rho}^i(\tilde{y}_3) = \rho^i(t) e^{-\frac{3}{2}\rho(t)}$.

3 Reduction of the Navier-Stokes equations to systems of PDEs in two independent variables

3.1 Ansatzes of codimension two

In this subsection we give ansatzes that reduce the NSEs to systems of PDEs in two independent variables. The ansatzes are constructed with the subalgebrical analysis of $A(NS)$ (see Subsec. A.3) by means of the method discribed in Sec. B .

$$\begin{aligned} 1.\quad u^1 &= (rR)^{-1}((x_1-\varkappa x_2)w^1-x_2w^2+x_1x_3r^{-1}w^3),\\ u^2 &= (rR)^{-1}((x_2+\varkappa x_1)w^1+x_1w^2+x_2x_3r^{-1}w^3),\\ u^3 &= x_3(rR)^{-1}w^1-R^{-1}w^3,\\ p &= R^{-2}s, \end{aligned}\tag{3.1}$$

where $z_1=\arctan x_2/x_1-\varkappa\ln R$, $z_2=\arctan r/x_3$, $\varkappa\geq 0$.

Here and below $w^a=w^a(z_1,z_2)$, $s=s(z_1,z_2)$, $r=(x_1^2+x_2^2)^{1/2}$, $R=(x_1^2+x_2^2+x_3^2)^{1/2}$, $\varkappa$, ε, σ, μ, and ν are real constants.

$$\begin{aligned} 2.\quad u^1 &= |t|^{-1/2}r^{-1}(x_1w^1-x_2w^2)+\tfrac{1}{2}t^{-1}x_1+x_1r^{-2},\\ u^2 &= |t|^{-1/2}r^{-1}(x_2w^1+x_1w^2)+\tfrac{1}{2}t^{-1}x_2+x_2r^{-2},\\ u^3 &= |t|^{-1/2}w^3+\varkappa r^{-1}w^2+\tfrac{1}{2}t^{-1}x_3,\\ p &= |t|^{-1}s-\tfrac{1}{2}r^{-2}+\tfrac{1}{8}t^{-2}R^2+\varepsilon|t|^{-1}\arctan x_2/x_1, \end{aligned}\tag{3.2}$$

where $z_1=|t|^{-1/2}r$, $z_2=|t|^{-1/2}x_3-\varkappa\arctan x_2/x_1$, $\varkappa\geq 0$, $\varepsilon\geq 0$.

$$\begin{aligned} 3.\quad u^1 &= r^{-1}(x_1w^1-x_2w^2)+x_1r^{-2},\\ u^2 &= r^{-1}(x_2w^1+x_1w^2)+x_2r^{-2},\\ u^3 &= w^3+\varkappa r^{-1}w^2,\\ p &= s-\tfrac{1}{2}r^{-2}+\varepsilon\arctan x_2/x_1, \end{aligned}\tag{3.3}$$

where $z_1 = r$, $z_2 = x_3 - \varkappa \arctan x_2/x_1$, $\varkappa \in \{0;1\}$, $\varepsilon \geq 0$ if $\varkappa = 1$ and $\varepsilon \in \{0;1\}$ if $\varkappa = 0$.

4. $$\begin{aligned} u^1 &= |t|^{-1/2}(\mu w^1 + \nu w^3)\cos\tau - |t|^{-1/2}w^2\sin\tau + \\ &+\nu\xi t^{-1}\cos\tau + \tfrac{1}{2}t^{-1}x_1 - \varkappa t^{-1}x_2, \\ u^2 &= |t|^{-1/2}(\mu w^1 + \nu w^3)\sin\tau + |t|^{-1/2}w^2\cos\tau + \\ &+\nu\xi t^{-1}\sin\tau + \tfrac{1}{2}t^{-1}x_2 + \varkappa t^{-1}x_1, \\ u^3 &= |t|^{-1/2}(-\nu w^1 + \mu w^3) + \mu\xi t^{-1} + \tfrac{1}{2}t^{-1}x_3, \\ p &= |t|^{-1}s - \tfrac{1}{2}t^{-2}\xi^2 + \tfrac{1}{8}t^{-2}R^2 + \tfrac{1}{2}\varkappa^2 t^{-2}r^2 + \\ &+\varepsilon|t|^{-3/2}(\nu x_1\cos\tau + \nu x_2\sin\tau + \mu x_3), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} z_1 &= |t|^{-1/2}(\mu x_1\cos\tau + \mu x_2\sin\tau - \nu x_3), \\ z_2 &= |t|^{-1/2}(x_2\cos\tau - x_1\sin\tau), \\ \xi &= \sigma(\nu x_1\cos\tau + \nu x_2\sin\tau + \mu x_3) + 2\varkappa\nu(x_2\cos\tau - x_1\sin\tau), \end{aligned}$$

$$\tau = \varkappa\ln|t|, \quad \varkappa > 0, \quad \mu \geq 0, \quad \nu \geq 0, \quad \mu^2 + \nu^2 = 1, \quad \sigma\varepsilon = 0, \quad \varepsilon \geq 0.$$

5. $$\begin{aligned} u^1 &= |t|^{-1/2}w^1 + \tfrac{1}{2}t^{-1}x_1, \\ u^2 &= |t|^{-1/2}w^2 + \tfrac{1}{2}t^{-1}x_2, \\ u^3 &= |t|^{-1/2}w^3 + (\sigma + \tfrac{1}{2})t^{-1}x_3, \\ p &= |t|^{-1}s - \tfrac{1}{2}\sigma^2 t^{-2}x_3^2 + \tfrac{1}{8}t^{-2}R^2 + \varepsilon|t|^{-3/2}x_3, \end{aligned} \tag{3.5}$$

where

$$z_1 = |t|^{-1/2}x_1, \quad z_2 = |t|^{-1/2}x_2, \quad \sigma\varepsilon = 0, \quad \varepsilon \geq 0.$$

6. $$\begin{aligned} u^1 &= (\mu w^1 + \nu w^3)\cos t - w^2\sin t + \nu\xi\cos t - x_2, \\ u^2 &= (\mu w^1 + \nu w^3)\sin t + w^2\cos t + \nu\xi\sin t + x_1, \\ u^3 &= (-\nu w^1 + \mu w^3) + \mu\xi, \\ p &= s - \tfrac{1}{2}\xi^2 + \tfrac{1}{2}r^2 + \varepsilon(\nu x_1\cos t + \nu x_2\sin t + \mu x_3), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} z_1 &= (\mu x_1 \cos t + \mu x_2 \sin t - \nu x_3), \\ z_2 &= (x_2 \cos t - x_1 \sin t), \\ \xi &= \sigma(\nu x_1 \cos t + \nu x_2 \sin t + \mu x_3) + 2\nu(x_2 \cos t - x_1 \sin t), \end{aligned}$$

$$\mu \geq 0, \quad \nu \geq 0, \quad \mu^2 + \nu^2 = 1, \quad \sigma\varepsilon = 0, \quad \varepsilon \geq 0.$$

7. $$\begin{aligned} u^1 &= w^1, \quad u^2 = w^2, \quad u^3 = w^3 + \sigma x_3, \\ p &= s - \tfrac{1}{2}\sigma^2 x_3^2 + \varepsilon x_3, \end{aligned} \qquad (3.7)$$

where

$$z_1 = x_1, \quad z_2 = x_2, \quad \sigma\varepsilon = 0, \quad \varepsilon \in \{0; 1\}.$$

8. $$\begin{aligned} u^1 &= x_1 w^1 - x_2 r^{-2}(w^2 - \chi(t)), \\ u^2 &= x_2 w^1 + x_1 r^{-2}(w^2 - \chi(t)), \\ u^3 &= (\rho(t))^{-1}(w^3 + \rho_t(t) x_3 + \varepsilon \arctan x_2/x_1), \\ p &= s - \tfrac{1}{2}\rho_{tt}(t)(\rho(t))^{-1} x_3^2 + \chi_t(t) \arctan x_2/x_1, \end{aligned} \qquad (3.8)$$

where

$$z_1 = t, \quad z_2 = r, \quad \varepsilon \in \{0; 1\}, \quad \chi, \rho \in C^\infty((t_0, t_1), \mathbb{R}).$$

9. $$\begin{aligned} \vec{u} &= \vec{w} + \lambda^{-1}(\vec{n}^i \cdot \vec{x})\vec{m}^i_t - \lambda^{-1}(\vec{k} \cdot \vec{x})\vec{k}_t, \\ p &= s - \tfrac{1}{2}\lambda^{-1}(\vec{m}^i_{tt} \cdot \vec{x})(\vec{n}^i \cdot \vec{x}) - \tfrac{1}{2}\lambda^{-2}(m^i_{tt} \cdot \vec{k})(\vec{n}^i \cdot \vec{x})(\vec{k} \cdot \vec{x}), \end{aligned} \qquad (3.9)$$

where

$$z_1 = t, \quad z_2 = (\vec{k} \cdot \vec{x}), \quad \vec{m}^i \in C^\infty((t_0, t_1), \mathbb{R}^3),$$

$$\vec{m}^1_{tt} \cdot \vec{m}^2 - \vec{m}^1 \cdot \vec{m}^2_{tt} = 0, \quad \vec{k} = \vec{m}^1 \times \vec{m}^2, \quad \vec{n}^1 = \vec{m}^2 \times \vec{k},$$

$$\vec{n}^2 = \vec{k} \times \vec{m}^1, \quad \lambda = \lambda(t) = \vec{k} \cdot \vec{k} \neq 0 \quad \forall t \in (t_0, t_1).$$

3.2 Reduced systems

Substituting ansatzes (3.1)–(3.9) into the NSEs (1.1), we obtain the following systems of reduced equations:

$$\begin{aligned}
1.\quad & w^2w^1_1+w^3w^1_2-w^1w^3\cot z_2-(w^1)^2-(w^2+\varkappa w^1)^2\sin^2 z_2-\\
&-(w^3)^2-((\varkappa^2+\sin^{-2} z_2)w^1_{11}+w^1_{22}-\varkappa w^1_1-2w^3_2-2w^2_1-\\
&-2w^1)\sin z_2+w^1_2\cos z_2-w^1\sin^{-1} z_2-(2s+\varkappa s_1)\sin^2 z_2=0,\\
&w^2w^2_1+w^3w^2_2+w^3(w^2+2\varkappa w^1)\cot z_2-\\
&-\varkappa((w^1)^2+(w^3)^2+(w^2+\varkappa w^1)^2\sin^2 z_2)-\\
&-((\varkappa^2+\sin^{-2} z_2)w^2_{11}+w^2_{22}+3\varkappa w^2_1+2\varkappa(w^3_2+\varkappa w^1_1+w^1))\cdot\\
&\cdot\sin z_2+(2w^1_1+2w^3_1\cot z_2-w^2-2\varkappa w^1)\sin^{-1} z_2-\\
&-(w^2_2+2\varkappa w^1_2)\cos z_2+2\varkappa s\sin^2 z_2+(1+\varkappa^2\sin^2 z_2)s_1=0,\\
&w^2w^3_1+w^3w^3_2-(w^3)^2\cot z_2-(w^2+\varkappa w^1)^2\sin z_2\cos z_2-\\
&-((\varkappa^2+\sin^{-2} z_2)w^3_{11}+w^3_{22}+\varkappa w^3_1+2w^1_2)\sin z_2+\\
&+(2w^1+w^3_2+w^2_1+\varkappa w^1_1)\cos z_2+s_2\sin^2 z_2=0,\\
&w^1+w^2_1+w^3_2=0.
\end{aligned}\tag{3.10}$$

Hereafter numeration of the reduced systems corresponds to that of the ansatzes in Subsec. 3.1. Subscripts 1 and 2 denote differentiation with respect to the variables z_1 and z_2, accordingly.

$$\begin{aligned}
2\text{–}3.\quad & w^1w^1_1+w^3w^1_2-z_1^{-1}w^2w^2-(w^1_{11}+(1+\varkappa^2z_1^{-2})w^1_{22})-\\
&-2\varkappa z_1^{-2}w^2_2+s_1=0,\\
&w^1w^2_1+w^3w^2_2+z_1^{-1}w^1w^2-(w^2_{11}+(1+\varkappa^2z_1^{-2})w^2_{22})+\\
&+2\varkappa z_1^{-2}w^1_2+2z_1^{-2}w^2-\varkappa z_1^{-1}s_2+\varepsilon z_1^{-1}=0,\\
&w^1w^3_1+w^3w^3_2-2\varkappa z_1^{-2}w^1w^2-(w^3_{11}+(1+\varkappa^2z_1^{-2})w^3_{22})+\\
&+2\varkappa(z_1^{-2}w^2)_1-2\varkappa^2z_1^{-3}w^1_2+(1+\varkappa^2z_1^{-2})s_2-\varepsilon\varkappa z_1^{-2}=0,\\
&w^1_1+w^3_2+z_1^{-1}w^1+\gamma=0,
\end{aligned}\tag{3.11}$$

where $\gamma=\pm3/2$ for ansatz (3.2) and $\gamma=0$ for ansatz (3.3). Here and below the upper and lower sign in the symbols "$\pm$" and "$\mp$" are associated with $t>0$ and $t<0$, respectively.

4–7.For ansatzes (3.4)–(3.7) the reduced equations can be written in the form

$$
\begin{aligned}
&w^iw^1_i - w^1_{ii} + s_1 + \alpha_2 w^2 = 0,\\
&w^iw^2_i - w^2_{ii} + s_2 - \alpha_2 w^1 + \alpha_1 w^3 = 0,\\
&w^iw^3_i - w^3_{ii} + \alpha_4 w^3 + \alpha_5 = 0,\\
&w^i_i = \alpha_3
\end{aligned}
\tag{3.12}
$$

where the constants α_n $(n = \overline{1,5})$, take on the values

4. $\alpha_1 = \pm 2\varkappa\nu$, $\alpha_2 = \mp 2\varkappa\mu$, $\alpha_3 = \mp(\sigma + 3/2)$, $\alpha_4 = \pm\sigma$, $\alpha_5 = \varepsilon$.
5. $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = \mp(\sigma + 3/2)$, $\alpha_4 = \pm\sigma$, $\alpha_5 = \varepsilon$.
6. $\alpha_1 = 2\nu$, $\alpha_2 = -2\mu$, $\alpha_3 = -\sigma$, $\alpha_4 = \sigma$, $\alpha_5 = \varepsilon$.
7. $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = -\sigma$, $\alpha_4 = \sigma$, $\alpha_5 = \varepsilon$.

8. $$
\begin{aligned}
&w^1_1 + (w^1)^2 - z_2^{-4}(w^2 - \chi)^2 + z_2 w^1 w^1_2 - w^1_{22} -\\
&-3z_2 w^1_2 + z_2^{-1} s_2 = 0,
\end{aligned}
\tag{3.13}
$$

$$w^2_1 + z_2 w^1 w^2_2 - w^2_{22} + z_2^{-1} w^2_2 = 0, \tag{3.14}$$

$$w^3_1 + z_2 w^1 w^3_2 - w^3_{22} - z_2^{-1} w^3_2 + z_2^{-2}(w^2 - \chi) = 0, \tag{3.15}$$

$$2w^1 + z_2 w^1_2 + \rho_1/\rho = 0. \tag{3.16}$$

9. $$\vec{w}_1 - \lambda \vec{w}_{22} + s_2 \vec{k} + \lambda^{-1}(\vec{n}^i \cdot \vec{w})\vec{m}^i_t + z_2 \vec{e} = \vec{0}, \tag{3.17}$$

$$\vec{k} \cdot \vec{w}_2 = 0, \tag{3.18}$$

where $y_1 = t$ and

$$\vec{e} = \vec{e}(t) = 2\lambda^{-2}(\vec{m}^1_t \cdot \vec{m}^2 - \vec{m}^1 \cdot \vec{m}^2_t)\vec{k}_t \times \vec{k} + \lambda^{-2}(2\vec{k}_t \cdot \vec{k}_t - \vec{k}_{tt} \cdot \vec{k}).$$

Let us study symmetry properties of reduced systems (3.10) and (3.11).

Theorem 3.1 *The MIA of (3.10) is given by the algebra* $< \partial_1 >$.

Theorem 3.2 *The MIA of (3.11) is given by the following algebras:*

a) $< \partial_2, \partial_s, D^2_1 = z_i\partial_i - w^a\partial_{w^a} - 2s\partial_s >$ *if* $\gamma = \varkappa = \varepsilon = 0$;

b) $< \partial_2, \partial_s >$ *if* $(\gamma, \varkappa, \varepsilon) \neq (0, 0, 0)$.

All the Lie symmetry operators of systems (3.10) and (3.11) are induced by elements of $A(NS)$. So, for system (3.10) the operator ∂_1 is induced by J_{12}. For system (3.11), when $\gamma = 0$ ($\gamma = \pm 3/2$), the operators D^2_1, ∂_2, and ∂_s (∂_2 and ∂_s) are induced by D, $R(0, 0, 1)$, and $Z(1)$ ($R(0, 0, |t|^{-1/2})$ and $Z(|t|^{-1})$), accordingly. Therefore, the Lie reductions of systems (3.10) and (3.11) give only solutions that can be obtained by reducing the NSEs with three-dimensional subalgebras of $A(NS)$ immediately to ODEs.

Investigation of reduced systems (3.13)–(3.16), (3.17)–(3.18), and (3.12) is given in Sec. 5 and 6.

4 Reduction of the Navier-Stokes equations to ordinary differential equations

4.1 Ansatzes of codimension three

By means of subalgebraic analysis of $A(NS)$ (see Subsec. A.3) and the method described in Sec. B one can obtain the following ansatzes that reduce the NSEs to ODEs:

$$\begin{aligned} 1.\quad u^1 &= x_1R^{-2}\varphi^1 - x_2(Rr)^{-1}\varphi^2 + x_1x_3r^{-1}R^{-2}\varphi^3,\\ u^2 &= x_2R^{-2}\varphi^1 + x_1(Rr)^{-1}\varphi^2 + x_2x_3r^{-1}R^{-2}\varphi^3,\\ u^3 &= x_3R^{-2}\varphi^1 - rR^{-2}\varphi^3,\\ p &= R^{-2}h, \end{aligned} \tag{4.1}$$

where $\omega = \arctan r/x_3$. Here and below $\varphi^a = \varphi^a(\omega)$, $h = h(\omega)$, $r = (x_1^2 + x_2^2)^{1/2}$, $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

$$\begin{aligned} 2.\quad & u^1 = r^{-2}(x_1\varphi^1 - x_2\varphi^2), \quad u^2 = r^{-2}(x_2\varphi^1 + x_1\varphi^2),\\ & u^3 = r^{-1}\varphi^3, \quad p = r^{-2}h, \end{aligned} \tag{4.2}$$

where $\omega = \arctan x_2/x_1 - \varkappa \ln r$, $\varkappa \geq 0$.

$$\begin{aligned} 3.\quad u^1 &= x_1|t|^{-1}\varphi^1 - x_2r^{-2}\varphi^2 + \tfrac{1}{2}x_1t^{-1},\\ u^2 &= x_2|t|^{-1}\varphi^1 + x_1r^{-2}\varphi^2 + \tfrac{1}{2}x_2t^{-1},\\ u^3 &= |t|^{-1/2}\varphi^3 + (\sigma + \tfrac{1}{2})x_3t^{-1} + \nu|t|^{1/2}t^{-1}\arctan x_2/x_1,\\ p &= |t|^{-1}h + \tfrac{1}{8}t^{-2}R^2 - \tfrac{1}{2}\sigma^2x_3^2t^{-2} +\\ &\quad +\varepsilon_1|t|^{-1}\arctan x_2/x_1 + \varepsilon_2x_3|t|^{-3/2}, \end{aligned} \tag{4.3}$$

where $\omega = |t|^{-1/2}r$, $\nu\sigma = 0$, $\varepsilon_2\sigma = 0$, $\varepsilon_1 \geq 0$, $\nu \geq 0$.

$$\begin{aligned} 4.\quad u^1 &= x_1\varphi^1 - x_2r^{-2}\varphi^2,\\ u^2 &= x_2\varphi^1 + x_1r^{-2}\varphi^2,\\ u^3 &= \varphi^3 + \sigma x_3 + \nu\arctan x_2/x_1,\\ p &= h - \tfrac{1}{2}\sigma^2x_3^2 + \varepsilon_1\arctan x_2/x_1 + \varepsilon_2x_3, \end{aligned} \tag{4.4}$$

where $\omega = r$, $\nu\sigma = 0$, $\varepsilon_2\sigma = 0$, and for $\sigma = 0$ one of the conditions

$$\nu = 1,\ \varepsilon_1 \geq 0; \quad \nu = 0,\ \varepsilon_1 = 1,\ \varepsilon_2 \geq 0; \quad \nu = \varepsilon_1 = 0,\ \varepsilon_2 \in \{0;1\}$$

is satisfied.

Two ansatzes are described better in the following way:

5. The expressions for u^a and p are determined by (2.1), where

$$\begin{aligned} v^1 &= a_1\varphi^1 + a_2\varphi^3 + b_{1i}\omega_i, \\ v^2 &= \varphi^2 + b_{2i}\omega_i, \\ v^3 &= a_2\varphi^1 - a_1\varphi^3 + b_{3i}\omega_i, \\ p &= h + c_{1i}\omega_i + c_{2i}\omega\omega_i + \tfrac{1}{2}d_{ij}\omega_i\omega_j. \end{aligned} \tag{4.5}$$

In formulas (4.5) we use the following definitions:

$$\omega_1 = a_1y_1 + a_2y_3, \quad \omega_2 = y_2, \quad \omega = \omega_3 = a_2y_1 - a_1y_3;$$

$$a_i = \text{const}, \quad a_1^2 + a_2^2 = 1; \quad a_2 = 0 \text{ if } \gamma_1 = 0;$$

$$\gamma_1 = -2\varkappa,\ \gamma_2 = -\tfrac{3}{2} \quad \text{if} \quad t > 0 \quad \text{and} \quad \gamma_1 = 2\varkappa,\ \gamma_2 = \tfrac{3}{2} \quad \text{if} \quad t < 0.$$

b_{ai}, B_i, c_{ij}, and d_{ij} are real constants that satisfy the equations

$$\begin{aligned} & b_{1i} = a_1B_i, \quad b_{3i} = a_2B_i, \quad c_{2i} + a_2\gamma_1b_{2i} = 0, \\ & b_{21}B_i + b_{22}b_{2i} - \gamma_1a_1B_i + d_{2i} = 0, \\ & B_1B_i + B_2b_{2i} + \gamma_1a_1B_i + d_{1i} = 0, \\ & (B_1 + b_{22})(B_2 + a_1\gamma_1 - b_{21}) = 0. \end{aligned} \tag{4.6}$$

6. The expressions for u^a and p have form (2.2), where v^a and q are determined by (4.5), (4.6), and $\gamma_1 = -2\varkappa$, $\gamma_2 = 0$.

Note 4.1 *Formulas (4.5) and (4.6) determine an ansatz for system (2.7), where equations (4.6) are the necessary and sufficient condition to reduce system (2.7) by means of an ansatz of form (4.5).*

7.
$$\begin{aligned} u^1 &= \varphi^1\cos x_3/\eta^3 - \varphi^2\sin x_3/\eta^3 + x_1\theta^1(t) + x_2\theta^2(t), \\ u^2 &= \varphi^1\sin x_3/\eta^3 + \varphi^2\cos x_3/\eta^3 - x_1\theta^2(t) + x_2\theta^1(t), \\ u^3 &= \varphi^3 + \eta^3_t(\eta^3)^{-1}x_3, \\ p &= h - \tfrac{1}{2}\eta^3_{tt}(\eta^3)^{-1}x_3^2 - \tfrac{1}{2}\eta^j_{tt}\eta^j(\eta^i\eta^i)^{-1}r^2, \end{aligned} \tag{4.7}$$

where $\omega = t$,

$$\eta^a \in C^\infty((t_0,t_1),\mathbb{R}), \quad \eta^3 \neq 0, \quad \eta^i\eta^i \neq 0, \quad \eta^1_t\eta^2 - \eta^1\eta^2_t \in \{0; \tfrac{1}{2}\},$$

$$\theta^1 = \eta^i_t\eta^i(\eta^j\eta^j)^{-1}, \quad \theta^2 = (\eta^1_t\eta^2 - \eta^1\eta^2_t)(\eta^j\eta^j)^{-1}.$$

8.
$$\begin{aligned} \vec{u} &= \vec{\varphi} + \lambda^{-1}(\vec{n}^a\cdot\vec{x})\vec{m}^a_t, \\ p &= h - \lambda^{-1}(\vec{m}^a_{tt}\cdot\vec{x})(\vec{n}^a\cdot\vec{x}) + \\ &\quad +\tfrac{1}{2}\lambda^{-2}(\vec{m}^b_{tt}\cdot\vec{m}^a)(\vec{n}^a\cdot\vec{x})(\vec{n}^b\cdot\vec{x}), \end{aligned} \tag{4.8}$$

where $\omega = t$, $\vec{m}^a \in C^\infty((t_0,t_1),\mathbb{R})$, $\vec{m}^a_{tt}\cdot\vec{m}^b - \vec{m}^a\cdot\vec{m}^b_{tt} = 0$,

$$\lambda = \lambda(t) = (\vec{m}^1\times\vec{m}^2)\cdot\vec{m}^3 \neq 0 \quad \forall t \in (t_0,t_1),$$

$$\vec{n}^1 = \vec{m}^2\times\vec{m}^3, \quad \vec{n}^2 = \vec{m}^3\times\vec{m}^1, \quad \vec{n}^3 = \vec{m}^1\times\vec{m}^2.$$

4.2 Reduced systems

Substituting the ansatzes 1–8 into the NSEs (1.1), we obtain the following systems of ODE in the functions φ^a and h:

1.
$$\begin{aligned} &\varphi^3\varphi^1_\omega - \varphi^a\varphi^a - \varphi^1_{\omega\omega} - \varphi^1_\omega\cot\omega - 2h = 0, \\ &\varphi^3\varphi^2_\omega + \varphi^2\varphi^3\cot\omega - \varphi^2_{\omega\omega} - \varphi^2_\omega\cot\omega + \varphi^2\sin^{-2}\omega = 0, \\ &\varphi^3\varphi^3_\omega - \varphi^2\varphi^2\cot\omega - \varphi^3_{\omega\omega} - \varphi^3_\omega\cot\omega + \varphi^3\sin^{-2}\omega + \\ &-2\varphi^1_\omega + h_\omega = 0, \\ &\varphi^1 + \varphi^3_\omega + \varphi^3\cot\omega = 0. \end{aligned} \tag{4.9}$$

2.
$$\begin{aligned} &(\varphi^2 - \varkappa\varphi^1)\varphi^1_\omega - (1+\varkappa^2)\varphi^1_{\omega\omega} - \varphi^1\varphi^1 - \varphi^2\varphi^2 - \varkappa h_\omega - 2h = 0, \\ &(\varphi^2 - \varkappa\varphi^1)\varphi^2_\omega - (1+\varkappa^2)\varphi^2_{\omega\omega} - 2(\varkappa\varphi^2_\omega + \varphi^1_\omega) + h_\omega = 0, \\ &(\varphi^2 - \varkappa\varphi^1)\varphi^3_\omega - (1+\varkappa^2)\varphi^3_{\omega\omega} - \varphi^1\varphi^3 - \varphi^3 - 2\varkappa\varphi^3_\omega = 0, \\ &\varphi^2_\omega - \varkappa\varphi^1_\omega = 0. \end{aligned} \tag{4.10}$$

3–4.
$$\begin{aligned} &\varphi^1\varphi^1 - \omega^{-4}\varphi^2\varphi^2 + \omega\varphi^1\varphi^1_\omega - \varphi^1_{\omega\omega} - 3\omega^{-1}\varphi^1_\omega + \omega^{-1}h_\omega = 0, \\ &\omega\varphi^1\varphi^2_\omega - \varphi^2_{\omega\omega} + \omega^{-1}\varphi^2_\omega + \varepsilon_1 = 0, \\ &\omega\varphi^1\varphi^3_\omega + \sigma_1\varphi^3 + \nu\omega^{-2}\varphi^2 - \varphi^3_{\omega\omega} - \omega^{-1}\varphi^3_\omega + \varepsilon_2 = 0, \\ &2\varphi^1 + \omega\varphi^1_\omega + \sigma_2 = 0, \end{aligned} \tag{4.11}$$

where

3. $\sigma_1 = \sigma, \qquad \sigma_2 = (\sigma + \frac{3}{2}) \qquad \text{if} \quad t > 0,$
$\sigma_1 = -\sigma, \qquad \sigma_2 = -(\sigma + \frac{3}{2}) \qquad \text{if} \quad t < 0.$

4. $\sigma_1 = \sigma_2 = \sigma.$

5–6.
$$\begin{aligned} &\varphi^3\varphi^1_\omega - \varphi^1_{\omega\omega} - \mu_{1i}\varphi^i + c_{11} + c_{21}\omega = 0,\\ &\varphi^3\varphi^2_\omega - \varphi^2_{\omega\omega} - \mu_{2i}\varphi^i + c_{12} + c_{22}\omega + \gamma_2 a_2\varphi^3 = 0,\\ &\varphi^3\varphi^3_\omega - \varphi^3_{\omega\omega} + \gamma_1 a_2\varphi^2 + h_\omega = 0,\\ &\varphi^3_\omega = \sigma, \end{aligned} \tag{4.12}$$

where $\mu_{11} = -B_1$, $\mu_{12} = -B_2 - \gamma_1 a_1$, $\mu_{21} = -b_{21} + \gamma_1 a_1$, $\mu_{22} = -b_{22}$, $\sigma = \gamma_1 - B_1 - b_{22}$.

7.
$$\begin{aligned} &\varphi^1_\omega + \theta^1\varphi^1 + \theta^2\varphi^2 - (\eta^3)^{-1}\varphi^3\varphi^2 + (\eta^3)^{-2}\varphi^1 = 0,\\ &\varphi^2_\omega - \theta^2\varphi^1 + \theta^1\varphi^2 + (\eta^3)^{-1}\varphi^3\varphi^1 + (\eta^3)^{-2}\varphi^2 = 0,\\ &\varphi^3_\omega + \eta^3_t(\eta^3)^{-1}\varphi^3 = 0,\\ &2\theta^1 + \eta^3_t(\eta^3)^{-1} = 0. \end{aligned} \tag{4.13}$$

8.
$$\begin{aligned} &\vec{\varphi}_\omega + \lambda^{-1}(\vec{n}^b\cdot\vec{\varphi})\vec{m}^b_t = 0,\\ &\vec{n}^a\cdot\vec{m}^a_t = 0. \end{aligned} \tag{4.14}$$

4.3 Exact solutions of the reduced systems

1. Ansatz (4.1) and system (4.9) determine the class of solutions of the NSEs (1.1) that are called the steady axially symmetric conically similar flows of a viscous fluid in hydrodynamics. This class of solutions was studied in a number of works (for example, see references in [16]). For $\varphi^2 = 0$ it was shown, by N.A.Slezkin [34], that system (4.9) is reduced to a Riccati equation. The general solution of this equation was expressed in terms of hypergeometric functions. Later similar calculations were made by V.I.Yatseev [38] and H.B.Squire [35]. The particular case in the class of solutions with $\varphi^2 = 0$ is formed by the Landau jets [24]. For swirling flows, where $\varphi^2 \neq 0$, the order of system (4.9) can be reduced too. For example [33], an arbitrary solution of (4.9) satisfies the equation

$$\varphi^2\varphi^2\sin^2\omega - \sin\omega(\Phi_\omega\sin^{-1}\omega)_\omega + 2\Phi_\omega\cot\omega + 2\Phi = \text{const},$$

where $\Phi = (\varphi^3_\omega - \frac{1}{2}\varphi^3\varphi^3)\sin^2\omega - \varphi^3\cos\omega\sin\omega$, and the Yatseev results [38] are completely extended to the case $\varphi^2\sin\omega = \text{const}$.

2. System (4.10) implies that

$$\begin{aligned}
&\varphi^2 = \varkappa\varphi^1 + C_1,\\
&h = \varkappa(1+\varkappa^2)\varphi^1_\omega + (2\varkappa^2 + 2 - \varkappa C_1)\varphi^1 + C_2,\\
&(1+\varkappa^2)\varphi^1_{\omega\omega} + (4\varkappa - C_1)\varphi^1_\omega + \varphi^1\varphi^1 + 4\varphi^1 +\\
&+(1+\varkappa^2)^{-1}(C_1^2 + 2C_2) = 0,\\
&(1+\varkappa^2)\varphi^3_{\omega\omega} - (C_1 - 2\varkappa)\varphi^3_\omega + (1+\varphi^1)\varphi^3 = 0.
\end{aligned} \tag{4.15}$$

If $\varphi^3 = 0$, the solution determined by ansatz (4.10) and formulas (4.15) coincides with the Hamel solution [18, 23]. In Sec. 6 we consider system (6.14) which is more general than system (4.10).

3–4. Let us integrate the last equation of system (4.11), i.e.,

$$\varphi^1 = C_1\omega^{-2} - \tfrac{1}{2}\sigma_2. \tag{4.16}$$

Taking into account the integration result, the other equations of system (4.11) can be written in the form

$$\begin{aligned}
&h_\omega = \omega^{-3}\varphi^2\varphi^2 + C_1^2\omega^{-3} - \tfrac{1}{4}\sigma_2^2\omega,\\
&\varphi^2_{\omega\omega} - ((C_1+1)\omega^{-1} - \tfrac{1}{2}\sigma_2\omega)\varphi^2_\omega = \varepsilon_1,\\
&\varphi^3_{\omega\omega} - ((C_1-1)\omega^{-1} - \tfrac{1}{2}\sigma_2\omega)\varphi^3_\omega - \sigma_1\varphi^3 = \nu\omega^{-2}\varphi^2 + \varepsilon_2.
\end{aligned} \tag{4.17}$$

Therefore,

$$h = \textstyle\int \omega^{-3}\varphi^2\varphi^2 d\omega - \tfrac{1}{2}C_1^2\omega^{-2} - \tfrac{1}{8}\sigma_2^2\omega^2, \tag{4.18}$$

$$\begin{aligned}
\varphi^2 = {}& C_2 + C_3\textstyle\int |\omega|^{C_1+1}e^{-\frac{1}{4}\sigma_2\omega^2}d\omega +\\
&+\varepsilon_1\textstyle\int |\omega|^{C_1+1}e^{-\frac{1}{4}\sigma_2\omega^2}\Big(\int |\omega|^{-C_1-1}e^{\frac{1}{4}\sigma_2\omega^2}d\omega\Big)d\omega.
\end{aligned} \tag{4.19}$$

If $\sigma_1 = 0$, it follows that

$$\begin{aligned}
\varphi^3 = {}& C_4 + C_5\textstyle\int |\omega|^{C_1-1}e^{-\frac{1}{4}\sigma_2\omega^2}d\omega +\\
&+\textstyle\int |\omega|^{C_1-1}e^{-\frac{1}{4}\sigma_2\omega^2}\Big(\int |\omega|^{-C_1+1}e^{\frac{1}{4}\sigma_2\omega^2}(\varepsilon_2 + \nu\omega^{-2}\varphi^2)d\omega\Big)d\omega.
\end{aligned} \tag{4.20}$$

Let $\sigma_1 \neq 0$ (and, therefore, $\nu = 0$). Then, if $\sigma_2 \neq 0$, the general solution of equation (4.17) is expressed in terms of Whittaker functions:

$$\varphi^3 = |\omega|^{\frac{1}{2}C_1-1}e^{-\frac{1}{8}\sigma_2\omega^2}W(-\sigma_1\sigma_2^{-1} + \tfrac{1}{4}C_1 - \tfrac{1}{2}, \tfrac{1}{4}C_1, \tfrac{1}{4}\sigma_2\omega^2),$$

where $W(\varkappa,\mu,\tau)$ is the general solution of the Whittaker equation

$$4\tau^2 W_{\tau\tau} = (\tau^2 - 4\varkappa\tau + 4\mu^2 - 1)W. \tag{4.21}$$

If $\sigma_2 = 0$, the general solution of equation (4.16) is expressed in terms of Bessel functions:

$$\varphi^3 = |\omega|^{\frac{1}{2}C_1} Z_{\frac{1}{2}C_1}((-\sigma_1)^{1/2}\omega),$$

where $Z_\nu(\tau)$ is the general solution of the Bessel equation

$$\tau^2 Z_{\tau\tau} + \tau Z_\tau + (\tau^2 - \nu^2)Z = 0. \tag{4.22}$$

Note 4.2 *If* $\sigma_2 = 0$, *all quadratures in formulas (4.18)–(4.20) are easily integrated. For example,*

$$\varphi^2 = \begin{cases} C_2 + C_3 \ln|\omega| + \frac{1}{4}\varepsilon_1\omega^2 & \text{if } C_1 = -2, \\ C_2 + C_3\frac{1}{2}\omega^2 + \frac{1}{2}\varepsilon_1\omega^2(\ln\omega - \frac{1}{2}) & \text{if } C_1 = 0, \\ C_2 + C_3(C_1+2)^{-1}|\omega|^{C_1+2} - \frac{1}{2}\varepsilon_1 C_1^{-1}\omega^2 & \text{if } C_1 \neq -2, 0. \end{cases}$$

5–6. Let $\sigma = 0$. Then the last equation of system (4.12) implies that $\varphi^3 = C_0 = \mathrm{const}$. The other equations of system (4.12) can be written in the form

$$h = -\gamma_1 a_2 \textstyle\int \varphi^2(\omega)d\omega,$$

$$\varphi^i_{\omega\omega} - C_0\varphi^i_\omega + \mu_{ij}\varphi^j = \nu_{1i} + \nu_{2i}\omega, \tag{4.23}$$

where $\nu_{11} = c_{11}$, $\nu_{21} = c_{21}$, $\nu_{12} = c_{12} + \gamma_2 a_2 C_0$, $\nu_{22} = c_{22}$. System (4.23) is a linear nonhomogeneous system of ODEs with constant coefficients. The form of its general solution depends on the Jordan form of the matrix $\mathrm{M} = \{\mu_{ij}\}$. Now let us transform the dependent variables

$$\varphi^i = e_{ij}\psi^j,$$

where the constants e_{ij} are determined by means of the system of linear algebraic equations

$$e_{ij}\tilde\mu_{jk} = \mu_{ij}e_{jk} \quad (i,j,k = 1,2)$$

with the condition $\det\{e_{ij}\} \neq 0$. Here $\tilde{\mathrm{M}} = \{\tilde\mu_{ij}\}$ is the real Jordan form of the matrix M. The new unknown functions ψ^i have to satisfy the following system

$$\psi^i_{\omega\omega} - C_0\psi^i_\omega + \tilde\mu_{ij}\psi^j = \tilde\nu_{1i} + \tilde\nu_{2i}\omega, \tag{4.24}$$

where $\nu_{1i} = e_{ij}\tilde\nu_{1j}$, $\nu_{2i} = e_{ij}\tilde\nu_{2j}$. Depending on the form of $\tilde{\mathrm{M}}$, we consider the following cases:

A. $\det\tilde{\mathrm{M}} = 0$ (this is equivalent to the condition $\det \mathrm{M} = 0$).

i. $\tilde{M} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$, where $\varepsilon \in \{0;1\}$. Then

$$\psi^2 = C_1 + C_2 e^{C_0\omega} - \tfrac{1}{2}\tilde{\nu}_{22}C_0^{-1}\omega^2 - (\tilde{\nu}_{12} - \tilde{\nu}_{22}C_0^{-1})C_0^{-1}\omega, \tag{4.25}$$

$$\psi^1 = C_3 + C_4 e^{C_0\omega} - \tfrac{1}{2}\tilde{\nu}_{21}C_0^{-1}\omega^2 - (\tilde{\nu}_{11} - \tilde{\nu}_{21}C_0^{-1})C_0^{-1}\omega +$$

$$+\varepsilon\Big(-\tfrac{1}{6}\tilde{\nu}_{22}C_0^{-2}\omega^3 - \tfrac{1}{2}(\tilde{\nu}_{12} - 2\tilde{\nu}_{22}C_0^{-1})C_0^{-2}\omega^2 +$$

$$+(C_1 + (\tilde{\nu}_{21} - 2\tilde{\nu}_{22}C_0^{-1})C_0^{-2})C_0^{-1}\omega - C_2C_0^{-1}\omega e^{C_0\omega}\Big)$$

for $C_0 \neq 0$, and

$$\psi^2 = C_1 + C_2\omega + \tfrac{1}{6}\tilde{\nu}_{22}\omega^3 + \tfrac{1}{2}\tilde{\nu}_{12}\omega^2, \tag{4.26}$$

$$\psi^1 = C_3 + C_4\omega + \tfrac{1}{6}(\tilde{\nu}_{21} - C_2)\omega^3 + \tfrac{1}{2}(\tilde{\nu}_{11} - C_1)\omega^2 -$$

$$-\tfrac{1}{120}\tilde{\nu}_{22}\omega^5 - \tfrac{1}{24}\tilde{\nu}_{12}\omega^4$$

for $C_0 = 0$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0\}$. Then the form of ψ^2 is given either by formula (4.25) for $C_0 \neq 0$ or by formula (4.26) for $C_0 = 0$. The form of ψ^1 is given by formula (4.28) (see below).

B. $\det\tilde{\mathrm{M}} \neq 0$ (this is equivalent to the condition $\det\mathrm{M} \neq 0$).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}\backslash\{0\}$. Then

$$\psi^2 = \tilde{\nu}_{22}\varkappa_2^{-1}\omega + (\tilde{\nu}_{12} - C_0\tilde{\nu}_{22}\varkappa_2^{-1})\varkappa_2^{-1} + C_1\theta^{21}(\omega) + C_2\theta^{22}(\omega), \tag{4.27}$$

$$\psi^1 = \tilde{\nu}_{21}\varkappa_1^{-1}\omega + (\tilde{\nu}_{11} - C_0\tilde{\nu}_{21}\varkappa_1^{-1})\varkappa_1^{-1} + C_3\theta^{11}(\omega) + C_4\theta^{12}(\omega), \tag{4.28}$$

where

$$\theta^{i1}(\omega) = \exp(\tfrac{1}{2}(C_0 - \sqrt{D_i})\omega), \quad \theta^{i2}(\omega) = \exp(\tfrac{1}{2}(C_0 + \sqrt{D_i})\omega)$$

if $D_i = C_0^2 - 4\varkappa_i > 0$,

$$\theta^{i1}(\omega) = e^{\frac{1}{2}C_0\omega}\cos(\tfrac{1}{2}\sqrt{-D_i}\omega), \quad \theta^{i2}(\omega) = e^{\frac{1}{2}C_0\omega}\sin(\tfrac{1}{2}\sqrt{-D_i}\omega)$$

if $D_i < 0$,

$$\theta^{i1}(\omega) = e^{\frac{1}{2}C_0\omega}, \qquad \theta^{i2}(\omega) = \omega e^{\frac{1}{2}C_0\omega}$$

if $D_i = 0$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_2 & 1 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_2 \in \mathbb{R}\backslash\{0\}$. Then the form of ψ^2 is given by formula (4.27), and

$$\psi^1 = (\tilde{\nu}_{11} - (\tilde{\nu}_{12} - C_0\tilde{\nu}_{22}\varkappa_2^{-1})\varkappa_2^{-1} - C_0(\tilde{\nu}_{21} - \tilde{\nu}_{22}\varkappa_2^{-1})\varkappa_2^{-1})\varkappa_2^{-1} +$$

$$+(\tilde{\nu}_{21} - \tilde{\nu}_{22}\varkappa_2^{-1})\varkappa_2^{-1}\omega + C_3\theta^{21}(\omega) + C_4\theta^{22}(\omega) - C_i\eta^i(\omega),$$

where

$$\eta^j(\omega) = D_2^{-1}\omega(2\theta^{2j}_\omega(\omega) - C_0\theta^{2j}(\omega)) \qquad \text{if} \quad D_2 \neq 0,$$

$$\eta^1(\omega) = \tfrac{1}{2}\omega^2 e^{\frac{1}{2}C_0\omega}, \quad \eta^2(\omega) = \tfrac{1}{6}\omega^3 e^{\frac{1}{2}C_0\omega} \quad \text{if} \quad D_2 = 0.$$

iii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & -\varkappa_2 \\ \varkappa_2 & \varkappa_1 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}$, $\varkappa_2 \neq 0$. Then

$$\psi^1 = (\varkappa_i\varkappa_i)^{-1}(\tilde{\nu}_{21}\varkappa_1 + \tilde{\nu}_{22}\varkappa_2)\omega + (\varkappa_i\varkappa_i)^{-1}(\tilde{\nu}_{11}\varkappa_1 + \tilde{\nu}_{12}\varkappa_2) -$$

$$-C_0(\varkappa_i\varkappa_i)^{-2}(\tilde{\nu}_{21}(\varkappa_2^2 - \varkappa_1^2) - \tilde{\nu}_{22}2\varkappa_1\varkappa_2) + C_n\theta^{1n}(\omega),$$

$$\psi^2 = (\varkappa_i\varkappa_i)^{-1}(-\tilde{\nu}_{21}\varkappa_2 + \tilde{\nu}_{22}\varkappa_1)\omega + (\varkappa_i\varkappa_i)^{-1}(-\tilde{\nu}_{11}\varkappa_2 + \tilde{\nu}_{12}\varkappa_1) -$$

$$-C_0(\varkappa_i\varkappa_i)^{-2}(\tilde{\nu}_{21}2\varkappa_1\varkappa_2 + \tilde{\nu}_{22}(\varkappa_2^2 - \varkappa_1^2)) + C_n\theta^{2n}(\omega),$$

where $n = \overline{1,4}$,

$$\gamma = \sqrt{(C_0^2 - 4\varkappa_1)^2 + (4\varkappa_2)^2},$$

$$\beta_1 = \tfrac{1}{4}\sqrt{2(\gamma + C_0^2 - 4\varkappa_1)}, \quad \beta_2 = \tfrac{1}{4}\tfrac{|\varkappa_2|}{\varkappa_2}\sqrt{2(\gamma - C_0^2 + 4\varkappa_1)},$$

$$\begin{aligned} \theta^{11}(\omega) &= \theta^{22}(\omega) = \exp((\tfrac{1}{2}C_0 - \beta_1)\omega)\cos\beta_2\omega, \\ -\theta^{21}(\omega) &= \theta^{12}(\omega) = \exp((\tfrac{1}{2}C_0 - \beta_1)\omega)\sin\beta_2\omega, \\ \theta^{13}(\omega) &= \theta^{24}(\omega) = \exp((\tfrac{1}{2}C_0 + \beta_1)\omega)\cos\beta_2\omega, \\ \theta^{23}(\omega) &= -\theta^{14}(\omega) = \exp((\tfrac{1}{2}C_0 + \beta_1)\omega)\sin\beta_2\omega. \end{aligned}$$

If $\sigma \neq 0$, the last equation of system (4.12) implies that $\psi^3 = \sigma\omega$ (translating ω, the integration constant can be made to vanish). The other equations of system (4.12) can be written in the form

$$h = -\gamma_1 a_2 \int \varphi^2(\omega)d\omega - \tfrac{1}{2}\sigma^2\omega^2,$$

$$\varphi^i_{\omega\omega} - \sigma\omega\varphi^i_\omega + \mu_{ij}\varphi^j = \nu_{1i} + \nu_{2i}\omega, \tag{4.29}$$

where $\nu_{11} = c_{11}$, $\nu_{21} = c_{21}$, $\nu_{12} = c_{12}$, $\nu_{22} = c_{22} + \gamma_2 a_2 \sigma$. The form of the general solution of system (4.29) depends on the Jordan form of the matrix $\mathrm{M} = \{\mu_{ij}\}$. Now, let us transform the dependent variables

$$\varphi^i = e_{ij}\psi^j,$$

where the constants e_{ij} are determined by means of the system of linear algebraic equations

$$e_{ij}\tilde{\mu}_{jk} = \mu_{ij}e_{jk} \quad (i,j,k = 1,2)$$

with the condition $\det\{e_{ij}\} \neq 0$. Here $\tilde{\mathrm{M}} = \{\tilde{\mu}_{ij}\}$ is the real Jordan form of the matrix M. The new unknown functions ψ^i have to satisfy the following system

$$\psi^i_{\omega\omega} - \sigma\omega\psi^i_\omega + \tilde{\mu}_{ij}\psi^j = \tilde{\nu}_{1i} + \tilde{\nu}_{2i}\omega, \tag{4.30}$$

where $\nu_{1i} = e_{ij}\tilde{\nu}_{1j}$, $\nu_{2i} = e_{ij}\tilde{\nu}_{2j}$. Depending on the form of $\tilde{\mathrm{M}}$, we consider the following cases:

A. $\det\tilde{\mathrm{M}} = 0$ (this is equivalent to the condition $\det\mathrm{M} = 0$).

i. $\tilde{M} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$, where $\varepsilon \in \{0;1\}$. Then

$$\begin{aligned} \psi^2 = {} & C_1 + C_2 \textstyle\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}\tilde{\nu}_{22}\omega + \\ & + \tilde{\nu}_{12}\textstyle\int e^{\frac{1}{2}\sigma\omega^2}(\int e^{-\frac{1}{2}\sigma\omega^2}d\omega)d\omega, \end{aligned} \tag{4.31}$$

$$\begin{aligned} \psi^1 = {} & C_3 + C_4 \textstyle\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}\tilde{\nu}_{21}\omega + \\ & + \textstyle\int e^{\frac{1}{2}\sigma\omega^2}(\int e^{-\frac{1}{2}\sigma\omega^2}(\tilde{\nu}_{11} - \varepsilon\psi^2)d\omega)d\omega. \end{aligned}$$

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$. Then the form of ψ^2 is given by formula (4.31), and

$$\begin{aligned} \psi^1 = {} & C_3\omega + C_4(\omega \textstyle\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac{1}{2}\sigma\omega^2}) + \sigma^{-1}\tilde{\nu}_{11} + \\ & + \sigma^{-1}\tilde{\nu}_{21}(\sigma\omega\textstyle\int e^{\frac{1}{2}\sigma\omega^2}\lambda^1(\omega)d\omega - e^{\frac{1}{2}\sigma\omega^2}\lambda^1(\omega)), \end{aligned}$$

where $\lambda^1(\omega) = \int e^{-\frac{1}{2}\sigma\omega^2}d\omega$.

iii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0;\sigma\}$. Then ψ^2 is determited by (4.31), and the form of ψ^1 is given by (4.33) (see below).

B. $\det\tilde{\mathrm{M}} \neq 0$, $\det\{\tilde{\mu}_{ij} - \sigma\delta_{ij}\} = 0$ (this is equivalent to the conditions $\det\mathrm{M} \neq 0$, $\det\{\mu_{ij} - \sigma\delta_{ij}\} = 0$; here δ_{ij} is the Kronecker symbol).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \sigma & \varepsilon \\ 0 & \sigma \end{pmatrix}$, where $\varepsilon \in \{0;1\}$. Then

$$\begin{aligned}
\psi^2 &= C_1\omega + C_2(\omega \textstyle\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac{1}{2}\sigma\omega^2}) + \sigma^{-1}\tilde{\nu}_{12}+ \\
&+\sigma^{-1}\tilde{\nu}_{22}(\sigma\omega \textstyle\int e^{\frac{1}{2}\sigma\omega^2}\lambda^1(\omega)d\omega - e^{\frac{1}{2}\sigma\omega^2}\lambda^1(\omega)),
\end{aligned} \tag{4.32}$$

$$\begin{aligned}
\psi^1 &= C_3\omega + C_4(\omega \textstyle\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac{1}{2}\sigma\omega^2}) + \sigma^{-1}\tilde{\nu}_{11}+ \\
&+\sigma\omega \textstyle\int e^{\frac{1}{2}\sigma\omega^2}\lambda^2(\omega)d\omega - e^{\frac{1}{2}\sigma\omega^2}\lambda^2(\omega) + \sigma^{-1}(\tilde{\nu}_{21}\omega - \varepsilon\psi^2),
\end{aligned}$$

where $\lambda^1(\omega) = \int e^{-\frac{1}{2}\sigma\omega^2}d\omega$, $\lambda^2(\omega) = \sigma^{-1}\int e^{-\frac{1}{2}\sigma\omega^2}(\tilde{\nu}_{21} - \varepsilon\psi^2_\omega)d\omega$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \sigma \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0;\sigma\}$. In this case ψ^2 is determined by (4.32), and the form of ψ^1 is given by (4.33) (see below).

C. $\det\tilde{\mathrm{M}} \neq 0$, $\det\{\tilde{\mu}_{ij} - \sigma\delta_{ij}\} \neq 0$ (this is equivalent to the condition $\det\mathrm{M} \neq 0$, $\det\{\mu_{ij} - \sigma\delta_{ij}\} \neq 0$: here δ_{ij} is the Kronecker symbol).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}\backslash\{0;\sigma\}$. Then

$$\begin{aligned}
\psi^1 &= \varkappa_1^{-1}\tilde{\nu}_{11} + (\varkappa_1 - \sigma)^{-1}\tilde{\nu}_{21}\omega + |\omega|^{-1/2}e^{\frac{1}{4}\sigma\omega^2}\cdot \\
&\cdot\Big(C_3M(\tfrac{1}{2}\varkappa_1\sigma^{-1} + \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}\sigma\omega^2) + C_4M(\tfrac{1}{2}\varkappa_1\sigma^{-1} + \tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{2}\sigma\omega^2)\Big),
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
\psi^2 &= \varkappa_2^{-1}\tilde{\nu}_{12} + (\varkappa_2 - \sigma)^{-1}\tilde{\nu}_{22}\omega + |\omega|^{-1/2}e^{\frac{1}{4}\sigma\omega^2}\cdot \\
&\cdot\Big(C_1M(\tfrac{1}{2}\varkappa_2\sigma^{-1} + \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}\sigma\omega^2) + C_2M(\tfrac{1}{2}\varkappa_2\sigma^{-1} + \tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{2}\sigma\omega^2)\Big),
\end{aligned} \tag{4.34}$$

where $M(\varkappa, \mu, \tau)$ is the Whittaker function:

$$M(\varkappa, \mu, \tau) = \tau^{\frac{1}{2}+\mu}e^{-\frac{1}{2}\tau}{}_1F_1(\tfrac{1}{2} + \mu - \varkappa, 2\mu + 1, \tau), \tag{4.35}$$

and ${}_1F_1(a, b, \tau)$ is the degenerate hypergeometric function defined by means of the series:

$${}_1F_1(a, b, \tau) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\dots(a+n-1)}{b(b+1)\dots(b+n-1)}\frac{\tau^n}{n!},$$

$b \neq 0, -1, -2, \dots$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & -\varkappa_2 \\ \varkappa_2 & \varkappa_1 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}$, $\varkappa_2 \neq 0$. Then

$$\psi^1 = (\varkappa_j\varkappa_j)^{-1}(\varkappa_1\tilde{\nu}_{11} + \varkappa_2\tilde{\nu}_{12})+$$
$$+((\varkappa_1-\sigma)^2+\varkappa_2^2)^{-1}((\varkappa_1-\sigma)\tilde{\nu}_{21}+\varkappa_2\tilde{\nu}_{22})\omega+$$
$$+C_1\mathrm{Re}\eta^1(\omega) - C_2\mathrm{Im}\eta^1(\omega) + C_3\mathrm{Re}\eta^2(\omega) - C_4\mathrm{Im}\eta^2(\omega),$$
$$\psi^2 = (\varkappa_j\varkappa_j)^{-1}(-\varkappa_2\tilde{\nu}_{11} + \varkappa_1\tilde{\nu}_{12})+$$
$$+((\varkappa_1-\sigma)^2+\varkappa_2^2)^{-1}(-\varkappa_2\tilde{\nu}_{21}+(\varkappa_1-\sigma)\tilde{\nu}_{22})\omega+$$
$$+C_1\mathrm{Im}\eta^1(\omega) + C_2\mathrm{Re}\eta^1(\omega) + C_3\mathrm{Im}\eta^2(\omega) + C_4\mathrm{Re}\eta^2(\omega),$$

where

$$\eta^1(\omega) = M(\tfrac{1}{2}(\varkappa_1+\varkappa_2\mathrm{i})\sigma^{-1}+\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}\sigma\omega^2),$$
$$\eta^2(\omega) = M(\tfrac{1}{2}(\varkappa_1+\varkappa_2\mathrm{i})\sigma^{-1}+\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{2}\sigma\omega^2), \quad \mathrm{i}^2=-1.$$

iii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_2 & 1 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_2 \in \mathbb{R}\backslash\{0;\sigma\}$. Here the form of ψ^2 is given by (4.34), and

$$\psi^1 = (\tilde{\nu}_{11} - \tilde{\nu}_{12}\varkappa_2^{-1})\varkappa_2^{-1} + (\tilde{\nu}_{21} - \tilde{\nu}_{22}(\varkappa_2-\sigma)^{-1})(\varkappa_2-\sigma)^{-1}\omega+$$
$$+|\omega|^{-1/2}e^{\frac{1}{4}\sigma\omega^2}\Big(C_3\theta^1(\tau) + C_4\theta^2(\tau) - \sigma^{-1}\theta^1(\tau)\textstyle\int \tau^{-1}\theta^2(\tau)C_i\theta^i(\tau)d\tau+$$
$$+\sigma^{-1}\theta^2(\tau)\textstyle\int \tau^{-1}\theta^1(\tau)C_i\theta^i(\tau)d\tau\Big),$$

where $\tau = \frac{1}{2}\sigma\omega^2$,

$$\theta^1(\tau) = M(\tfrac{1}{2}\varkappa_2\sigma^{-1}+\tfrac{1}{4}, \tfrac{1}{4}, \tau), \quad \theta^2(\tau) = M(\tfrac{1}{2}\varkappa_2\sigma^{-1}+\tfrac{1}{4}, -\tfrac{1}{4}, \tau).$$

Note 4.3 *The general solution of the equation*

$$\psi_{\omega\omega} - \sigma\omega\psi_\omega - (n+1)\sigma\psi = 0,$$

where n is an integer and $n \geq 0$, is determined by the formula

$$\psi = \left(\frac{d^n}{d\omega^n}e^{\frac{1}{2}\sigma\omega^2}\right)\left(C_1 + C_2\int e^{\frac{1}{2}\sigma\omega^2}\left(\frac{d^n}{d\omega^n}e^{\frac{1}{2}\sigma\omega^2}\right)^{-2}d\omega\right).$$

Note 4.4 *If function ψ satisfies the equation*

$$\psi_{\omega\omega} - \sigma\omega\psi_\omega + \varkappa\psi = 0 \quad (\varkappa \neq -\sigma),$$

then $\int \psi(\omega)d\omega = (\varkappa+\sigma)^{-1}(\sigma\omega\psi - \psi_\omega) + C_1$.

7. The last equation of system (4.13) is the compatibility condition of the NSEs (1.1) and ansatz (4.7). Integrating this equation, we obtain that

$$\eta^3 = C_0(\eta^i\eta^i)^{-1}, \quad C_0 \neq 0.$$

As $\varphi^3_\omega = -\eta^3_\omega(\eta^3)^{-1}\varphi^3 = 2\theta^1\varphi^3$, $\varphi^3 = C_3\eta^i\eta^i$. Then system (4.13) is reduced to the equations

$$\begin{aligned} \varphi^1_\omega &= \chi^1(\omega)\varphi^1 - \chi^2(\omega)\varphi^2, \\ \varphi^2_\omega &= \chi^2(\omega)\varphi^1 + \chi^1(\omega)\varphi^2, \end{aligned} \tag{4.36}$$

where $\chi^1 = -C_0^{-2}(\eta^i\eta^i)^2 - \theta^1$ and $\chi^2 = \theta^2 - C_3C_0^{-1}(\eta^i\eta^i)^2$. System (4.36) implies that

$$\varphi^1 = \exp(\textstyle\int \chi^1(\omega)d\omega)\Big(C_1\cos(\int \chi^2(\omega)d\omega) - C_2\sin(\int \chi^2(\omega)d\omega)\Big),$$

$$\varphi^2 = \exp(\textstyle\int \chi^1(\omega)d\omega)\Big(C_1\sin(\int \chi^2(\omega)d\omega) + C_2\cos(\int \chi^2(\omega)d\omega)\Big).$$

8. Let us apply the trasformation generated by the operator $R(\vec{k}(t))$, where

$$\vec{k}_t = \lambda^{-1}(\vec{n}^b\cdot\vec{k})\vec{m}^b_t - \vec{\varphi},$$

to ansatz (4.8). As a result we obtain an ansatz of the same form, where the functions $\vec{\varphi}$ and h are replaced by the new functions $\tilde{\vec{\varphi}}$ and $\tilde{h}$:

$$\tilde{\vec{\varphi}} = \vec{\varphi} - \lambda^{-1}(\vec{n}^a\cdot\vec{k})\vec{m}^a_t + \vec{k}_t = 0,$$

$$\tilde{h} = h - \lambda^{-1}(\vec{m}^a_{tt}\cdot\vec{k})(\vec{n}^a\cdot\vec{k}) + \tfrac{1}{2}\lambda^{-2}(\vec{m}^b_{tt}\cdot\vec{m}^a)(\vec{n}^a\cdot\vec{k})(\vec{n}^b\cdot\vec{k}).$$

Let us make $\tilde{h}$ vanish by means of the transformation generated by the operator $Z(-\tilde{h}(t))$. Therefore, the functions φ^a and h can be considered to vanish. The equation $(\vec{n}^a\cdot\vec{m}^a_t) = 0$ is the compatibility condition of ansatz (4.8) and the NSEs (1.1).

Note 4.5 *The solutions of the NSEs obtained by means of ansatzes 5–8 are equivalent to either solutions (5.1) or solutions (5.5).*

5 Reduction of the Navier-Stokes equations to linear systems of PDEs

Let us show that non-linear systems 8 and 9, from Subsec. 3.2, are reduced to linear systems of PDEs.

5.1 Investigation of system (3.17)–(3.18)

Consider system 9 from Subsec. 3.2, i.e., equations (3.17) and (3.18). Equation (3.18) integrates with respect to z_2 to the following expression:

$$\vec{k}\cdot\vec{w}=\psi(t).$$

Here $\psi=\psi(t)$ is an arbitrary smooth function of $z_1=t$. Let us make the transformation from the symmetry group of the NSEs:

$$\tilde{\vec{u}}(t,\vec{x})=\vec{u}(t,\vec{x}-\vec{l})+\vec{l}_t(t),$$

$$\tilde{p}(t,\vec{x})=p(t,\vec{x}-\vec{l})-\vec{l}_{tt}(t)\cdot\vec{x},$$

where $\vec{l}_{tt}\cdot\vec{m}^i-\vec{l}\cdot\vec{m}^i_{tt}=0$ and

$$\vec{k}\cdot(\vec{l}_t-\lambda^{-1}(\vec{n}^i\cdot\vec{l})m^i_t+\lambda^{-1}(\vec{k}\cdot\vec{l})\vec{k}_t)+\psi=0.$$

This transformation does not modify ansatz (3.9), but it makes the function $\psi(t)$ vanish, i.e., $\vec{k}\cdot\tilde{\vec{w}}=0$. Therefore, without loss of generality we may assume, at once, that $\vec{k}\cdot\vec{w}=0$.

Let $f^i=f^i(z_1,z_2)=\vec{m}^i\cdot\vec{w}$. Since $\vec{m}^1_{tt}\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_{tt}=0$, it follows that $\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=C=\mathrm{const}$. Let us multiply the scalar equation (3.17) by $\vec{m}^i$ and $\vec{k}$. As a result we obtain the linear system of PDEs with variable coefficients in the functions f^i and s:

$$f^i_1-\lambda f^i_{22}+C\lambda^{-1}((\vec{m}^i\cdot\vec{m}^2)f^1-(\vec{m}^i\cdot\vec{m}^1)f^2)-$$
$$-2C\lambda^{-2}((\vec{k}\times\vec{k}_t)\cdot\vec{m}^i)z_2=0,$$

$$s_2=2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)f^i+\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k}-2\vec{k}_t\cdot\vec{k}_t)z_2.$$

Consider two possible cases.

A. Let $C=0$. Then there exist functions $g^i=g^i(\tau,\omega)$, where $\tau=\int\lambda(t)dt$ and $\omega=z_2$, such that $f^i=g^i_\omega$ and $g^i_\tau-g^i_{\omega\omega}=0$. Therefore,

$$\vec{u}=\lambda^{-1}(g^i_\omega(\tau,\omega)+\vec{m}^i_t\cdot\vec{x})\vec{n}^i-\lambda^{-1}(\vec{k}_t\cdot\vec{x})k,$$

$$p=2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)g^i(\tau,\omega)+\tfrac{1}{2}\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k}-2\vec{k}_t\cdot\vec{k}_t)\omega^2- \tag{5.1}$$
$$-\tfrac{1}{2}\lambda^{-1}(\vec{n}^i\cdot\vec{x})(\vec{m}^i_{tt}\cdot\vec{x})-\tfrac{1}{2}\lambda^{-2}(\vec{k}\cdot\vec{m}^i_{tt})(\vec{n}^i\cdot\vec{x})(\vec{k}\cdot\vec{x}),$$

where $\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=0$, $\vec{k}=\vec{m}^1\times\vec{m}^2$, $\vec{n}^1=\vec{m}^2\times\vec{k}$, $\vec{n}^2=\vec{k}\times\vec{m}^1$, $\lambda=|\vec{k}|^2$, $\omega=\vec{k}\cdot\vec{x}$, $\tau=\int\lambda(t)dt$, and $g^i_\tau-g^i_{\omega\omega}=0$.

For example, if $\vec{m}=(\eta^1(t),0,0)$ and $\vec{n}=(0,\eta^2(t),0)$ with $\eta^i(t)\neq0$, it follows that

$$u^1=(\eta^1)^{-1}(f^1+\eta^1_tx_1),\quad u^2=(\eta^2)^{-1}(f^2+\eta^2_tx_2),$$

$$u^3 = -(\eta^1\eta^2)_t(\eta^1\eta^2)^{-1}x_3,$$

$$p = -\tfrac{1}{2}\eta^1_{tt}(\eta^1)^{-1}x_1^2 - \tfrac{1}{2}\eta^2_{tt}(\eta^2)^{-1}x_2^2 +$$

$$+\Big(\tfrac{1}{2}(\eta^1\eta^2)_{tt}(\eta^1\eta^2)^{-1} - ((\eta^1\eta^2)_t(\eta^1\eta^2)^{-1})^2\Big)x_3^2,$$

where $f^i = f^i(\tau,\omega)$, $f^i_\tau - f^i_{\omega\omega} = 0$, $\tau = \int(\eta^1\eta^2)^2dt$, and $\omega = \eta^1\eta^2x_3$. If $\vec{m}^1 = (\eta^1(t),\eta^2(t),0)$ and $\vec{m}^2 = (0,0,\eta^3(t))$ with $\eta^3(t)\neq 0$ and $\eta^i(t)\eta^i(t)\neq 0$, we obtain that

$$u^1 = (\eta^i\eta^i)^{-1}\Big\{\eta^1(g_\omega + \eta^i_tx_i) - \eta^2(\eta^3_t(\eta^3)^{-2}\omega + \eta^2_tx_1 - \eta^1_tx_2)\Big\},$$

$$u^2 = (\eta^i\eta^i)^{-1}\Big\{\eta^2(g_\omega + \eta^i_tx_i) + \eta^1(\eta^3_t(\eta^3)^{-2}\omega + \eta^2_tx_1 - \eta^1_tx_2)\Big\},$$

$$u^3 = (\eta^3)^{-1}(f + \eta^3_tx_3),$$

$$p = 2(\eta^3)^{-1}(\eta^1\eta^2_t - \eta^1_t\eta^2)(\eta^i\eta^i)^{-2}g + \tfrac{1}{2}\lambda^{-1}\cdot$$

$$\cdot\Big\{\lambda^{-1}((\eta^3_{tt}\eta^3 - 2\eta^3_t\eta^3_t)\eta^i\eta^i - 2\eta^3\eta^3_t\eta^i\eta^i_t - 2(\eta^3)^2\eta^i_t\eta^i_t)\omega^2 +$$

$$+(\eta^3)^2((\eta^2\eta^2_{tt} - \eta^1\eta^1_{tt})(x_1^2 - x_2^2) - 2(\eta^1_{tt}\eta^2 + \eta^1\eta^2_{tt})x_1x_2) -$$

$$-\eta^i\eta^i\eta^3\eta^3_{tt}x_3^2\Big\}.$$

Here $f = f(\tau,\omega)$, $f_\tau - f_{\omega\omega} = 0$, $g = g(\tau,\omega)$, $g_\tau - g_{\omega\omega} = 0$, $\tau = \int(\eta^3)^2\eta^i\eta^idt$, $\omega = \eta^3(\eta^2x_1 - \eta^1x_2)$, and $\lambda = (\eta^3)^2\eta^i\eta^i$.

Note 5.1 *The equation*

$$\vec{m}^1_t\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_t = 0 \tag{5.2}$$

can easily be solved in the following way: Let us fix arbitrary smooth vector-functions $\vec{m}^1, \vec{l} \in C^\infty((t_0,t_1),\mathbb{R}^3)$ *such that* $\vec{m}^1(t)\neq\vec{0}$, $\vec{l}(t)\neq\vec{0}$, *and* $\vec{m}^1(t)\cdot\vec{l}(t) = 0$ *for all* $t\in(t_0,t_1)$. *Then the vector-function* $\vec{m}^2 = \vec{m}^2(t)$ *is taken in the form*

$$\vec{m}^2(t) = \rho(t)\vec{m}^1 + \vec{l}(t). \tag{5.3}$$

Equation (5.2) implies

$$\rho(t) = \int(\vec{m}^1\cdot\vec{m}^1)^{-1}(\vec{m}^1_t\cdot\vec{l} - \vec{m}^1\cdot\vec{l}_t)dt. \tag{5.4}$$

B. Let $C \neq 0$. By means of the transformation $\vec{m}^i \to a_{ij}\vec{m}^j$, where $a_{ij} = \mathrm{const}$ and $\det\{a_{ij}\} = C$, we make $C = 1$. Then we obtain the following solution of the NSEs (1.1)

$$
\begin{aligned}
\vec{u} = {} & \lambda^{-1}\Big(\theta^{ij}(t)g^j_\omega(\tau,\omega) + \theta^{i0}(t)\omega + \vec{m}^i_t\cdot\vec{x} - \lambda^{-1}((\vec{k}\times\vec{m}^i)\cdot\vec{x})\Big)\vec{n}^i - \\
& -\lambda^{-1}(\vec{k}_t\cdot\vec{x})\vec{k}, \\
p = {} & 2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)(\theta^{ij}(t)g^i(\tau,\omega) + \tfrac{1}{2}\theta^{i0}(t)\omega^2) + \\
& +\tfrac{1}{2}\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k} - 2\vec{k}_t\cdot\vec{k}_t)\omega^2 - \tfrac{1}{2}\lambda^{-1}(\vec{n}^i\cdot\vec{x})(\vec{m}^i_{tt}\cdot\vec{x}) - \\
& -\tfrac{1}{2}\lambda^{-2}(\vec{k}\cdot\vec{m}^i_{tt})(\vec{n}^i\cdot\vec{x})(\vec{k}\cdot\vec{x}).
\end{aligned}
\tag{5.5}
$$

Here $\vec{m}^1_t\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_t = 1$, $\vec{k} = \vec{m}^1\times\vec{m}^2$, $\vec{n}^1 = \vec{m}^2\times\vec{k}$, $\vec{n}^2 = \vec{k}\times\vec{m}^1$, $\lambda = |\vec{k}|^2$, $\omega = \vec{k}\cdot\vec{x}$, $\tau = \int\lambda(t)dt$, and $g^i_\tau - g^i_{\omega\omega} = 0$. $(\theta^{1i}(t),\theta^{2i}(t))$ $(i = 1,2)$ are linearly independent solutions of the system

$$
\theta^i_t + \lambda^{-1}(\vec{m}^i\cdot\vec{m}^2)\theta^1 - \lambda^{-1}(\vec{m}^i\cdot\vec{m}^1)\theta^2 = 0, \tag{5.6}
$$

and $(\theta^{10}(t),\theta^{20}(t))$ is a particular solution of the nonhomogeneous system

$$
\theta^i_t + \lambda^{-1}(\vec{m}^i\cdot\vec{m}^2)\theta^1 - \lambda^{-1}(\vec{m}^i\cdot\vec{m}^1)\theta^2 = 2\lambda^{-2}((\vec{k}\times\vec{k}_t)\cdot\vec{m}^i). \tag{5.7}
$$

For example, if $\vec{m}^1 = (\eta\cos\psi, \eta\sin\psi, 0)$ and $\vec{m}^2 = (-\eta\sin\psi, \eta\cos\psi, 0)$, where $\eta = \eta(t) \neq 0$ and $\psi = -\frac{1}{2}\int(\eta)^{-2}dt$ (therefore, $\vec{m}^1_t\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_t = 1$), we obtain

$$
\begin{aligned}
u^1 &= \eta^{-1}(f^1\cos\psi - f^2\sin\psi + \eta_t x_1 - \tfrac{1}{2}\eta^{-1}x_2), \\
u^2 &= \eta^{-1}(f^1\sin\psi + f^2\cos\psi + \eta_t x_2 + \tfrac{1}{2}\eta^{-1}x_1), \\
u^3 &= -2\eta_t\eta^{-1}x_3, \\
p &= (\eta_{tt}\eta - 3\eta_t\eta_t)\eta^{-2}x_3^2 - \tfrac{1}{2}(\eta_{tt}\eta^{-1} - \tfrac{1}{4}\eta^{-4})x_ix_i.
\end{aligned}
$$

Here $f^i = f^i(\tau,\omega)$, $f^i_\tau - f^i_{\omega\omega} = 0$, $\tau = \int(\eta)^4dt$, and $\omega = (\eta)^2x_3$.

Note 5.2 *As in the case $C = 0$, the solutions of the equation*

$$
\vec{m}^1_t\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_t = 1 \tag{5.8}
$$

can be sought in form (5.3). As a result we obtain that

$$
\rho(t) = \int|\vec{m}^1|^{-2}(\vec{m}^1_t\cdot\vec{l} - \vec{m}^1\cdot\vec{l}_t - 1)dt. \tag{5.9}
$$

Note 5.3 *System (5.6) can be reduced to a second-order homogeneous differential equation either in* θ^1, *i.e.,*

$$\left(\lambda|\vec{m}^1|^{-2}\theta^1_t\right)_t + \left(((\vec{m}^1\cdot\vec{m}^2)|\vec{m}^1|^{-2})_t + |\vec{m}^1|^{-2}\right)\theta^1 = 0 \tag{5.10}$$

(then $\theta^2 = |\vec{m}^1|^{-2}(\lambda\theta^1_t + (\vec{m}^1\cdot\vec{m}^2)\theta^1)$*), or in* θ^2, *i.e.,*

$$\left(\lambda|\vec{m}^2|^{-2}\theta^2_t\right)_t + \left(-((\vec{m}^1\cdot\vec{m}^2)|\vec{m}^2|^{-2})_t + |\vec{m}^2|^{-2}\right)\theta^2 = 0 \tag{5.11}$$

(then $\theta^1 = |\vec{m}^2|^{-2}(-\lambda\theta^2_t + (\vec{m}^1\cdot\vec{m}^2)\theta^2)$*). Under the notation of Note 5.1 equation (5.10) has the form:*

$$((\vec{l}\cdot\vec{l})\theta^1_t)_t + |\vec{m}^1|^{-2}(\vec{m}^1_t\cdot\vec{l} - \vec{m}^1\cdot\vec{l}_t)\theta^1 = 0. \tag{5.12}$$

The vector-functions $\vec{m}^1$ *and* $\vec{l}$ *are chosen in such a way that one can find a fundamental set of solutions for equation (5.12). For example, let* $\vec{m}\times\vec{m}_t \neq 0\ \forall t\in(t_0,t_1)$. *Let us introduce the notation* $\vec{m} := \vec{m}^1$ *and put* $\vec{l} = \eta(t)\vec{m}\times\vec{m}_t$, *where* $\eta\in C^\infty((t_0,t_1),\mathbb{R})$, $\eta(t)\neq 0\ \forall t\in(t_0,t_1)$. *Then*

$$\vec{m}\cdot\vec{l} = 0,\quad \vec{m}_t\cdot\vec{l} - \vec{m}\cdot\vec{l}_t = 0,\quad \vec{m}^2 = -(\textstyle\int|\vec{m}|^{-2}dt)\vec{m} + \eta\vec{m}\times\vec{m}_t,$$

$$\vec{k} = \eta\vec{m}\times(\vec{m}\times\vec{m}_t),\quad \lambda = (\eta)^2|\vec{m}|^2|\vec{m}\times\vec{m}_t|^{-2},$$

$$\vec{n}^2 = \eta|\vec{m}|^2\vec{m}\times\vec{m}_t,\quad \vec{n}^1 = (\textstyle\int|\vec{m}|^{-2}dt)\vec{n}^2 + (\eta)^2|\vec{m}\times\vec{m}_t|^{-2}\vec{m},$$

$$\theta^{11}(t) = \textstyle\int(\eta)^{-2}|\vec{m}\times\vec{m}_t|^{-2}dt,\quad \theta^{21}(t) = 1 - \theta^{11}\int|\vec{m}|^{-2}dt,$$

$$\theta^{12}(t) = 1,\quad \theta^{22}(t) = -\textstyle\int|\vec{m}|^{-2}dt,$$

$$\begin{aligned}\theta^{10}(t) = \ & 2\textstyle\int(((\vec{m}\times\vec{m}_t)\cdot\vec{m}_{tt})|\vec{m}\times\vec{m}_t|^{-2} + \int\eta^{-1}|\vec{m}|^{-4}dt)\cdot\\ & \cdot\eta^{-2}|\vec{m}\times\vec{m}_t|^{-2}dt,\end{aligned}$$

$$\theta^{20}(t) = -\theta^{10}(t)\textstyle\int|\vec{m}|^{-2}dt + 2\int\eta^{-1}|\vec{m}|^{-4}dt.$$

Consider the following cases: $\vec{m}\times\vec{m}_t \equiv \vec{0}$, *i.e.,* $\vec{m} = \chi(t)\vec{a}$, *where* $\chi(t)\in C^\infty((t_0,t_1),\mathbb{R})$, $\chi(t)\neq 0\ \forall t\in(t_0,t_1)$, $\vec{a} = \mathrm{const}$, *and* $|\vec{a}| = 1$. *Let us put*

$$\vec{l}(t) = \eta^1(t)\vec{b} + \eta^2(t)\vec{c},$$

where $\eta^1,\eta^2\in C^\infty((t_0,t_1),\mathbb{R})$, $(\eta^1(t),\eta^2(t))\neq(0,0)\ \forall t\in(t_0,t_1)$, $\vec{b} = \mathrm{const}$, $|\vec{b}| = 1$, $\vec{a}\cdot\vec{b} = 0$, *and* $\vec{c} = \vec{a}\times\vec{b}$. *Then*

$$\vec{m}^2 = -(\chi\textstyle\int\chi^{-2}dt)\vec{a} + \eta^1\vec{b} + \eta^2\vec{c},\quad \vec{k} = \chi\eta^1\vec{c} - \chi\eta^2\vec{b},$$

$$\lambda = (\chi)^2\eta^i\eta^i,\quad \vec{n}^2 = (\chi)^2(\eta^1\vec{b} + \eta^2\vec{c}),\quad \vec{n}^1 = (\textstyle\int\chi^{-2}dt)\vec{n}^2 + \chi\eta^i\eta^i\vec{a},$$

$$\theta^{11} = \textstyle\int(\eta^i\eta^i)^{-1}dt,\quad \theta^{21} = 1 - \theta^{11}\int\chi^{-2}dt,$$

$$\theta^{12} = 1,\quad \theta^{22} = -\textstyle\int\chi^{-2}dt,$$

$$\theta^{10} = 2\textstyle\int(\eta^2_t\eta^1 - \eta^2\eta^1_t)\chi^{-1}(\eta^i\eta^i)^{-1}dt,\quad \theta^{20} = -\theta^{10}\int\chi^{-2}dt.$$

Note 5.4 *In formulas (5.1) and (5.5) solutions of the NSEs (1.1) are expressed in terms of solutions of the decomposed system of two linear one-dimensional heat equations (LOHEs) that have the form:*

$$g^i_\tau = g^i_{\omega\omega}. \tag{5.13}$$

The Lie symmetry of the LOHE are known. Large sets of its exact solutions were constructed [27, 3]. The Q-conditional symmetries of LOHE were investigated in [14]. Moreover, being decomposed system (5.13) admits transformations of the form

$$\tilde g^1(\tau',\omega') = F^1(\tau,\omega,g^1(\tau,\omega)), \quad \tau' = G^1(\tau,\omega), \quad \omega' = H^1(\tau,\omega),$$

$$\tilde g^2(\tau'',\omega'') = F^2(\tau,\omega,g^2(\tau,\omega)), \quad \tau'' = G^2(\tau,\omega), \quad \omega'' = H^2(\tau,\omega),$$

where $(G^1,H^1) \neq (G^2,H^2)$, i.e. the independent variables can be transformed in the functions g^1 and g^2 in different ways. A similar statement is true for system (5.19)–(5.20) (see below) if $\varepsilon = 0$.

Note 5.5 *It can be proved that an arbitrary Navier-Stokes field $(\vec u, p)$, where*

$$\vec u = \vec w(t,\omega) + (\vec k^i(t)\cdot\vec x)\vec l^i(t)$$

with $\vec k^i, \vec l^i \in C^\infty((t_0,t_1),\mathbb{R}^3)$, $\vec k^1 \times \vec k^2 \neq 0$, and $\omega = (\vec k^1\times\vec k^2)\cdot\vec x$, is equivalent to either a solution from family (5.1) or a solution from family (5.5). The equivalence transformation is generated by $R(\vec m)$ and $Z(\chi)$.

5.2 Investigation of system (3.13)–(3.16)

Consider system 8 from Subsec. 3.2, i.e., equations (3.13)–(3.16). Equation (3.16) immediately gives

$$w^1 = -\tfrac{1}{2}\rho_t\rho^{-1} + (\eta-1)z_2^{-2}, \tag{5.14}$$

where $\eta = \eta(t)$ is an arbitrary smooth function of $z_1 = t$. Substituting (5.14) into remaining equations (5.13)–(5.15), we get

$$\begin{aligned} q_2 = {} & \tfrac{1}{2}((\rho_t\rho^{-1})_t - \tfrac{1}{2}(\rho_t\rho^{-1})^2)z_2 - \eta_t z_2^{-1} - (\eta-1)^2 z_2^{-3} + \\ & +(w^2-\chi)^2 z_2^{-3}, \end{aligned} \tag{5.15}$$

$$w^2_1 - w^2_{22} + (\eta z_2^{-1} - \tfrac{1}{2}\rho_t\rho^{-1}z_2)w^2_2 = 0, \tag{5.16}$$

$$w^3_1 - w^3_{22} + (\eta z_2^{-1} - \tfrac{1}{2}\rho_t\rho^{-1}z_2)w^3_2 + \varepsilon(w^2-\chi)z_2^{-2} = 0. \tag{5.17}$$

Recall that $\rho = \rho(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t; $\varepsilon \in \{0;1\}$. After the change of the independent variables

$$\tau = \textstyle\int |\rho(t)|dt, \quad z = |\rho(t)|^{1/2} z_2 \tag{5.18}$$

in equations (5.16) and (5.17), we obtain a linear system of a simpler form:

$$w^2_\tau - w^2_{zz} + \hat\eta(\tau)z^{-1}w^2_z = 0, \tag{5.19}$$

$$w^3_\tau - w^3_{zz} + (\hat\eta(\tau)-2)z^{-1}w^3_z + \varepsilon(w^2-\hat\chi(\tau))z^{-2} = 0, \tag{5.20}$$

where $\hat\eta(\tau)=\eta(t)$ and $\hat\chi(\tau)=\chi(t)$. Equation (5.15) implies

$$\begin{aligned} q = {} & \tfrac14((\rho_t\rho^{-1})_t - \tfrac12(\rho_t\rho^{-1})^2)z_2^2 - \eta_t \ln|z_2| - \\ & -\tfrac12(\eta-1)^2 z_2^{-2} + \textstyle\int (w^2(\tau,z)-\hat\chi(\tau))^2 z_2^{-3}dz_2. \end{aligned} \tag{5.21}$$

Formulas (5.14), (5.18)–(5.21), and ansatz (3.8) determine a solution of the NSEs (1.1).

If $\varepsilon=0$ system (5.19)–(5.20) is decomposed and consists of two translational linear equations of the general form

$$f_\tau + \tilde\eta(\tau)z^{-1}f_z - f_{zz} = 0, \tag{5.22}$$

where $\tilde\eta=\hat\eta$ ($\tilde\eta=\hat\eta-2$) for equation (5.19) ((5.20)). Tilde over η is omitted below. Let us investigate symmetry properties of equation (5.22) and construct some of its exact solutions.

Theorem 5.1 *The MIA of (5.22) is given by the following algebras*

a) $L_1 = < f\partial_f,\ g(\tau,z)\partial_f >$ *if* $\eta(\tau)\neq \text{const}$*;*

b) $L_2 = < \partial_\tau,\ \hat D,\ \Pi, f\partial_f,\ g(\tau,z)\partial_f >$ *if* $\eta(\tau)=\text{const},\ \eta\notin\{0;-2\}$*;*

c) $L_3 = < \partial_\tau,\ \hat D,\ \Pi,\ \partial_z+\frac12\eta z^{-1}f\partial_f,\ G=2\tau\partial_\tau-(z-\eta z^{-1}\tau)f\partial_f,\ f\partial_f,$ $g(\tau,z)\partial_f >$ *if* $\eta\in\{0;-2\}$*.*

Here $\hat D = 2\tau\partial_\tau + z\partial_z$*,* $\Pi = 4\tau^2\partial_\tau + 4\tau z\partial_z - (z^2+2(1-\eta)\tau)f\partial_f$*;* $g=g(\tau,z)$ *is an arbitrary solution of (5.22).*

When $\eta=0$, equation (5.22) is the heat equation, and, when $\eta=-2$, it is reduced to the heat equation by means of the change $\tilde f = zf$.

For the case $\eta=\text{const}$ equation (5.22) can be reduced by inequivalent one-dimensional subalgebras of L_2. We construct the following solutions:

For the subalgebra $< \partial_\tau + af\partial_f >$, where $a\in\{-1;0;1\}$, it follows that

$$f = e^{-\tau}z^\nu(C_1J_\nu(z)+C_2Y_\nu(z)) \quad \text{if} \quad a=-1,$$

$$f = e^{\tau}z^\nu(C_1I_\nu(z)+C_2K_\nu(z)) \quad \text{if} \quad a=1,$$

$$f = C_1z^{\eta+1}+C_2 \quad \text{if} \quad a=0 \quad \text{and} \quad \eta\neq-1,$$

$$f = C_1\ln z + C_2 \quad \text{if} \quad a=0 \quad \text{and} \quad \eta=-1.$$

Here J_ν and Y_ν are the Bessel functions of a real variable, whereas I_ν and K_ν are the Bessel functions of an imaginary variable, and $\nu = \frac{1}{2}(\eta + 1)$.

For the subalgebra $< \hat{D} + 2af\partial_f >$, where $a \in \mathbb{R}$, it follows that

$$f = |\tau|^a e^{-\frac{1}{2}\omega} |\omega|^{\frac{1}{2}(\eta-1)} W(\tfrac{1}{4}(\eta-1) - a, \tfrac{1}{4}(\eta+1), \omega)$$

with $\omega = \frac{1}{4}z^2\tau^{-1}$. Here $W(\varkappa, \mu, \omega)$ is the general solution of the Whittaker equation

$$4\omega^2 W_{\omega\omega} = (\omega^2 - 4\varkappa\omega + 4\mu^2 - 1)W.$$

For the subalgebra $< \partial_\tau + \Pi + af\partial_f >$, where $a \in \mathbb{R}$, it follows that

$$f = (4\tau^2+1)^{\frac{1}{4}(\eta-1)} \exp(-\tau\omega + \tfrac{1}{2}a \arctan 2\tau)\varphi(\omega)$$

with $\omega = z^2(4\tau^2+1)^{-1}$. The function φ is a solution of the equation

$$4\omega\varphi_{\omega\omega} + 2(1-\eta)\varphi_\omega + (\omega - a)\varphi = 0.$$

For example if $a = 0$, then $\varphi(\omega) = \omega^\mu\Big(C_1 J_\mu(\frac{1}{2}\omega) + C_2 Y_\mu(\frac{1}{2}\omega)\Big)$, where $\mu = \frac{1}{4}(\eta + 1)$.

Consider equation (5.22), where η is an arbitrary smooth function of τ.

Theorem 5.2 *Equation (5.22) is Q-conditional invariant under the operators*

$$Q^1 = \partial_\tau + g^1(\tau, z)\partial_z + (g^2(\tau, z)f + g^3(\tau, z))\partial_f \tag{5.23}$$

if and only if

$$\begin{gathered} g^1_\tau - \eta z^{-1} g^1_z + \eta z^{-2} g^1 - g^1_{zz} + 2g^1_z g^1 - \eta_\tau z^{-1} + 2g^2_z = 0, \\ g^k_\tau + \eta z^{-1} g^k_z - g^k_{zz} + 2g^1_z g^k = 0, \quad k = 2, 3, \end{gathered} \tag{5.24}$$

and

$$Q^2 = \partial_z + B(\tau, z, f)\partial_f \tag{5.25}$$

if and only if

$$B_\tau - \eta z^{-2} B + \eta z^{-1} B_z - B_{zz} - 2BB_{zf} - B^2 B_{ff} = 0. \tag{5.26}$$

An arbitrary operator of Q-conditional symmetry of equation (5.22) is equivalent to either an operator of form (5.23) or an operator of form (5.25).

Theorem 5.2 is proved by means of the method described in [13].

Note 5.6 *It can be shown (in a way analogous to one in [13]) that system (5.24) is reduced to the decomposed linear system*

$$f^a_\tau + \eta z^{-1} f^a_z - f^a_{zz} = 0 \tag{5.27}$$

by means of the following non-local transformation

$$\begin{aligned} g^1 &= -\frac{f^1_{zz} f^2 - f^1 f^2_{zz}}{f^1_z f^2 - f^1 f^2_z} + \eta z^{-1}, \\ g^2 &= -\frac{f^1_{zz} f^2_z - f^1_z f^2_{zz}}{f^1_z f^2 - f^1 f^2_z}, \\ g^3 &= f^3_{zz} - \eta z^{-1} f^3_z + g^1 f^3_z - g^2 f^3. \end{aligned} \tag{5.28}$$

Equation (5.26) is reduced, by means of the change

$$B = -\Phi_\tau / \Phi_f, \quad \Phi = \Phi(\tau, z, f)$$

and the hodograph transformation

$$y_0 = \tau, \quad y_1 = z, \quad y_2 = \Phi, \quad \Psi = f,$$

to the following equation in the function $\Psi = \Psi(y_0, y_1, y_2)$*:*

$$\Psi_{y_0} + \eta(y_0) y_1^{-1} \Psi_{y_1} - \Psi_{y_1 y_1} = 0.$$

Therefore, unlike Lie symmetries Q-conditional symmetries of (5.22) are more extended for an arbitrary smooth function $\eta = \eta(\tau)$. Thus, Theorem 5.2 implies that equation (5.22) is Q-conditional invariant under the operators

$$\partial_z, \quad X = \partial_\tau + (\eta - 1) z^{-1} \partial_z, \quad G = (2\tau + C)\partial_z - z f \partial_f$$

with $C = \text{const}$. Reducing equation (5.22) by means of the operator G, we obtain the following solution:

$$f = C_2 (z^2 - 2 \textstyle\int (\eta(\tau) - 1) d\tau) + C_1. \tag{5.29}$$

In generalizing this we can construct solutions of the form

$$f = \sum_{k=0}^{N} T^k(\tau) z^{2k}, \tag{5.30}$$

where the coefficients $T^k = T^k(\tau)$ $(k = \overline{0, N})$ satisfy the system of ODEs:

$$\begin{aligned} &T^k_\tau + (2k + 2)(\eta(\tau) - 2k - 1) T^{k+1} = 0, \\ &k = \overline{0, N-1}, \quad T^N_\tau = 0. \end{aligned} \tag{5.31}$$

Equation (5.31) is easily integrated for arbitrary $N \in \mathbb{N}$. For example if $N = 2$, it follows that

$$f = C_3\Big\{z^4 - 4z^2\!\int(\eta(\tau) - 3)d\tau + 8\int\Big((\eta(\tau) - 1)\int(\eta(\tau) - 3)d\tau\Big)d\tau\Big\} +$$

$$+C_2\Big\{z^2 - 2\int(\eta(\tau) - 1)d\tau\Big\} + C_1.$$

An explicit form for solution (5.30) with $N = 1$ is given by (5.29).

Generalizing the solution

$$f = C_0 \exp\{-z^2(4\tau + 2C)^{-1} + \int(\eta(\tau) - 1)(2\tau + C)^{-1}d\tau\} \tag{5.32}$$

obtained by means of reduction of (5.22) by the operator G, we can construct solutions of the general form

$$\begin{aligned} f = & \sum_{k=0}^{N} S^k(\tau)(z(2\tau + C)^{-1})^{2k}\cdot \\ & \cdot\exp\Big\{-z^2(4\tau + 2C)^{-1} + \int(\eta(\tau) - 1)(2\tau + C)^{-1}d\tau\Big\}, \end{aligned} \tag{5.33}$$

where the coefficients $S^k = S^k(\tau)$ $(k = \overline{0, N})$ satisfy the system of ODEs:

$$\begin{aligned} & S^k_\tau + (2k + 2)(\eta(\tau) - 2k - 1)(2\tau + C)^{-2}S^{k+1} = 0, \\ & k = \overline{0, N - 1}, \quad S^N_\tau = 0. \end{aligned} \tag{5.34}$$

For example if $N = 1$, then

$$\begin{aligned} f = & \Big\{C_1\Big(z^2(2\tau + C)^{-2} - 2\int(\eta(\tau) - 1)(2\tau + C)^{-2}d\tau\Big) + C_0\Big\}\cdot \\ & \cdot\exp\Big\{-z^2(4\tau + 2C)^{-1} + \int(\eta(\tau) - 1)(2\tau + C)^{-1}d\tau\Big\}. \end{aligned}$$

Here we do not present results for arbitrary N as they are very cumbersome.

Putting $g^2 = g^3 = 0$ in system (5.24), we obtain one equation in the function g^1:

$$g^1_\tau - \eta z^{-1}g^1_z + \eta z^{-2}g^1 - g^1_{zz} + 2g^1_z g^1 - \eta_\tau z^{-1} = 0.$$

It follows that $g^1 = -g_z/g + (\eta - 1)/z$, where $g = g(\tau, z)$ is a solution of the equation

$$g_\tau + (\eta - 2)z^{-1}g_z - g_{zz} = 0. \tag{5.35}$$

Q-conditional symmetry of (5.22) under the operator

$$Q = \partial_\tau + (-g_z/g + (\eta - 1)/z)\partial_z \tag{5.36}$$

gives rise to the following

Theorem 5.3 *If g is a solution of equation (5.35) and*

$$\begin{aligned} f(\tau,z) = \ & \int_{z_0}^{z} z' g(\tau,z')dz' + \\ & + \int_{\tau_0}^{\tau} \Big(z_0 g_z(\tau',z_0) - (\eta(\tau')-1)g(\tau',z_0) \Big) d\tau', \end{aligned} \tag{5.37}$$

where (τ_0, z_0) is a fixed point, then f is a solution of equation (5.22).

Proof. Equation (5.35) implies

$$(zg)_\tau = (zg_z - (\eta-1)g)_z$$

Therefore, $f_z = zg$, $f_\tau = zg_z - (\eta-1)g$ and

$$f_\tau + \eta z^{-1} f_z - f_{zz} = zg_z - (\eta-1)g + \eta g - (zg)_z = 0. \quad \text{QED.}$$

The converse of Theorem 5.3 is the following obvious

Theorem 5.4 *If f is a solution of (5.22), the function*

$$g = z^{-1} f_z \tag{5.38}$$

satisfies (5.35).

Theorems 5.3 and 5.4 imply that, when $\eta = 2n$ ($n \in \mathbf{Z}$), solutions of (5.22) can be constructed from known solutions of the heat equation by means of applying either formula (5.37) (for $n > 0$) or formula (5.38) (for $n < 0$) $|n|$ times.

Let us investigate symmetry properties and construct some exact solutions of system (5.19)–(5.20) for $\varepsilon = 1$, i.e., the system

$$w^1_\tau - w^1_{zz} + \hat\eta(\tau) z^{-1} w^1_z = 0, \tag{5.39}$$

$$w^2_\tau - w^2_{zz} + (\hat\eta(\tau) - 2) z^{-1} w^2_z + (w^1 - \hat\chi(\tau)) z^{-2} = 0. \tag{5.40}$$

If (w^1, w^2) is a solution of system (5.39)–(5.40), then $(w^1, w^2 + g)$ (where $g = g(\tau, z)$) is also a solution of (5.39)–(5.40) if and only if the function g satisfies the following equation

$$g_\tau - g_{zz} + (\hat\eta(\tau) - 2) z^{-1} g_z = 0 \tag{5.41}$$

System (5.39)–(5.40), for some $\hat\chi = \hat\chi(\tau)$, has particular solutions of the form

$$w^1 = \sum_{k=0}^{N} T^k(\tau) z^{2k}, \quad w^2 = \sum_{k=0}^{N-1} S^k(\tau) z^{2k},$$

where $T^0(\tau) = \hat\chi(\tau)$. For example, if $\hat\chi(\tau) = -2C_1 \int (\hat\eta(\tau) - 1) d\tau + C_2$ and $N = 1$, then

$$w^1 = C_1(z^2 - 2\int(\hat\eta(\tau) - 1)d\tau) + C_2, \quad w^2 = -C_1 \tau.$$

Let $\hat\chi(\tau) = 0$.

Theorem 5.5 *The MIA of system (5.39)–(5.40) with* $\hat\chi(\tau)=0$ *is given by the following algebras*

a) $<w^i\partial_{w^i},\ \tilde w^i(\tau,z)\partial_{w^i}>$ *if* $\hat\eta(\tau)\neq\mathrm{const}$*;*

b) $<2\tau\partial_\tau+z\partial_z,\ \partial_\tau,\ w^i\partial_{w^i},\ \tilde w^i(\tau,z)\partial_{w^i}>$ *if* $\hat\eta(\tau)=\mathrm{const},\quad \hat\eta\neq0$*;*

c) $<2\tau\partial_\tau+z\partial_z,\ \partial_\tau,\ w^1z^{-1}\partial_{w^2},\ w^i\partial_{w^i},\ \tilde w^i(\tau,z)\partial_{w^i}>$ *if* $\hat\eta\equiv0$.

Here $(\tilde w^1,\tilde w^2)$ *is an arbitrary solution of (5.39)–(5.40) with* $\hat\chi(\tau)=0$.

For the case $\hat\chi(\tau)=0$ and $\hat\eta(\tau)=\mathrm{const}$ system (5.39)–(5.40) can be reduced by inequivalent one-dimensional subalgebras of its MIA. We obtain the following solutions:

For the subalgebra $<\partial_\tau>$ it follows that

$$w^1=C_1\ln z+C_2,$$
$$w^2=\tfrac14C_1(\ln^2z-\ln z)+\tfrac12C_2\ln z+C_3z^{-2}+C_4$$

if $\hat\eta=-1$;

$$w^1=C_1z^2+C_2,$$
$$w^2=\tfrac14C_1z^2+\tfrac12C_2\ln^2z+C_3\ln z+C_4$$

if $\hat\eta=1$;

$$w^1=C_1z^{\hat\eta+1}+C_2,$$
$$w^2=\tfrac12C_1(\hat\eta+1)^{-1}z^{\hat\eta+1}+C_2(\hat\eta-1)^{-1}\ln z+C_3z^{\hat\eta-1}+C_4$$

if $\hat\eta\notin\{-1;1\}$.

For the subalgebra $<\partial_\tau-w^i\partial_{w^i}>$ it follows that

$$w^1=e^{-\tau}z^{\frac12(\hat\eta+1)}\psi^1(z),\quad w^2=e^{-\tau}z^{\frac12(\hat\eta-1)}\psi^2(z),$$

where the functions ψ^1 and ψ^2 satisfy the system

$$z^2\psi^1_{zz}+z\psi^1_z+(z^2-\tfrac14(\hat\eta+1)^2)\psi^1=0,\tag{5.42}$$

$$z^2\psi^2_{zz}+z\psi^2_z+(z^2-\tfrac14(\hat\eta-1)^2)\psi^2=z\psi^1.\tag{5.43}$$

The general solution of system (5.42)–(5.43) can be expressed by quadratures in terms of the Bessel functions of a real variable $J_\nu(z)$ and $Y_\nu(z)$:

$$\psi^1=C_1J_{\nu+1}(z)+C_2Y_{\nu+1}(z),$$

$$\psi^2=C_3J_\nu(z)+C_4Y_\nu(z)+$$
$$+\tfrac{\pi}{2}Y_\nu(z)\int J_\nu(z)\psi^1(z)dz-\tfrac{\pi}{2}J_\nu(z)\int Y_\nu(z)\psi^1(z)dz$$

with $\nu = \frac{1}{2}(\hat{\eta} - 1)$;

For the subalgebra $< \partial_\tau + w^i\partial_{w^i} >$ it follows that

$$w^1 = e^\tau z^{\frac{1}{2}(\hat{\eta}+1)}\psi^1(z), \quad w^2 = e^\tau z^{\frac{1}{2}(\hat{\eta}-1)}\psi^2(z),$$

where the functions ψ^1 and ψ^2 satisfy the system

$$z^2\psi^1_{zz} + z\psi^1_z - (z^2 + \tfrac{1}{4}(\hat{\eta}+1)^2)\psi^1 = 0, \tag{5.44}$$

$$z^2\psi^2_{zz} + z\psi^2_z - (z^2 + \tfrac{1}{4}(\hat{\eta}-1)^2)\psi^2 = z\psi^1. \tag{5.45}$$

The general solution of system (5.44)–(5.45) can be expressed by quadratures in terms of the Bessel functions of an imaginary variable $I_\nu(z)$ and $K_\nu(z)$:

$$\psi^1 = C_1 I_{\nu+1}(z) + C_2 K_{\nu+1}(z),$$

$$\psi^2 = C_3 I_\nu(z) + C_4 K_\nu(z) + \\ K_\nu(z)\int I_\nu(z)\psi^1(z)dz - I_\nu(z)\int K_\nu(z)\psi^1(z)dz$$

with $\nu = \frac{1}{2}(\hat{\eta} - 1)$.

For the subalgebra $< 2\tau\partial_\tau + z\partial_z + aw^i\partial_{w^i} >$ it follows that

$$w^1 = |\tau|^a e^{-\frac{1}{2}\omega}|\omega|^{\frac{1}{4}(\hat{\eta}-1)}\psi^1(\omega), \quad w^2 = |\tau|^a e^{-\frac{1}{2}\omega}|\omega|^{\frac{1}{4}(\hat{\eta}-3)}\psi^2(\omega)$$

with $\omega = \frac{1}{4}z^2\tau^{-1}$, where the functions ψ^1 and ψ^2 satisfy the system

$$4\omega^2\psi^1_{\omega\omega} = \Big(\omega^2 + (a - \tfrac{1}{4}(\hat{\eta}-1))\omega + \tfrac{1}{4}(\hat{\eta}+1)^2 - 1\Big)\psi^1, \tag{5.46}$$

$$4\omega^2\psi^2_{\omega\omega} = \Big(\omega^2 + (a - \tfrac{1}{4}(\hat{\eta}-3))\omega + \tfrac{1}{4}(\hat{\eta}-1)^2 - 1\Big)\psi^2 + \\ +2|\omega|^{1/2}\psi^1. \tag{5.47}$$

The general solution of system (5.46)–(5.47) can be expressed by quadratures in terms of the Whittaker functions.

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