Classical Poisson Structure for a Hierarchy of One–Dimensional Particle Systems Separable in Parabolic Coordinates

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Abstract

We consider a hierarchy of many-particle systems on the line with polynomial potentials separable in parabolic coordinates. The first non-trivial member of this hierarchy is a generalization of an integrable case of the Hénon-Heiles system. We give a Lax representation in terms of 2×2 matrices for the whole hierarchy and construct the associated linear r-matrix algebra with the r-matrix dependent on the dynamical variables. A Yang-Baxter equation of dynamical type is proposed. Classical integration in a particular case is carried out and quantization of the system is discussed with the help of separation variables.

1 Introduction

In the last decade, much work has been carried out in the study of completely integrable systems admitting a classical r-matrix Poisson structure (c.f. [1, 2]) where the r-matrices depend *only* on the spectral parameters. More recently, interest has developed in the study of completely integrable systems where the r-matrices depend also on dynamical variables [3–6]. It is remarkable that the celebrated Calogero–Moser system, whose complete integrability was shown a number of years ago (see e.g. [7]), has been found only recently to possess a classical r-matrix of this dynamical type [8]. In this paper we study another example of a dynamical r-matrix structure. It is already well known that it is possible to provide 2×2 Lax operator satisfying the standard linear r-matrix algebra for the Hamiltonian system of natural form where the potential $\mathcal{U}$ is of second degree by the

coordinates (see e.g. [6, 9–12]). We consider a hierarchy of one-dimensional manyparticle systems with polynomial potentials of higher degree whose complete integrability in the framework of the Lax representation is known, but for which the associated r-matrix Poisson structure has not been discussed. The systems represent a generalization of the known hierarchy of two-particles systems with polynomial potentials separable in parabolic coordinates, which have the form (see, e.g. [13, 14])

$$\mathcal{V}_N = \sum_{k=0}^{[N/2]} 2^{1-2k} \binom{N-k}{k} q_1^{2k} q_2^{N-2k}, \tag{1.1}$$

where the positive integer N enumerates the members of the hierarchy. We study an integrable many particle generalizations of (1.1) in the framework of the Lax representation, written in terms of 2×2 matrices,

$$\dot{L}^{(N)}(z) = [M_N(z), L_N(z)],$$
$$L_N(z) = \begin{pmatrix} V(z) & U(z) \\ W_N(z) & -V(z) \end{pmatrix}, \quad M^{(N)}(z) = \begin{pmatrix} 0 & 1 \\ Q_N(z) & 0 \end{pmatrix}, \tag{1.2}$$

where the functions $U(z)$, $V(z)$, $W_N(z)$, $Q_N(z)$ depend rationally in the spectral parameter z and some constraints are imposed on them. In the first nontrivial case ($N = 3$) the system can be considered as a many-particle generalization of a known integrable case of the Hénon-Heiles system [15]. We emphasize, that the ansatz (1.2) is a natural generalization of the Lax representation found in [16] to describe an integrable case of the Hénon-Heiles system. See also [11, 12] for the link to the $su(1,1)$–Gaudin magnet which corresponds to a free n-dimensional particle separable in parabolic coordinates.

The polynomial second order spectral problem associated with the Lax representation was studied in [17] in the framework of a $(K \times K)$ Lax representation, and the recursion relations between different members of a hierarchy of Lax matrices was also given, but no explicit treatment of any associated dynamical system was discussed. The (2×2) Lax representation we describe appears to us to be more useful to investigate the class of integrable systems under consideration and to elucidate the classical r-matrix Poisson structure of the system.

We consider the system within the method of variable separation [18, 10, 11] that permits us to develop the classical theta-functional integration theory and to consider the associated quantum problem. The last problem is reduced to a set of multiparameter spectral problems which are a confluent form of ordinary differential equations of the Fuchsian type.

The central result of the paper is the description of the Poisson structure of the system by a dynamical linear r-matrix algebra defined in $V^2 \otimes V^2$ [3, 5, 6],

$$\{L_1^{(N)}(x) \overset{\otimes}{,} L_2^{(N)}(y)\} = [d_{12}^{(N)}(x,y), L_1^{(N)}(x)] - [d_{21}^{(N)}(x,y) L_2^{(N)}(y)], \tag{1.3}$$

where $\overset{\otimes}{,}$ is the direct product $\otimes$ of two matrices, but with the product of two matrix elements replaced by their Poisson bracket. The L matrices are defined in $V^2 \otimes V^2$ by $L_1^{(N)}(x) = L^{(N)}(x) \otimes I$, $L_2^{(N)}(y) = I \otimes L^{(N)}(y)$, I is the 2×2 unit matrix, $d_{12}^{(N)}$ and $d_{21}^{(N)}$ are matrices depending both on spectral parameters and on dynamical variables through the factor $(Q_N(x) - Q_N(y))/(x - y)$.

In the space $V^2 \otimes V^2 \otimes V^2$, the matrices $d_{ij}^{(N)}$ satisfy the constraints (classical Yang–Baxter equation of dynamical type)

$$
\begin{aligned}
&[d_{12}^{(N)}(x,y), d_{13}^{(N)}(x,z)] + [d_{12}^{(N)}(x,y), d_{23}^{(N)}(y,z)] + [d_{32}^{(N)}(z,y), d_{13}^{(N)}(x,z)] + \\
&\quad \{L_2^{(N)}(y) \overset{\otimes}{?} d_{13}^{(N)}(x,z)\} + \{L_3^{(N)}(z) \overset{\otimes}{?} d_{12}^{(N)}(x,y)\} + \\
&\quad [c^{(N)}(x,y,z), L_2^{(N)}(y) - L_3^{(N)}(z)] = 0
\end{aligned}
\tag{1.4}
$$

plus cyclic permutations. The matrices $d_{ij}^{(N)}$ and c in (1.4) involve the dynamical variables through the factor Q_N. We follow the now standard but somewhat confusing notation that $L_i^{(N)}, d_{ij}^{(N)}$ refer to different matrices depend on whether the current space is $V^2 \otimes V^2$ or $V^2 \otimes V^2 \otimes V^2$. In $V^2 \otimes V^2 \otimes V^2$, $L_1^{(N)}(x) = L^{(N)}(x) \otimes I \otimes id$, $L_2^{(N)} = I \otimes L^{(N)}(y) \otimes I$, $L_3^{(N)}(z) = L^{(N)}(z) \otimes I \otimes I$. The matrix $d_{ij}^{(N)}$ in $V^2 \otimes V^2 \otimes V^2$ acts like $d_{ij}^{(N)}$ in ith and jth space and as I in the third space. Except for the final term, eqn. (1.4) is the same as that given in [5].

A main theme of the paper is to present a new solution of the dynamical Yang-Baxter equations (1.4) which is associated with a hierarchy of one-dimensional dynamical systems. At the same time the r-matrix structure of parabolic coordinates is elucidated.

The main results were announced in the paper [19].

2   The Hierarchy of Separable Systems

In this section we describe a hierarchy of completely integrable one-dimensional many-particle systems with polynomial potentials. The different members of the hierarchy are connected by recurrent relations. We give the Lax representation in terms of (2×2) matrices for all the hierarchy, describe the associated algebraic curves and present explicit formulae for the integrals of motion.

2.1   Hierarchy of Hamiltonian Systems with Polynomial Potentials

Let us consider the hierarchy of the Hamiltonian systems of $n+1$ particles defined by the Hamiltonians

$$
H_N(p_1, \ldots, p_{n+1}; q_1 \ldots, q_{n+1}) = \frac{1}{2} \sum_{i=1}^{n+1} p_i^2 + \mathcal{U}_N(q_1, \ldots, q_{n+1}),
\tag{2.1}
$$

where the potentials $\mathcal{U}_N$ for each member of the hierarchy are given by the recurrence relation

$$
\mathcal{U}_N = (q_{n+1} - B)\mathcal{U}_{N-1} + \frac{1}{4} \sum_{i=1}^{n} \sum_{j=2}^{N} (-1)^j q_i^2 \mathcal{U}_{N-j} A_i^{j-2}
\tag{2.2}
$$

with the first trivial potentials given as

$$
\begin{aligned}
\mathcal{U}_0 &= 0, \\
\mathcal{U}_1 &= -2q_{n+1} - 2B, \\
\mathcal{U}_2 &= -2q_{n+1}^2 - \frac{1}{2} \sum_{i=1} q_i^2.
\end{aligned}
\tag{2.3}
$$

At $A_i = 0$, $i = 1, \ldots, n$, $B = 0$ the recurrent relations (2.2) can be solved as

$$\mathcal{V}_N(q_1, \ldots, q_{n+1}) = - \sum_{k=0}^{[N/2]} 2^{1-2k} \binom{N-k}{k} \left(\sum_{i=1}^{n} q_i^2 \right)^k q_{n+1}^{N-2k} \tag{2.4}$$

yielding the principal term of $\mathcal{U}_N(q_1, \ldots, q_{n+1})$ which is a many-particle generalization of (1.1). The lower degree terms are introduced so as to make the many-particle generalization non-degenerate. The expression for the potentials $\mathcal{U}_N$ at $N > 2$ can be written in the form of $(N-2) \times (N-2)$ determinant

$$\mathcal{U}_N = (-1)^{N-1} \det \begin{pmatrix} f_N & g_{-1} & g_0 & g_1 & \cdots & g_{N-5} \\ f_{N-1} & -1 & g_{-1} & g_0 & \cdots & g_{N-6} \\ f_{N-2} & 0 & -1 & g_{-1} & \cdots & g_{N-7} \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ f_4 & 0 & \cdots & 0 & -1 & g_{-1} \\ f_3 & 0 & \cdots & \cdots & 0 & -1 \end{pmatrix}, \tag{2.5}$$

where

$$g_{-1} = q_{n+1} - B, \quad g_m = \frac{(-1)^m}{4} \sum_{i=1}^{n} A_i^m q_i^2, \quad m = 0, \ldots,$$

$$f_k = \sum_{l=0}^{2} \mathcal{U}_l g_{k-l-2}, \quad k = 3, \ldots, N, \tag{2.6}$$

and the $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{U}_2$ are given by (2.3). The first nontrivial potentials are

$$\mathcal{U}_3 = -2q_{n+1}^3 - q_{n+1} \sum_{i=1}^{n} q_i^2 + \frac{1}{2} \sum_{i=1}^{n} A_i q_i^2 + 2B q_{n+1}^2, \tag{2.7}$$

$$\mathcal{U}_4 = -\frac{1}{8} \left(\sum_{i=1}^{n} q_i^2 \right)^2 - \frac{3}{2} q_{n+1}^2 \sum_{i=1}^{n} q_i^2 - 2q_{n+1}^4 + \sum_{i=1}^{n} A_i q_i^2 \left(q_{n+1} - \frac{1}{2} A_i \right) +$$

$$B \left(4q_{n+1}^3 - 2B q_{n+1}^2 + q_{n+1} \sum_{j=1}^{n} q_j^2 \right). \tag{2.8}$$

The potential (2.7) is exactly the many-particle generalization of a known integrable case of the Hénon-Heiles system for which $n = 1$ (c.f. [20, 16]). Analogously the potential (2.8) is a many-particle generalization of a "(1 : 12 : 16)" system known to be separable in parabolic coordinates (see e.g. [13, 14]).

The system with the potential (2.7) possesses the following interesting reductions: a) at $q_{n+1} = \text{const}$, it reduces to the Neuman system, which describes the motion of a particle on a sphere in the field of a second order potential, and b) at $q_{n+1} = \sum_{i=1}^{n} q_i^2$ it reduces to an anisotropic oscillator in a fourth order potential (see, for instance [14]),

$$\ddot{q}_i - 2 \sum_{k=1}^{n} q_k^2 q_i + A_i q_i = 0, \quad i = 1, \ldots, n. \tag{2.9}$$

The same reduction can be carried out for other members of the hierarchy.

2.2 The Lax Representation

We look at a Lax representation of the form (1.2) with

$$U(z) = 4z - 4q_{n+1} + 4B - \sum_{i=1}^{n} \frac{q_i^2}{z + A_i}, \tag{2.10}$$

$$V(z) = -\frac{1}{2}\dot{U}(z),$$

$$= 2p_{n+1} + \sum_{i=1}^{n} \frac{p_i q_i}{z + A_i}, \tag{2.11}$$

$$W_N(z) = -\frac{1}{2}\ddot{U}(z) + U(z)Q_N(z), \tag{2.12}$$

where $Q_N(z)$ is a polynomial of degree $N-2$. The ansatz for the functions $U(z), V(z), W(z)$ is a generalization of the corresponding ansatz constructed by Newell et al. [16] to give the Lax representation for the integrable Hénon-Heiles system. Here we introduce additional degrees of freedom $n > 1$, and consider higher degrees of the polynomial Q_N. See also [11, 12] for the link of such ansatz to the $su(1,1)$–Gaudin magnet which corresponds to a free n-dimensional particle separable in parabolic coordinates.

Proposition 1 *The Lax representation (1.2) is valid for all the hierarchy of Hamiltonian systems (2.1), (2.5) with the polynomial $Q_N(z)$ and the function $W_N(z)$ given by the formulae*

$$Q_N(z) = zQ_{N-1}(z) - \frac{1}{2}\frac{\partial \mathcal{U}_{N-1}(q_1, \ldots, q_{n+1})}{\partial q_{n+1}}, \tag{2.13}$$

$$W_N(z) = W_N^+(z) + W^-(z), \tag{2.14}$$

$$W_N^+(z) = zW_{N-1}^+ - 2\mathcal{U}_{N-1}, \quad N = 2, \ldots, \tag{2.15}$$

$$W^-(z) = \sum_{i=1}^{n} \frac{p_i^2}{z + A_i}, \tag{2.16}$$

where $\mathcal{U}_{N-1}$ is the potential fixing the $N-1$-th member of the hierarchy.

P r o o f The Lax representation (1.2) with the functions (2.10)–(2.12) is equivalent to the two equations

$$W_N(z) = \dot{V}(z) + Q_N(z)U(z), \tag{2.17}$$

$$\dot{W}_N(z) = 2Q_N(z)V(z). \tag{2.18}$$

Substituting (2.10), (2.11) and (2.13) into (2.17) we obtain after some simplification

$$W_N(z) = \sum_{i=1}^{n} \frac{p_i^2}{z + A_i} + z\left(U(z)Q_{N-1}(z) - 2\frac{\partial \mathcal{U}_{N-1}}{\partial q_{n+1}} - \sum_{i=1}^{n} \frac{q_i^2}{z + A_i}\frac{\partial \mathcal{U}_{N-1}}{\partial q_i}\right) +$$

$$z\sum_{i=1}^{n} \frac{q_i^2}{z + A_i}\frac{\partial \mathcal{U}_{N-1}}{\partial q_i} - \sum_{i=1}^{n} \frac{q_i}{z + A_i}\frac{\partial \mathcal{U}_N}{\partial q_i} - 2\frac{\partial \mathcal{U}_N}{\partial q_{n+1}} +$$

$$2(q_{n+1} - B)\frac{\partial \mathcal{U}_{N-1}}{\partial q_{n+1}} + \frac{1}{2}\frac{\partial \mathcal{U}_{N-1}}{\partial q_{n+1}}\sum_{i=1}^{n} \frac{q_i^2}{z + A_i}. \tag{2.19}$$

Imposing the conditions (2.14)–(2.16) we arrive at the following recurrence relations for the potentials $\mathcal{U}_N(q_1, \ldots, q_{n+1})$

$$\frac{\partial \mathcal{U}_N}{\partial q_i} = \frac{1}{2} \frac{\partial \mathcal{U}_{N-1}}{\partial q_{n+1}} q_i - A_i \frac{\partial \mathcal{U}_{N-1}}{\partial q_i}, \quad i = 1, \ldots, n, \tag{2.20}$$

$$\frac{\partial \mathcal{U}_N}{\partial q_{n+1}} = \mathcal{U}_{N-1} + \frac{1}{2} \sum_{i=1}^{n} q_i \frac{\partial \mathcal{U}_{N-1}}{\partial q_i} + (q_{n+1} - B) \frac{\partial \mathcal{U}_{N-1}}{\partial q_{n+1}}, \tag{2.21}$$

where we use (2.3) to start the recurrence relations.

To prove that the equation (2.18) is also satisfied, we proceed by induction. First assume that it is valid for $N - 1$. Then we have

$$\dot{W}_N(z) = 2zQ_{N-1}(z)V(z) - \frac{\partial \mathcal{U}_{N-1}}{\partial q_{n+1}} \left(2p_{n+1} + \sum_{i=1}^{n} \frac{q_i p_i}{z + A_i} \right). \tag{2.22}$$

The left hand side of (2.22) can be rewritten in the form

$$\frac{d}{dt} \left(zW_{N-1}^+(z) - 2\mathcal{U}_{N-1} + \sum \frac{p_i^2}{z + A_i} \right) =$$

$$z\dot{W}_{N-1} + 2z \sum_{i=1}^{n} \frac{p_i}{z + A_i} \frac{\partial \mathcal{U}_{N-1}}{\partial q_i} - 2p_{n+1} \frac{\partial \mathcal{U}_{N-1}}{\partial q_{n+1}} -$$

$$2 \sum_{i=1}^{n} p_i \frac{\partial \mathcal{U}_{N-1}}{\partial q_i} - 2 \sum_{i=1}^{n} \frac{p_i}{z + A_i} \frac{\partial \mathcal{U}_N}{\partial q_i}.$$

Using induction and the equality (2.20) completes the proof.

We can obtain explicit formulae for the functions $W_N(z)$ and $Q_N(z)$

$$Q_N(z) = z^{N-2} - \frac{1}{2} \sum_{k=0}^{N-3} \frac{\partial \mathcal{U}_{N-k-1}}{\partial q_{n+1}} z^k, \tag{2.23}$$

$$W_N(z) = 4z^{N-1} - 2 \sum_{k=0}^{N-2} \mathcal{U}_{N-k-1} z^k + \sum_{i=1}^{n} \frac{p_i^2}{z + A_i}. \tag{2.24}$$

It easy to see that

$$Q_N(x) = \frac{1}{4} \frac{\partial W_N(x)}{\partial q_{n+1}}. \tag{2.25}$$

For example, for the first nontrivial cases we have

$$Q_3(z) = z + 2q_{n+1}, \tag{2.26}$$

$$W_3(z) = 4z^2 + 4zq_{n+1} + 4Bz + 4q_{n+1}^2 + \sum_{k=1}^{n} q_k^2 + \sum_{i=1}^{n} \frac{p_i^2}{z + A_i} \tag{2.27}$$

for the many-particle Hénon-Heiles system and

$$Q_4(z) = z^2 + 2zq_{n+1} + 3q_{n+1}^2 + \frac{1}{2} \sum_{i=1}^{n} q_i^2 - 2Bq_{n+1}, \tag{2.28}$$

$$
W_4(z) = 4z^3 + 4Bz^2 + 4z^2 q_{n+1} + 4z q_{n+1}^2 + z \sum_{i=1}^{n} q_i^2 +
$$

$$
4q_{n+1}^3 + 2q_{n+1} \sum_{i=1}^{n} q_i^2 - \sum_{i=1}^{n} A_i q_i^2 - 4B q_{n+1}^2 + \sum_{i=1}^{n} \frac{p_i^2}{z + A_i} \qquad (2.29)
$$

for the system with $N = 4$.

2.3   Integrability

The Lax representation yields the hyperelliptic curve $C^{(N)} = (w, z)$,

$$
\mathrm{Det}\,(L^{(N)}(z) - wI) = 0 \qquad (2.30)
$$

generating the integrals of motion $H_N, F_N^{(i)}, \quad i = 1, \ldots, n$. We have

$$
w^2 = \frac{1}{2}\mathrm{Tr}(L^{(N)}(z))^2 = V^2(z) + U(z)W(z). \qquad (2.31)
$$

From (2.31) and (2.10)–(2.12) we obtain

$$
w^2 = 16z^{N-2}(z + B)^2 + 8H_N + \sum_{i=1}^{n} \frac{F_N^{(i)}}{z + A_i}, \quad N = 3, \ldots, \qquad (2.32)
$$

where

$$
F_N^{(i)} = 2q_i^2 \sum_{j=1}^{N-1} (-1)^{j-1} A_i^j \mathcal{U}_{N-j} + 4p_{n+1}p_i q_i - p_i^2(A_i + 4q_{n+1} - 4B) +
$$

$$
\sum_{k,m=1,\, k\neq m}^{k,m=n} \frac{l_{mk}^2}{A_m - A_k}, \quad i = 1, \ldots, n \qquad (2.33)
$$

with $l_{km} = q_k p_m - q_m p_k$. The coefficients of the curve are the integrals of motion, so noting that the coefficient of z^{N-1} is $32B$ and the coefficient of z^{N-2} is $16B^2$ we obtain the expressions for the first potentials (2.3). Equating all the other coefficients of powers of z to zero, i.e. the coefficients of z^j, $j = N - 3, \ldots, 1$ we reproduce all the hierarchy of integrable potentials in the form (2.2).

Proposition 2   *The integrals H_N, $F_N^{(i)}$, $i = 1, \ldots, n$ are independent and Poisson commute with respect to the standard Poisson bracket,*

$$
\{H_N, F_N^{(i)}\} = 0, \ \{F_N^{(j)}, F_N^{(k)}\} = 0, \quad i, j, k = 1, \ldots, n. \qquad (2.34)
$$

The independence of the integrals is evident. But we postpone the proof of (2.34) until §4 where the classical r-matrix structure will be elucidated, making the proof trivial.

3 r-Matrix Representation

In this section we describe the Poisson structure associated with the Lax representation for the hierarchy under consideration. It is found to be a linear r-matrix algebra with the r-matrices dependent on dynamical variables and satisfying some compatibility conditions which are classical Yang–Baxter equations of dynamical type. We also discuss in this section a way to describe the system within the standard linear r-matrix algebra by embedding into a larger dynamical system.

Through all the section we use the standard notation for the Pauli matrices

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}$$

Denote $\sigma_\pm = \sigma_1 \pm i\sigma_2$. Denote as P the permutation matrix $P = \frac{1}{2}\sum_{k=0}^{3}\sigma_k \otimes \sigma_k$ acting in the product of two spaces $V^2 \otimes V^2$. We denote $P_{12} = P \otimes I$, $P_{13} = \frac{1}{2}\sum_{k=0}^{3}\sigma_k \otimes I \otimes \sigma_k$ and $P_{23} = I \otimes P$ the permutation matrices acting in the product of three spaces $V^2 \otimes V^2 \otimes V^2$. The notation m_{ik} will be used to denote an (8×8) matrix acting as a unit matrix in j-th space ($i \neq j \neq k$). For example, let $S = \sigma_- \otimes \sigma_-$, then $S_{12} = S \otimes I$, $S_{23} = I \otimes S$, $S_{13} = \sigma_- \otimes I \otimes \sigma_-$. In addition we introduce the notation $r_{i,j}(x_i - x_j) = 2P_{i,j}/(x_i - x_j)$, $s_{i,j}^{(N)}(x_i, x_j) = 2\alpha(x_i, x_j)S_{i,j}$, where $\alpha(x_i, x_j)$ is a scalar function depending on dynamical variables.

3.1   The Classical Poisson Structure

The following proposition holds:

Proposition 3   *The classical Poisson structure for the hierarchy (2.10)–(2.12) is written in the form*

$$\{L_1^{(N)}(x), L_2^{(N)}(y)\} = [r(x - y), L_1^{(N)}(x) + L_2^{(N)}(y)] + [s(x,y), L_1^{(N)}(x) - L_2^{(N)}(y)], \tag{3.2}$$

where $L_1^{(N)}(x) = I \otimes L^{(N)}(x)$, $L_2^{(N)} = L^{(N)}(x) \otimes I$, the matrices $r(x - y)$ and $s_N(x,y)$ are given by the formulae

$$r(x - y) = \frac{2}{x - y}P, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.3}$$

$$s_N(x,y) = 2\alpha_N(x,y)S, \quad S = \sigma_- \otimes \sigma_- \tag{3.4}$$

with

$$\alpha_N(x,y) = \frac{Q_N(x) - Q_N(y)}{x - y} = \frac{x^{N-1} - y^{N-1}}{x - y} - \frac{1}{2}\sum_{k=o}^{N-3}\frac{x^{N-k-1} - y^{N-k-1}}{x - y}\frac{\partial \mathcal{U}_{N-k-1}}{\partial q_{n+1}}. \tag{3.5}$$

P r o o f Let us rewrite the relation (3.2) in extended form

$$\{U(x), U(y)\} = \{V(x), V(y)\} = 0, \tag{3.6}$$

$$\{W_N(x), W_N(y)\} = 4\alpha_N(x, y)(V(x) - V(y)), \tag{3.7}$$

$$\{V(x), U(y)\} = \frac{2}{y - x}(U(x) - U(y)), \tag{3.8}$$

$$\{W_N(x), U(y)\} = \frac{4}{x - y}(V(x) - V(y)), \tag{3.9}$$

$$\{V(x), W_N(y)\} = \frac{2}{x - y}(W_N(x) - W_N(y)) - 2\alpha_N(x, y)U(x) \tag{3.10}$$

with respect to the standard Poisson bracket. The equalities (3.6), (3.8), (3.9) can be proved directly from the definitions (2.10)–(2.12). Let us prove (3.7) by induction i.e. let (3.7) be valid for the number N. Then for $N + 1$ we have

$$\begin{aligned}
\{W_{N+1}(x), W_{N+1}(y)\} &= \{W_{N+1}^+(x), W^-(y)\} - \{W_{N+1}^+(y), W^-(x)\} = \\
&\{xW_N^+(x) - 2\mathcal{U}_N, W^-(y)\} - \{yW_N^+(y) - 2\mathcal{U}_N, W^-(x)\} = \\
&x(\{W_N^+(x), W^-(y)\} - \{W_N^+(y), W^-(x)\}) - \\
&y(\{W_N^+(y), W^-(x)\} - \{W_N^+(x), W^-(y)\}) = \\
&x\{W_N^+(y), W^-(x)\} - y\{W_N^+(x), W^-(y)\} - 2\{\mathcal{U}_N, W^-(y)\} \tag{3.11}
\end{aligned}$$

the inductive hypothesis gives

$$\{W_{N+1}(x), W_{N+1}(y)\} - 4\alpha_{N+1}(x, y)(V(x) - V(y)) = \Delta \tag{3.12}$$

with

$$\begin{aligned}
\Delta &= 2\sum_{i=1}^n \frac{1}{(x + A_i)(y + A_i)}\left(2p_iq_i(xQ_N(y) - yQ_N(x)) + 2(x - y)\frac{\partial \mathcal{U}_N}{\partial q_i} - \right. \\
&\left. p_i\left(x(y + A_i)\frac{\partial W_N^+(y)}{\partial q_i} - y(x + A_i)\frac{\partial W_N^+(x)}{\partial q_i}\right)\right) = \sum_{i=1}^n \Delta_i. \tag{3.13}
\end{aligned}$$

To prove that $\Delta = 0$ we substitute the expressions (2.23), (2.24) into Δ_i. We have

$$\begin{aligned}
\Delta_i &= 4q_i\left(xy^{N-2} - yx^{N-2} - \frac{1}{2}\sum_{k=0}^{N-3}(xy^k - yx^k)\frac{\partial \mathcal{U}_{N-k-1}}{\partial q_{n+1}}\right) + \\
&4\sum_{k=0}^{N-2}\left(x(y + A_i)y^k - y(x + A_i)x^k\right)\frac{\partial \mathcal{U}_{N-k-1}}{\partial q_i} + 4(x - y)\frac{\partial \mathcal{U}_N}{\partial q_i}. \tag{3.14}
\end{aligned}$$

From (2.20) it follows that all $\Delta_i = 0$ and the derivation of (3.7) is complete.

Eq. (3.10) we also prove by induction. Assume that it is valid at N. Then

$$\begin{aligned}
\{V(x), W_{N+1}(y)\} &= y\{V(x), W_N(y)\} + (1 - y)\{V(x), W^-(y)\} - 2\{V(x), \mathcal{U}_N\} = \\
&\frac{2y}{x - y}(W_N^+(x) - W_N^+(y)) + \frac{2y}{x - y}(W^-(x) - W^-(y)) - \\
&2yU(x)\alpha_N(x, y) + (1 - y)\{V(x), W^-(y)\} - 2\{V(x), \mathcal{U}_N\}.
\end{aligned}$$

Using the inductive proposition and the formulae (2.14)–(2.16) we find

$$\{V(x), W_{N+1}(y)\} \quad - \quad \frac{2}{y-x}(W_N(x) - W_N(y)) + 2\alpha_{N+1}(x,y)U(x) = \widetilde{\Delta}$$

with

$$\widetilde{\Delta} = \frac{2(y-1)}{x-y}(W^-(x) - W^-(y)) + (1-y)\{V(x), W^-(y)\} -$$
$$2W^+ + 2U(x)Q_N(x) - 2\{V(x), \mathcal{U}_N\}. \tag{3.15}$$

The first two terms in (3.15) cancel due to the definitions (2.11), (2.16). To cancel the rest we compute the Poisson bracket directly and use (2.12). Therefore (3.10) is proven.

The equality (3.2) contains all the information concerning the hierarchy of dynamical system under consideration.

Let us write the relation (3.2) in the form

$$\{L_1^{(N)}(x) \overset{\otimes}{,} L_2^{(N)}(y)\} = [d_{12}^{(N)}(x,y), L_1^{(N)}(x)] - [d_{21}^{(N)}(x,y)L_2^{(N)}(y)], \tag{3.16}$$

with $d_{ij}^{(N)} = r_{ij} + s_{ij}$, $d_{ji}^{(N)} = s_{ij} - r_{ij}$ at $i < j$.

Then [3]

$$\{(L_1^{(N)}(x))^k \overset{\otimes}{,} (L_2^{(N)}(y))^l\} = [d_{12}^{(N,k,l)}(x,y), L_1^{(N)}(x)] - [d_{21}^{(N,k,l)}(x,y)L_2^{(N)}(y)], \tag{3.17}$$

with

$$d_{12}^{(N,k,l)}(x,y) = \sum_{p=0}^{k-1}\sum_{q=0}^{l-1}(L_1^{(N)}(x))^{k-p-1}(L_2^{(N)}(y))^{k-q-1}d_{12}^{(N)}(x,y)(L_1^{(N)}(x))^p(L_2^{(N)}(y))^q,$$
$$\tag{3.18}$$

$$d_{21}^{(N,k,l)}(x,y) = \sum_{p=0}^{k-1}\sum_{q=0}^{l-1}(L_1^{(N)}(x))^{k-p-1}(L_2^{(N)}(y))^{k-q-1}d_{21}^{(N)}(x,y)(L_1^{(N)}(x))^p(L_2^{(N)}(y))^q.$$

As an immediate consequence of (3.17) and (3.18) we obtain the proof that the conserved quantities H_N, $F_N^{(i)}$ are in involution. We have that

$$\{\text{Tr}(L_1^{(N)}(x))^2, \text{Tr}(L_2^{(N)}(y))^2\} = \text{Tr}\{(L_1^{(N)}(x))^2 \overset{\otimes}{,} (L_2^{(N)}(y))^2\}. \tag{3.19}$$

Applying to the last equation the equality (3.17) at $k = l = 2$ and taking the trace we obtain the desired involutivity.

The Lax representation (1.2) also can be recovered from (3.2). We have

$$\dot{L}^{(N)}(x) = \frac{1}{8}\lim_{y\to\infty}\text{Tr}_2\{L_1^{(N)}(x) \overset{\otimes}{,} (L_2^{(N)}(y))^2\}, \tag{3.20}$$

where the trace is taken over the second space. Applying the identity (3.17) at $k = 1, l = 2$ to (3.20) we obtain the Lax representation $\dot{L}^{(N)}(x) = [M^{(N)}(x), L^{(N)}(x)]$ with the matrix $M^{(N)}(x)$ given as

$$M^{(N)}(x) = 2\lim_{y\to\infty}\text{Tr}_2 L_1^{(N)}(y)(r(x-y) - s^{(N)}(x,y)). \tag{3.21}$$

After the calculation in which we take into the account the asymptotic assumptions

$$\lim_{y\to\infty}\frac{U(y)}{y} = 4, \quad \lim_{y\to\infty}\frac{V(y)}{y} = 0,$$
$$\lim_{y\to\infty}(W_N(y) - U(y)Q_N(y)) = 0$$

we obtain the Lax representation in the form (1.2). Analogously the Lax representation for the higher flows can be obtained.

3.2 Jacobi Identity

To prove that the algebra (1.3) is associative we write the Jacobi identity in the $V^2 \otimes V^2 \otimes V^2$ as

$$\{L_1^{(N)}(x) \overset{\otimes}{,} \{L_2^{(N)}(x) \overset{\otimes}{,} L_3^{(N)}(y)\}\} + \{L_3^{(N)}(z) \overset{\otimes}{,} \{L_1^{(N)}(x) \overset{\otimes}{,} L_2^{(N)}(y)\}\} +$$
$$\{L_2^{(N)}(y) \overset{\otimes}{,} \{L_3^{(N)}(z) \overset{\otimes}{,} L_1^{(N)}(x)\}\} = 0 \tag{3.22}$$

with $L_1^{(N)}(x) = L^{(N)}(x) \otimes I \otimes I$, $L_2^{(N)}(y) = I \otimes L^{(N)}(x) \otimes id$, $L_3^{(N)}(z) = I \otimes I \otimes L^{(N)}(z)$. Let us rewrite (3.22) in the form [5],

$$[L_1^{(N)}(x), [d_{12}^{(N)}(x,y), d_{13}^{(N)}(x,z)] + [d_{12}^{(N)}(x,y), d_{23}^{(N)}(y,z)] + [d_{32}^{(N)}(z,y), d_{13}^{(N)}(x,z)]] +$$
$$[L_1^{(N)}(x), \{L_2^{(N)}(y) \overset{\otimes}{,} d_{13}^{(N)}(x,z)\} - \{L_3^{(N)}(z) \overset{\otimes}{,} d_{12}^{(N)}(x,y)\}] +$$
$$\text{cyclic permutations} = 0. \tag{3.23}$$

We shall show below, that in the cases considered the solutions (3.23) satisfy the three equations

$$[d_{12}^{(N)}(x,y), d_{13}^{(N)}(x,z)] + [d_{12}^{(N)}(x,y), d_{23}^{(N)}(y,z)] + [d_{32}^{(N)}(z,y), d_{13}^{(N)}(x,z)] +$$
$$\{L_2^{(N)}(y) \overset{\otimes}{,} d_{13}^{(N)}(x,z)\} - \{L_3^{(N)}(z) \overset{\otimes}{,} d_{12}^{(N)}(x,y)\} +$$
$$[c^{(N)}(x,y,z), L_2^{(N)}(y) - L_3^{(N)}(z)] = 0 \tag{3.24}$$

where $c^{(N)}(x,y,z)$ is some matrix dependent on dynamical variables. The other two equations are obtained from (3.24) by cyclic permutations. We remark, that validity of the equations (3.24) with an arbitrary matrix $c^{(N)}(x,y,z)$ is sufficient for the validity of (3.22) and therefore (3.24) can be interpreted as some dynamical classical Yang–Baxter equations. These equations have an extra term $[c, L_i^{(N)} - L_j^{(N)}]$ in comparison with the extended Yang–Baxter equations in [5].

Proposition 4 *The following equality is valid for all the members of the hierarchy of dynamical systems*

$$\{L_2^{(N)}(y) \overset{\otimes}{,} s_{13}(x,z)\} - \{L_3^{(N)}(z) \overset{\otimes}{,} s_{12}(x,y)\} = 2\beta_N(x,y,z)[P_{23}, S_{13} + S_{12}] -$$
$$\frac{\partial \beta_N(x,y,z)}{\partial q_{n+1}}[s, L_2^{(N)}(y) - L_3^{(N)}(z)] \tag{3.25}$$

with cyclic permutations. In (3.25) the matrix $s = \sigma_- \otimes \sigma_- \otimes \sigma_-$ and

$$\beta_N(x,y,z) = \frac{Q_N(x)(y-z) + Q_N(y)(z-x) + Q_N(z)(x-y)}{(x-y)(y-z)(z-x)}. \tag{3.26}$$

P r o o f Let us write the equality (3.25) in the extended form

$$\{Q(x), Q(y)\} \;=\; \{U(x), Q(y)\} = 0, \tag{3.27}$$

$$\{V(x), Q(y)\} \;=\; 4\alpha_N(x, y) - \frac{1}{2}\frac{\partial \alpha_N(x, y)}{\partial q_{n+1}}U(x), \tag{3.28}$$

$$\{Q(x), W(y)\} \;+\; \{W(x), Q(y)\} = \frac{\partial \alpha_N(x, y)}{\partial q_{n+1}}(V(x) - V(y)), \tag{3.29}$$

$$\{V(z), \alpha_N(x, y)\} \;=\; \frac{4}{x - y}(\alpha_N(z, x) - \alpha_N(z, y)) -$$
$$\frac{1}{2}\frac{U(z)}{x - y}\left(\frac{\partial \alpha_N(z, y)}{\partial q_{n+1}} - \frac{\partial \alpha_N(z, x)}{\partial q_{n+1}}\right), \tag{3.30}$$

(3.27) is trivial and (3.28) and (3.29) follows immediately from (3.10), (3.7) and (2.25). To prove (3.30) we write it using the properties of $W_N(x)$ and $V(x)$ in the form

$$\sum_{i=1}^{n} \frac{p_i}{(y + A_i)(x + A_i)}\left((x + A_i)\frac{\partial \alpha_N(z, x)}{\partial q_i} - (y + A_i)\frac{\partial \alpha_N(z, y)}{\partial q_i}\right) =$$
$$\frac{1}{2}\sum_{i=1}^{n} \frac{p_i q_i}{(x + A_i)(y + A_i)}\left(\frac{\partial \alpha_N(z, x)}{\partial q_{n+1}} - \frac{\partial \alpha_N(z, y)}{\partial q_{n+1}}\right). \tag{3.31}$$

Using the decomposition (3.32), the equality

$$x\frac{z^k - x^k}{z - x} - y\frac{z^k - y^k}{z - y} = \frac{z^{k+1} - x^{k+1}}{z - x} - \frac{z^{k+1} - y^{k+1}}{z - y}$$

and the recurrence relation (2.20) we find that the equality (3.31) is valid.

   Note that (3.28) can we rewritten in the form

$$\{W_N(y), \alpha_N(z, x)\} - \{W_N(x), \alpha_N(z, y)\} =$$
$$\frac{V(x) - V(y)}{x - y}\left(\frac{\partial \alpha_N(z, y)}{\partial q_{n+1}} - \frac{\partial \alpha_N(z, x)}{\partial q_{n+1}}\right). \tag{3.32}$$

   All the preceding material can be formulated in the following theorem

Theorem 1 *Let*

$$\{L_1^{(N)}(x) \overset{\otimes}{,} L_2^{(N)}(y)\} = [d_{12}^{(N)}(x, y), L_1^{(N)}(x)] - [d_{21}^{(N)}(x, y), L_2^{(N)}(y)] \tag{3.33}$$

be the r-matrix algebra constrained by the three conditions on the matrices $d_{ij}^{(N)}$ the first of which is written as

$$[d_{12}^{(N)}(x, y), d_{13}^{(N)}(x, z)] + [d_{12}^{(N)}(x, y), d_{23}^{(N)}(y, z)] + [d_{32}^{(N)}(z, y), d_{13}^{(N)}(x, z)] +$$
$$\{L_2^{(N)}(y), d_{13}^{(N)}(x, z)\} - \{L_3^{(N)}(z) \overset{\otimes}{,} d_{12}^{(N)}(x, y)\} +$$
$$[c^{(N)}(x, y, z) \overset{\otimes}{,} L_2^{(N)}(y) - L_3^{(N)}(z)] = 0 \tag{3.34}$$

and the two other are obtained from (3.34) by cyclic permutations. Then there exists a solution of the equations (3.34) describing the dynamics of the hierarchy of completely

integrable systems in the $2(n+1)$ dimensional phase space $(p_1, q_1, \ldots, p_{n+1}, q_{n+1})$ given by the formulae

$$
\begin{aligned}
d_{ij}^{(N)}(x,y) &= \frac{P_{ij}}{x-y} + \frac{Q_N(x) - Q_N(y)}{x-y} S_{ij}, & &(3.35)\\
d_{ji}^{(N)}(x,y) &= -\frac{P_{ij}}{x-y} + \frac{Q_N(x) - Q_N(y)}{x-y} S_{ij}, & i < j, \quad i,j = 1,2,3,&\\
c^{(N)}(x,y,z) &= \frac{\partial}{\partial q_{n+1}} \frac{Q_N(x)(y-z) + Q_N(y)(z-x) + Q_N(z)(x-y)}{(x-y)(y-z)(z-x)} s&
\end{aligned}
$$

with

$$
Q_N(z) = z^{N-2} - \frac{1}{2} \sum_{k=0}^{N-3} \frac{\partial \mathcal{U}_{N-k-1}}{\partial q_{n+1}} z^k \tag{3.36}
$$

and the potentials $\mathcal{U}_N$ defined as (2.5). The associated Lax representation has the form (1.2) with the matrix elements given by (2.10)–(2.12).

3.3 Embedding into a Larger System

In conclusion we point out another direction in the algebraic interpretation of the hierarchy of dynamical systems considered. The hierarchy can be embedded into a system with more degrees of freedom, which admits the standard linear r-matrix algebra ($s = 0$),

$$
\{L_1^{(N)}(x) \overset{\otimes}{,} L_2^{(N)}(y)\} = [r(x-y), L_1^{(N)}(x) + L_2^{(N)}(y)]. \tag{3.37}
$$

To describe this embedding we introduce the following ansatz for the functions $U(z)$, $V(z)$, $W(z)$

$$
U(z) = \sum_{i=0}^{N-2} U_i z^i - \sum_{i=1}^{n} \frac{q_i^2}{z + A_i}, \tag{3.38}
$$

$$
V(z) = \sum_{i=0}^{N-2} V_i z^i + \sum_{i=1}^{n} \frac{q_i p_i}{z + A_i}, \tag{3.39}
$$

$$
W_N(z) = 4z^{N-1} + \sum_{i=0}^{N-2} W_i^{(N)} z^i + \sum_{i=1}^{n} \frac{p_i^2}{z + A_i}, \tag{3.40}
$$

where the coefficients $U_i, V_i, W_i^{(N)}$ depend on the variables q_{n+1}, p_{n+1} and in addition $N - 2$ new canonically conjugated variables $\mathcal{Q}_j, \mathcal{P}_j$, $j = 1, \ldots N - 2$. The substitution of the ansatz (3.38)–(3.40) into (3.37) leads to the Lie algebra

$$
\begin{aligned}
\{U_i, U_j\} &= \{V_i, V_j\} = \{W_i^{(N)}, W_j^{(N)}\} = 0, \quad i,j = 0, \ldots, N-2,\\
\{U_l, V_k\} &= -2U_{l+k+1}\theta(k+l-N-1),\\
\{V_l, W_k^{(N)}\} &= 2W_{l+k+1}\theta(k+l-N-1),\\
\{W_l^{(N)}, U_k\} &= 4U_{l+k+1}\theta(k+l-N-1), \quad k,l = 0, \ldots, N-2,\\
\{V_i, W_{N-i-2}^{(N)}\} &= 8, \quad i = 0, \ldots, N-2, \tag{3.41}
\end{aligned}
$$

where $\theta(n) = 1$ if $n > 0$ and $\theta(n) = 0$ if $n \leq 0$. The representations of this algebra give the algebraic description of the hierarchy of integrable systems considered.

Let us discuss, as an example, the case $N = 3$ – many-particle Hénon-Heiles system. We have only the coefficients $U_0, U_1, V_0, V_1, W_0^{(3)}, W_1^{(3)}$ in the ansatz (3.38)–(3.40) and the Lie algebra for them is

$$
\begin{aligned}
\{U_0, U_1\} &= \{V_0, V_1\} = \{W_0^{(3)}, W_1^{(3)}\} = 0, \\
\{U_1, V_1\} &= \{U_1, W_1^{(3)}\} = \{V_1, W_1^{(3)}\} = 0, \\
\{V_0, U_0\} &= -2U_1, \\
\{W_0^{(3)}, U_0\} &= 4V_1, \\
\{V_0, W_0^{(3)}\} &= 2W_1, \\
\{V_1, W_0^{(3)}\} &= \{V_0, W_1^{(3)}\} = 8.
\end{aligned}
\tag{3.42}
$$

The following representation for the algebra (3.42) can be found

$$
\begin{aligned}
U_0 &= 4B - 4q_{n+1}, \; U_1 = 4, \\
V_0 &= 2B\mathcal{P}_1 + 2p_{n+1} - 2\mathcal{P}_1 q_{n+1}, \; V_1 = 2\mathcal{P}_1, \\
W_0^{(3)} &= 4q_{n+1}^2 + \mathcal{Q}_1 - 2\mathcal{P}_1 p_{n+1} + \mathcal{P}_1^2 q_{n+1}, \\
W_1^{(3)} &= 4B + 4q_{n+1} - \mathcal{P}_1^2.
\end{aligned}
\tag{3.43}
$$

The corresponding enlarged dynamical system admits the integrals of motion $\mathcal{I}_3, \mathcal{H}_3, \mathcal{F}_3^i, \quad i = 1, \ldots, n$ given by the formulae

$$
\mathcal{I}_3 = 16\mathcal{Q}_1 - 4\sum_{m=1}^{n} q_m^2 + 4B\mathcal{P}_1^2,
\tag{3.44}
$$

$$
\mathcal{H}_3 = H_3 + \frac{1}{8}\left[\mathcal{I}_3(B - q_{n+1}) + \mathcal{P}_1\left(4\sum_{i=1}^{n} P_i q_i + \mathcal{P}_1 \sum_{i=1}^{n} q_i^2\right)\right],
\tag{3.45}
$$

$$
\begin{aligned}
\mathcal{F}_3^{(i)} = &F_3^{(i)} - \frac{1}{4}\mathcal{I}_3 \sum_{i=1}^{n} q_i + \mathcal{P}_1^2\left(B \sum_{i=1}^{n} q_i^2 - \frac{1}{8}\sum_{i=1}^{n} A_i q_i^2 - \frac{1}{8} q_{n+1} \sum_{i=1}^{n} q_i^2\right) + \\
&\frac{1}{2}\mathcal{P}_1\left((B - q_{n+1}) \sum_{i=1}^{n} p_i q_i - \sum_{i=1}^{n} A_i p_i q_i + \frac{1}{4} p_{n+1} \sum_{i=1}^{n} q_i^2\right)
\end{aligned}
\tag{3.46}
$$

where in (3.45), (3.46) H_3 and $F_3^{(i)}$ are the integrals of motions of the many-particle Hénon-Heiles system calculated by the formula (2.33). We can see, that at $\mathcal{P}_1 = 0, \mathcal{I}_3 = 0$ the system is reduced to the many-particle Henon–Heiles system.

The question of the explicit description of the other members of the hierarchy within this approach will be given elsewhere.

4   Separation of Variables

The separation of variables (c.f. [10, 11]) is understood in the context of the given hierarchy of Hamiltonian system as the construction of $n + 1$ pairs of canonical variables

$\pi_i, \mu_i, i = 1, \ldots, n+1,$

$$\{\mu_i, \mu_k\} = \{\pi_i, \pi_k\} = 0, \quad \{\pi_i, \mu_k\} = \delta_{ik} \tag{4.1}$$

and $n+1$ functions Φ_j such that

$$\Phi_j \left(\mu_j, \pi_j, H_N, F_N^{(1)}, \ldots, F_N^{(n)} \right) = 0, \quad j = 1, 2, \ldots, n+1, \tag{4.2}$$

where $H_N, F_N^{(i)}$ are the integrals of motion in involution. The equations (4.2) are the separation equations. The considered integrable system admits a Lax representation in the form of (2×2) matrices (1.2) and we will introduce the separation variables π_i, μ_i as

$$\pi_i = V(\mu_i), \quad U(\mu_i) = 0, \quad i = 1, \ldots, n+1. \tag{4.3}$$

Below we write these formulae explicitly for our system.

4.1 Parabolic Coordinates

The set of zeros $\mu_j, j = 1, \ldots, n+1$ of the function $U(z)$ in the Lax representation (1.2) defines the parabolic coordinates given by the formulae [18, 11]

$$
\begin{aligned}
q_{n+1} &= \sum_{i=1}^{n} A_i + B + \sum_{i=1}^{n+1} \mu_i, \\
q_m^2 &= -4 \frac{\prod_{j=1}^{n+1}(\mu_j + A_m)}{\prod_{k \neq m}(A_m - A_k)}, \quad m = 1, \ldots, n, \quad \text{if} \quad n > 1
\end{aligned}
\tag{4.4}
$$

and

$$q_1^2 = -4(\mu_1 + A)(\mu_2 + A), \quad A_1 = A, \quad \text{if} \quad n = 1. \tag{4.5}$$

Let us denote by π_m,

$$\pi_m = V(\mu_i) = \dot{\mu}_m \prod_{\substack{i \neq m \\ i=1,\ldots,n+1}} \frac{\mu_m - \mu_i}{\mu_m + A_i}, \quad m = 1, \ldots, n+1. \tag{4.6}$$

Proposition 5 *The coordinates μ_i, π_i given by (4.4),(4.6) are canonically conjugated.*

P r o o f Let us list the commutation relations betveen $U(z)$ and $V(z)$,

$$
\begin{aligned}
\{U(x), U(y)\} &= \{V(x), V(y)\} = 0, \tag{4.7} \\
\{V(x), U(y)\} &= \frac{2}{y - x}(U(x) - U(y)). \tag{4.8}
\end{aligned}
$$

The equalities $\{\mu_i, \mu_j\} = 0, i, j = 1, \ldots, n+1$ follows from (4.7). To derive the equality $\{\mu_i, \pi_j\} = \delta_{ij}, i, j = 1, \ldots, n+1$ we substitute $x = \mu_j$ in (4.8), obtaining thus

$$\{\pi_j, U(y)\} = -\frac{2}{y - \mu_j} U(y)), \tag{4.9}$$

which together with the equation

$$0 = \{\pi_j, U(\mu_i)\} = \{\pi_j, U(y)\} \mid_{y=\mu_i} +U'(\mu_i)\{\pi_j, \mu_i\},$$

gives

$$\{\pi_j, \mu_i\} = -\frac{1}{U'(\mu_i)}\{\pi_j, U(y)\} \mid_{y=\mu_i} = \delta_{ij}.$$

Equalities $\{\pi_i, \pi_j\} = 0, i, j = 1, \ldots, n+1$ can be verified by the similar way:

$$-\{\pi_i, \pi_j\} = \{V(\mu_i), V(\mu_j)\}\{V(\mu_j), V(y)\} \mid_{y=\mu_i} +V'(\mu_i)\{\mu_i, V(\mu_j)\} \times$$
$$V'(\mu_j)\{V(\mu_j), \mu_j\} \mid_{y=\mu_i} +V'(\mu_i)\{\mu_i, V(\mu_j)\} = 0.$$

The separation of variable equations have the form

$$\pi_i^2 = w^2(\mu_i), \quad i = 1, \ldots, n+1, \tag{4.10}$$

where the function $w^2(z)$ is given by (2.32). Our use below of the separation equations is two-fold – to integrate the equations of motion in terms of theta functions and to quantize the system.

4.2 Theta Functional Integration of the Many Particle Hénon–Heiles System

The curve (2.32) has genus $g = n + \left[\frac{N-1}{2}\right]$. So only in the cases $N = 3, 4$ does the number of degrees of freedom coincide with the genus. We consider here only the case $N = 3$ (the Many-Particle Hénon-Heiles System) for which the curve (2.32) has a branching point at infinity. With this in mind we reduce (4.10) to the Jacobi inversion problem

$$\sum_{i=1}^{n+1} \int_{\mu_0}^{\mu_i} \frac{\mu^k d\mu}{y(\mu)} = C_k, \quad k = 0, \ldots, n-1,$$
$$\sum_{i=1}^{n+1} \int_{\mu_0}^{\mu_i} \frac{\mu^n d\mu}{y(\mu)} = it + C_n, \tag{4.11}$$

where $C_{n+1} = (y, z)$ is a nonsingular hyperelliptic curve of genus $n+1$ given by the formula

$$y^2 = w^2 \prod_{m=1}^{n+1} (z + A_m) = \prod_{k=1}^{2n+3} (z - z(Q_k)), \quad Q_i \neq Q_j \tag{4.12}$$

with the function w^2 given in (2.32) and $Q_1, \ldots, Q_{2n+3}, Q_{2n+4}, z(Q_{2n+4}) = \infty$ being the branching points.

Let $(\mathbf{a}, \mathbf{b}) = (a_1, \ldots, a_{n+1}; b_1, \ldots, b_{n+1})$ be a canonical base of the first homology group $H_1(C_{n+1}, \mathbf{Z})$ with the intersection matrix,

$$\mathbf{I}_{n+1} = \begin{pmatrix} \mathbf{0}_{n+1} & \mathbf{1}_{n+1} \\ -\mathbf{1}_{n+1} & \mathbf{0}_{n+1} \end{pmatrix}$$

and let $\mathbf{v} = (v_1, \ldots, v_{n+1})$,

$$v_j = \sum_{k=1}^{n+1} c_{jk} w_{g-k+1}, \qquad w_k = \frac{z^{k-1}dz}{y}, \qquad j, k = 1, \ldots, n+1 \qquad (4.13)$$

be the basis of holomorphic differentials normalized in such a way, that the period matrix of $v_{n+1}, \ldots, v_{n+1}$ with respect to $(\mathbf{a}, \mathbf{b})$ has the form $\Pi^T = (\mathbf{1}_{n+1}; \tau)$, where $\mathbf{1}_{n+1}$ is the $(n+1) \times (n+1)$ unit matrix, and the $(n+1) \times (n+1)$ matrix τ belongs to the Siegel upper half–space of the degree $n+1$, $\mathcal{S}_{n+1} = \{\tau \mid \tau = \tau^T, \operatorname{Im}\tau > 0\}$. Let $\mathbf{D} = (\mathcal{A}, \mathcal{B})$ be the divisor, where $\mathcal{A}$ and $\mathcal{B}$ are positive divisors of degree g; let us denote the Jacobian variety of the curve C_{n+1} by $J(C_{n+1}) = \mathbf{C}^{n+1}/(1_{n+1}, \tau)$ and the mapping (*Abel map*) which establishes the correspondence between the point $\mathbf{e} = \left(\int_{\mathcal{A}}^{\mathcal{B}} \mathbf{v}\right) \in J(C_{n+1})$ and divisors $\mathcal{D} \in C_{n+1}$ by $\mathcal{D} \Rightarrow J(C_{n+1})$. We write $\mathbf{e} = \frac{1}{2}(\varepsilon', \varepsilon'') \begin{bmatrix} \mathbf{1}_{n+1} \\ \tau \end{bmatrix}$, $(\varepsilon', \varepsilon'') \in \mathbf{C}^{n+1}$, where $[\varepsilon] = \begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix}$ is the characteristic of the point. If $\varepsilon_i', \varepsilon_j''$ is 0 or 1, the characteristics $[\varepsilon]$ are the characteristics of half-periods.

We identify the branching points Q_j, $j = 1, \ldots, 2n+4$ with the characteristic of half-periods $\mathbf{e}_j \in J(C_{n+1})$ and fix for definiteness the basis of homologies $(\mathbf{a}, \mathbf{b})$ as follows

$$\mathbf{e}_1 = \begin{bmatrix} 0\ldots0 \\ 0\ldots0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0\,0\ldots0 \\ 1\,0\ldots0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 1\,1\,0\ldots0 \\ 1\,0\,0\ldots0 \end{bmatrix}, \ldots,$$

$$\mathbf{e}_{2k+1} = \begin{bmatrix} 0\,0\ldots0\,1\,0\ldots0 \\ 1\,1\ldots1\,0\,0\ldots0 \end{bmatrix}, \quad \mathbf{e}_{2k+2} = \begin{bmatrix} 0\,0\ldots0\,1\,0\ldots0 \\ 1\,1\ldots1\,1\,0\ldots0 \end{bmatrix}, \ldots,$$

$$\mathbf{e}_{2n+3} = \begin{bmatrix} 1\,0\ldots0 \\ 1\,1\ldots1 \end{bmatrix}, \quad \mathbf{e}_{2n+4} = \begin{bmatrix} 1\,0\ldots0 \\ 0\,0\ldots0 \end{bmatrix}. \qquad (4.14)$$

It is naturally to chose the **a**-cycles to be homological to the trajectories of μ variables under the evolution of the system. The vector of Riemann constants $\mathbf{K}$ in the basis (4.14) is $\mathbf{K} = \begin{bmatrix} n+1\,1\ldots1 \\ n+1\,n\ldots1 \end{bmatrix}$.

The Riemann theta function $\Theta[\varepsilon](\mathbf{z}|\tau)$ with the characteristic $[\varepsilon] = \begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix}$ is determined on $\mathbf{C}^{n+1} \times \mathcal{S}_{n+1}$ as

$$\Theta[\varepsilon](\mathbf{z}|\tau) = \sum_{\mathbf{m} \in \mathbf{Z}^{n+1}} \exp \pi\sqrt{-1}\{\langle (\mathbf{m} + \frac{\varepsilon'}{2})\tau, (\mathbf{m} + \frac{\varepsilon'}{2})\rangle +$$

$$2\langle (\mathbf{m} + \frac{\varepsilon'}{2}), \mathbf{z} + \frac{\varepsilon''}{2}\rangle\}, \qquad (4.15)$$

where $\langle \cdot, \cdot \rangle$ means the Euclidean scalar product. Let us consider the curve with the ordering of the branching points

$$Q_{2k} = A_k, \quad k = 1, \ldots, n. \qquad (4.16)$$

Using known hyperelliptic theta formulae [21],

$$\prod_{j=1}^{n+1} (\mu_j + A_k) = h_k \frac{\Theta(\int_{Q_0}^{Q_k} \mathbf{v} + (\int_{Q_0}^{\mu_1} + \ldots + \int_{Q_0}^{\mu_{n+1}})\mathbf{v} + \mathbf{K}; \tau)}{\Theta(\int_{Q_0}^{Q_{2n+4}} \mathbf{v} + (\int_{Q_0}^{\mu_1} + \ldots + \int_{Q_0}^{\mu_{n+1}})\mathbf{v} + \mathbf{K}; \tau)}, \qquad (4.17)$$

where $k = 1, \ldots, n$ and h_k are the constants we obtain

$$q_k^2(t) = q_k^2(0) \frac{\Theta^2[\mathbf{e}_{2n+4} + \mathbf{K}](\omega_0; \tau)\Theta^2[\mathbf{e}_{2k+2} + \mathbf{K}](\omega t + \omega_0; \tau)}{\Theta^2[\mathbf{e}_{2k+2} + \mathbf{K}](\omega_0; \tau)\Theta^2[\mathbf{e}_{2n+4} + \mathbf{K}](\omega t + \omega_0; \tau)},$$

$$q_{n+1}(t) = B + \sum_{j=1}^{n} A_j + \sum_{i=1}^{n+1} \oint_{a_i} z v_i(z) -$$

$$\frac{\partial^2}{\partial t^2} \ln\Theta[\mathbf{e}_{2n+4} + \mathbf{K}](\omega t + \omega_0; \tau), \tag{4.18}$$

where the "winding vector" ω is expressed in terms of $\mathbf{b}$-periods of the differential of the second kind Ω with zero $\mathbf{a}$-periods and a unique pole at infinity which is expanded as $\Omega(z) = d\zeta/\zeta^2 + o(1)d\zeta$, $z = 1/\zeta^2 = \infty$ as

$$\omega = \frac{1}{16\pi i} \left(\oint_{b_1} \Omega, \ldots, \oint_{b_{n+1}} \Omega \right)$$

and $\omega_0 \in \mathbf{C}^{g+1}$ is a constant.

The set of angles $\theta_i = \omega_i t + \omega_{i0}$, $i = 1, \ldots, n+1$ and the $n+1$ $\mathbf{a}$-periods of Abelian differentials

$$J_k = \oint_{a_k} \pi_k d\mu_k = \oint_{a_k} w\,dz, \quad k = 1, \ldots, n+1, \tag{4.19}$$

constitute the "action-angle" variables J_k, θ_k for the system. The standard equality

$$H_3 = \sum_{i=1}^{n+1} \omega_i J_i \tag{4.20}$$

follows from the Riemann bilinear identity.

The theta functional formulae for the evolution of the system can also be given for the case $N = 4$, where the curve (2.32) has no branching points at infinity, but its genus coincides with the number of degrees of freedom (see, e.g. [22]). The algebro-geometric integration of the systems corresponding to the higher members of hierarchy ($N > 4$) requires special consideration.

4.3   Quantization

The separation of variables has a direct quantum counterpart [23, 12]. To pass to quantum mechanics we change the variables π_i, μ_i to operators and the Poisson brackets (4.1) to the commutators

$$[\mu_i, \mu_k] = [\pi_i, \pi_k] = 0, \qquad [\pi_i, \mu_k] = -i\delta_{ik}. \tag{4.21}$$

Suppose that the common spectrum of μ_i is simple and the momenta π_i are realized as the differentials $\pi_j = -i\frac{\partial}{\partial\mu_j}$. The separation equations (4.10) become the operator equations, where the noncommuting operators are assumed to be ordered precisely in the order as those listed in (4.2), that is $\pi_i, \mu_i, H_N, F_N^{(1)}, \ldots, F_N^{(n)}$. Let $\Psi(\mu_1, \ldots, \mu_{n+1})$ be a common eigenfunction of the quantum integrals of motion:

$$H_N\Psi = \lambda_{n+1}\Psi, \qquad F_N^{(i)}\Psi = \lambda_i\Psi, \quad i = 1, \ldots, n. \tag{4.22}$$

Then the operator separation equations lead to the set of differential equations

$$\Phi_j(-i\frac{\partial}{\partial\mu_i}, \mu_i, H_N, F_N^{(1)}, \ldots, F_N^{(n)})\Psi(\mu_1, \ldots, \mu_{n+1}) = 0, \quad j = 1, \ldots, n+1, \tag{4.23}$$

which allows the separation of variables

$$\Psi(\mu_1, \ldots, \mu_{n+1}) = \prod_{j=1}^{n+1} \psi_j(\mu_j). \qquad (4.24)$$

The original multidimensional spectral problem is therefore reduced to the set of one-dimensional multiparametric spectral problems which have the following form in the context of the problem under consideration

$$\left(\frac{d^2}{dx^2} + 16x^{N-2}(x+B)^2 + 8\lambda_{n+1} + \sum_{i=2}^{n} \frac{\lambda_i}{x+A_i} \right) \psi_j(x; \lambda_1, \ldots, \lambda_{n+1}) = 0 \qquad (4.25)$$

with the spectral parameters $\lambda_1, \ldots, \lambda_{n+1}$. The problem (4.25) must be solved on the $n+1$ different intervals – "permitted zones": $x \in [Q_1, Q_2], \ldots, [Q_{2n+1}, Q_{2n+2}]$ for the variable x.

Acknowledgements

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